1. Preface
2. Foundations
    1. Introduction
    2. Introduction to Whole Numbers
    3. Use the Language of Algebra
    4. Add and Subtract Integers
    5. Multiply and Divide Integers
    6. Visualize Fractions
    7. Add and Subtract Fractions
    8. Decimals
    9. Properties of Real Numbers
    10. The Real Numbers
    11. Converting Units
3. Solving Linear Equations and Inequalities
    1. Introduction
    2. Solve Equations Using the Subtraction and Addition Properties of Equality
    3. Solve Equations using the Division and Multiplication Properties of Equality
    4. Solve Equations with Variables and Constants on Both Sides
    5. Use a General Strategy to Solve Linear Equations
    6. Solve Equations with Fractions or Decimals
    7. Solve a Formula for a Specific Variable
    8. Solve Linear Inequalities
4. Math Models
    1. Introduction
    2. Use a Problem-Solving Strategy
    3. Solve Percent Applications
    4. Solve Geometry Applications: Triangles, Rectangles, and the Pythagorean Theorem

5. Graphs of Linear Equations
    1. Introduction
    2. Use the Rectangular Coordinate System
    3. Graph Linear Equations in Two Variables
    4. Graph with Intercepts
    5. Understand Slope of a Line
    6. Use the Slope–Intercept Form of an Equation of a Line
    7. Find the Equation of a Line
6. Systems of Linear Equations
    1. Introduction
    2. Solve Systems of Equations by Graphing
    3. Solve Systems of Equations by Substitution
    4. Solve Systems of Equations by Elimination
    5. Solve Applications with Systems of Equations
    6. Solve Mixture Applications with Systems of Equations
7. Polynomials
    1. Introduction
    2. Add and Subtract Polynomials
    3. Use Multiplication Properties of Exponents
    4. Multiply Polynomials
    5. Special Products
    6. Divide Monomials
    7. Divide Polynomials
    8. Integer Exponents and Scientific Notation
8. Factoring
    1. Introduction
    2. Greatest Common Factor and Factor by Grouping
    3. Factor Quadratic Trinomials with Leading Coefficient 1
    4. Factor Quadratic Trinomials with Leading Coefficient Other than 1
    5. Factor Special Products
    6. General Strategy for Factoring Polynomials

    7. Quadratic Equations
9. Rational Expressions and Equations
    1. Introduction
    2. Simplify Rational Expressions
    3. Multiply and Divide Rational Expressions
    4. Add and Subtract Rational Expressions with a Common Denominator
    5. Add and Subtract Rational Expressions with Unlike Denominators
    6. Simplify Complex Rational Expressions
    7. Solve Rational Equations
    8. Solve Proportion and Similar Figure Applications
    9. Solve Uniform Motion and Work Applications
    10. Use Direct and Inverse Variation
10. Roots and Radicals
    1. Introduction
    2. Simplify and Use Square Roots
    3. Simplify Square Roots
    4. Add and Subtract Square Roots
    5. Multiply Square Roots
    6. Divide Square Roots
    7. Solve Equations with Square Roots
    8. Higher Roots
    9. Rational Exponents
11. Quadratic Equations
    1. Introduction
    2. Solve Quadratic Equations Using the Square Root Property
    3. Solve Quadratic Equations by Completing the Square
    4. Solve Quadratic Equations Using the Quadratic Formula
    5. Solve Applications Modeled by Quadratic Equations
    6. Graphing Quadratic Equations

12. Exponential and Logarithmic Functions
    1. Introduction
    2. Evaluate and Graph Exponential Functions
    3. Logarithmic Functions
    4. Using the Change-of-Base Formula to Evaluate Logarithms

Preface

Welcome to *Elementary Algebra*, an OpenStax resource. This textbook was written to increase student access to high-quality learning materials, maintaining highest standards of academic rigor at little to no cost.

About OpenStax

OpenStax is a nonprofit based at Rice University, and it’s our mission to improve student access to education. Our first openly licensed college textbook was published in 2012, and our library has since scaled to over 25 books for college and AP courses used by hundreds of thousands of students. Our adaptive learning technology, designed to improve learning outcomes through personalized educational paths, is being piloted in college courses throughout the country. Through our partnerships with philanthropic foundations and our alliance with other educational resource organizations, OpenStax is breaking down the most common barriers to learning and empowering students and instructors to succeed.

About OpenStax Resources

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Instructors also have the option of creating a customized version of their OpenStax book. The custom version can be made available to students in low-cost print or digital form through their campus bookstore. Visit your book page on openstax.org for more information.

Errata

All OpenStax textbooks undergo a rigorous review process. However, like any professional-grade textbook, errors sometimes occur. Since our books are web based, we can make updates periodically when deemed pedagogically necessary. If you have a correction to suggest, submit it through the link on your book page on openstax.org. Subject matter experts review all errata suggestions. OpenStax is committed to remaining transparent about all updates, so you will also find a list of past errata changes on your book page on openstax.org.

Format

You can access this textbook for free in web view or PDF through openstax.org, and for a low cost in print.

About *Elementary Algebra*

Elementary Algebra is designed to meet the scope and sequence requirements of a one-semester elementary algebra course. The book’s organization makes it easy to adapt to a variety of course syllabi. The text expands on the fundamental concepts of algebra while addressing the needs of students with diverse backgrounds and learning styles. Each topic builds upon previously developed material to demonstrate the cohesiveness and structure of mathematics.

Coverage and Scope

Elementary Algebra follows a nontraditional approach in its presentation of content. Building on the content in *Prealgebra*, the material is presented as a sequence of small steps so that students gain confidence in their ability to succeed in the course. The order of topics was carefully planned to emphasize the logical progression through the course and to facilitate a thorough understanding of each concept. As new ideas are presented, they are explicitly related to previous topics.

Chapter 1: Foundations
Chapter 1 reviews arithmetic operations with whole numbers, integers, fractions, and decimals, to give the student a solid base that will support their study of algebra.

Chapter 2: Solving Linear Equations and Inequalities
In Chapter 2, students learn to verify a solution of an equation, solve equations using the Subtraction and Addition Properties of Equality, solve equations using the Multiplication and Division Properties of Equality, solve equations with variables and constants on both sides, use a general strategy to solve linear equations, solve equations with fractions or decimals, solve a formula for a specific variable, and solve linear inequalities.

Chapter 3: Math Models
Once students have learned the skills needed to solve equations, they apply these skills in Chapter 3 to solve word and number problems.

Chapter 4: Graphs
Chapter 4 covers the rectangular coordinate system, which is the basis for most consumer graphs. Students learn to plot points on a rectangular coordinate system, graph linear equations in two variables, graph with intercepts, understand slope of a line, use the slope-intercept form of an equation of a line, find the equation of a line, and create graphs of linear inequalities.

Chapter 5: Systems of Linear Equations
Chapter 5 covers solving systems of equations by graphing, substitution, and elimination; solving applications with systems of equations, solving mixture applications with systems of equations, and graphing systems of linear inequalities.

Chapter 6: Polynomials
In Chapter 6, students learn how to add and subtract polynomials, use

multiplication properties of exponents, multiply polynomials, use special products, divide monomials and polynomials, and understand integer exponents and scientific notation.

Chapter 7: Factoring

In Chapter 7, students explore the process of factoring expressions and see how factoring is used to solve certain types of equations.

Chapter 8: Rational Expressions and Equations

In Chapter 8, students work with rational expressions, solve rational equations, and use them to solve problems in a variety of applications.

Chapter 9: Roots and Radical

In Chapter 9, students are introduced to and learn to apply the properties of square roots, and extend these concepts to higher order roots and rational exponents.

Chapter 10: Quadratic Equations

In Chapter 10, students study the properties of quadratic equations, solve and graph them. They also learn how to apply them as models of various situations.

All chapters are broken down into multiple sections, the titles of which can be viewed in the **Table of Contents**.

Key Features and Boxes

Examples Each learning objective is supported by one or more worked examples that demonstrate the problem-solving approaches that students must master. Typically, we include multiple Examples for each learning objective to model different approaches to the same type of problem, or to introduce similar problems of increasing complexity.

All Examples follow a simple two- or three-part format. First, we pose a problem or question. Next, we demonstrate the solution, spelling out the steps along the way. Finally (for select Examples), we show students how to check the solution. Most Examples are written in a two-column format, with explanation on the left and math on the right to mimic the way that instructors “talk through” examples as they write on the board in class.

Be Prepared! Each section, beginning with Section 2.1, starts with a few "Be Prepared!" exercises so that students can determine if they have mastered the prerequisite skills for the section. Reference is made to specific Examples from previous sections so students who need further review can easily find explanations. Answers to these exercises can be found in the supplemental resources that accompany this title.

Try It

The Try It feature includes a pair of exercises that immediately follow an Example, providing the student with an immediate opportunity to solve a similar problem. In the Web View version of the text, students can click an Answer link directly below the question to check their understanding. In the PDF, answers to the Try It exercises are located in the Answer Key.

How To

How To feature typically follows the Try It exercises and outlines the series of steps for how to solve the problem in the preceding Example.

Media

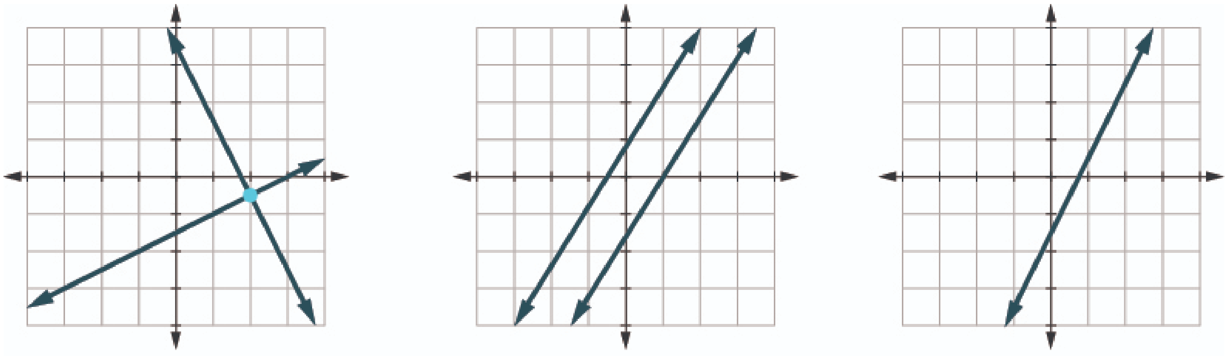
The Media icon appears at the conclusion of each section, just prior to the Self Check. This icon marks a list of links to online video tutorials that reinforce the concepts and skills introduced in the section.

Disclaimer: While we have selected tutorials that closely align to our learning objectives, we did not produce these tutorials, nor were they specifically produced or tailored to accompany *Elementary Algebra*.

Self Check The Self Check includes the learning objectives for the section so that students can self-assess their mastery and make concrete plans to improve.

Art Program

Elementary Algebra contains many figures and illustrations. Art throughout the text adheres to a clear, understated style, drawing the eye to the most important information in each figure while minimizing visual distractions.



Intersecting Parallel Coincident

Section Exercises and Chapter Review

Section Exercises Each section of every chapter concludes with a well-rounded set of exercises that can be assigned as homework or used selectively for guided practice. Exercise sets are named *Practice Makes Perfect* to encourage completion of homework assignments.

- Exercises correlate to the learning objectives. This facilitates assignment of personalized study plans based on individual student needs.
- Exercises are carefully sequenced to promote building of skills.
- Values for constants and coefficients were chosen to practice and reinforce arithmetic facts.
- Even and odd-numbered exercises are paired.

- Exercises parallel and extend the text examples and use the same instructions as the examples to help students easily recognize the connection.
- Applications are drawn from many everyday experiences, as well as those traditionally found in college math texts.
- **Everyday Math** highlights practical situations using the concepts from that particular section
- **Writing Exercises** are included in every exercise set to encourage conceptual understanding, critical thinking, and literacy.

Chapter Review Each chapter concludes with a review of the most important takeaways, as well as additional practice problems that students can use to prepare for exams.

- **Key Terms** provide a formal definition for each bold-faced term in the chapter.
- **Key Concepts** summarize the most important ideas introduced in each section, linking back to the relevant Example(s) in case students need to review.
- **Chapter Review Exercises** include practice problems that recall the most important concepts from each section.
- **Practice Test** includes additional problems assessing the most important learning objectives from the chapter.
- **Answer Key** includes the answers to all Try It exercises and every other exercise from the Section Exercises, Chapter Review Exercises, and Practice Test.

Additional Resources

Student and Instructor Resources

We’ve compiled additional resources for both students and instructors, including Getting Started Guides, manipulative mathematics worksheets, and an answer key to Be Prepared Exercises. Instructor resources require a verified instructor account, which can be requested on your openstax.org log-in. Take advantage of these resources to supplement your OpenStax book.

Partner Resources

OpenStax Partners are our allies in the mission to make high-quality learning materials affordable and accessible to students and instructors everywhere. Their tools integrate seamlessly with our OpenStax titles at a low cost. To access the partner resources for your text, visit your book page on openstax.org.

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Introduction class="introduction"

In order to be structurally sound, the foundation of a building must be carefully constructed.

Just like a building needs a firm foundation to support it, your study of algebra needs to have a firm foundation. To ensure this, we begin this book with a review of arithmetic operations with whole numbers, integers, fractions, and decimals, so that you have a solid base that will support your study of algebra.

Introduction to Whole Numbers
By the end of this section, you will be able to:

- Use place value with whole numbers
- Identify multiples and and apply divisibility tests
- Find prime factorizations and least common multiples

Note:
A more thorough introduction to the topics covered in this section can be found in *Prealgebra* in the chapters **Whole Numbers** and **The Language of Algebra**.

As we begin our study of elementary algebra, we need to refresh some of our skills and vocabulary. This chapter will focus on whole numbers, integers, fractions, decimals, and real numbers. We will also begin our use of algebraic notation and vocabulary.

Use Place Value with Whole Numbers

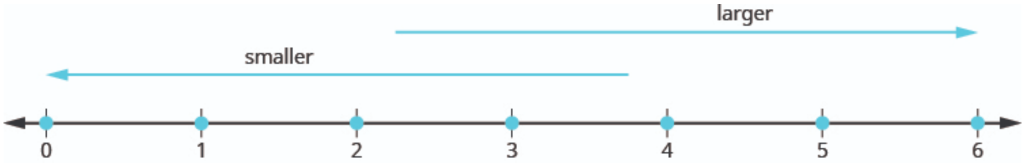
The most basic numbers used in algebra are the numbers we use to count objects in our world: 1, 2, 3, 4, and so on. These are called the **counting numbers**. Counting numbers are also called *natural numbers*. If we add zero to the counting numbers, we get the set of **whole numbers**.

Counting Numbers: 1, 2, 3, …

Whole Numbers: 0, 1, 2, 3, …

The notation "…" is called ellipsis and means "and so on," or that the pattern continues endlessly.

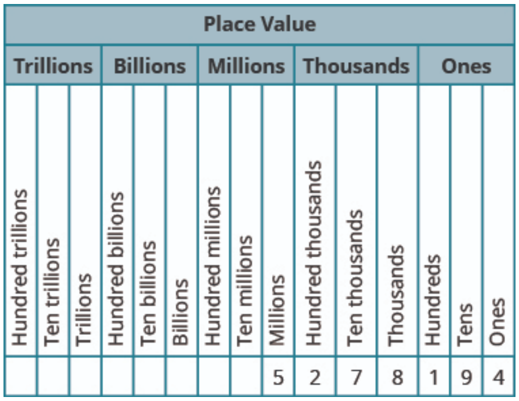
We can visualize counting numbers and whole numbers on a **number line** (see [link]).



The numbers on the number line get larger as they go from left to right, and smaller as they go from right to left. While this number line shows only the whole numbers 0 through 6, the numbers keep going without end.

Note: Doing the Manipulative Mathematics activity "Number Line-Part 1" will help you develop a better understanding of the counting numbers and the whole numbers.

Our number system is called a place value system, because the value of a digit depends on its position in a number. [link] shows the place values. The place values are separated into groups of three, which are called periods. The periods are *ones, thousands, millions, billions, trillions*, and so on. In a written number, commas separate the periods.



Place Value

Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
								5	2	7	8	1	9	4

The number 5,278,194 is shown in the chart. The digit 5 is in the millions place. The digit 2 is in the hundred-thousands place. The digit 7 is in the ten-thousands place. The digit 8 is in the thousands place. The digit 1 is in the hundreds place. The digit 9 is in the tens place. The digit 4 is in the ones place.

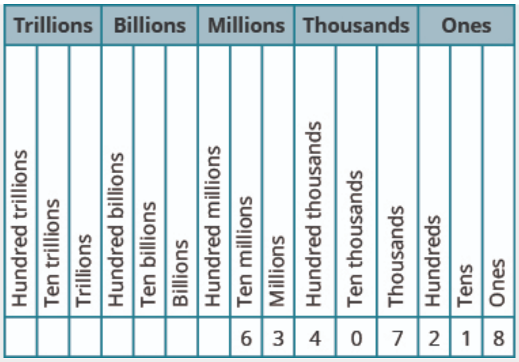
Example:
Exercise:

Problem: In the number 63,407,218, find the place value of each digit:

ⓐ 7
ⓑ 0
ⓒ 1
ⓓ 6
ⓔ 3

Solution:
Solution

Place the number in the place value chart:



Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
							6	3	4	0	7	2	1	8

ⓐ The 7 is in the thousands place.
ⓑ The 0 is in the ten thousands place.
ⓒ The 1 is in the tens place.
ⓓ The 6 is in the ten-millions place.
ⓔ The 3 is in the millions place.

Note:
Exercise:

Problem: For the number 27,493,615, find the place value of each digit:

ⓐ 2 ⓑ 1 ⓒ 4 ⓓ 7 ⓔ 5

Solution:

ⓐ ten millions ⓑ tens ⓒ hundred thousands ⓓ millions ⓔ ones

Note:
Exercise:

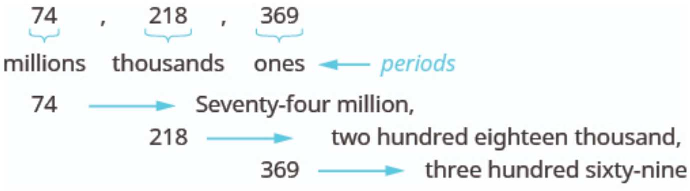
Problem: For the number 519,711,641,328, find the place value of each digit:

ⓐ 9 ⓑ 4 ⓒ 2 ⓓ 6 ⓔ 7

Solution:

ⓐ billions ⓑ ten thousands ⓒ tens ⓓ hundred thousands ⓔ hundred millions

When you write a check, you write out the number in words as well as in digits. To write a number in words, write the number in each period, followed by the name of the period, without the *s* at the end. Start at the left, where the periods have the largest value. The ones period is not named. The commas separate the periods, so wherever there is a comma in the number, put a comma between the words (see [link]). The number 74,218,369 is written as seventy-four million, two hundred eighteen thousand, three hundred sixty-nine.



Note:
Name a Whole Number in Words.

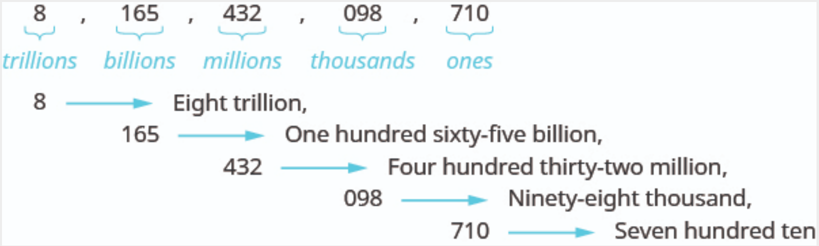
Start at the left and name the number in each period, followed by the period name.
Put commas in the number to separate the periods.
Do not name the ones period.

Example:
Exercise:

Problem: Name the number 8,165,432,098,710 using words.

Solution:
Solution

Name the number in each period, followed by the period name.



Put the commas in to separate the periods.

So, $8,165,432,098,710$ is named as eight trillion, one hundred sixty-five billion, four hundred thirty-two million, ninety-eight thousand, seven hundred ten.

Note:
Exercise:

Problem: Name the number $9,258,137,904,061$ using words.

Solution:

nine trillion, two hundred fifty-eight billion, one hundred thirty-seven million, nine hundred four thousand, sixty-one

Note:
Exercise:

Problem: Name the number $17,864,325,619,004$ using words.

Solution:

seventeen trillion, eight hundred sixty-four billion, three hundred twenty-five million, six hundred nineteen thousand, four

We are now going to reverse the process by writing the digits from the name of the number. To write the number in digits, we first look for the clue words that indicate the periods. It is helpful to draw three blanks for the needed periods and then fill in the blanks with the numbers, separating the periods with commas.

Note:
Write a Whole Number Using Digits.

Identify the words that indicate periods. (Remember, the ones period is never named.)
Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.
Name the number in each period and place the digits in the correct place value position.

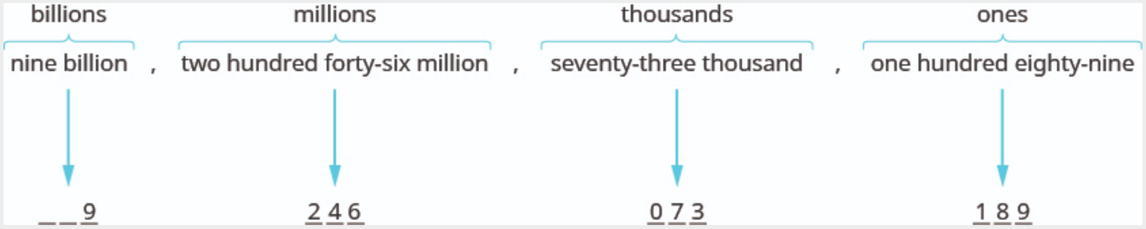
Example:
Exercise:

Problem:

Write *nine billion, two hundred forty-six million, seventy-three thousand, one hundred eighty-nine* as a whole number using digits.

Solution:
Solution

Identify the words that indicate periods.
Except for the first period, all other periods must have three places. Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.
Then write the digits in each period.



The number is 9,246,073,189.

Note:
Exercise:

Problem:

Write the number two billion, four hundred sixty-six million, seven hundred fourteen thousand, fifty-one as a whole number using digits.

Solution:

2,466,714,051

Note:
Exercise:

Problem:

Write the number eleven billion, nine hundred twenty-one million, eight hundred thirty thousand, one hundred six as a whole number using digits.

Solution:

11,921,830,106

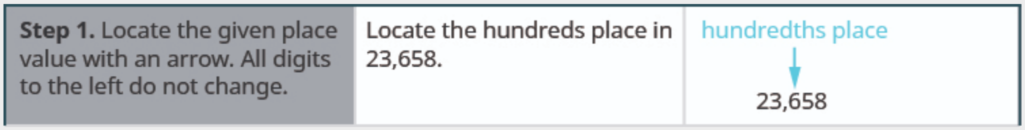
In 2013, the U.S. Census Bureau estimated the population of the state of New York as 19,651,127. We could say the population of New York was approximately 20 million. In many cases, you don't need the exact value; an approximate number is good enough.

The process of approximating a number is called rounding. Numbers are rounded to a specific place value, depending on how much accuracy is needed. Saying that the population of New York is approximately 20 million means that we rounded to the millions place.

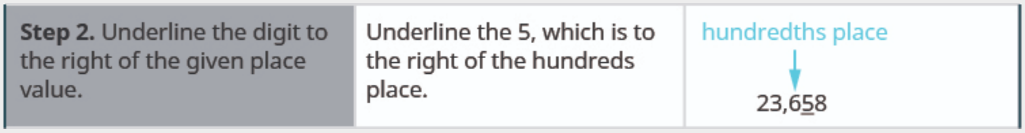
Example:
How to Round Whole Numbers
Exercise:

Problem: Round 23,658 to the nearest hundred.

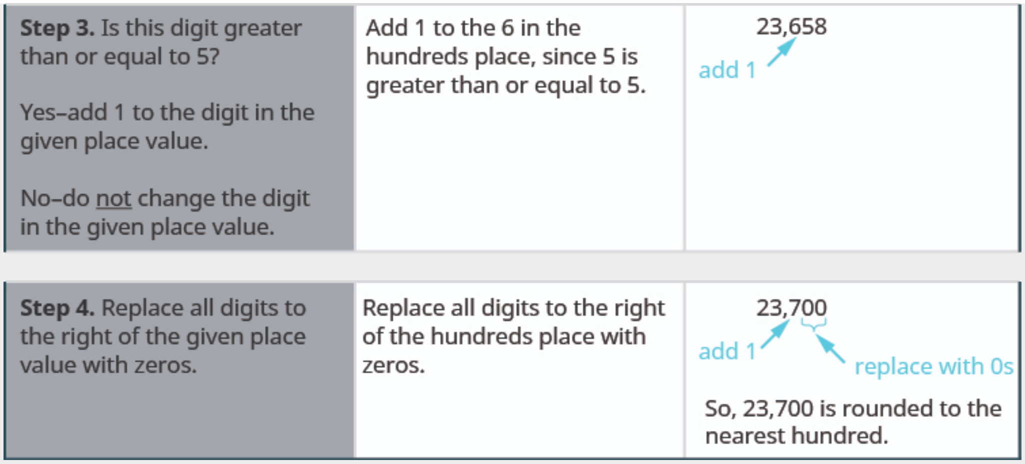
Solution:
Solution



Step 1. Locate the given place value with an arrow. All digits to the left do not change.	Locate the hundreds place in 23,658.	hundredths place ↓ 23,658



Step 2. Underline the digit to the right of the given place value.	Underline the 5, which is to the right of the hundreds place.	hundredths place ↓ 23,6$\underline{5}$8



Step 3. Is this digit greater than or equal to 5? Yes–add 1 to the digit in the given place value. No–do not change the digit in the given place value.	Add 1 to the 6 in the hundreds place, since 5 is greater than or equal to 5.	23,658 add 1
Step 4. Replace all digits to the right of the given place value with zeros.	Replace all digits to the right of the hundreds place with zeros.	23,700 add 1 replace with 0s So, 23,700 is rounded to the nearest hundred.

Note:
Exercise:

Problem: Round to the nearest hundred: 17,852.

Solution:

17,900

Note:
Exercise:

Problem: Round to the nearest hundred: 468,751.

Solution:

468,800

Note:
Round Whole Numbers.

Locate the given place value and mark it with an arrow. All digits to the left of the arrow do not change.
Underline the digit to the right of the given place value.
Is this digit greater than or equal to 5?
  - Yes–add 1 to the digit in the given place value.
  - No–do not change the digit in the given place value.

Replace all digits to the right of the given place value with zeros.

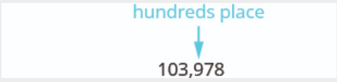
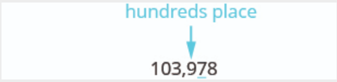
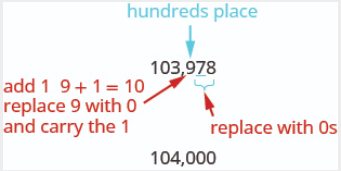
Example:
Exercise:

Problem: Round 103,978 to the nearest:

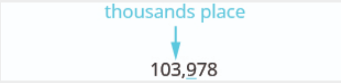
ⓐ hundred
ⓑ thousand
ⓒ ten thousand

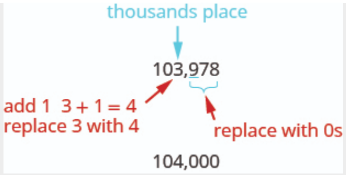
Solution:
Solution

ⓐ

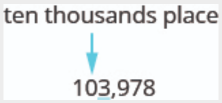
Locate the hundreds place in 103,978.	
Underline the digit to the right of the hundreds place.	
Since 7 is greater than or equal to 5, add 1 to the 9. Replace all digits to the right of the hundreds place with zeros.	 hundreds place 103,978 add 1 9 + 1 = 10 replace 9 with 0 and carry the 1 replace with 0s 104,000
	So, 104,000 is 103,978 rounded to the nearest hundred.

ⓑ

Locate the thousands place and underline the digit to the right of the thousands place.	

Since 9 is greater than or equal to 5, add 1 to the 3. Replace all digits to the right of the hundreds place with zeros.	 thousands place 103,978 add 1 3 + 1 = 4 replace 3 with 4 replace with 0s 104,000
	So, 104,000 is 103,978 rounded to the nearest thousand.

ⓒ

Locate the ten thousands place and underline the digit to the right of the ten thousands place.	
Since 3 is less than 5, we leave the 0 as is, and then replace the digits to the right with zeros.	100,000
	So, 100,000 is 103,978 rounded to the nearest ten thousand.

Note:
Exercise:

Problem: Round 206,981 to the nearest: ⓐ hundred ⓑ thousand ⓒ ten thousand.

Solution:

ⓐ 207,000 ⓑ 207,000 ⓒ 210,000

Note:
Exercise:

Problem: Round 784,951 to the nearest: ⓐ hundred ⓑ thousand ⓒ ten thousand.

Solution:

ⓐ 785,000 ⓑ 785,000 ⓒ 780,000

Identify Multiples and Apply Divisibility Tests

The numbers 2, 4, 6, 8, 10, and 12 are called **multiples** of 2. A multiple of 2 can be written as the product of a counting number and 2.

$$\begin{array}{cccccc} 2, & 4, & 6, & 8, & 10, & 12, \ldots \\ 2 \cdot 1, & 2 \cdot 2, & 2 \cdot 3, & 2 \cdot 4, & 2 \cdot 5, & 2 \cdot 6 \end{array}$$

Similarly, a multiple of 3 would be the product of a counting number and 3.

$$\begin{array}{cccccc} 3, & 6, & 9, & 12, & 15, & 18, \ldots \\ 3 \cdot 1, & 3 \cdot 2, & 3 \cdot 3, & 3 \cdot 4, & 3 \cdot 5, & 3 \cdot 6 \end{array}$$

We could find the multiples of any number by continuing this process.

Note: Doing the Manipulative Mathematics activity “Multiples” will help you develop a better understanding of multiples.

[link] shows the multiples of 2 through 10 for the first 12 counting numbers.

Counting Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Multiples of 2	2	4	6	8	10	12	14	16	18	20	22	24
Multiples of 3	3	6	9	12	15	18	21	24	27	30	33	36
Multiples of 4	4	8	12	16	20	24	28	32	36	40	44	48
Multiples of 5	5	10	15	20	25	30	35	40	45	50	55	60
Multiples of 6	6	12	18	24	30	36	42	48	54	60	66	72
Multiples of 7	7	14	21	28	35	42	49	56	63	70	77	84
Multiples of 8	8	16	24	32	40	48	56	64	72	80	88	96

Counting Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Multiples of 9	9	18	27	36	45	54	63	72	81	90	99	108
Multiples of 10	10	20	30	40	50	60	0	80	90	100	110	120

Note:
Multiple of a Number
A number is a **multiple** of *n* if it is the product of a counting number and *n*.

Another way to say that 15 is a multiple of 3 is to say that 15 is **divisible** by 3. That means that when we divide 3 into 15, we get a counting number. In fact, $15 \div 3$ is 5, so 15 is $5 \cdot 3$.

Note:
Divisible by a Number
If a number *m* is a multiple of *n*, then *m* is **divisible** by *n*.

Look at the multiples of 5 in [link]. They all end in 5 or 0. Numbers with last digit of 5 or 0 are divisible by 5. Looking for other patterns in [link] that shows multiples of the numbers 2 through 9, we can discover the following divisibility tests:

Note:
Divisibility Tests
A number is divisible by:

- 2 if the last digit is 0, 2, 4, 6, or 8.
- 3 if the sum of the digits is divisible by 3.
- 5 if the last digit is 5 or 0.
- 6 if it is divisible by both 2 and 3.
- 10 if it ends with 0.

Example:
Exercise:

Problem: Is 5,625 divisible by 2? By 3? By 5? By 6? By 10?

Solution:
Solution

Is 5,625 divisible by 2?	
Does it end in 0, 2, 4, 6, or 8?	No.
	5,625 is not divisible by 2.
Is 5,625 divisible by 3?	
What is the sum of the digits?	$5+6+2+5=18$
Is the sum divisible by 3?	Yes. 5,625 is divisible by 3.
Is 5,625 divisible by 5 or 10?	
What is the last digit? It is 5.	5,625 is divisible by 5 but not by 10.
Is 5,625 divisible by 6?	
Is it divisible by both 2 and 3?	No, 5,625 is not divisible by 2, so 5,625 is not divisible by 6.

Note:
Exercise:

Problem: Determine whether 4,962 is divisible by 2, by 3, by 5, by 6, and by 10.

Solution:

by 2, 3, and 6

Note:
Exercise:

Problem: Determine whether 3,765 is divisible by 2, by 3, by 5, by 6, and by 10.

Solution:

by 3 and 5

Find Prime Factorizations and Least Common Multiples

In mathematics, there are often several ways to talk about the same ideas. So far, we’ve seen that if *m* is a multiple of *n*, we can say that *m* is divisible by *n*. For example, since 72 is a multiple of 8, we say 72 is divisible by 8. Since 72 is a multiple of 9, we say 72 is divisible by 9. We can express this still another way.

Since $8\cdot 9=72$, we say that 8 and 9 are **factors** of 72. When we write $72=8\cdot 9$, we say we have factored 72.

$$\underbrace{8\cdot 9}_{\text{factors}} = \underbrace{72}_{\text{product}}$$

Other ways to factor 72 are $1\cdot 72, 2\cdot 36, 3\cdot 24, 4\cdot 18,$ and $6\cdot 12$. Seventy-two has many factors: 1, 2, 3, 4, 6, 8, 9, 12, 18, 36, and 72.

Note:
Factors
If $a \cdot b = m$, then a and b are **factors** of m.

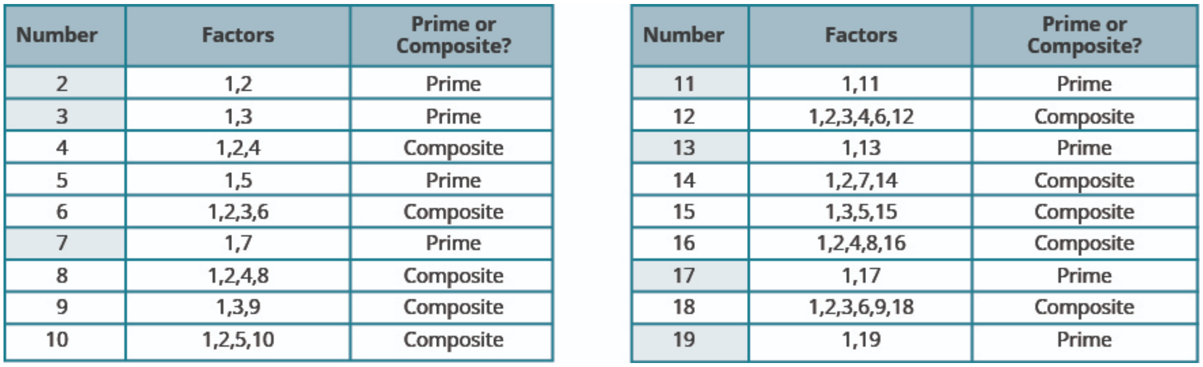
Some numbers, like 72, have many factors. Other numbers have only two factors.

Note: Doing the Manipulative Mathematics activity “Model Multiplication and Factoring” will help you develop a better understanding of multiplication and factoring.

Note:
Prime Number and Composite Number
A **prime number** is a counting number greater than 1, whose only factors are 1 and itself.
A **composite number** is a counting number that is not prime. A composite number has factors other than 1 and itself.

Note: Doing the Manipulative Mathematics activity “Prime Numbers” will help you develop a better understanding of prime numbers.

The counting numbers from 2 to 19 are listed in [link], with their factors. Make sure to agree with the “prime” or “composite” label for each!



Number	Factors	Prime or Composite?
2	1,2	Prime
3	1,3	Prime
4	1,2,4	Composite
5	1,5	Prime
6	1,2,3,6	Composite
7	1,7	Prime
8	1,2,4,8	Composite
9	1,3,9	Composite
10	1,2,5,10	Composite

Number	Factors	Prime or Composite?
11	1,11	Prime
12	1,2,3,4,6,12	Composite
13	1,13	Prime
14	1,2,7,14	Composite
15	1,3,5,15	Composite
16	1,2,4,8,16	Composite
17	1,17	Prime
18	1,2,3,6,9,18	Composite
19	1,19	Prime

The **prime numbers** less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19. Notice that the only even prime number is 2.

A composite number can be written as a unique product of primes. This is called the **prime factorization** of the number. Finding the prime factorization of a composite number will be useful later in this course.

Note:
Prime Factorization
The **prime factorization** of a number is the product of prime numbers that equals the number.

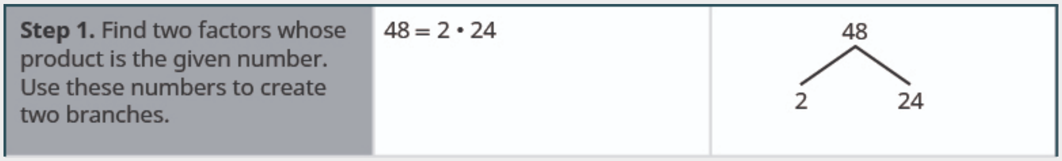
To find the prime factorization of a composite number, find any two factors of the number and use them to create two branches. If a factor is prime, that branch is complete. Circle that prime!

If the factor is not prime, find two factors of the number and continue the process. Once all the branches have circled primes at the end, the factorization is complete. The composite number can now be written as a product of prime numbers.

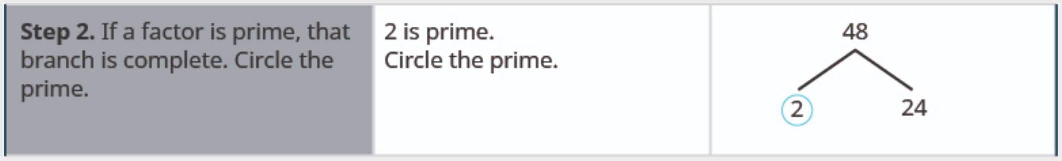
Example:
How to Find the Prime Factorization of a Composite Number
Exercise:

Problem: Factor 48.

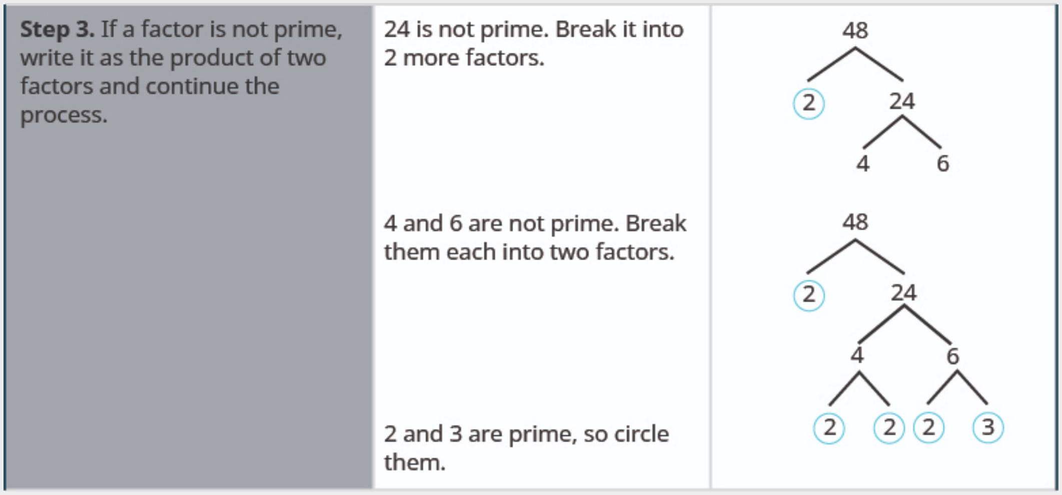
Solution:
Solution



Step 1. Find two factors whose product is the given number. Use these numbers to create two branches.	48 = 2 • 24	

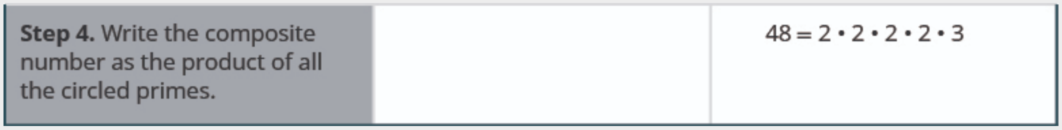


Step 2. If a factor is prime, that branch is complete. Circle the prime.	2 is prime. Circle the prime.	



Step 3. If a factor is not prime, write it as the product of two factors and continue the process.	24 is not prime. Break it into 2 more factors. 4 and 6 are not prime. Break them each into two factors. 2 and 3 are prime, so circle them.	

48
2
24
4
6
48
2
24
4
6
2
2
2
3



Step 4. Write the composite number as the product of all the circled primes.		48 = 2 • 2 • 2 • 2 • 3

We say $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$ is the prime factorization of 48. We generally write the primes in ascending order. Be sure to multiply the factors to verify your answer!

If we first factored 48 in a different way, for example as $6 \cdot 8$, the result would still be the same. Finish the prime factorization and verify this for yourself.

Note:
Exercise:

Problem: Find the prime factorization of 80.

Solution:

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$

Note:
Exercise:

Problem: Find the prime factorization of 60.

Solution:

$2 \cdot 2 \cdot 3 \cdot 5$

Note:
Find the Prime Factorization of a Composite Number.

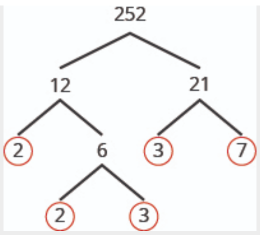
Find two factors whose product is the given number, and use these numbers to create two branches.
If a factor is prime, that branch is complete. Circle the prime, like a bud on the tree.
If a factor is not prime, write it as the product of two factors and continue the process.
Write the composite number as the product of all the circled primes.

Example:
Exercise:

Problem: Find the prime factorization of 252.

Solution:
Solution

Step 1. Find two factors whose product is 252. 12 and 21 are not prime.	

Break 12 and 21 into two more factors. Continue until all primes are factored.	
Step 2. Write 252 as the product of all the circled primes.	$252 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7$

Note:
Exercise:

Problem: Find the prime factorization of 126.

Solution:

$2 \cdot 3 \cdot 3 \cdot 7$

Note:
Exercise:

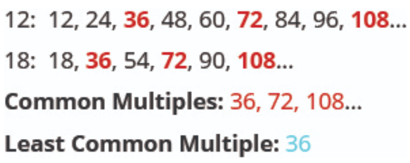
Problem: Find the prime factorization of 294.

Solution:

$2 \cdot 3 \cdot 7 \cdot 7$

One of the reasons we look at multiples and primes is to use these techniques to find the **least common multiple** of two numbers. This will be useful when we add and subtract fractions with different denominators. Two methods are used most often to find the least common multiple and we will look at both of them.

The first method is the Listing Multiples Method. To find the least common multiple of 12 and 18, we list the first few multiples of 12 and 18:



12: 12, 24, **36**, 48, 60, **72**, 84, 96, **108**...

18: 18, **36**, 54, **72**, 90, **108**...

Common Multiples: 36, 72, 108...

Least Common Multiple: 36

Notice that some numbers appear in both lists. They are the **common multiples** of 12 and 18.

We see that the first few common multiples of 12 and 18 are 36, 72, and 108. Since 36 is the smallest of the common multiples, we call it the *least common multiple*. We often use the abbreviation LCM.

Note:
Least Common Multiple
The **least common multiple** (LCM) of two numbers is the smallest number that is a multiple of both numbers.

The procedure box lists the steps to take to find the LCM using the prime factors method we used above for 12 and 18.

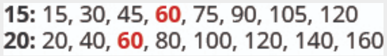
Note:
Find the Least Common Multiple by Listing Multiples.

List several multiples of each number.
Look for the smallest number that appears on both lists.
This number is the LCM.

Example:
Exercise:

Problem: Find the least common multiple of 15 and 20 by listing multiples.

Solution:
Solution

Make lists of the first few multiples of 15 and of 20, and use them to find the least common multiple.	 **15:** 15, 30, 45, **60**, 75, 90, 105, 120 **20:** 20, 40, **60**, 80, 100, 120, 140, 160
Look for the smallest number that appears in both lists.	The first number to appear on both lists is 60, so 60 is the least common multiple of 15 and 20.

Notice that 120 is in both lists, too. It is a common multiple, but it is not the *least* common multiple.

Note:
Exercise:

Problem: Find the least common multiple by listing multiples: 9 and 12.

Solution:

36

Note:
Exercise:

Problem: Find the least common multiple by listing multiples: 18 and 24.

Solution:

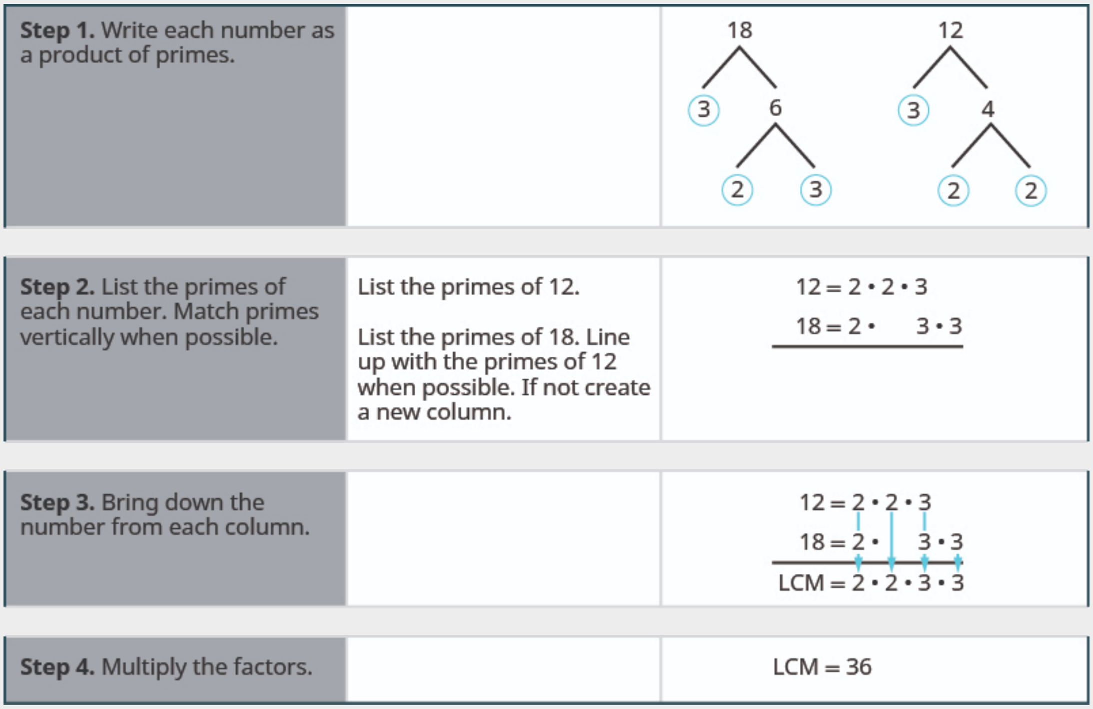
72

Our second method to find the least common multiple of two numbers is to use The Prime Factors Method. Let’s find the LCM of 12 and 18 again, this time using their prime factors.

Example:
How to Find the Least Common Multiple Using the Prime Factors Method
Exercise:

Problem: Find the Least Common Multiple (LCM) of 12 and 18 using the prime factors method.

Solution:
Solution


18
3
6
2
3
12
3
4
2
2

Step 1. Write each number as a product of primes.		
Step 2. List the primes of each number. Match primes vertically when possible.	List the primes of 12. List the primes of 18. Line up with the primes of 12 when possible. If not create a new column.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$
Step 3. Bring down the number from each column.		$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$ $\text{LCM} = 2 \cdot 2 \cdot 3 \cdot 3$
Step 4. Multiply the factors.		$\text{LCM} = 36$

Notice that the prime factors of 12 $(2 \cdot 2 \cdot 3)$ and the prime factors of 18 $(2 \cdot 3 \cdot 3)$ are included in the LCM $(2 \cdot 2 \cdot 3 \cdot 3)$. So 36 is the least common multiple of 12 and 18.

By matching up the common primes, each common prime factor is used only once. This way you are sure that 36 is the *least* common multiple.

Note:
Exercise:

Problem: Find the LCM using the prime factors method: 9 and 12.

Solution:

36

Note:
Exercise:

Problem: Find the LCM using the prime factors method: 18 and 24.

Solution:

72

Note:
Find the Least Common Multiple Using the Prime Factors Method.

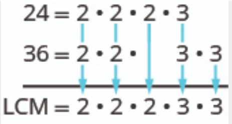
Write each number as a product of primes.
List the primes of each number. Match primes vertically when possible.
Bring down the columns.
Multiply the factors.

Example:
Exercise:

Problem: Find the Least Common Multiple (LCM) of 24 and 36 using the prime factors method.

Solution:
Solution

Find the primes of 24 and 36. Match primes vertically when possible.	

Bring down all columns.	 $24 = 2 \cdot 2 \cdot 2 \cdot 3$ $36 = 2 \cdot 2 \cdot \quad 3 \cdot 3$ $\text{LCM} = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$
Multiply the factors.	$\text{LCM} = 72$
	The LCM of 24 and 36 is 72.

Note:
Exercise:

Problem: Find the LCM using the prime factors method: 21 and 28.

Solution:

84

Note:
Exercise:

Problem: Find the LCM using the prime factors method: 24 and 32.

Solution:

96

Note:
Access this online resource for additional instruction and practice with using whole numbers. You will need to enable Java in your web browser to use the application.

- Sieve of Eratosthenes

Key Concepts

- **Place Value** as in [link].
- **Name a Whole Number in Words**

  Start at the left and name the number in each period, followed by the period name.
  Put commas in the number to separate the periods.
  Do not name the ones period.

- **Write a Whole Number Using Digits**

  Identify the words that indicate periods. (Remember the ones period is never named.)
  Draw 3 blanks to indicate the number of places needed in each period. Separate the periods by commas.
  Name the number in each period and place the digits in the correct place value position.

- **Round Whole Numbers**

  Locate the given place value and mark it with an arrow. All digits to the left of the arrow do not change.
  Underline the digit to the right of the given place value.
  Is this digit greater than or equal to 5?

    - Yes—add 1 to the digit in the given place value.
    - No—do not change the digit in the given place value.

  Replace all digits to the right of the given place value with zeros.

- **Divisibility Tests:** A number is divisible by:

    - 2 if the last digit is 0, 2, 4, 6, or 8.
    - 3 if the sum of the digits is divisible by 3.
    - 5 if the last digit is 5 or 0.
    - 6 if it is divisible by both 2 and 3.
    - 10 if it ends with 0.

- **Find the Prime Factorization of a Composite Number**

  Find two factors whose product is the given number, and use these numbers to create two branches.
  If a factor is prime, that branch is complete. Circle the prime, like a bud on the tree.
  If a factor is not prime, write it as the product of two factors and continue the process.
  Write the composite number as the product of all the circled primes.

- **Find the Least Common Multiple by Listing Multiples**

  List several multiples of each number.
  Look for the smallest number that appears on both lists.
  This number is the LCM.

- **Find the Least Common Multiple Using the Prime Factors Method**

  Write each number as a product of primes.
  List the primes of each number. Match primes vertically when possible.
  Bring down the columns.
  Multiply the factors.

Practice Makes Perfect

Use Place Value with Whole Numbers

In the following exercises, find the place value of each digit in the given numbers.
Exercise:

51,493
ⓐ 1
ⓑ 4
ⓒ 9
ⓓ 5
Problem: ⓔ 3

Solution:

ⓐ thousands ⓑ hundreds ⓒ tens ⓓ ten thousands ⓔ ones

Exercise:

87,210
ⓐ 2
ⓑ 8
ⓒ 0
ⓓ 7
Problem: ⓔ 1

Exercise:

164,285
ⓐ 5
ⓑ 6
ⓒ 1
ⓓ 8
Problem: ⓔ 2

Solution:

ⓐ ones ⓑ ten thousands ⓒ hundred thousands ⓓ tens ⓔ hundreds

Exercise:

395,076
ⓐ 5
ⓑ 3
ⓒ 7
ⓓ 0
Problem: ⓔ 9

Exercise:

93,285,170
ⓐ 9
ⓑ 8
ⓒ 7
ⓓ 5
Problem: ⓔ 3

Solution:

ⓐ ten millions ⓑ ten thousands ⓒ tens ⓓ thousands ⓔ millions

Exercise:

36,084,215
ⓐ 8
ⓑ 6
ⓒ 5
ⓓ 4
Problem: ⓔ 3

Exercise:

7,284,915,860,132
ⓐ 7
ⓑ 4
ⓒ 5
ⓓ 3
Problem: ⓔ 0

Solution:

ⓐ trillions ⓑ billions ⓒ millions ⓓ tens ⓔ thousands

Exercise:

2,850,361,159,433
ⓐ 9
ⓑ 8
ⓒ 6
ⓓ 4
Problem: ⓔ 2

In the following exercises, name each number using words.

Exercise:

Problem: 1,078

Solution:

one thousand, seventy-eight

Exercise:

Problem: 5,902

Exercise:

Problem: 364,510

Solution:

three hundred sixty-four thousand, five hundred ten

Exercise:

Problem: 146,023

Exercise:

Problem: 5,846,103

Solution:

five million, eight hundred forty-six thousand, one hundred three

Exercise:

Problem: 1,458,398

Exercise:

Problem: 37,889,005

Solution:

thirty-seven million, eight hundred eighty-nine thousand, five

Exercise:

Problem: 62,008,465

In the following exercises, write each number as a whole number using digits.

Exercise:

Problem: four hundred twelve

Solution:

412

Exercise:

Problem: two hundred fifty-three

Exercise:

Problem: thirty-five thousand, nine hundred seventy-five

Solution:

35,975

Exercise:

Problem: sixty-one thousand, four hundred fifteen

Exercise:

Problem: eleven million, forty-four thousand, one hundred sixty-seven

---

Solution:

11,044,167

Exercise:

Problem: eighteen million, one hundred two thousand, seven hundred eighty-three

Exercise:

Problem: three billion, two hundred twenty-six million, five hundred twelve thousand, seventeen

---

Solution:

3,226,512,017

Exercise:

Problem: eleven billion, four hundred seventy-one million, thirty-six thousand, one hundred six

In the following, round to the indicated place value.

Exercise:

Problem: Round to the nearest ten.

ⓐ 386 ⓑ 2,931

Solution:

ⓐ 390 ⓑ 2,930

Exercise:

Problem: Round to the nearest ten.

ⓐ 792 ⓑ 5,647

Exercise:

Problem: Round to the nearest hundred.

ⓐ 13,748 ⓑ 391,794

Solution:

ⓐ 13,700 ⓑ 391,800

Exercise:

Problem: Round to the nearest hundred.

ⓐ 28,166 ⓑ 481,628

Exercise:

Problem: Round to the nearest ten.

ⓐ 1,492 ⓑ 1,497

Solution:

ⓐ 1,490 ⓑ 1,500

Exercise:

Problem: Round to the nearest ten.

ⓐ 2,791 ⓑ 2,795

Exercise:

Problem: Round to the nearest hundred.

ⓐ 63,994 ⓑ 63,040

Solution:

ⓐ 64,000 ⓑ 63,000

Exercise:

Problem: Round to the nearest hundred.

ⓐ 49,584 ⓑ 49,548

In the following exercises, round each number to the nearest ⓐ hundred, ⓑ thousand, ⓒ ten thousand.

Exercise:

Problem: 392,546

Solution:

ⓐ 392,500 ⓑ 393,000 ⓒ 390,000

Exercise:

Problem: 619,348

Exercise:

Problem: 2,586,991

Solution:

ⓐ 2,587,000 ⓑ 2,587,000 ⓒ 2,590,000

Exercise:

Problem: 4,287,965

Identify Multiples and Factors

In the following exercises, use the divisibility tests to determine whether each number is divisible by 2, 3, 5, 6, and 10.

Exercise:

Problem: 84

Solution:

divisible by 2, 3, and 6

Exercise:

Problem: 9,696

Exercise:

Problem: 75

Solution:

divisible by 3 and 5

Exercise:

Problem: 78

Exercise:

Problem: 900

Solution:

divisible by 2, 3, 5, 6, and 10

Exercise:

Problem: 800

Exercise:

Problem: 986

Solution:

divisible by 2

Exercise:

Problem: 942

Exercise:

Problem: 350

Solution:

divisible by 2, 3, and 10

Exercise:

Problem: 550

Exercise:

Problem: 22,335

Solution:

divisible by 3 and 5

Exercise:

Problem: 39,075

Find Prime Factorizations and Least Common Multiples

In the following exercises, find the prime factorization.

Exercise:

Problem: 86

Solution:

$2 \cdot 43$

Exercise:

Problem: 78

Exercise:

Problem: 132

Solution:

$2 \cdot 2 \cdot 3 \cdot 11$

Exercise:

Problem: 455

Exercise:

Problem: 693

Solution:

$3 \cdot 3 \cdot 7 \cdot 11$

Exercise:

Problem: 400

Exercise:

Problem: 432

Solution:

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$

Exercise:

Problem: 627

Exercise:

Problem: 2,160

Solution:

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5$

Exercise:

Problem: 2,520

In the following exercises, find the least common multiple of each pair of numbers using the multiples method.

Exercise:

Problem: 8, 12

Solution:

24

Exercise:

Problem: 4, 3

Exercise:

Problem: 12, 16

Solution:

48

Exercise:

Problem: 30, 40

Exercise:

Problem: 20, 30

Solution:

60

Exercise:

Problem: 44, 55

In the following exercises, find the least common multiple of each pair of numbers using the prime factors method.

Exercise:

Problem: 8, 12

Solution:

24

Exercise:

Problem: 12, 16

Exercise:

Problem: 28, 40

Solution:

280

Exercise:

Problem: 84, 90

Exercise:

Problem: 55, 88

Solution:

440

Exercise:

Problem: 60, 72

Everyday Math

Exercise:

Problem:

Writing a Check Jorge bought a car for $24,493. He paid for the car with a check. Write the purchase price in words.

Solution:

twenty-four thousand, four hundred ninety-three dollars

Exercise:

Problem:

Writing a Check Marissa’s kitchen remodeling cost $18,549. She wrote a check to the contractor. Write the amount paid in words.

Exercise:

Problem:

Buying a Car Jorge bought a car for $24,493. Round the price to the nearest ⓐ ten ⓑ hundred ⓒ thousand; and ⓓ ten-thousand.

Solution:

ⓐ $24,490 ⓑ $24,500 ⓒ $24,000 ⓓ $20,000

Exercise:

Problem:

Remodeling a Kitchen Marissa’s kitchen remodeling cost $18,549. Round the cost to the nearest ⓐ ten ⓑ hundred ⓒ thousand and ⓓ ten-thousand.

Exercise:

Problem:

Population The population of China was 1,339,724,852 on November 1, 2010. Round the population to the nearest ⓐ billion ⓑ hundred-million; and ⓒ million.

Solution:

ⓐ 1,000,000,000 ⓑ 1,300,000,000 ⓒ 1,340,000,000

Exercise:

Problem:

Astronomy The average distance between Earth and the sun is 149,597,888 kilometers. Round the distance to the nearest ⓐ hundred-million ⓑ ten-million; and ⓒ million.

Exercise:

Problem:

Grocery Shopping Hot dogs are sold in packages of 10, but hot dog buns come in packs of eight. What is the smallest number that makes the hot dogs and buns come out even?

Solution:

40

Exercise:

Problem:

Grocery Shopping Paper plates are sold in packages of 12 and party cups come in packs of eight. What is the smallest number that makes the plates and cups come out even?

Writing Exercises

Exercise:

Problem: Give an everyday example where it helps to round numbers.

Exercise:

Problem: If a number is divisible by 2 and by 3 why is it also divisible by 6?

Exercise:

Problem: What is the difference between prime numbers and composite numbers?

Solution:

Answers may vary.

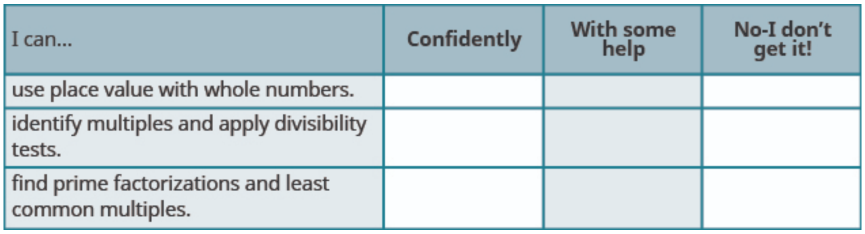
Exercise:

Problem:

Explain in your own words how to find the prime factorization of a composite number, using any method you prefer.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
use place value with whole numbers.			
identify multiples and apply divisibility tests.			
find prime factorizations and least common multiples.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

composite number
A composite number is a counting number that is not prime. A composite number has factors other than 1 and itself.

counting numbers
The counting numbers are the numbers 1, 2, 3, …

divisible by a number

If a number m is a multiple of n, then m is divisible by n. (If 6 is a multiple of 3, then 6 is divisible by 3.)

factors
If $a \cdot b = m$, then a and b are factors of m. Since 3 · 4 = 12, then 3 and 4 are factors of 12.

least common multiple
The least common multiple of two numbers is the smallest number that is a multiple of both numbers.

multiple of a number
A number is a multiple of n if it is the product of a counting number and n.

number line
A number line is used to visualize numbers. The numbers on the number line get larger as they go from left to right, and smaller as they go from right to left.

prime factorization
The prime factorization of a number is the product of prime numbers that equals the number.

prime number
A prime number is a counting number greater than 1, whose only factors are 1 and itself.

whole numbers
The whole numbers are the numbers 0, 1, 2, 3, ...

Use the Language of Algebra
By the end of this section, you will be able to:

- Use variables and algebraic symbols
- Simplify expressions using the order of operations
- Evaluate an expression
- Identify and combine like terms
- Translate an English phrase to an algebraic expression

Note:
A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **The Language of Algebra**.

Use Variables and Algebraic Symbols

Suppose this year Greg is 20 years old and Alex is 23. You know that Alex is 3 years older than Greg. When Greg was 12, Alex was 15. When Greg is 35, Alex will be 38. No matter what Greg's age is, Alex's age will always be 3 years more, right? In the language of algebra, we say that Greg's age and Alex's age are **variables** and the 3 is a **constant**. The ages change ("vary") but the 3 years between them always stays the same ("constant"). Since Greg's age and Alex's age will always differ by 3 years, 3 is the *constant*.

In algebra, we use letters of the alphabet to represent variables. So if we call Greg's age *g*, then we could use $g+3$ to represent Alex's age. See [link].

Greg's age	**Alex's age**
12	15
20	23
35	38
$\boldsymbol{g}$	$\boldsymbol{g}+3$

The letters used to represent these changing ages are called *variables*. The letters most commonly used for variables are *x*, *y*, *a*, *b*, and *c*.

Note:
Variable
A **variable** is a letter that represents a number whose value may change.

Note:
Constant
A **constant** is a number whose value always stays the same.

To write algebraically, we need some operation symbols as well as numbers and variables. There are several types of symbols we will be using.

There are four basic arithmetic operations: addition, subtraction, multiplication, and division. We’ll list the symbols used to indicate these operations below ([link]). You’ll probably recognize some of them.

Operation	**Notation**	**Say:**	**The result is…**
Addition	$a+b$	*a* plus *b*	the sum of *a* and *b*
Subtraction	$a-b$	*a* minus *b*	the difference of *a* and *b*
Multiplication	$a\cdot b, ab, (a)\,(b), (a)b, a\,(b)$	*a* times *b*	the product of *a* and *b*
Division	$a\div b, a/b, \frac{a}{b}, b\overline{)a}$	*a* divided by *b*	the quotient of *a* and *b*, *a* is called the dividend, and *b* is called the divisor

We perform these operations on two numbers. When translating from symbolic form to English, or from English to symbolic form, pay attention to the words “of” and “and.”

- The *difference of* 9 and 2 means subtract 9 and 2, in other words, 9 minus 2, which we write symbolically as $9-2$.
- The *product of* 4 and 8 means multiply 4 and 8, in other words 4 times 8, which we write symbolically as $4\cdot 8$.

In algebra, the cross symbol, $\times$, is not used to show multiplication because that symbol may cause confusion. Does 3*xy* mean $3\ \times\ y$ (‘three times *y*’) or $3\cdot x\cdot y$ (three times *x* times *y*)? To make it clear, use · or parentheses for multiplication.

When two quantities have the same value, we say they are equal and connect them with an **equal sign**.

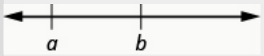
Note:
Equality Symbol
$a=b$ is read “*a* is equal to *b*”
The symbol “$=$” is called the **equal sign**.

On the number line, the numbers get larger as they go from left to right. The number line can be used to explain the symbols “<” and “>.”

Note:
Inequality

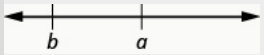
Equation:

$$a < b \text{ is read ``}a\text{ is less than }b\text{''}$$
$$a \text{ is to the left of } b \text{ on the number line}$$



Equation:

$$a > b \text{ is read ``}a\text{ is greater than }b\text{''}$$
$$a \text{ is to the right of } b \text{ on the number line}$$



The expressions $a < b$ or $a > b$ can be read from left to right or right to left, though in English we usually read from left to right ([link]). In general, $a < b$ is equivalent to $b > a$. For example $7 < 11$ is equivalent to $11 > 7$. And $a > b$ is equivalent to $b < a$. For example $17 > 4$ is equivalent to $4 < 17$.

Inequality Symbols	**Words**
$a \ne b$	*a* is *not equal to b*
$a < b$	*a* is *less than b*
$a \le b$	*a* is *less than or equal to b*
$a > b$	*a* is *greater than b*
$a \ge b$	*a* is *greater than or equal to b*

Example:
Exercise:

Problem: Translate from algebra into English:

ⓐ $17 \le 26$ ⓑ $8 \ne 17 - 8$ ⓒ $12 > 27 \div 3$ ⓓ $y + 7 < 19$

Solution:
Solution

ⓐ $17 \le 26$
17 is less than or equal to 26

ⓑ $8 \neq 17 - 8$
8 is not equal to 17 minus 8

ⓒ $12 > 27 \div 3$
12 is greater than 27 divided by 3

ⓓ $y + 7 < 19$
y plus 7 is less than 19

Note:
Exercise:

Problem: Translate from algebra into English:

ⓐ $14 \leq 27$ ⓑ $19 - 2 \neq 8$ ⓒ $12 > 4 \div 2$ ⓓ $x - 7 < 1$

Solution:

ⓐ 14 is less than or equal to 27 ⓑ 19 minus 2 is not equal to 8 ⓒ 12 is greater than 4 divided by 2 ⓓ *x* minus 7 is less than 1

Note:
Exercise:

Problem: Translate from algebra into English:

ⓐ $19 \geq 15$ ⓑ $7 = 12 - 5$ ⓒ $15 \div 3 < 8$ ⓓ $y + 3 > 6$

Solution:

ⓐ 19 is greater than or equal to 15 ⓑ 7 is equal to 12 minus 5 ⓒ 15 divided by 3 is less than 8 ⓓ *y* plus 3 is greater than 6

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in English. They help to make clear which expressions are to be kept together and separate from other expressions. We will introduce three types now.

Note:
Grouping Symbols
Equation:

$$\begin{array}{ll} \text{Parentheses} & () \\ \text{Brackets} & [] \\ \text{Braces} & \{\} \end{array}$$

Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.

Equation:

$$8(14-8) \qquad 21-3[2+4(9-8)] \qquad 24 \div \{13-2[1(6-5)+4]\}$$

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. “Running very fast” is a phrase, but “The football player was running very fast” is a sentence. A sentence has a subject and a verb. In algebra, we have *expressions* and *equations*.

Note:
Expression
An **expression** is a number, a variable, or a combination of numbers and variables using operation symbols.

An **expression** is like an English phrase. Here are some examples of expressions:

Expression	**Words**	**English Phrase**
$3+5$	3 plus 5	the sum of three and five
$n-1$	*n* minus one	the difference of *n* and one
$6 \cdot 7$	6 times 7	the product of six and seven
$\frac{x}{y}$	*x* divided by *y*	the quotient of *x* and *y*

Notice that the English phrases do not form a complete sentence because the phrase does not have a verb.

An **equation** is two expressions linked with an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb.

Note:
Equation
An **equation** is two expressions connected by an equal sign.

Here are some examples of equations.

Equation	**English Sentence**
$3+5=8$	The sum of three and five is equal to eight.
$n-1=14$	*n* minus one equals fourteen.
$6\cdot 7=42$	The product of six and seven is equal to forty-two.
$x=53$	*x* is equal to fifty-three.
$y+9=2y-3$	*y* plus nine is equal to two *y* minus three.

Example:
Exercise:

Problem: Determine if each is an expression or an equation:

ⓐ $2(x+3)=10$ ⓑ $4(y-1)+1$ ⓒ $x\div 25$ ⓓ $y+8=40$

Solution:
Solution

ⓐ $2(x+3)=10$ This is an *equation*—two expressions are connected with an equal sign.
ⓑ $4(y-1)+1$ This is an *expression*—no equal sign.
ⓒ $x\div 25$ This is an *expression*—no equal sign.
ⓓ $y+8=40$ This is an *equation*—two expressions are connected with an equal sign.

Note:
Exercise:

Problem: Determine if each is an expression or an equation: ⓐ $3(x-7)=27$ ⓑ $5(4y-2)-7$.

Solution:

ⓐ equation ⓑ expression

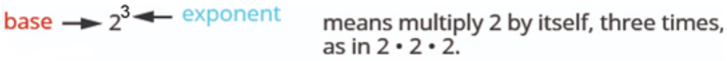
Note:
Exercise:

Problem: Determine if each is an expression or an equation: ⓐ $y^3\div 14$ ⓑ $4x-6=22$.

Solution:

ⓐ expression ⓑ equation

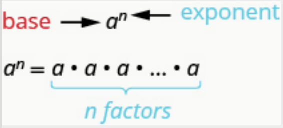
Suppose we need to multiply 2 nine times. We could write this as $2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2$. This is tedious and it can be hard to keep track of all those 2s, so we use exponents. We write $2\cdot2\cdot2$ as 2^3 and $2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2$ as 2^9. In expressions such as 2^3, the 2 is called the *base* and the 3 is called the *exponent*. The exponent tells us how many times we need to multiply the base.



We read 2^3 as “two to the third power” or “two cubed.”

We say 2^3 is in *exponential notation* and $2\cdot2\cdot2$ is in *expanded notation*.

Note:
Exponential Notation
a^n means multiply *a* by itself, *n* times.


base
a^n
exponent
$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n\ factors}$

The expression a^n is read *a* to the n^{th} power.

While we read a^n as “*a* to the n^{th} power,” we usually read:

- a^2 “*a* squared”
- a^3 “*a* cubed”

We’ll see later why a^2 and a^3 have special names.

[link] shows how we read some expressions with exponents.

Expression	**In Words**
7^2	7 to the second power or 7 squared
5^3	5 to the third power or 5 cubed
9^4	9 to the fourth power
12^5	12 to the fifth power

Example:
Exercise:

Problem: Simplify: 3^4.

Solution:
Solution

	3^4
Expand the expression.	$3 \cdot 3 \cdot 3 \cdot 3$
Multiply left to right.	$9 \cdot 3 \cdot 3$
Multiply.	$27 \cdot 3$
Multiply.	81

Note:
Exercise:

Problem: Simplify: ⓐ 5^3 ⓑ 1^7.

Solution:

ⓐ 125 ⓑ 1

Note:
Exercise:

Problem: Simplify: ⓐ 7^2 ⓑ 0^5.

Solution:

ⓐ 49 ⓑ 0

Simplify Expressions Using the Order of Operations

To **simplify an expression** means to do all the math possible. For example, to simplify $4 \cdot 2 + 1$ we'd first multiply $4 \cdot 2$ to get 8 and then add the 1 to get 9. A good habit to develop is to work down the page, writing each step of the process below the previous step. The example just described would look like this:
Equation:

$$\begin{array}{c} 4 \cdot 2 + 1 \\ 8 + 1 \\ 9 \end{array}$$

By not using an equal sign when you simplify an expression, you may avoid confusing expressions with equations.

Note:

Simplify an Expression

To **simplify an expression**, do all operations in the expression.

We’ve introduced most of the symbols and notation used in algebra, but now we need to clarify the order of operations. Otherwise, expressions may have different meanings, and they may result in different values. For example, consider the expression:

Equation:

$$4+3\cdot 7$$

If you simplify this expression, what do you get?

Some students say 49,

Equation:

$$\begin{array}{ll} & 4+3\cdot 7 \\ \text{Since } 4+3 \text{ gives } 7. & 7\cdot 7 \\ \text{And } 7\cdot 7 \text{ is } 49. & 49 \end{array}$$

Others say 25,

Equation:

$$\begin{array}{ll} & 4+3\cdot 7 \\ \text{Since } 3\cdot 7 \text{ is } 21. & 4+21 \\ \text{And } 21+4 \text{ makes } 25. & 25 \end{array}$$

Imagine the confusion in our banking system if every problem had several different correct answers!

The same expression should give the same result. So mathematicians early on established some guidelines that are called the Order of Operations.

Note:

Perform the Order of Operations.

Parentheses and Other Grouping Symbols	◦ Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
Exponents	◦ Simplify all expressions with exponents.
Multiplication and Division	◦ Perform all multiplication and division in order from left to right. These operations have equal priority.
Addition and Subtraction	◦ Perform all addition and subtraction in order from left to right. These operations have equal priority.

Note: Doing the Manipulative Mathematics activity “Game of 24” give you practice using the order of operations.

Students often ask, “How will I remember the order?” Here is a way to help you remember: Take the first letter of each key word and substitute the silly phrase: “Please Excuse My Dear Aunt Sally.”

Equation:

Parentheses	**P**lease
Exponents	**E**xcuse
Multiplication **D**ivision	**M**y **D**ear
Addition **S**ubtraction	**A**unt **S**ally

It’s good that “**M**y **D**ear” goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority. We do not always do multiplication before division or always do division before multiplication. We do them in order from left to right.

Similarly, “**A**unt **S**ally” goes together and so reminds us that **a**ddition and **s**ubtraction also have equal priority and we do them in order from left to right.

Let’s try an example.

Example:
Exercise:

Problem: Simplify: ⓐ $4+3\cdot 7$ ⓑ $(4+3)\cdot 7$.

Solution:
Solution

ⓐ

	$4+3\cdot 7$
Are there any **p**arentheses? No.	
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply first.	$4+3\cdot 7$
Add.	$4+21$

	25

ⓑ

	$(4+3) \cdot 7$
Are there any **p**arentheses? Yes.	$(4+3) \cdot 7$
Simplify inside the parentheses.	$(7)7$
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply.	49

Note:
Exercise:

Problem: Simplify: ⓐ $12 - 5 \cdot 2$ ⓑ $(12 - 5) \cdot 2$.

Solution:

ⓐ 2 ⓑ 14

Note:
Exercise:

Problem: Simplify: ⓐ $8 + 3 \cdot 9$ ⓑ $(8 + 3) \cdot 9$.

Solution:

ⓐ 35 ⓑ 99

Example:
Exercise:

Problem: Simplify: $18 \div 6 + 4\left(5 - 2\right)$.

Solution:
Solution

Parentheses? Yes, subtract first.	$18 \div 6 + 4\left(5 - 2\right)$ $18 \div 6 + 4(3)$
Exponents? No.	
Multiplication or division? Yes.	$18 \div 6 + 4(3)$
Divide first because we multiply and divide left to right.	$3 + 4(3)$
Any other multiplication or division? Yes.	
Multiply.	$3 + 12$
Any other multiplication or division? No.	
Any addition or subtraction? Yes.	15

Note:
Exercise:

Problem: Simplify: $30 \div 5 + 10\left(3 - 2\right)$.

Solution:

16

Note:
Exercise:

Problem: Simplify: $70 \div 10 + 4\left(6 - 2\right)$.

Solution:

23

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

Example:
Exercise:

Problem: Simplify: $5+2^3+3\left[6-3\left(4-2\right)\right]$.

Solution:
Solution

	$5+2^3+3[6-3(4-2)]$
Are there any parentheses (or other grouping symbol)? Yes.	
Focus on the parentheses that are inside the brackets.	$5+2^3+3[6-3(4-2)]$
Subtract.	$5+2^3+3[6-3(2)]$
Continue inside the brackets and multiply.	$5+2^3+3[6-6]$
Continue inside the brackets and subtract.	$5+2^3+3[0]$
The expression inside the brackets requires no further simplification.	
Are there any exponents? Yes.	$5+2^3+3[0]$
Simplify exponents.	$5+8+3[0]$
Is there any multiplication or division? Yes.	
Multiply.	$5+8+0$

Is there any addition or subtraction? Yes.	
Add.	$13+0$
Add.	13

Note:
Exercise:

Problem: Simplify: $9+5^3-\left[4\left(9+3\right)\right]$.

Solution:

86

Note:
Exercise:

Problem: Simplify: $7^2-2\left[4\left(5+1\right)\right]$.

Solution:

1

Evaluate an Expression

In the last few examples, we simplified expressions using the order of operations. Now we’ll evaluate some expressions—again following the order of operations. To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

Note:
Evaluate an Expression
To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

To evaluate an expression, substitute that number for the variable in the expression and then simplify the expression.

Example:

Exercise:

Problem: Evaluate $7x - 4$, when ⓐ $x = 5$ and ⓑ $x = 1$.

Solution:
Solution

ⓐ

when $x = 5$	$7x - 4$
	$7(5) - 4$
Multiply.	$35 - 4$
Subtract.	31

ⓑ

when $x = 1$	$7x - 4$
	$7(1) - 4$
Multiply.	$7 - 4$
Subtract.	3

Note:
Exercise:

Problem: Evaluate $8x - 3$, when ⓐ $x = 2$ and ⓑ $x = 1$.

Solution:

ⓐ 13 ⓑ 5

Note:
Exercise:

Problem: Evaluate $4y-4$, when ⓐ $y=3$ and ⓑ $y=5$.

Solution:

ⓐ 8 ⓑ 16

Example:
Exercise:

Problem: Evaluate each expression when $x=4$: ⓐ x^2 ⓑ 3^x.

Solution:
Solution

ⓐ

	x^2
Replace *x* with 4.	4^2
Use definition of exponent.	$4 \cdot 4$
Simplify.	16

ⓑ

	3^x
Replace *x* with 4.	3^4

Use definition of exponent.	$3 \cdot 3 \cdot 3 \cdot 3$
Simplify.	81

Note:
Exercise:

Problem: Evaluate each expression when $x = 3$: ⓐ x^2 ⓑ 4^x.

Solution:

ⓐ 9 ⓑ 64

Note:
Exercise:

Problem: Evaluate each expression when $x = 6$: ⓐ x^3 ⓑ 2^x.

Solution:

ⓐ 216 ⓑ 64

Example:
Exercise:

Problem: Evaluate $2x^2 + 3x + 8$ when $x = 4$.

Solution:
Solution

	$2x^2 + 3x + 8$
Substitute $x = 4$.	$2(4)^2 + 3(4) + 8$
Follow the order of operations.	$2(16) + 3(4) + 8$
	$32 + 12 + 8$
	52

Note:
Exercise:

Problem: Evaluate $3x^2 + 4x + 1$ when $x = 3$.

Solution:

40

Note:
Exercise:

Problem: Evaluate $6x^2 - 4x - 7$ when $x = 2$.

Solution:

9

Identify and Combine Like Terms

Algebraic expressions are made up of terms. A **term** is a constant, or the product of a constant and one or more variables.

Note:
Term
A **term** is a constant, or the product of a constant and one or more variables.

Examples of terms are $7, y, 5x^2, 9a,$ and b^5.

The constant that multiplies the variable is called the **coefficient**.

Note:
Coefficient
The **coefficient** of a term is the constant that multiplies the variable in a term.

Think of the coefficient as the number in front of the variable. The coefficient of the term $3x$ is 3. When we write x, the coefficient is 1, since $x = 1 \cdot x$.

Example:

Exercise:

Problem: Identify the coefficient of each term: ⓐ $14y$ ⓑ $15x^2$ ⓒ a.

Solution:
Solution

ⓐ The coefficient of $14y$ is 14.

ⓑ The coefficient of $15x^2$ is 15.

ⓒ The coefficient of a is 1 since $a = 1a$.

Note:
Exercise:

Problem: Identify the coefficient of each term: ⓐ $17x$ ⓑ $41b^2$ ⓒ z.

Solution:

ⓐ 17 ⓑ 41 ⓒ 1

Note:
Exercise:

Problem: Identify the coefficient of each term: ⓐ $9p$ ⓑ $13a^3$ ⓒ y^3.

Solution:

ⓐ 9 ⓑ 13 ⓒ 1

Some terms share common traits. Look at the following 6 terms. Which ones seem to have traits in common?
Equation:

$$5x \qquad 7 \qquad n^2 \qquad 4 \qquad 3x \qquad 9n^2$$

The 7 and the 4 are both constant terms.

The *5x* and the 3*x* are both terms with *x*.

The n^2 and the $9n^2$ are both terms with n^2.

When two terms are constants or have the same variable and exponent, we say they are **like terms**.

- 7 and 4 are like terms.
- 5*x* and 3*x* are like terms.
- x^2 and $9x^2$ are like terms.

Note:
Like Terms
Terms that are either constants or have the same variables raised to the same powers are called **like terms**.

Example:
Exercise:

Problem: Identify the like terms: y^3, $7x^2$, 14, 23, $4y^3$, 9*x*, $5x^2$.

Solution:
Solution

y^3 and $4y^3$ are like terms because both have y^3; the variable and the exponent match.

$7x^2$ and $5x^2$ are like terms because both have x^2; the variable and the exponent match.

14 and 23 are like terms because both are constants.

There is no other term like 9*x*.

Note:
Exercise:

Problem: Identify the like terms: 9, $2x^3$, y^2, $8x^3$, 15, $9y$, $11y^2$.

Solution:

9 and 15, y^2 and $11y^2$, $2x^3$ and $8x^3$

Note:
Exercise:

Problem: Identify the like terms: $4x^3$, $8x^2$, 19, $3x^2$, 24, $6x^3$.

Solution:

19 and 24, $8x^2$ and $3x^2$, $4x^3$ and $6x^3$

Adding or subtracting terms forms an expression. In the expression $2x^2 + 3x + 8$, from [link], the three terms are $2x^2$, $3x$, and 8.

Example:
Exercise:

Problem: Identify the terms in each expression.

ⓐ $9x^2 + 7x + 12$
ⓑ $8x + 3y$

Solution:
Solution

ⓐ The terms of $9x^2 + 7x + 12$ are $9x^2$, $7x$, and 12.

ⓑ The terms of $8x + 3y$ are $8x$ and $3y$.

Note:
Exercise:

Problem: Identify the terms in the expression $4x^2 + 5x + 17$.

Solution:

$4x^2, 5x, 17$

Note:
Exercise:

Problem: Identify the terms in the expression $5x + 2y$.

Solution:

$5x$, $2y$

If there are like terms in an expression, you can simplify the expression by combining the like terms. What do you think $4x + 7x + x$ would simplify to? If you thought $12x$, you would be right!
Equation:

$$\begin{gathered} 4x + 7x + x \\ x + x + x + x \quad + x + x + x + x + x + x + x \quad + x \\ 12x \end{gathered}$$

Add the coefficients and keep the same variable. It doesn't matter what x is—if you have 4 of something and add 7 more of the same thing and then add 1 more, the result is 12 of them. For example, 4 oranges plus 7 oranges plus 1 orange is 12 oranges. We will discuss the mathematical properties behind this later.

Simplify: $4x + 7x + x$.

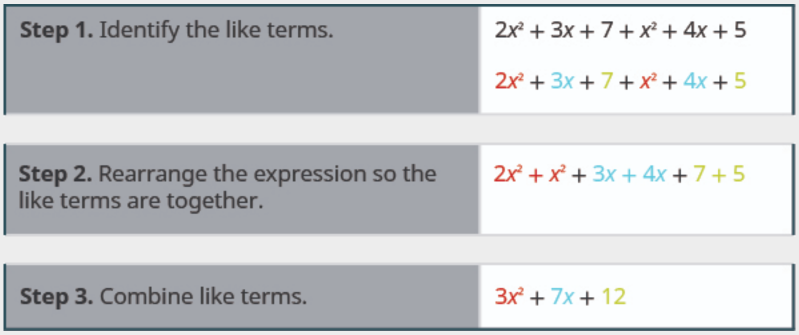
Add the coefficients. $12x$

Example:

How To Combine Like Terms
Exercise:

Problem: Simplify: $2x^2 + 3x + 7 + x^2 + 4x + 5$.

Solution:
Solution



Step 1. Identify the like terms.	$2x^2 + 3x + 7 + x^2 + 4x + 5$ $2x^2 + 3x + 7 + x^2 + 4x + 5$
Step 2. Rearrange the expression so the like terms are together.	$2x^2 + x^2 + 3x + 4x + 7 + 5$
Step 3. Combine like terms.	$3x^2 + 7x + 12$

Note:
Exercise:

Problem: Simplify: $3x^2 + 7x + 9 + 7x^2 + 9x + 8$.

Solution:

$10x^2 + 16x + 17$

Note:
Exercise:

Problem: Simplify: $4y^2 + 5y + 2 + 8y^2 + 4y + 5$.

Solution:

$12y^2 + 9y + 7$

Note:
Combine Like Terms.

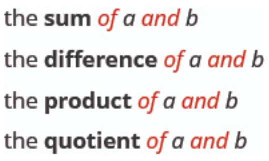
Identify like terms.
Rearrange the expression so like terms are together.
Add or subtract the coefficients and keep the same variable for each group of like terms.

Translate an English Phrase to an Algebraic Expression

In the last section, we listed many operation symbols that are used in algebra, then we translated expressions and equations into English phrases and sentences. Now we'll reverse the process. We'll translate English phrases into algebraic expressions. The symbols and variables we've talked about will help us do that. [link] summarizes them.

Operation	**Phrase**	**Expression**
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a	$a+b$
Subtraction	a minus b the difference of a and b a decreased by b b less than a b subtracted from a	$a-b$
Multiplication	a times b the product of a and b twice a	$a\cdot b, ab, a(b), (a)(b)$ $2a$
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a	$a\div b, a/b, \frac{a}{b}, b\overline{)a}$

Look closely at these phrases using the four operations:


the **sum** *of a and b*
the **difference** *of a and b*
the **product** *of a and b*
the **quotient** *of a and b*

Each phrase tells us to operate on two numbers. Look for the words *of* and *and* to find the numbers.

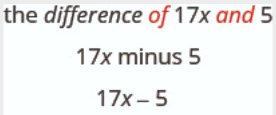
Example:
Exercise:

Problem:

Translate each English phrase into an algebraic expression: ⓐ the difference of $17x$ and 5 ⓑ the quotient of $10x^2$ and 7.

Solution:
Solution

ⓐ The key word is *difference*, which tells us the operation is subtraction. Look for the words *of* and *and* to find the numbers to subtract.

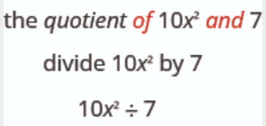


the *difference of* 17*x and* 5

17*x* minus 5

17*x* – 5

ⓑ The key word is “quotient,” which tells us the operation is division.



the *quotient of* 10*x*² *and* 7

divide 10*x*² by 7

10*x*² ÷ 7

This can also be written $10x^2/7$ or $\frac{10x^2}{7}$.

Note:
Exercise:

Problem:

Translate the English phrase into an algebraic expression: ⓐ the difference of $14x^2$ and 13 ⓑ the quotient of 12*x* and 2.

Solution:

ⓐ $14x^2 - 13$ ⓑ $12x \div 2$

Note:
Exercise:

Problem:

Translate the English phrase into an algebraic expression: ⓐ the sum of $17y^2$ and 19 ⓑ the product of 7 and y.

Solution:

ⓐ $17y^2 + 19$ ⓑ $7y$

How old will you be in eight years? What age is eight more years than your age now? Did you add 8 to your present age? Eight “more than” means 8 added to your present age. How old were you seven years ago? This is 7 years less than your age now. You subtract 7 from your present age. Seven “less than” means 7 subtracted from your present age.

Example:

Exercise:

Problem:

Translate the English phrase into an algebraic expression: ⓐ Seventeen more than *y* ⓑ Nine less than $9x^2$.

Solution:
Solution

ⓐ The key words are *more than*. They tell us the operation is addition. *More than* means “added to.”
Equation:

$$\begin{array}{c}\text{Seventeen more than } y \\ \text{Seventeen added to } y \\ y+17\end{array}$$

ⓑ The key words are *less than*. They tell us to subtract. *Less than* means “subtracted from.”
Equation:

$$\begin{array}{c}\text{Nine less than } 9x^2 \\ \text{Nine subtracted from } 9x^2 \\ 9x^2-9\end{array}$$

Note:
Exercise:

Problem:

Translate the English phrase into an algebraic expression: ⓐ Eleven more than x ⓑ Fourteen less than $11a$.

Solution:

ⓐ $x+11$ ⓑ $11a-14$

Note:
Exercise:

Problem: Translate the English phrase into an algebraic expression: ⓐ 13 more than *z* ⓑ 18 less than 8*x*.

Solution:

ⓐ $z+13$ ⓑ $8x-18$

Example:
Exercise:

Problem:

Translate the English phrase into an algebraic expression: ⓐ five times the sum of *m* and *n* ⓑ the sum of five times *m* and *n*.

Solution:
Solution

There are two operation words—*times* tells us to multiply and *sum* tells us to add.

ⓐ Because we are multiplying 5 times the sum we need parentheses around the sum of *m* and *n*, $(m+n)$. This forces us to determine the sum first. (Remember the order of operations.)

$$\begin{array}{c}\text{five times the sum of } m \text{ and } n\\ 5\left(m+n\right)\end{array}$$

ⓑ To take a sum, we look for the words "of" and "and" to see what is being added. Here we are taking the sum *of* five times *m* and *n*.

$$\begin{array}{c}\text{the sum of five times } m \text{ and } n\\ 5m+n\end{array}$$

Note:
Exercise:

Problem:

Translate the English phrase into an algebraic expression: ⓐ four times the sum of *p* and *q* ⓑ the sum of four times *p* and *q*.

Solution:

ⓐ $4\left(p+q\right)$ ⓑ $4p+q$

Note:
Exercise:

Problem:

Translate the English phrase into an algebraic expression: ⓐ the difference of two times *x* and 8, ⓑ two times the difference of *x* and 8.

Solution:

ⓐ $2x-8$ ⓑ $2\left(x-8\right)$

Later in this course, we'll apply our skills in algebra to solving applications. The first step will be to translate an English phrase to an algebraic expression. We'll see how to do this in the next two examples.

Example:
Exercise:

Problem:

The length of a rectangle is 6 less than the width. Let *w* represent the width of the rectangle. Write an expression for the length of the rectangle.

Solution:
Solution

Write a phrase about the length of the rectangle.	6 less than the width
Substitute w for "the width."	6 less than w
Rewrite "less than" as "subtracted from."	6 subtracted from w
Translate the phrase into algebra.	$w - 6$

Note:
Exercise:

Problem:

The length of a rectangle is 7 less than the width. Let *w* represent the width of the rectangle. Write an expression for the length of the rectangle.

Solution:

$w - 7$

Note:
Exercise:

Problem:

The width of a rectangle is 6 less than the length. Let *l* represent the length of the rectangle. Write an expression for the width of the rectangle.

Solution:

$l - 6$

Example:
Exercise:

Problem:

June has dimes and quarters in her purse. The number of dimes is three less than four times the number of quarters. Let *q* represent the number of quarters. Write an expression for the number of dimes.

Solution:
Solution

Write the phrase about the number of dimes.	three less than four times the number of quarte
Substitute q for the number of quarters.	3 less than 4 times q
Translate “4 times q.”	3 less than $4q$
Translate the phrase into algebra.	$4q-3$

Note:
Exercise:

Problem:

Geoffrey has dimes and quarters in his pocket. The number of dimes is eight less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Solution:

$4q-8$

Note:
Exercise:

Problem:

Lauren has dimes and nickels in her purse. The number of dimes is three more than seven times the number of nickels. Let n represent the number of nickels. Write an expression for the number of dimes.

Solution:

$7n+3$

Key Concepts

- **Notation**

Notation	**The result is…**
$a+b$	the sum of a and b
$a-b$	the difference of a and b
$a\cdot b, ab, (a)(b)\ (a)b, a(b)$	the product of a and b
$a\div b, a/b, \frac{a}{b}, b\overline{)a}$	the quotient of a and b

- **Inequality**

$a<b$ is read “a is less than b”	a is to the left of b on the number line
$a>b$ is read “a is greater than b”	a is to the right of b on the number line

- **Inequality Symbols** / **Words**

Inequality Symbols	**Words**
$a \neq b$	a is **not equal to** b
$a < b$	a is **less than** b
$a \leq b$	a is **less than or equal to** b
$a > b$	a is **greater than** b
$a \geq b$	a is **greater than or equal to** b

- **Grouping Symbols**
  - Parentheses ( )
  - Brackets [ ]
  - Braces { }
- **Exponential Notation**
  - a^n means multiply a by itself, n times. The expression a^n is read a to the n^{th} power.
- **Order of Operations:** When simplifying mathematical expressions perform the operations in the following order:

  Parentheses and other Grouping Symbols: Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
  Exponents: Simplify all expressions with exponents.
  Multiplication and Division: Perform all multiplication and division in order from left to right. These operations have equal priority.
  Addition and Subtraction: Perform all addition and subtraction in order from left to right. These operations have equal priority.

- **Combine Like Terms**

  Identify like terms.
  Rearrange the expression so like terms are together.
  Add or subtract the coefficients and keep the same variable for each group of like terms.

Practice Makes Perfect

Use Variables and Algebraic Symbols

In the following exercises, translate from algebra to English.

Exercise:

Problem: $16 - 9$

Solution:

16 minus 9, the difference of sixteen and nine

Exercise:

Problem: $3 \cdot 9$

Exercise:

Problem: $28 \div 4$

Solution:

28 divided by 4, the quotient of twenty-eight and four

Exercise:

Problem: $x + 11$

Exercise:

Problem: $(2)\,(7)$

Solution:

2 times 7, the product of two and seven

Exercise:

Problem: $(4)\,(8)$

Exercise:

Problem: $14 < 21$

Solution:

fourteen is less than twenty-one

Exercise:

Problem: $17 < 35$

Exercise:

Problem: $36 \geq 19$

Solution:

thirty-six is greater than or equal to nineteen

Exercise:

Problem: $6n = 36$

Exercise:

Problem: $y - 1 > 6$

Solution:

y minus 1 is greater than 6, the difference of *y* and one is greater than six

Exercise:

Problem: $y - 4 > 8$

Exercise:

Problem: $2 \leq 18 \div 6$

Solution:

2 is less than or equal to 18 divided by 6; 2 is less than or equal to the quotient of eighteen and six

Exercise:

Problem: $a \neq 1 \cdot 12$

In the following exercises, determine if each is an expression or an equation.

Exercise:

Problem: $9 \cdot 6 = 54$

Solution:

equation

Exercise:

Problem: $7 \cdot 9 = 63$

Exercise:

Problem: $5 \cdot 4 + 3$

Solution:

expression

Exercise:

Problem: $x + 7$

Exercise:

Problem: $x + 9$

Solution:

expression

Exercise:

Problem: $y - 5 = 25$

Simplify Expressions Using the Order of Operations

In the following exercises, simplify each expression.

Exercise:

Problem: 5^3

Solution:

125

Exercise:

Problem: 8^3

Exercise:

Problem: 2^8

Solution:

256

Exercise:

Problem: 10^5

In the following exercises, simplify using the order of operations.

Exercise:

Problem: ⓐ $3+8\cdot 5$ ⓑ $(3+8)\cdot 5$

Solution:

ⓐ 43 ⓑ 55

Exercise:

Problem: ⓐ $2+6\cdot 3$ ⓑ $(2+6)\cdot 3$

Exercise:

Problem: $2^3-12\div(9-5)$

Solution:

5

Exercise:

Problem: $3^2-18\div(11-5)$

Exercise:

Problem: $3\cdot 8+5\cdot 2$

Solution:

34

Exercise:

Problem: $4\cdot 7+3\cdot 5$

Exercise:

Problem: $2+8(6+1)$

Solution:

58

Exercise:

Problem: $4 + 6(3 + 6)$

Exercise:

Problem: $4 \cdot 12/8$

Solution:

6

Exercise:

Problem: $2 \cdot 36/6$

Exercise:

Problem: $(6 + 10) \div (2 + 2)$

Solution:

4

Exercise:

Problem: $(9 + 12) \div (3 + 4)$

Exercise:

Problem: $20 \div 4 + 6 \cdot 5$

Solution:

35

Exercise:

Problem: $33 \div 3 + 8 \cdot 2$

Exercise:

Problem: $3^2 + 7^2$

Solution:

58

Exercise:

Problem: $(3 + 7)^2$

Exercise:

Problem: $3(1+9\cdot 6)-4^2$

Solution:

149

Exercise:

Problem: $5(2+8\cdot 4)-7^2$

Exercise:

Problem: $2\left[1+3\left(10-2\right)\right]$

Solution:

50

Exercise:

Problem: $5\left[2+4\left(3-2\right)\right]$

Evaluate an Expression

In the following exercises, evaluate the following expressions.

Exercise:

Problem: $7x+8$ when $x=2$

Solution:

22

Exercise:

Problem: $8x-6$ when $x=7$

Exercise:

Problem: x^2 when $x=12$

Solution:

144

Exercise:

Problem: x^3 when $x=5$

Exercise:

Problem: x^5 when $x=2$

Solution:

32

Exercise:

Problem: 4^x when $x = 2$

Exercise:

Problem: $x^2 + 3x - 7$ when $x = 4$

Solution:

21

Exercise:

Problem: $6x + 3y - 9$ when $x = 6, y = 9$

Exercise:

Problem: $(x - y)^2$ when $x = 10, y = 7$

Solution:

9

Exercise:

Problem: $(x + y)^2$ when $x = 6, y = 9$

Exercise:

Problem: $a^2 + b^2$ when $a = 3, b = 8$

Solution:

73

Exercise:

Problem: $r^2 - s^2$ when $r = 12, s = 5$

Exercise:

Problem: $2l + 2w$ when $l = 15, w = 12$

Solution:

54

Exercise:

Problem: $2l + 2w$ when $l = 18, w = 14$

Simplify Expressions by Combining Like Terms

In the following exercises, identify the coefficient of each term.

Exercise:

Problem: $8a$

Solution:

8

Exercise:

Problem: $13m$

Exercise:

Problem: $5r^2$

Solution:

5

Exercise:

Problem: $6x^3$

In the following exercises, identify the like terms.

Exercise:

Problem: $x^3, 8x, 14, 8y, 5, 8x^3$

Solution:

x^3 and $8x^3$, 14 and 5

Exercise:

Problem: $6z, 3w^2, 1, 6z^2, 4z, w^2$

Exercise:

Problem: $9a, a^2, 16, 16b^2, 4, 9b^2$

Solution:

16 and 4, $16b^2$ and $9b^2$

Exercise:

Problem: $3, 25r^2, 10s, 10r, 4r^2, 3s$

In the following exercises, identify the terms in each expression.

Exercise:

Problem: $15x^2 + 6x + 2$

Solution:

$15x^2, 6x, 2$

Exercise:

Problem: $11x^2 + 8x + 5$

Exercise:

Problem: $10y^3 + y + 2$

Solution:

$10y^3, y, 2$

Exercise:

Problem: $9y^3 + y + 5$

In the following exercises, simplify the following expressions by combining like terms.

Exercise:

Problem: $10x + 3x$

Solution:

$13x$

Exercise:

Problem: $15x + 4x$

Exercise:

Problem: $4c + 2c + c$

Solution:

$7c$

Exercise:

Problem: $6y + 4y + y$

Exercise:

Problem: $7u + 2 + 3u + 1$

Solution:

$10u + 3$

Exercise:

Problem: $8d + 6 + 2d + 5$

Exercise:

Problem: $10a + 7 + 5a - 2 + 7a - 4$

Solution:

$22a + 1$

Exercise:

Problem: $7c + 4 + 6c - 3 + 9c - 1$

Exercise:

Problem: $3x^2 + 12x + 11 + 14x^2 + 8x + 5$

Solution:

$17x^2 + 20x + 16$

Exercise:

Problem: $5b^2 + 9b + 10 + 2b^2 + 3b - 4$

Translate an English Phrase to an Algebraic Expression

In the following exercises, translate the phrases into algebraic expressions.

Exercise:

Problem: the difference of 14 and 9

Solution:

$14 - 9$

Exercise:

Problem: the difference of 19 and 8

Exercise:

Problem: the product of 9 and 7

Solution:

$9 \cdot 7$

Exercise:

Problem: the product of 8 and 7

Exercise:

Problem: the quotient of 36 and 9

Solution:

$36 \div 9$

Exercise:

Problem: the quotient of 42 and 7

Exercise:

Problem: the sum of *8x* and 3*x*

Solution:

$8x+3x$

Exercise:

Problem: the sum of *13x* and 3*x*

Exercise:

Problem: the quotient of *y* and 3

Solution:

$\frac{y}{3}$

Exercise:

Problem: the quotient of *y* and 8

Exercise:

Problem: eight times the difference of *y* and nine

Solution:

$8\left(y-9\right)$

Exercise:

Problem: seven times the difference of *y* and one

Exercise:

Problem:

Eric has rock and classical CDs in his car. The number of rock CDs is 3 more than the number of classical CDs. Let *c* represent the number of classical CDs. Write an expression for the number of rock CDs.

Solution:

$c-4$

Exercise:

Problem:

The number of girls in a second-grade class is 4 less than the number of boys. Let *b* represent the number of boys. Write an expression for the number of girls.

Exercise:

Problem:

Greg has nickels and pennies in his pocket. The number of pennies is seven less than twice the number of nickels. Let n represent the number of nickels. Write an expression for the number of pennies.

Solution:

$2n-7$

Exercise:

Problem:

Jeannette has \$5 and \$10 bills in her wallet. The number of fives is three more than six times the number of tens. Let *t* represent the number of tens. Write an expression for the number of fives.

Everyday Math

Exercise:

Problem:

Car insurance Justin’s car insurance has a \$750 deductible per incident. This means that he pays \$750 and his insurance company will pay all costs beyond \$750. If Justin files a claim for \$2,100.

ⓐ how much will he pay?
ⓑ how much will his insurance company pay?

Solution:

ⓐ \$750 ⓑ \$1,350

Exercise:

Problem:

Home insurance Armando’s home insurance has a \$2,500 deductible per incident. This means that he pays \$2,500 and the insurance company will pay all costs beyond \$2,500. If Armando files a claim for \$19,400.

ⓐ how much will he pay?
ⓑ how much will the insurance company pay?

Writing Exercises

Exercise:

Problem: Explain the difference between an expression and an equation.

Solution:

Answers may vary

Exercise:

Problem: Why is it important to use the order of operations to simplify an expression?

Exercise:

Problem: Explain how you identify the like terms in the expression $8a^2+4a+9-a^2-1$.

Solution:

Answers may vary

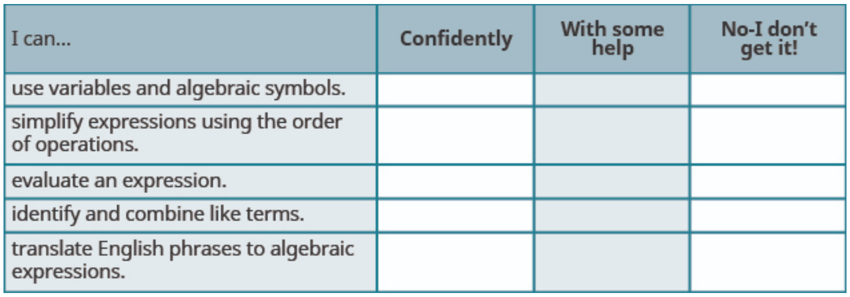
Exercise:

Problem:

Explain the difference between the phrases “4 times the sum of *x* and *y*” and “the sum of 4 times *x* and *y*.”

Self Check

ⓐ Use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
use variables and algebraic symbols.			
simplify expressions using the order of operations.			
evaluate an expression.			
identify and combine like terms.			
translate English phrases to algebraic expressions.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

coefficient
The coefficient of a term is the constant that multiplies the variable in a term.

constant
A constant is a number whose value always stays the same.

equality symbol
The symbol “$=$” is called the equal sign. We read $a = b$ as “a is equal to b.”

equation
An equation is two expressions connected by an equal sign.

evaluate an expression
To evaluate an expression means to find the value of the expression when the variable is replaced by a given number.

expression
An expression is a number, a variable, or a combination of numbers and variables using operation symbols.

like terms
Terms that are either constants or have the same variables raised to the same powers are called like terms.

simplify an expression
To simplify an expression, do all operations in the expression.

term
A term is a constant or the product of a constant and one or more variables.

variable

A variable is a letter that represents a number whose value may change.

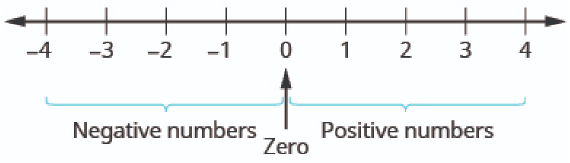
Add and Subtract Integers
By the end of this section, you will be able to:

- Use negatives and opposites
- Simplify: expressions with absolute value
- Add integers
- Subtract integers

Note:
A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **Integers**.

Use Negatives and Opposites

Our work so far has only included the counting numbers and the whole numbers. But if you have ever experienced a temperature below zero or accidentally overdrawn your checking account, you are already familiar with negative numbers. **Negative numbers** are numbers less than 0. The negative numbers are to the left of zero on the number line. See [link].

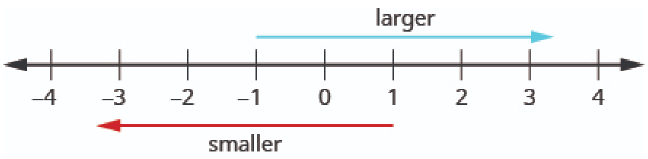


The number line shows the location of positive and negative numbers.

The arrows on the ends of the number line indicate that the numbers keep going forever. There is no biggest positive number, and there is no smallest negative number.

Is zero a positive or a negative number? Numbers larger than zero are positive, and numbers smaller than zero are negative. Zero is neither positive nor negative.

Consider how numbers are ordered on the number line. Going from left to right, the numbers increase in value. Going from right to left, the numbers decrease in value. See [link].



The numbers on a number line increase in value going from left to right and decrease in value going from right to left.

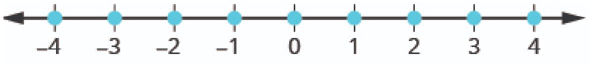
Note: Doing the Manipulative Mathematics activity “Number Line-part 2” will help you develop a better understanding of integers.

Remember that we use the notation:

$a < b$ (read “a is less than b”) when a is to the left of b on the number line.

$a > b$ (read “a is greater than b”) when a is to the right of b on the number line.

Now we need to extend the number line which showed the whole numbers to include negative numbers, too. The numbers marked by points in [link] are called the integers. The integers are the numbers $\ldots -3, -2, -1, 0, 1, 2, 3\ldots$



All the marked numbers are called *integers*.

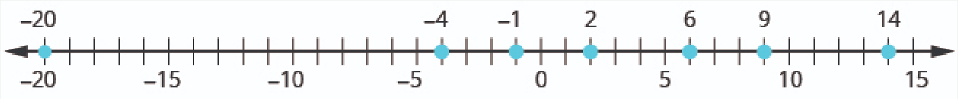
Example:
Exercise:

Problem:

Order each of the following pairs of numbers, using < or >: ⓐ 14 ___ 6 ⓑ -1 ___ 9 ⓒ -1 ___ -4 ⓓ 2 ___ -20.

Solution:
Solution

It may be helpful to refer to the number line shown.



ⓐ

$$14 \_\_\_ 6$$

14 is to the right of 6 on the number line. $14 > 6$

ⓑ

$$-1 \_\_\_ 9$$

-1 is to the left of 9 on the number line. $-1 < 9$

ⓒ

$$-1 \_\_\_ -4$$

-1 is to the right of -4 on the number line. $-1 > -4$

ⓓ

$$2 \_\_\_ -20$$

2 is to the right of -20 on the number line. $2 > -20$

Note:
Exercise:

Problem:

Order each of the following pairs of numbers, using $<$ or $>$: ⓐ $15 \_\_\_ 7$ ⓑ $-2 \_\_\_ 5$ ⓒ
$-3 \_\_\_ -7$
ⓓ $5 \_\_\_ -17$.

Solution:

ⓐ > ⓑ < ⓒ > ⓓ >

Note:
Exercise:

Problem:

Order each of the following pairs of numbers, using $<$ or $>$: ⓐ $8 \_\_\_ 13$ ⓑ $3 \_\_\_ -4$ ⓒ
$-5 \_\_\_ -2$
ⓓ $9 \_\_\_ -21$.

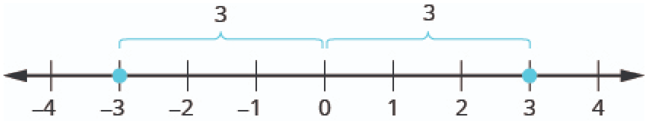
Solution:

ⓐ < ⓑ > ⓒ < ⓓ >

You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, they are called **opposites**. The opposite of 2 is -2, and the opposite of -2 is 2.

Note:
Opposite
The **opposite** of a number is the number that is the same distance from zero on the number line, but on the opposite side of zero.

[link] illustrates the definition.



The opposite of 3 is -3.

Sometimes in algebra the same symbol has different meanings. Just like some words in English, the specific meaning becomes clear by looking at how it is used. You have seen the symbol “–” used in three different ways.

Equation:

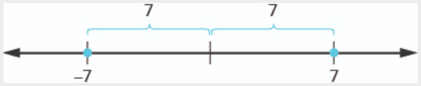
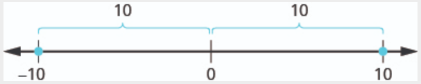
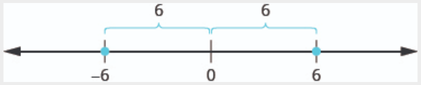
$$\begin{array}{ll}
10-4 & \text{Between two numbers, it indicates the operation of } \textit{subtraction}. \\
 & \text{We read } 10-4 \text{ as “10 minus 4.”} \\
-8 & \text{In front of a number, it indicates a } \textit{negative} \text{ number.} \\
 & \text{We read } -8 \text{ as “negative eight.”} \\
-x & \text{In front of a variable, it indicates the } \textit{opposite}. \text{ We read } -x \text{ as “the opposite of } x.\text{”} \\
-(-2) & \text{Here there are two “ } - \text{ ” signs. The one in the parentheses tells us the number is} \\
 & \text{negative } 2. \text{ The one outside the parentheses tells us to take the } \textit{opposite} \text{ of } -2. \\
 & \text{We read } -(-2) \text{ as “the opposite of negative two.”}
\end{array}$$

Note:
Opposite Notation
$-a$ means the opposite of the number a.
The notation $-a$ is read as “the opposite of a.”

Example:
Exercise:

Problem: Find: ⓐ the opposite of 7 ⓑ the opposite of -10 ⓒ $-(-6)$.

Solution:
Solution

ⓐ -7 is the same distance from 0 as 7, but on the opposite side of 0.	 The opposite of 7 is -7.
ⓑ 10 is the same distance from 0 as -10, but on the opposite side of 0.	 The opposite of -10 is 10.
ⓒ 6 is the same distance from 0 as -6, but on the opposite side of 0.	 $-(-6)$, or the opposite of -6, is 6.

Note:
Exercise:

Problem: Find: ⓐ the opposite of 4 ⓑ the opposite of -3 ⓒ $-(-1)$.

Solution:

ⓐ -4 ⓑ 3 ⓒ 1

Note:
Exercise:

Problem: Find: ⓐ the opposite of 8 ⓑ the opposite of -5 ⓒ $-(-5)$.

Solution:

ⓐ -8 ⓑ 5 ⓒ 5

Our work with opposites gives us a way to define the integers.The whole numbers and their opposites are called the **integers**. The integers are the numbers $\ldots -3, -2, -1, 0, 1, 2, 3\ldots$

Note:
Integers
The whole numbers and their opposites are called the **integers**.
The integers are the numbers
Equation:

$$\ldots -3, -2, -1, 0, 1, 2, 3\ldots$$

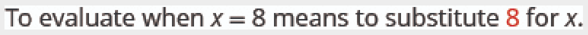
When evaluating the opposite of a variable, we must be very careful. Without knowing whether the variable represents a positive or negative number, we don't know whether $-x$ is positive or negative. We can see this in [link].

Example:
Exercise:

Problem: Evaluate ⓐ $-x$, when $x=8$ ⓑ $-x$, when $x=-8$.

Solution:
Solution

ⓐ

 To evaluate when $x = 8$ means to substitute 8 for x.	
	$-x$

Substitute 8 for *x*.	$-(8)$
Write the opposite of 8.	-8

ⓑ

To evaluate when $x = -8$ means to substitute -8 for x.	
	$-x$
Substitute -8 for *x*.	$-(-8)$
Write the opposite of −8.	8

Note:
Exercise:

Problem: Evaluate $-n$, when ⓐ $n = 4$ ⓑ $n = -4$.

Solution:

ⓐ -4 ⓑ 4

Note:
Exercise:

Problem: Evaluate $-m$, when ⓐ $m = 11$ ⓑ $m = -11$.

Solution:

ⓐ -11 ⓑ 11

Simplify: Expressions with Absolute Value

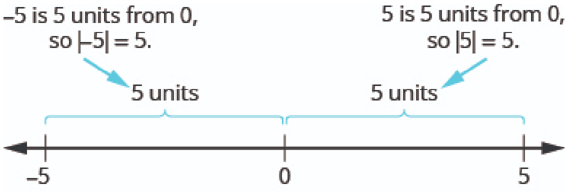
We saw that numbers such as 2 and -2 are opposites because they are the same distance from 0 on the number line. They are both two units from 0. The distance between 0 and any number on the number line is called the **absolute value** of that number.

Note:
Absolute Value
The **absolute value** of a number is its distance from 0 on the number line.
The absolute value of a number n is written as $|n|$.

For example,

- -5 is 5 units away from 0, so $|-5|=5$.
- 5 is 5 units away from 0, so $|5|=5$.

[link] illustrates this idea.


–5 is 5 units from 0, so |–5| = 5.
5 is 5 units from 0, so |5| = 5.
5 units
5 units
–5
0
5

The integers -5 and 5 are both 5 units away from 0.

The absolute value of a number is never negative (because distance cannot be negative). The only number with absolute value equal to zero is the number zero itself, because the distance from 0 to 0 on the number line is zero units.

Note:
Property of Absolute Value
$|n|\geq 0$ for all numbers
Absolute values are always greater than or equal to zero!

Mathematicians say it more precisely, “absolute values are always non-negative.” Non-negative means greater than or equal to zero.

Example:
Exercise:

Problem: Simplify: ⓐ $|3|$ ⓑ $|-44|$ ⓒ $|0|$.

Solution:
Solution

The absolute value of a number is the distance between the number and zero. Distance is never negative, so the absolute value is never negative.

ⓐ $|3|$
3

ⓑ $|-44|$
44

ⓒ $|0|$
0

Note:
Exercise:

Problem: Simplify: ⓐ $|4|$ ⓑ $|-28|$ ⓒ $|0|$.

Solution:

ⓐ 4 ⓑ 28 ⓒ 0

Note:
Exercise:

Problem: Simplify: ⓐ $|-13|$ ⓑ $|47|$ ⓒ $|0|$.

Solution:

ⓐ 13 ⓑ 47 ⓒ 0

In the next example, we’ll order expressions with absolute values. Remember, positive numbers are always greater than negative numbers!

Example:

Exercise:

Problem: Fill in $<, >,$ or $=$ for each of the following pairs of numbers:

ⓐ $|-5|$ ___ $-|-5|$ ⓑ 8 ___ $-|-8|$ ⓒ -9 ___ $-|-9|$ ⓓ $-(-16)$ ___ $-|-16|$

Solution:
Solution

ⓐ

$$\begin{array}{lrcl} & |-5| & \_\_\_ & -|-5| \\ \text{Simplify.} & 5 & \_\_\_ & -5 \\ \text{Order.} & 5 & > & -5 \\ & |-5| & > & -|-5| \end{array}$$

ⓑ

$$\begin{array}{lrcl} & 8 & \_\_\_ & -|-8| \\ \text{Simplify.} & 8 & \_\_\_ & -8 \\ \text{Order.} & 8 & > & -8 \\ & 8 & > & -|-8| \end{array}$$

ⓒ

$$\begin{array}{lrcl} & -9 & \_\_\_ & -|-9| \\ \text{Simplify.} & -9 & \_\_\_ & -9 \\ \text{Order.} & -9 & = & -9 \\ & -9 & = & -|-9| \end{array}$$

ⓓ

$$\begin{array}{lrcl} & -(-16) & \_\_\_ & -|-16| \\ \text{Simplify.} & 16 & \_\_\_ & -16 \\ \text{Order.} & 16 & > & -16 \\ & -(-16) & > & -|-16| \end{array}$$

Note:
Exercise:

Problem:

Fill in <, >, or = for each of the following pairs of numbers: ⓐ $|-9|$ ___ $-|-9|$ ⓑ 2 ___ $-|-2|$
ⓒ -8 ___ $|-8|$
ⓓ $-(-9)$ ___ $-|-9|$.

Solution:

ⓐ > ⓑ > ⓒ < ⓓ >

Note:
Exercise:

Problem:

Fill in <, >, or = for each of the following pairs of numbers: ⓐ 7 ___ $-|-7|$ ⓑ $-(-10)$ ___ $-|-10|$
ⓒ $|-4|$ ___ $-|-4|$ ⓓ -1 ___ $|-1|$.

Solution:

ⓐ > ⓑ > ⓒ > ⓓ <

We now add absolute value bars to our list of grouping symbols. When we use the order of operations, first we simplify inside the absolute value bars as much as possible, then we take the absolute value of the resulting number.

Note:
Grouping Symbols
Equation:

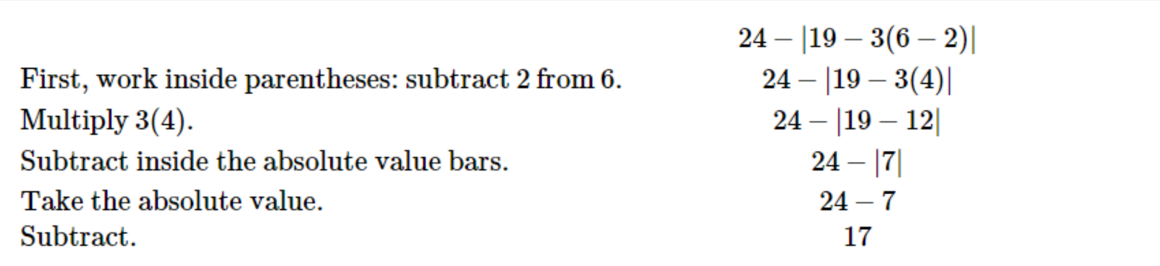
$$\begin{array}{llll} \text{Parentheses} & () & \text{Braces} & \{\} \\ \text{Brackets} & [] & \text{Absolute value} & |\,| \end{array}$$

In the next example, we simplify the expressions inside absolute value bars first, just like we do with parentheses.

Example:
Exercise:

Problem: Simplify: $24 - |19 - 3(6 - 2)|$.

Solution:
Solution



	$24 - |19 - 3(6 - 2)|$
First, work inside parentheses: subtract 2 from 6.	$24 - |19 - 3(4)|$
Multiply 3(4).	$24 - |19 - 12|$
Subtract inside the absolute value bars.	$24 - |7|$
Take the absolute value.	$24 - 7$
Subtract.	17

Note:
Exercise:

Problem: Simplify: $19 - |11 - 4(3 - 1)|$.

Solution:

16

Note:
Exercise:

Problem: Simplify: $9 - |8 - 4(7 - 5)|$.

Solution:

9

Example:
Exercise:

Problem:

Evaluate: ⓐ $|x|$ when $x = -35$ ⓑ $|-y|$ when $y = -20$ ⓒ $-|u|$ when $u = 12$ ⓓ $-|p|$ when $p = -14$.

Solution:
Solution

ⓐ $|x|$ when $x = -35$

	$|x|$
Substitute −35 for *x*.	$|-35|$
Take the absolute value.	35

ⓑ $|-y|$ when $y = -20$

	$\|-y\|$
Substitute -20 for y.	$\|-(-20)\|$
Simplify.	$\|20\|$
Take the absolute value.	20

ⓒ $-|u|$ when $u = 12$

	$-\|u\|$
Substitute 12 for u.	$-\|12\|$
Take the absolute value.	-12

ⓓ $-|p|$ when $p = -14$

	$-\|p\|$
Substitute -14 for p.	$-\|-14\|$

Take the absolute value.	-14

Note:
Exercise:

Problem:

Evaluate: ⓐ $|x|$ when $x=-17$ ⓑ $|-y|$ when $y=-39$ ⓒ $-|m|$ when $m=22$ ⓓ $-|p|$ when $p=-11$.

Solution:

ⓐ 17 ⓑ 39 ⓒ -22 ⓓ -11

Note:
Exercise:

Problem:

Evaluate: ⓐ $|y|$ when $y=-23$ ⓑ $|-y|$ when $y=-21$ ⓒ $-|n|$ when $n=37$ ⓓ $-|q|$ when $q=-49$.

Solution:

ⓐ 23 ⓑ 21 ⓒ -37 ⓓ -49

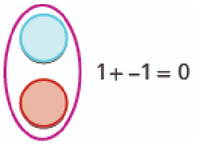
Add Integers

Most students are comfortable with the addition and subtraction facts for positive numbers. But doing addition or subtraction with both positive and negative numbers may be more challenging.

Note: Doing the Manipulative Mathematics activity “Addition of Signed Numbers” will help you develop a better understanding of adding integers.”

We will use two color counters to model addition and subtraction of negatives so that you can visualize the procedures instead of memorizing the rules.

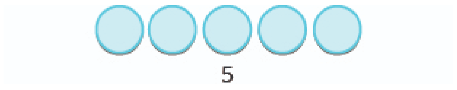
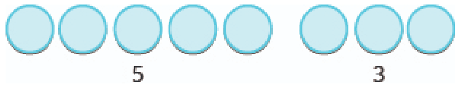
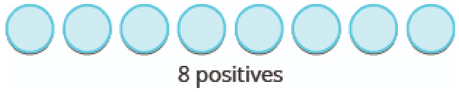
We let one color (blue) represent positive. The other color (red) will represent the negatives. If we have one positive counter and one negative counter, the value of the pair is zero. They form a neutral pair. The value of this neutral pair is zero.



We will use the counters to show how to add the four addition facts using the numbers $5, -5$ and $3, -3$.
Equation:

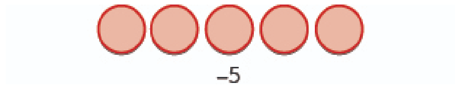
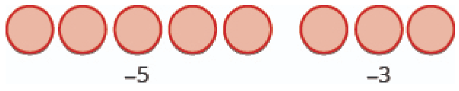
$$5+3 \qquad -5+(-3) \qquad -5+3 \qquad 5+(-3)$$

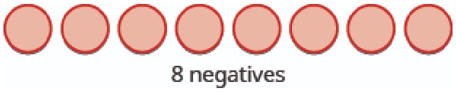
To add $5+3$, we realize that $5+3$ means the sum of 5 and 3.

We start with 5 positives.	
And then we add 3 positives.	
We now have 8 positives. The sum of 5 and 3 is 8.	

Now we will add $-5+(-3)$. Watch for similarities to the last example $5+3=8$.

To add $-5+(-3)$, we realize this means the sum of -5 and -3.

We start with 5 negatives.	
And then we add 3 negatives.	

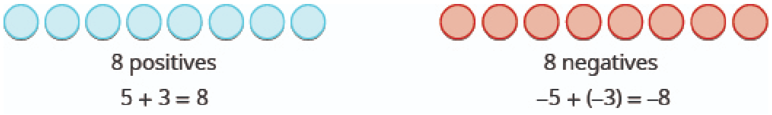
We now have 8 negatives. The sum of −5 and −3 is −8.	

In what ways were these first two examples similar?

- The first example adds 5 positives and 3 positives—both positives.
- The second example adds 5 negatives and 3 negatives—both negatives.

In each case we got 8—either 8 positives or 8 negatives.

When the signs were the same, the counters were all the same color, and so we added them.

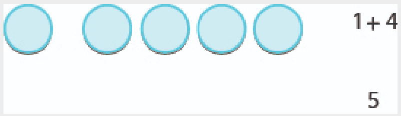


Example:
Exercise:

Problem: Add: ⓐ $1 + 4$ ⓑ $-1 + (-4)$.

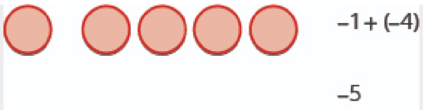
Solution:
Solution

ⓐ



1 positive plus 4 positives is 5 positives.

ⓑ



1 negative plus 4 negatives is 5 negatives.

Note:
Exercise:

Problem: Add: ⓐ $2+4$ ⓑ $-2+(-4)$.

Solution:

ⓐ 6 ⓑ -6

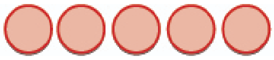
Note:
Exercise:

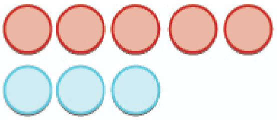
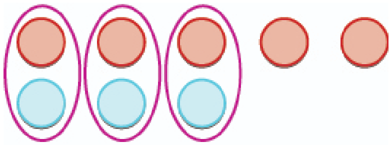
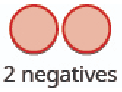
Problem: Add: ⓐ $2+5$ ⓑ $-2+(-5)$.

Solution:

ⓐ 7 ⓑ -7

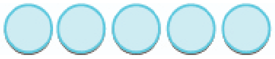
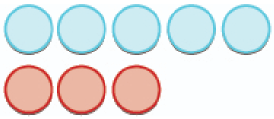
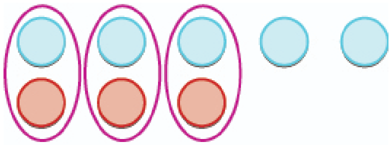
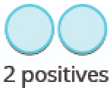
So what happens when the signs are different? Let’s add $-5+3$. We realize this means the sum of -5 and 3. When the counters were the same color, we put them in a row. When the counters are a different color, we line them up under each other.

	$-5+3$ means the sum of -5 and 3.
We start with 5 negatives.	
And then we add 3 positives.	

	
We remove any neutral pairs.	
We have 2 negatives left.	
The sum of -5 and 3 is -2.	$-5+3=-2$

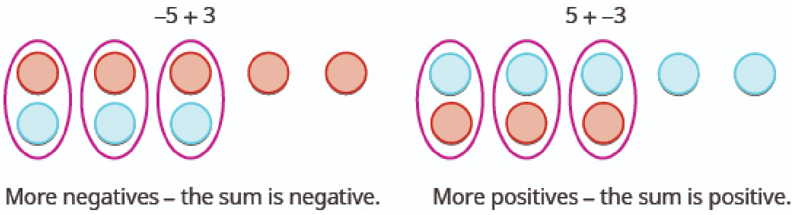
Notice that there were more negatives than positives, so the result was negative.

Let’s now add the last combination, $5+(-3)$.

	$5+(-3)$. means the sum of 5 and -3.
We start with 5 positives.	
And then we add 3 negatives.	
We remove any neutral pairs.	
We have 2 positives left.	

The sum of 5 and -3 is 2.	$5+(-3)=2$

When we use counters to model addition of positive and negative integers, it is easy to see whether there are more positive or more negative counters. So we know whether the sum will be positive or negative.

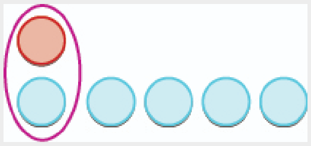

–5 + 3
More negatives – the sum is negative.
5 + –3
More positives – the sum is positive.

Example:
Exercise:

Problem: Add: ⓐ $-1+5$ ⓑ $1+(-5)$.

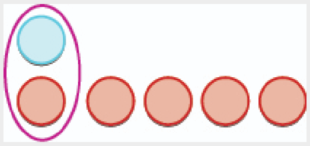
Solution:
Solution

ⓐ

	$-1+5$
	
There are more positives, so the sum is positive.	4

ⓑ

	$1+(-5)$.

	
There are more negatives, so the sum is negative.	-4

Note:
Exercise:

Problem: Add: ⓐ $-2+4$ ⓑ $2+(-4)$.

Solution:

ⓐ 2 ⓑ -2

Note:
Exercise:

Problem: Add: ⓐ $-2+5$ ⓑ $2+(-5)$.

Solution:

ⓐ 3 ⓑ -3

Now that we have added small positive and negative integers with a model, we can visualize the model in our minds to simplify problems with any numbers.

When you need to add numbers such as $37+(-53)$, you really don’t want to have to count out 37 blue counters and 53 red counters. With the model in your mind, can you visualize what you would do to solve the problem?

Picture 37 blue counters with 53 red counters lined up underneath. Since there would be more red (negative) counters than blue (positive) counters, the sum would be *negative*. How many more red counters would there be? Because $53-37=16$, there are 16 more red counters.

Therefore, the sum of $37+(-53)$ is -16.
Equation:

$$37+(-53)=-16$$

Let’s try another one. We’ll add $-74+(-27)$. Again, imagine 74 red counters and 27 more red counters, so we’d have 101 red counters. This means the sum is -101.

Equation:

$$-74+(-27)=-101$$

Let’s look again at the results of adding the different combinations of 5, -5 and 3, -3.

Note:

Addition of Positive and Negative Integers

Equation:

$$\begin{array}{cc} 5+3 & -5+(-3) \\ 8 & -8 \\ \text{both positive, sum positive} & \text{both negative, sum negative} \end{array}$$

When the signs are the same, the counters would be all the same color, so add them.

Equation:

$$\begin{array}{cc} -5+3 & 5+(-3) \\ -2 & 2 \\ \text{different signs, more negatives, sum negative} & \text{different signs, more positives, sum positive} \end{array}$$

When the signs are different, some of the counters would make neutral pairs, so subtract to see how many are left.

Visualize the model as you simplify the expressions in the following examples.

Example:

Exercise:

Problem: Simplify: ⓐ $19+(-47)$ ⓑ $-14+(-36)$.

Solution:

ⓐ Since the signs are different, we subtract 19 from 47. The answer will be negative because there are more negatives than positives.

$$\begin{array}{ll} & 19+(-47) \\ \text{Add.} & -28 \end{array}$$

ⓑ Since the signs are the same, we add. The answer will be negative because there are only negatives.

$$-14 + (-36)$$
Add. -50

Note:
Exercise:

Problem: Simplify: ⓐ $-31 + (-19)$ ⓑ $15 + (-32)$.

Solution:

ⓐ -50 ⓑ -17

Note:
Exercise:

Problem: Simplify: ⓐ $-42 + (-28)$ ⓑ $25 + (-61)$.

Solution:

ⓐ -70 ⓑ -36

The techniques used up to now extend to more complicated problems, like the ones we've seen before. Remember to follow the order of operations!

Example:
Exercise:

Problem: Simplify: $-5 + 3(-2 + 7)$.

Solution:
Solution

	$-5 + 3(-2 + 7)$
Simplify inside the parentheses.	$-5 + 3(5)$
Multiply.	$-5 + 15$
Add left to right.	10

Note:
Exercise:

Problem: Simplify: $-2+5\,(-4+7)$.

Solution:

13

Note:
Exercise:

Problem: Simplify: $-4+2\,(-3+5)$.

Solution:

0

Subtract Integers

Note: Doing the Manipulative Mathematics activity “Subtraction of Signed Numbers” will help you develop a better understanding of subtracting integers.

We will continue to use counters to model the subtraction. Remember, the blue counters represent positive numbers and the red counters represent negative numbers.

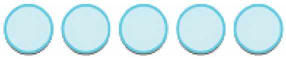
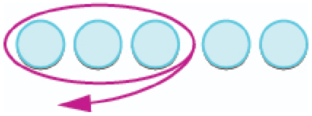
Perhaps when you were younger, you read “$5-3$” as “5 take away 3.” When you use counters, you can think of subtraction the same way!

We will model the four subtraction facts using the numbers 5 and 3.
Equation:

$$5-3 \qquad -5-(-3) \qquad -5-3 \qquad 5-(-3)$$

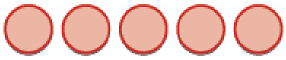
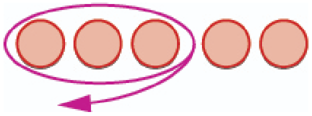
To subtract $5-3$, we restate the problem as “5 take away 3.”

We start with 5 positives.	

	
We ‘take away’ 3 positives.	
We have 2 positives left.	
The difference of 5 and 3 is 2.	2

Now we will subtract $-5-(-3)$. Watch for similarities to the last example $5-3=2$.

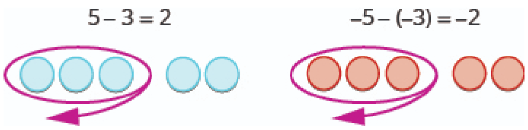
To subtract $-5-(-3)$, we restate this as “–5 take away –3”

We start with 5 negatives.	
We ‘take away’ 3 negatives.	
We have 2 negatives left.	
The difference of −5 and −3 is −2.	−2

Notice that these two examples are much alike: The first example, we subtract 3 positives from 5 positives and end up with 2 positives.

In the second example, we subtract 3 negatives from 5 negatives and end up with 2 negatives.

Each example used counters of only one color, and the “take away” model of subtraction was easy to apply.



Example:
Exercise:

Problem: Subtract: ⓐ $7-5$ ⓑ $-7-(-5)$.

Solution:

ⓐ

	$7-5$
Take 5 positives from 7 positives and get 2 positives.	2

ⓑ

	$-7-(-5)$
Take 5 negatives from 7 negatives and get 2 negatives.	-2

Note:
Exercise:

Problem: Subtract: ⓐ $6-4$ ⓑ $-6-(-4)$.

Solution:

ⓐ 2 ⓑ -2

Note:
Exercise:

Problem: Subtract: ⓐ $7-4$ ⓑ $-7-(-4)$.

Solution:

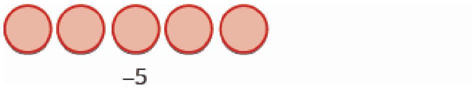
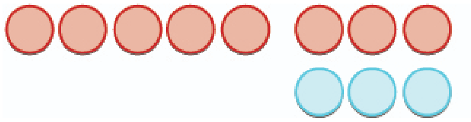
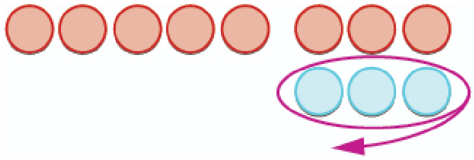
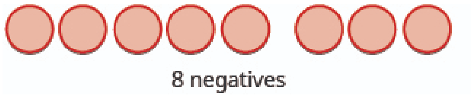
ⓐ 3 ⓑ -3

What happens when we have to subtract one positive and one negative number? We'll need to use both blue and red counters as well as some neutral pairs. Adding a neutral pair does not change the value. It is like changing quarters to nickels—the value is the same, but it looks different.

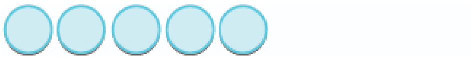
- To subtract $-5-3$, we restate it as -5 take away 3.

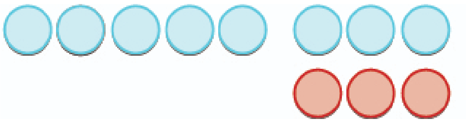
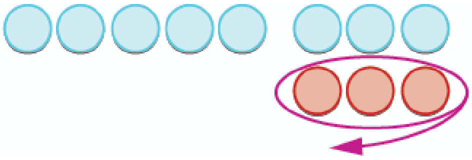
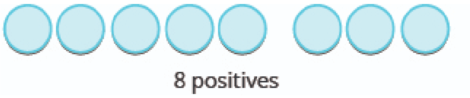
We start with 5 negatives. We need to take away 3 positives, but we do not have any positives to take away.

Remember, a neutral pair has value zero. If we add 0 to 5 its value is still 5. We add neutral pairs to the 5 negatives until we get 3 positives to take away.

	$-5-3$ means -5 take away 3.
We start with 5 negatives.	
We now add the neutrals needed to get 3 positives.	
We remove the 3 positives.	
We are left with 8 negatives.	
The difference of -5 and 3 is -8.	$-5-3=-8$

And now, the fourth case, $5-(-3)$. We start with 5 positives. We need to take away 3 negatives, but there are no negatives to take away. So we add neutral pairs until we have 3 negatives to take away.

	$5-(-3)$. means 5 take away -3.
We start with 5 positives.	
We now add the needed neutrals pairs.	

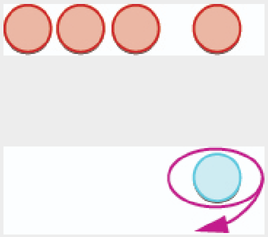
	
We remove the 3 negatives.	
We are left with 8 positives.	
The difference of 5 and -3 is 8.	$5-(-3)=8.$

Example:
Exercise:

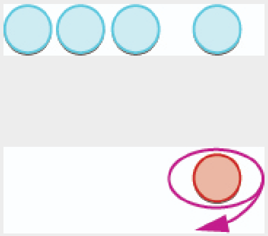
Problem: Subtract: ⓐ $-3-1$ ⓑ $3-(-1)$.

Solution:
Solution

ⓐ

Take 1 positive from the one added neutral pair.		$-3-1$ -4

ⓑ

Take 1 negative from the one added neutral pair.		$3-(-1)$. 4

Note:
Exercise:

Problem: Subtract: ⓐ $-6-4$ ⓑ $6-(-4)$.

Solution:

ⓐ -10 ⓑ 10

Note:
Exercise:

Problem: Subtract: ⓐ $-7-4$ ⓑ $7-(-4)$.

Solution:

ⓐ -11 ⓑ 11

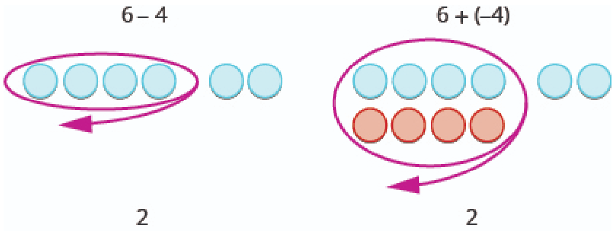
Have you noticed that *subtraction of integers can be done by adding the opposite*? In [link], $-3-1$ is the same as $-3+(-1)$ and $3-(-1)$ is the same as $3+1$. You will often see this idea, the **subtraction property**, written as follows:

Note:
Subtraction Property
Equation:

$$a-b=a+(-b)$$

Subtracting a number is the same as adding its opposite.

Look at these two examples.



Equation:

$$6 - 4 \text{ gives the same answer as } 6 + (-4).$$

Of course, when you have a subtraction problem that has only positive numbers, like $6 - 4$, you just do the subtraction. You already knew how to subtract $6 - 4$ long ago. But *knowing* that $6 - 4$ gives the same answer as $6 + (-4)$ helps when you are subtracting negative numbers. Make sure that you understand how $6 - 4$ and $6 + (-4)$ give the same results!

Example:
Exercise:

Problem: Simplify: ⓐ $13 - 8$ and $13 + (-8)$ ⓑ $-17 - 9$ and $-17 + (-9)$.

Solution:
Solution

ⓐ

	$13 - 8$	and	$13 + (-8)$
Subtract.	5		5

ⓑ

	$-17 - 9$	and	$-17 + (-9)$
Subtract.	-26		-26

Note:
Exercise:

Problem: Simplify: ⓐ $21 - 13$ and $21 + (-13)$ ⓑ $-11 - 7$ and $-11 + (-7)$.

Solution:

ⓐ 8 ⓑ -18

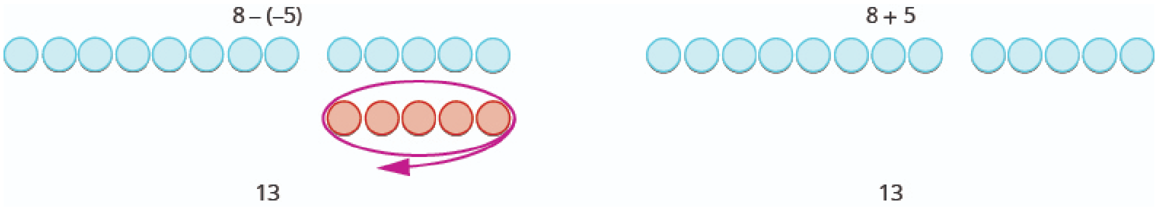
Note:
Exercise:

Problem: Simplify: ⓐ $15 - 7$ and $15 + (-7)$ ⓑ $-14 - 8$ and $-14 + (-8)$.

Solution:

ⓐ 8 ⓑ -22

Look at what happens when we subtract a negative.



Equation:

$$8 - (-5) \text{ gives the same answer as } 8 + 5$$

Subtracting a negative number is like adding a positive!

You will often see this written as $a - (-b) = a + b$.

Does that work for other numbers, too? Let’s do the following example and see.

Example:
Exercise:

Problem: Simplify: ⓐ $9 - (-15)$ and $9 + 15$ ⓑ $-7 - (-4)$ and $-7 + 4$.

Solution:
Solution

ⓐ

	$9 - (-15)$	$9 + 15$
Subtract.	24	24

ⓑ

	$-7 - (-4)$	$-7 + 4$
Subtract.	-3	-3

Note:
Exercise:

Problem: Simplify: ⓐ $6-(-13)$ and $6+13$ ⓑ $-5-(-1)$ and $-5+1$.

Solution:

ⓐ 19 ⓑ -4

Note:
Exercise:

Problem: Simplify: ⓐ $4-(-19)$ and $4+19$ ⓑ $-4-(-7)$ and $-4+7$.

Solution:

ⓐ 23 ⓑ 3

Let’s look again at the results of subtracting the different combinations of 5, -5 and 3, -3.

Note:
Subtraction of Integers
Equation:

$$\begin{array}{cc} 5-3 & -5-(-3) \\ 2 & -2 \\ \text{5 positives take away 3 positives} & \text{5 negatives take away 3 negatives} \\ \text{2 positives} & \text{2 negatives} \end{array}$$

When there would be enough counters of the color to take away, subtract.
Equation:

$$\begin{array}{cc} -5-3 & 5-(-3) \\ -8 & 8 \\ \text{5 negatives, want to take away 3 positives} & \text{5 positives, want to take away 3 negatives} \\ \text{need neutral pairs} & \text{need neutral pairs} \end{array}$$

When there would be not enough counters of the color to take away, add.

What happens when there are more than three integers? We just use the order of operations as usual.

Example:
Exercise:

Problem: Simplify: $7-(-4-3)-9$.

Solution:
Solution

	$7-(-4-3)-9$
Simplify inside the parentheses first.	$7-(-7)-9$
Subtract left to right.	$14-9$
Subtract.	5

Note:
Exercise:

Problem: Simplify: $8-(-3-1)-9$.

Solution:

3

Note:
Exercise:

Problem: Simplify: $12-(-9-6)-14$.

Solution:

13

Note:
Access these online resources for additional instruction and practice with adding and subtracting integers. You will need to enable Java in your web browser to use the applications.

- Add Colored Chip
- Subtract Colored Chip

Key Concepts

- **Addition of Positive and Negative Integers**

$5+3$ 8 both positive, sum positive	$-5+(-3)$ -8 both negative, sum negative
$-5+3$ -2 different signs, more negatives sum negative	$5+(-3)$ 2 different signs, more positives sum positive

- **Property of Absolute Value**: $|n| \geq 0$ for all numbers. Absolute values are always greater than or equal to zero!
- **Subtraction of Integers**

$5-3$ 2 5 positives take away 3 positives 2 positives	$-5-(-3)$ -2 5 negatives take away 3 negatives 2 negatives
$-5-3$ -8 5 negatives, want to subtract 3 positives need neutral pairs	$5-(-3)$ 8 5 positives, want to subtract 3 negatives need neutral pairs

- **Subtraction Property:** Subtracting a number is the same as adding its opposite.

Practice Makes Perfect

Use Negatives and Opposites of Integers

In the following exercises, order each of the following pairs of numbers, using < or >.

Exercise:

ⓐ 9 ___ 4
ⓑ -3 ___ 6
ⓒ -8 ___ -2
Problem: ⓓ 1 ___ -10

Solution:

ⓐ > ⓑ < ⓒ < ⓓ >

Exercise:

ⓐ -7 ___ 3
ⓑ -10 ___ -5
ⓒ 2 ___ -6
Problem: ⓓ 8 ___ 9

In the following exercises, find the opposite of each number.

Exercise:

ⓐ 2
Problem: ⓑ -6

Solution:

ⓐ -2 ⓑ 6

Exercise:

ⓐ 9
Problem: ⓑ -4

In the following exercises, simplify.

Exercise:

Problem: $-(-4)$

Solution:

4

Exercise:

Problem: $-(-8)$

Exercise:

Problem: $-(-15)$

Solution:

15

Exercise:

Problem: $-(-11)$

In the following exercises, evaluate.

Exercise:

Problem: $-c$ when
ⓐ $c = 12$
ⓑ $c = -12$

Solution:

ⓐ -12 ⓑ 12

Exercise:

Problem: $-d$ when
ⓐ $d = 21$
ⓑ $d = -21$

Simplify Expressions with Absolute Value

In the following exercises, simplify.

Exercise:

Problem:
ⓐ $|-32|$
ⓑ $|0|$
ⓒ $|16|$

Solution:

ⓐ 32 ⓑ 0 ⓒ 16

Exercise:

Problem:
ⓐ $|0|$
ⓑ $|-40|$
ⓒ $|22|$

In the following exercises, fill in <, >, or $=$ for each of the following pairs of numbers.

Exercise:

Problem:
ⓐ -6 ___ $|-6|$
ⓑ $-|-3|$ ___ -3

Solution:

ⓐ < ⓑ $=$

Exercise:

ⓐ $|-5|$ ___ $-|-5|$
Problem: ⓑ 9 ___ $-|-9|$

In the following exercises, simplify.

Exercise:

Problem: $-(-5)$ and $-|-5|$

Solution:

$5, -5$

Exercise:

Problem: $-|-9|$ and $-(-9)$

Exercise:

Problem: $8|-7|$

Solution:

56

Exercise:

Problem: $5|-5|$

Exercise:

Problem: $|15-7|-|14-6|$

Solution:

0

Exercise:

Problem: $|17-8|-|13-4|$

Exercise:

Problem: $18-|2(8-3)|$

Solution:

8

Exercise:

Problem: $18 - |3(8-5)|$

In the following exercises, evaluate.

Exercise:

Problem: ⓐ $-|p|$ when $p = 19$
ⓑ $-|q|$ when $q = -33$

Solution:

ⓐ -19 ⓑ -33

Exercise:

Problem: ⓐ $-|a|$ when $a = 60$
ⓑ $-|b|$ when $b = -12$

Add Integers

In the following exercises, simplify each expression.

Exercise:

Problem: $-21 + (-59)$

Solution:

-80

Exercise:

Problem: $-35 + (-47)$

Exercise:

Problem: $48 + (-16)$

Solution:

32

Exercise:

Problem: $34 + (-19)$

Exercise:

Problem: $-14 + (-12) + 4$

Solution:

-22

Exercise:

Problem: $-17 + (-18) + 6$

Exercise:

Problem: $135 + (-110) + 83$

Solution:

108

Exercise:

Problem: $6 - 38 + 27 + (-8) + 126$

Exercise:

Problem: $19 + 2\left(-3 + 8\right)$

Solution:

29

Exercise:

Problem: $24 + 3\left(-5 + 9\right)$

Subtract Integers

In the following exercises, simplify.

Exercise:

Problem: $2 - 8$

Solution:

-6

Exercise:

Problem: $-6 - (-4)$

Exercise:

Problem: $-5 - 4$

Solution:

-9

Exercise:

Problem: $-7-2$

Exercise:

Problem: $8-(-4)$

Solution:

12

Exercise:

Problem: $7-(-3)$

Exercise:

Problem: ⓐ $44-28$ ⓑ $44+(-28)$

Solution:

ⓐ 16 ⓑ 16

Exercise:

Problem: ⓐ $35-16$ ⓑ $35+(-16)$

Exercise:

Problem: ⓐ $27-(-18)$ ⓑ $27+18$

Solution:

ⓐ 45 ⓑ 45

Exercise:

Problem: ⓐ $46-(-37)$ ⓑ $46+37$

In the following exercises, simplify each expression.

Exercise:

Problem: $15-(-12)$

Solution:

27

Exercise:

Problem: $14-(-11)$

Exercise:

Problem: $48-87$

Solution:

-39

Exercise:

Problem: $45-69$

Exercise:

Problem: $-17-42$

Solution:

-59

Exercise:

Problem: $-19-46$

Exercise:

Problem: $-103-(-52)$

Solution:

-51

Exercise:

Problem: $-105-(-68)$

Exercise:

Problem: $-45-(54)$

Solution:

-99

Exercise:

Problem: $-58-(-67)$

Exercise:

Problem: $8-3-7$

Solution:

-2

Exercise:

Problem: $9-6-5$

Exercise:

Problem: $-5-4+7$

Solution:

-2

Exercise:

Problem: $-3-8+4$

Exercise:

Problem: $-14-(-27)+9$

Solution:

22

Exercise:

Problem: $64+(-17)-9$

Exercise:

Problem: $(2-7)-(3-8)$

Solution:

0

Exercise:

Problem: $(1-8)-(2-9)$

Exercise:

Problem: $-(6-8)-(2-4)$

Solution:

4

Exercise:

Problem: $-(4-5)-(7-8)$

Exercise:

Problem: $25-[10-(3-12)]$

Solution:

6

Exercise:

Problem: $32-[5-(15-20)]$

Exercise:

Problem: $6.3-4.3-7.2$

Solution:

-5.2

Exercise:

Problem: $5.7-8.2-4.9$

Exercise:

Problem: 5^2-6^2

Solution:

-11

Exercise:

Problem: 6^2-7^2

Everyday Math

Exercise:

Problem:

Elevation The highest elevation in the United States is Mount McKinley, Alaska, at 20,320 feet above sea level. The lowest elevation is Death Valley, California, at 282 feet below sea level.

Use integers to write the elevation of:

ⓐ Mount McKinley.
ⓑ Death Valley.

Solution:

ⓐ 20329 ⓑ -282

Exercise:

Problem:

Extreme temperatures The highest recorded temperature on Earth was 58° Celsius, recorded in the Sahara Desert in 1922. The lowest recorded temperature was 90° below 0° Celsius, recorded in Antarctica in 1983.

Use integers to write the:

ⓐ highest recorded temperature.
ⓑ lowest recorded temperature.

Exercise:

Problem:

State budgets In June, 2011, the state of Pennsylvania estimated it would have a budget surplus of $540 million. That same month, Texas estimated it would have a budget deficit of $27 billion.

Use integers to write the budget of:

ⓐ Pennsylvania.
ⓑ Texas.

Solution:

ⓐ $\$540$ million ⓑ $-\$27$ billion

Exercise:

Problem:

College enrollments Across the United States, community college enrollment grew by 1,400,000 students from Fall 2007 to Fall 2010. In California, community college enrollment declined by 110,171 students from Fall 2009 to Fall 2010.

Use integers to write the change in enrollment:

ⓐ in the U.S. from Fall 2007 to Fall 2010.
ⓑ in California from Fall 2009 to Fall 2010.

Exercise:

Problem:

Stock Market The week of September 15, 2008 was one of the most volatile weeks ever for the US stock market. The closing numbers of the Dow Jones Industrial Average each day were:

Monday	-504
Tuesday	$+142$
Wednesday	-449
Thursday	$+410$
Friday	$+369$

What was the overall change for the week? Was it positive or negative?

Solution:

-32

Exercise:

Problem:

Stock Market During the week of June 22, 2009, the closing numbers of the Dow Jones Industrial Average each day were:

Monday	-201
Tuesday	-16
Wednesday	-23
Thursday	$+172$
Friday	-34

What was the overall change for the week? Was it positive or negative?

Writing Exercises

Exercise:

Problem: Give an example of a negative number from your life experience.

Solution:

Answers may vary

Exercise:

Problem: What are the three uses of the “ $-$ ” sign in algebra? Explain how they differ.

Exercise:

Problem: Explain why the sum of -8 and 2 is negative, but the sum of 8 and -2 is positive.

Solution:

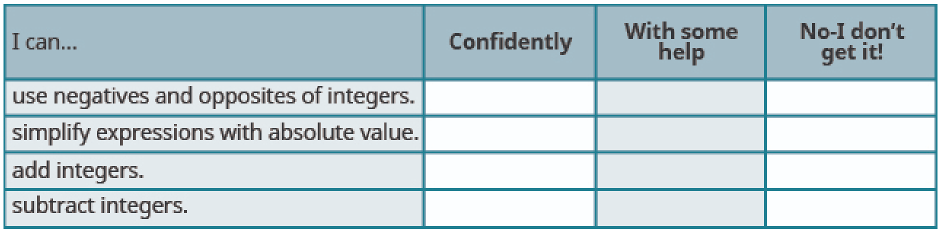
Answers may vary

Exercise:

Problem: Give an example from your life experience of adding two negative numbers.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don’t get it!
use negatives and opposites of integers.			
simplify expressions with absolute value.			
add integers.			
subtract integers.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

absolute value

The absolute value of a number is its distance from 0 on the number line. The absolute value of a number n is written as $|n|$.

integers

The whole numbers and their opposites are called the integers: ...−3, −2, −1, 0, 1, 2, 3...

opposite

The opposite of a number is the number that is the same distance from zero on the number line but on the opposite side of zero: $-a$ means the opposite of the number. The notation $-a$ is read “the opposite of a.”

Multiply and Divide Integers

By the end of this section, you will be able to:

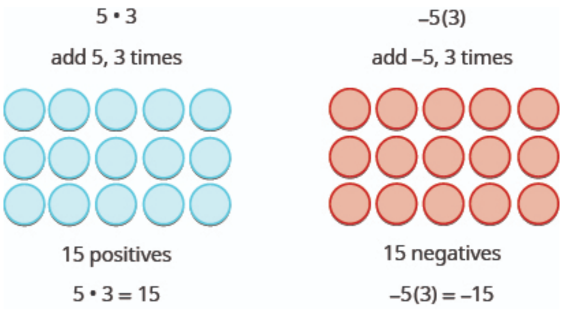
- Multiply integers
- Divide integers
- Simplify expressions with integers
- Evaluate variable expressions with integers
- Translate English phrases to algebraic expressions
- Use integers in applications

Note:
A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **Integers**.

Multiply Integers

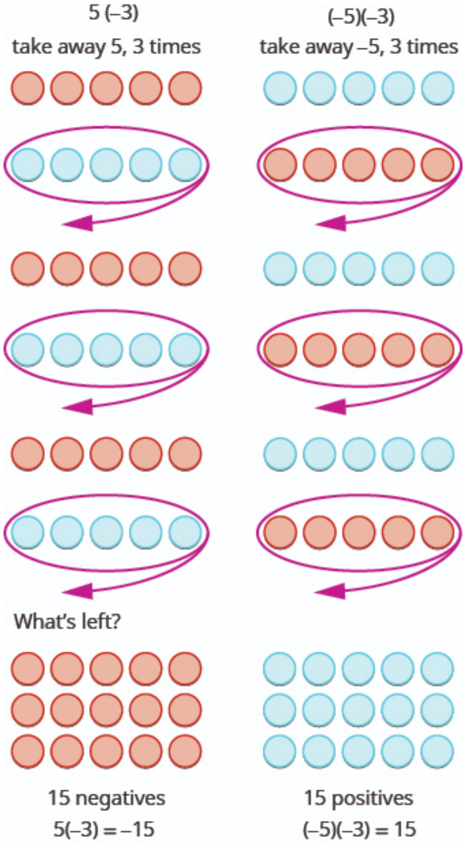
Since multiplication is mathematical shorthand for repeated addition, our model can easily be applied to show multiplication of integers. Let’s look at this concrete model to see what patterns we notice. We will use the same examples that we used for addition and subtraction. Here, we will use the model just to help us discover the pattern.

We remember that $a \cdot b$ means add a, b times. Here, we are using the model just to help us discover the pattern.



The next two examples are more interesting.

What does it mean to multiply 5 by -3? It means subtract 5, 3 times. Looking at subtraction as “taking away,” it means to take away 5, 3 times. But there is nothing to take away, so we start by adding neutral pairs on the workspace. Then we take away 5 three times.



In summary:
Equation:

$$\begin{array}{rclcrcl} 5\cdot 3 & = & 15 & \quad & -5\left(3\right) & = & -15 \\ 5\left(-3\right) & = & -15 & \quad & \left(-5\right)\left(-3\right) & = & 15 \end{array}$$

Notice that for multiplication of two integers, when the:

- signs are the *same*, the product is *positive*.
- signs are *different*, the product is *negative*.

We'll put this all together in the chart below.

Note:
Multiplication of Integers
For multiplication of two integers:

Same signs	**Product**	**Example**

Same signs	Product	Example
Two positives Two negatives	Positive Positive	$7 \cdot 4 = 28$ $-8(-6) = 48$

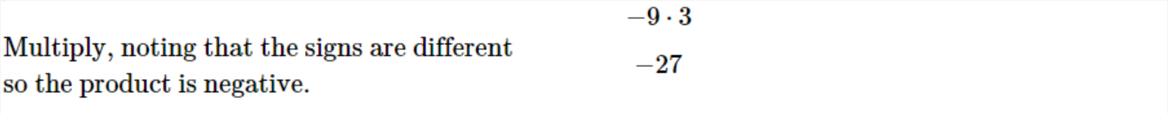
Different signs	Product	Example
Positive · negative Negative · positive	Negative Negative	$7(-9) = -63$ $-5 \cdot 10 = -50$

Example:
Exercise:

Problem: Multiply: ⓐ $-9 \cdot 3$ ⓑ $-2(-5)$ ⓒ $4(-8)$ ⓓ $7 \cdot 6$.

Solution:
Solution

ⓐ

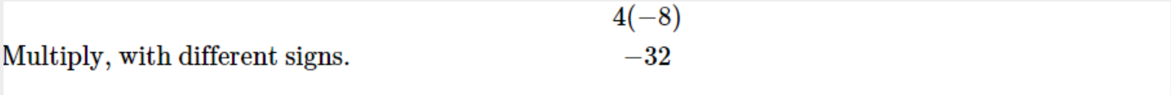


	$-9 \cdot 3$
Multiply, noting that the signs are different so the product is negative.	-27

ⓑ

	$-2(-5)$
Multiply, noting that the signs are the same so the product is positive.	10

ⓒ



	$4(-8)$
Multiply, with different signs.	-32

ⓓ

	$7 \cdot 6$
Multiply, with same signs.	42

Note:
Exercise:

Problem: Multiply: ⓐ $-6 \cdot 8$ ⓑ $-4\,(-7)$ ⓒ $9\,(-7)$ ⓓ $5 \cdot 12$.

Solution:

ⓐ -48 ⓑ 28 ⓒ -63 ⓓ 60

Note:
Exercise:

Problem: Multiply: ⓐ $-8 \cdot 7$ ⓑ $-6\,(-9)$ ⓒ $7\,(-4)$ ⓓ $3 \cdot 13$.

Solution:

ⓐ -56 ⓑ 54 ⓒ -28 ⓓ 39

When we multiply a number by 1, the result is the same number. What happens when we multiply a number by -1? Let's multiply a positive number and then a negative number by -1 to see what we get.
Equation:

$$\begin{array}{ccc} & -1 \cdot 4 & -1(-3) \\ \text{Multiply.} & -4 & 3 \\ & -4 \text{ is the opposite of } 4. & 3 \text{ is the opposite of } -3. \end{array}$$

Each time we multiply a number by -1, we get its opposite!

Note:
Multiplication by -1
Equation:

$$-1a = -a$$

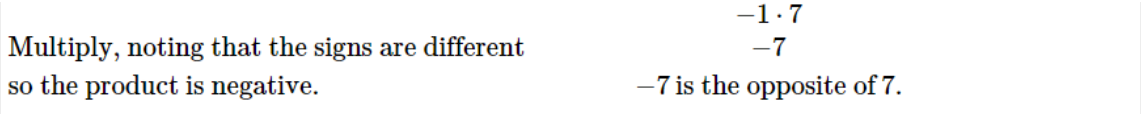
Multiplying a number by -1 gives its opposite.

Example:
Exercise:

Problem: Multiply: ⓐ $-1 \cdot 7$ ⓑ $-1\,(-11)$.

Solution:
Solution

ⓐ



	$-1 \cdot 7$
Multiply, noting that the signs are different	-7
so the product is negative.	-7 is the opposite of 7.

ⓑ

	$-1\,(-11)$
Multiply, noting that the signs are the same	11
so the product is positive.	11 is the opposite of -11.

Note:
Exercise:

Problem: Multiply: ⓐ $-1 \cdot 9$ ⓑ $-1 \cdot (-17)$.

Solution:

ⓐ -9 ⓑ 17

Note:
Exercise:

Problem: Multiply: ⓐ $-1 \cdot 8$ ⓑ $-1 \cdot (-16)$.

Solution:

ⓐ -8 ⓑ 16

Divide Integers

What about division? Division is the inverse operation of multiplication. So, $15 \div 3 = 5$ because $15 \cdot 3 = 5$. In words, this expression says that 15 can be divided into three groups of five each because adding five three times gives 15. Look at some examples of multiplying integers, to figure out the rules for dividing integers.
Equation:

$$\begin{array}{rclcrcl} 5 \cdot 3 & = & 15 \text{ so } 15 \div 3 & = & 5 & \qquad -5\,(3) & = & -15 \text{ so } -15 \div 3 & = & -5 \\ (-5)\,(-3) & = & 15 \text{ so } 15 \div (-3) & = & -5 & \qquad 5\,(-3) & = & -15 \text{ so } -15 \div (-3) & = & 5 \end{array}$$

Division follows the same rules as multiplication!

For division of two signed numbers, when the:

- signs are the *same*, the quotient is *positive*.
- signs are *different*, the quotient is *negative*.

And remember that we can always check the answer of a division problem by multiplying.

Note:
Multiplication and Division of Signed Numbers
For multiplication and division of two signed numbers:

- If the signs are the same, the result is positive.
- If the signs are different, the result is negative.

Same signs	**Result**
Two positives Two negatives	Positive Positive
If the signs are the same, the result is positive.	

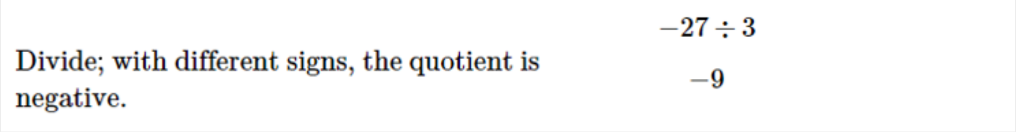
Different signs	**Result**
Positive and negative Negative and positive	Negative Negative
If the signs are different, the result is negative.	

Example:
Exercise:

Problem: Divide: ⓐ $-27 \div 3$ ⓑ $-100 \div (-4)$.

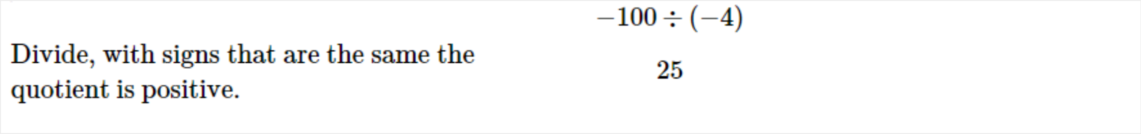
Solution:
Solution

ⓐ



	$-27 \div 3$
Divide; with different signs, the quotient is negative.	-9

ⓑ



	$-100 \div (-4)$
Divide, with signs that are the same the quotient is positive.	25

Note:
Exercise:

Problem: Divide: ⓐ $-42 \div 6$ ⓑ $-117 \div (-3)$.

Solution:

ⓐ -7 ⓑ 39

Note:
Exercise:

Problem: Divide: ⓐ $-63 \div 7$ ⓑ $-115 \div (-5)$.

Solution:

ⓐ -9 ⓑ 23

Simplify Expressions with Integers

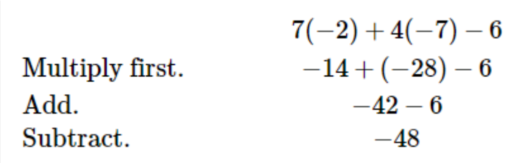
What happens when there are more than two numbers in an expression? The order of operations still applies when negatives are included. Remember My Dear Aunt Sally?

Let’s try some examples. We’ll simplify expressions that use all four operations with integers—addition, subtraction, multiplication, and division. Remember to follow the order of operations.

Example:
Exercise:

Problem: Simplify: $7(-2) + 4(-7) - 6$.

Solution:
Solution



	$7(-2)+4(-7)-6$
Multiply first.	$-14+(-28)-6$
Add.	$-42-6$
Subtract.	-48

Note:
Exercise:

Problem: Simplify: $8\left(-3\right)+5\left(-7\right)-4$.

Solution:

-63

Note:
Exercise:

Problem: Simplify: $9\left(-3\right)+7\left(-8\right)-1$.

Solution:

-84

Example:
Exercise:

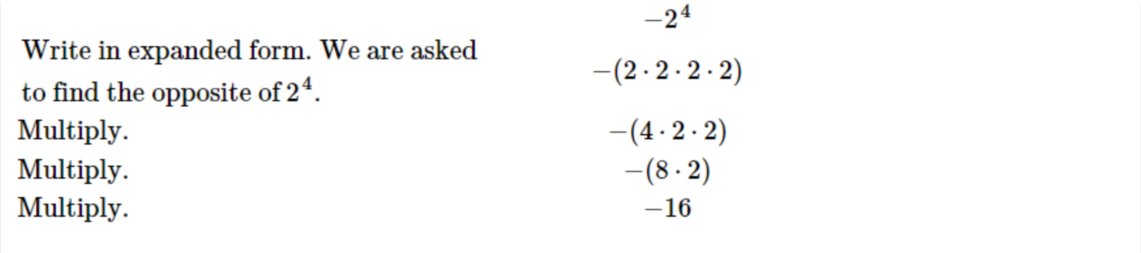
Problem: Simplify: ⓐ $\left(-2\right)^{4}$ ⓑ -2^{4}.

Solution:
Solution

ⓐ

	$\left(-2\right)^{4}$
Write in expanded form.	$\left(-2\right)\left(-2\right)\left(-2\right)\left(-2\right)$
Multiply.	$4\left(-2\right)\left(-2\right)$
Multiply.	$-8\left(-2\right)$
Multiply.	16

ⓑ



	-2^4
Write in expanded form. We are asked to find the opposite of 2^4.	$-(2\cdot2\cdot2\cdot2)$
Multiply.	$-(4\cdot2\cdot2)$
Multiply.	$-(8\cdot2)$
Multiply.	-16

Notice the difference in parts ⓐ and ⓑ. In part ⓐ, the exponent means to raise what is in the parentheses, the (-2) to the 4^{th} power. In part ⓑ, the exponent means to raise just the 2 to the 4^{th} power and then take the opposite.

Note:
Exercise:

Problem: Simplify: ⓐ $(-3)^4$ ⓑ -3^4.

Solution:

ⓐ 81 ⓑ -81

Note:
Exercise:

Problem: Simplify: ⓐ $(-7)^2$ ⓑ -7^2.

Solution:

ⓐ 49 ⓑ -49

The next example reminds us to simplify inside parentheses first.

Example:
Exercise:

Problem: Simplify: $12-3\,(9-12)$.

Solution:
Solution

	$12 - 3(9 - 12)$
Subtract in parentheses first.	$12 - 3(-3)$
Multiply.	$12 - (-9)$
Subtract.	21

Note:
Exercise:

Problem: Simplify: $17 - 4\,(8 - 11)$.

Solution:

29

Note:
Exercise:

Problem: Simplify: $16 - 6\,(7 - 13)$.

Solution:

52

Example:
Exercise:

Problem: Simplify: $8\,(-9) \div (-2)^3$.

Solution:
Solution

	$8(-9) \div (-2)^3$
Exponents first.	$8(-9) \div (-8)$
Multiply.	$-72 \div (-8)$
Divide.	9

Note:
Exercise:

Problem: Simplify: $12\left(-9\right)\div\left(-3\right)^{3}$.

Solution:

4

Note:
Exercise:

Problem: Simplify: $18\left(-4\right)\div\left(-2\right)^{3}$.

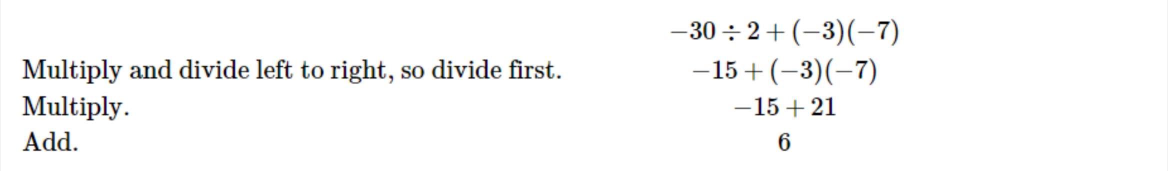
Solution:

9

Example:
Exercise:

Problem: Simplify: $-30\div2+\left(-3\right)\left(-7\right)$.

Solution:
Solution



	$-30\div2+(-3)(-7)$
Multiply and divide left to right, so divide first.	$-15+(-3)(-7)$
Multiply.	$-15+21$
Add.	6

Note:
Exercise:

Problem: Simplify: $-27\div3+\left(-5\right)\left(-6\right)$.

Solution:

21

Note:
Exercise:

Problem: Simplify: $-32 \div 4 + (-2)(-7)$.

Solution:

6

Evaluate Variable Expressions with Integers

Remember that to evaluate an expression means to substitute a number for the variable in the expression. Now we can use negative numbers as well as positive numbers.

Example:
Exercise:

Problem: When $n = -5$, evaluate: ⓐ $n + 1$ ⓑ $-n + 1$.

Solution:
Solution

ⓐ

	$n + 1$
Substitute -5 for n.	$-5 + 1$
Simplify.	-4

ⓑ

	$-n + 1$
Substitute -5 for n.	$-(-5) + 1$
Simplify.	

	$5+1$
Add.	6

Note:
Exercise:

Problem: When $n=-8$, evaluate ⓐ $n+2$ ⓑ $-n+2$.

Solution:

ⓐ -6 ⓑ 10

Note:
Exercise:

Problem: When $y=-9$, evaluate ⓐ $y+8$ ⓑ $-y+8$.

Solution:

ⓐ -1 ⓑ 17

Example:
Exercise:

Problem: Evaluate $(x+y)^2$ when $x=-18$ and $y=24$.

Solution:
Solution

	$(x+y)^2$
Substitute -18 for x and 24 for y.	$(-18+24)^2$
Add inside parenthesis.	$(6)^2$
Simplify.	36

Note:
Exercise:

Problem: Evaluate $(x+y)^2$ when $x=-15$ and $y=29$.

Solution:

196

Note:
Exercise:

Problem: Evaluate $(x+y)^3$ when $x=-8$ and $y=10$.

Solution:

8

Example:
Exercise:

Problem: Evaluate $20-z$ when ⓐ $z=12$ and ⓑ $z=-12$.

Solution:
Solution

ⓐ

	$20 - z$
Substitute 12 for z.	$20 - 12$
Subtract.	8

ⓑ

	$20 - z$
Substitute -12 for z.	$20 - (-12)$
Subtract.	32

Note:
Exercise:

Problem: Evaluate: $17 - k$ when ⓐ $k = 19$ and ⓑ $k = -19$.

Solution:

ⓐ -2 ⓑ 36

Note:
Exercise:

Problem: Evaluate: $-5 - b$ when ⓐ $b = 14$ and ⓑ $b = -14$.

Solution:

ⓐ -19 ⓑ 9

Example:
Exercise:

Problem: Evaluate: $2x^2 + 3x + 8$ when $x = 4$.

Solution:
Solution

Substitute 4 for x. Use parentheses to show multiplication.

	$2x^2 + 3x + 8$

Substitute.	$2(4)^2 + 3(4) + 8$
Evaluate exponents.	$2(16) + 3(4) + 8$
Multiply.	$32 + 12 + 8$
Add.	52

Note:
Exercise:

Problem: Evaluate: $3x^2 - 2x + 6$ when $x = -3$.

Solution:

39

Note:
Exercise:

Problem: Evaluate: $4x^2 - x - 5$ when $x = -2$.

Solution:

13

Translate Phrases to Expressions with Integers

Our earlier work translating English to algebra also applies to phrases that include both positive and negative numbers.

Example:
Exercise:

Problem: Translate and simplify: the sum of 8 and -12, increased by 3.

Solution:
Solution

	the **sum** of 8 and -12, increased by 3
Translate.	$[8+(-12)]+3$
Simplify. Be careful not to confuse the brackets with an absolute value sign.	$(-4)+3$
Add.	-1

Note:
Exercise:

Problem: Translate and simplify the sum of 9 and -16, increased by 4.

Solution:

$(9+(-16))+4=-3$

Note:
Exercise:

Problem: Translate and simplify the sum of -8 and -12, increased by 7.

Solution:

$(-8+(-12))+7=-13$

When we first introduced the operation symbols, we saw that the expression may be read in several ways. They are listed in the chart below.

$a-b$
a minus b the difference of a and b b subtracted from a b less than a

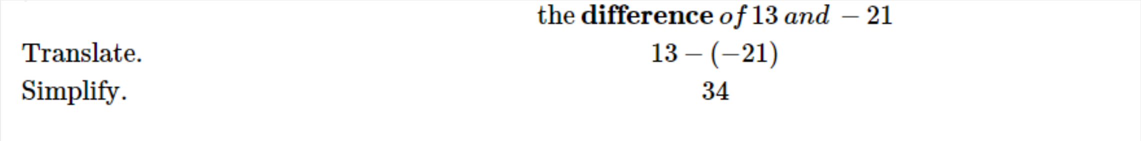
Be careful to get a and b in the right order!

Example:
Exercise:

Problem: Translate and then simplify ⓐ the difference of 13 and -21 ⓑ subtract 24 from -19.

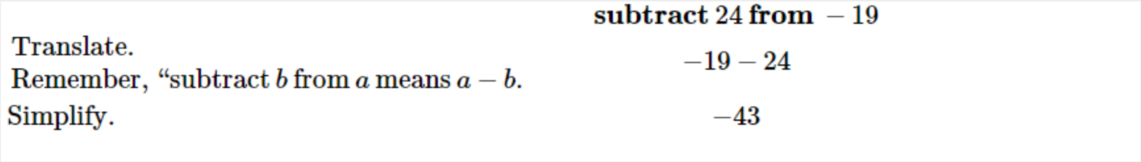
Solution:
Solution

ⓐ



	the **difference** of 13 and -21
Translate.	$13-(-21)$
Simplify.	34

ⓑ



	subtract 24 **from** -19
Translate. Remember, "subtract b from a means $a-b$.	$-19-24$
Simplify.	-43

Note:
Exercise:

Problem: Translate and simplify ⓐ the difference of 14 and -23 ⓑ subtract 21 from -17.

Solution:

ⓐ $14-(-23); 37$ ⓑ $-17-21; -38$

Note:
Exercise:

Problem: Translate and simplify ⓐ the difference of 11 and -19 ⓑ subtract 18 from -11.

Solution:

ⓐ $11-(-19); 30$ ⓑ $-11-18; -29$

Once again, our prior work translating English to algebra transfers to phrases that include both multiplying and dividing integers. Remember that the key word for multiplication is "product" and for division is "quotient."

Example:
Exercise:

Problem: Translate to an algebraic expression and simplify if possible: the product of -2 and 14.

Solution:
Solution

$$\text{the product } of \ -2 \ and \ 14$$

Translate. $(-2)(14)$
Simplify. -28

Note:
Exercise:

Problem: Translate to an algebraic expression and simplify if possible: the product of -5 and 12.

Solution:

$-5(12); -60$

Note:
Exercise:

Problem: Translate to an algebraic expression and simplify if possible: the product of 8 and -13.

Solution:

$-8(13); -104$

Example:
Exercise:

Problem: Translate to an algebraic expression and simplify if possible: the quotient of -56 and -7.

Solution:
Solution

$$\text{the quotient } of \ -56 \ and \ -7$$

Translate. $-56 \div (-7)$
Simplify. 8

Note:

Exercise:

Problem: Translate to an algebraic expression and simplify if possible: the quotient of -63 and -9.

Solution:

$-63 \div (-9); 7$

Note:
Exercise:

Problem: Translate to an algebraic expression and simplify if possible: the quotient of -72 and -9.

Solution:

$-72 \div (-9); 8$

Use Integers in Applications

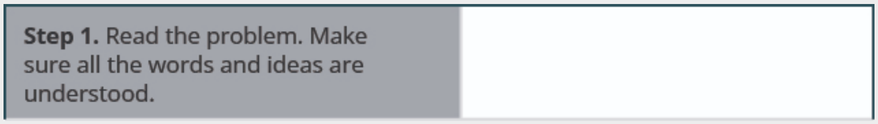
We'll outline a plan to solve applications. It's hard to find something if we don't know what we're looking for or what to call it! So when we solve an application, we first need to determine what the problem is asking us to find. Then we'll write a phrase that gives the information to find it. We'll translate the phrase into an expression and then simplify the expression to get the answer. Finally, we summarize the answer in a sentence to make sure it makes sense.

Example:
How to Apply a Strategy to Solve Applications with Integers
Exercise:

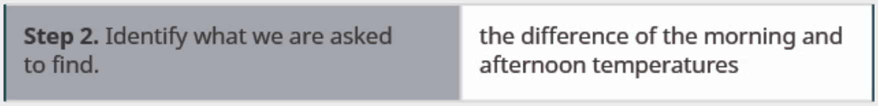
Problem:

The temperature in Urbana, Illinois one morning was 11 degrees. By mid-afternoon, the temperature had dropped to -9 degrees. What was the difference of the morning and afternoon temperatures?

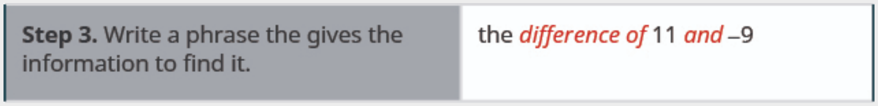
Solution:
Solution



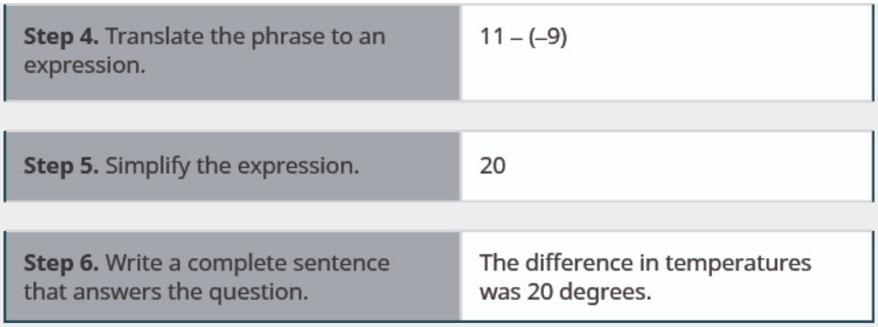
Step 1. Read the problem. Make sure all the words and ideas are understood.	



Step 2. Identify what we are asked to find.	the difference of the morning and afternoon temperatures



Step 3. Write a phrase the gives the information to find it.	the *difference of* 11 *and* –9



Step 4. Translate the phrase to an expression.	11 – (–9)
Step 5. Simplify the expression.	20
Step 6. Write a complete sentence that answers the question.	The difference in temperatures was 20 degrees.

Note:
Exercise:

Problem:

The temperature in Anchorage, Alaska one morning was 15 degrees. By mid-afternoon the temperature had dropped to 30 degrees below zero. What was the difference in the morning and afternoon temperatures?

Solution:

The difference in temperatures was 45 degrees.

Note:
Exercise:

Problem:

The temperature in Denver was -6 degrees at lunchtime. By sunset the temperature had dropped to -15 degrees. What was the difference in the lunchtime and sunset temperatures?

Solution:

The difference in temperatures was 9 degrees.

Note:
Apply a Strategy to Solve Applications with Integers.

Read the problem. Make sure all the words and ideas are understood
Identify what we are asked to find.
Write a phrase that gives the information to find it.
Translate the phrase to an expression.
Simplify the expression.
Answer the question with a complete sentence.

Example:
Exercise:

Problem:

The Mustangs football team received three penalties in the third quarter. Each penalty gave them a loss of fifteen yards. What is the number of yards lost?

Solution:
Solution

Step 1. Read the problem. Make sure all the words and ideas are understood.	
Step 2. Identify what we are asked to find.	the number of yards lost
Step 3. Write a phrase that gives the information to find it.	three times a 15-yard penalty
Step 4. Translate the phrase to an expression.	$3\left(-15\right)$
Step 5. Simplify the expression.	-45
Step 6. Answer the question with a complete sentence.	The team lost 45 yards.

Note:
Exercise:

Problem:

The Bears played poorly and had seven penalties in the game. Each penalty resulted in a loss of 15 yards. What is the number of yards lost due to penalties?

Solution:

The Bears lost 105 yards.

Note:
Exercise:

Problem:

Bill uses the ATM on campus because it is convenient. However, each time he uses it he is charged a $2 fee. Last month he used the ATM eight times. How much was his total fee for using the ATM?

Solution:

A $16 fee was deducted from his checking account.

Key Concepts

- **Multiplication and Division of Two Signed Numbers**
  - Same signs—Product is positive
  - Different signs—Product is negative

- **Strategy for Applications**

  Identify what you are asked to find.
  Write a phrase that gives the information to find it.
  Translate the phrase to an expression.
  Simplify the expression.
  Answer the question with a complete sentence.

Practice Makes Perfect

Multiply Integers

In the following exercises, multiply.

Exercise:

Problem: $-4 \cdot 8$

Solution:

-32

Exercise:

Problem: $-3 \cdot 9$

Exercise:

Problem: $9\left(-7\right)$

Solution:

-63

Exercise:

Problem: $13\left(-5\right)$

Exercise:

Problem: $-1 \cdot 6$

Solution:

-6

Exercise:

Problem: $-1 \cdot 3$

Exercise:

Problem: $-1\left(-14\right)$

Solution:

14

Exercise:

Problem: $-1\left(-19\right)$

Divide Integers

In the following exercises, divide.

Exercise:

Problem: $-24\div 6$

Solution:

-4

Exercise:

Problem: $35\div \left(-7\right)$

Exercise:

Problem: $-52\div \left(-4\right)$

Solution:

13

Exercise:

Problem: $-84\div \left(-6\right)$

Exercise:

Problem: $-180\div 15$

Solution:

-12

Exercise:

Problem: $-192\div 12$

Simplify Expressions with Integers

In the following exercises, simplify each expression.

Exercise:

Problem: $5\left(-6\right)+7\left(-2\right)-3$

Solution:

-47

Exercise:

Problem: $8\left(-4\right)+5\left(-4\right)-6$

Exercise:

Problem: ${\left(-2\right)}^{6}$

Solution:

64

Exercise:

Problem: ${\left(-3\right)}^{5}$

Exercise:

Problem: $-{4}^{2}$

Solution:

-16

Exercise:

Problem: $-{6}^{2}$

Exercise:

Problem: $-3\left(-5\right)\left(6\right)$

Solution:

90

Exercise:

Problem: $-4\left(-6\right)\left(3\right)$

Exercise:

Problem: $\left(8-11\right)\left(9-12\right)$

Solution:

9

Exercise:

Problem: $\left(6-11\right)\left(8-13\right)$

Exercise:

Problem: $26-3\left(2-7\right)$

Solution:

41

Exercise:

Problem: $23-2\left(4-6\right)$

Exercise:

Problem: $65\div\left(-5\right)+\left(-28\right)\div\left(-7\right)$

Solution:

-9

Exercise:

Problem: $52\div\left(-4\right)+\left(-32\right)\div\left(-8\right)$

Exercise:

Problem: $9-2\left[3-8\left(-2\right)\right]$

Solution:

-29

Exercise:

Problem: $11-3\left[7-4\left(-2\right)\right]$

Exercise:

Problem: ${\left(-3\right)}^{2}-24\div\left(8-2\right)$

Solution:

5

Exercise:

Problem: ${\left(-4\right)}^{2}-32\div\left(12-4\right)$

Evaluate Variable Expressions with Integers

In the following exercises, evaluate each expression.

Exercise:

$y+\left(-14\right)$ when
ⓐ $y=-33$
Problem: ⓑ $y=30$

Solution:

ⓐ -47 ⓑ 16

Exercise:

$x+(-21)$ when
ⓐ $x=-27$
Problem: ⓑ $x=44$

Exercise:

ⓐ $a+3$ when $a=-7$
Problem: ⓑ $-a+3$ when $a=-7$

Solution:

ⓐ -4 ⓑ 10

Exercise:

ⓐ $d+(-9)$ when $d=-8$
Problem: ⓑ $-d+(-9)$ when $d=-8$

Exercise:

$m+n$ when
Problem: $m=-15, n=7$

Solution:

-8

Exercise:

$p+q$ when
Problem: $p=-9, q=17$

Exercise:

Problem: $r+s$ when $r=-9, s=-7$

Solution:

-16

Exercise:

Problem: $t+u$ when $t=-6, u=-5$

Exercise:

$(x+y)^2$ when
Problem: $x=-3, y=14$

Solution:

121

Exercise:

Problem: $(y+z)^2$ when
$y=-3, z=15$

Exercise:

Problem: $-2x+17$ when
ⓐ $x=8$
ⓑ $x=-8$

Solution:

ⓐ 1 ⓑ 33

Exercise:

Problem: $-5y+14$ when
ⓐ $y=9$
ⓑ $y=-9$

Exercise:

Problem: $10-3m$ when
ⓐ $m=5$
ⓑ $m=-5$

Solution:

ⓐ -5 ⓑ 25

Exercise:

Problem: $18-4n$ when
ⓐ $n=3$
ⓑ $n=-3$

Exercise:

Problem: $2w^2-3w+7$ when
$w=-2$

Solution:

21

Exercise:

Problem: $3u^2-4u+5$ when $u=-3$

Exercise:

Problem: $9a-2b-8$ when
$a=-6$ and $b=-3$

Solution:

-56

Exercise:

Problem: $7m-4n-2$ when $m=-4$ and $n=-9$

Translate English Phrases to Algebraic Expressions

In the following exercises, translate to an algebraic expression and simplify if possible.

Exercise:

Problem: the sum of 3 and -15, increased by 7

Solution:

$(3+(-15))+7;-5$

Exercise:

Problem: the sum of -8 and -9, increased by 23

Exercise:

Problem: the difference of 10 and -18

Solution:

$10-(-18);28$

Exercise:

Problem: subtract 11 from -25

Exercise:

Problem: the difference of -5 and -30

Solution:

$-5-(-30);25$

Exercise:

Problem: subtract -6 from -13

Exercise:

Problem: the product of -3 and 15

Solution:

$-3\cdot 15;-45$

Exercise:

Problem: the product of -4 and 16

Exercise:

Problem: the quotient of -60 and -20

Solution:

$-60 \div (-20); 3$

Exercise:

Problem: the quotient of -40 and -20

Exercise:

Problem: the quotient of -6 and the sum of a and b

Solution:

$\frac{-6}{a+b}$

Exercise:

Problem: the quotient of -7 and the sum of m and n

Exercise:

Problem: the product of -10 and the difference of p and q

Solution:

$-10\left(p-q\right)$

Exercise:

Problem: the product of -13 and the difference of c and d

Use Integers in Applications

In the following exercises, solve.

Exercise:

Problem:

Temperature On January 15, the high temperature in Anaheim, California, was 84°. That same day, the high temperature in Embarrass, Minnesota was -12°. What was the difference between the temperature in Anaheim and the temperature in Embarrass?

Solution:

96°

Exercise:

Problem:

Temperature On January 21, the high temperature in Palm Springs, California, was 89°, and the high temperature in Whitefield, New Hampshire was -31°. What was the difference between the temperature in Palm Springs and the temperature in Whitefield?

Exercise:

Problem:

Football At the first down, the Chargers had the ball on their 25 yard line. On the next three downs, they lost 6 yards, gained 10 yards, and lost 8 yards. What was the yard line at the end of the fourth down?

Solution:

21

Exercise:

Problem:

Football At the first down, the Steelers had the ball on their 30 yard line. On the next three downs, they gained 9 yards, lost 14 yards, and lost 2 yards. What was the yard line at the end of the fourth down?

Exercise:

Problem:

Checking Account Mayra has \$124 in her checking account. She writes a check for \$152. What is the new balance in her checking account?

Solution:

$-\$28$

Exercise:

Problem:

Checking Account Selina has \$165 in her checking account. She writes a check for \$207. What is the new balance in her checking account?

Exercise:

Problem:

Checking Account Diontre has a balance of $-\$38$ in his checking account. He deposits \$225 to the account. What is the new balance?

Solution:

\$187

Exercise:

Problem:

Checking Account Reymonte has a balance of $-\$49$ in his checking account. He deposits \$281 to the account. What is the new balance?

Everyday Math

Exercise:

Problem:

Stock market Javier owns 300 shares of stock in one company. On Tuesday, the stock price dropped $12 per share. What was the total effect on Javier's portfolio?

Solution:

$-\$3600$

Exercise:

Problem:

Weight loss In the first week of a diet program, eight women lost an average of 3 pounds each. What was the total weight change for the eight women?

Writing Exercises

Exercise:

Problem: In your own words, state the rules for multiplying integers.

Solution:

Answers may vary

Exercise:

Problem: In your own words, state the rules for dividing integers.

Exercise:

Problem: Why is $-2^4 \neq (-2)^4$?

Solution:

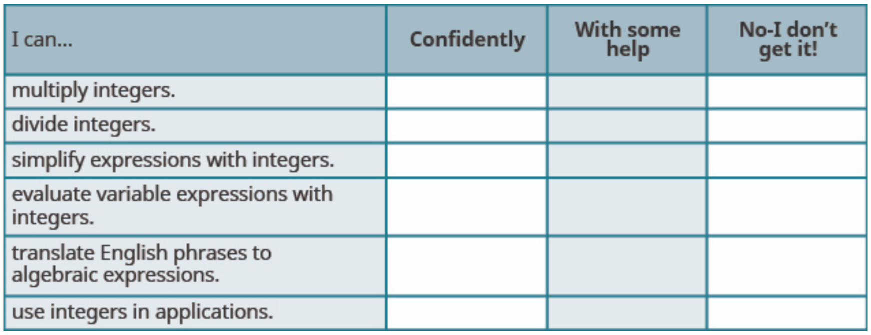
Answers may vary

Exercise:

Problem: Why is $-4^3 = (-4)^3$?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don’t get it!
multiply integers.			
divide integers.			
simplify expressions with integers.			
evaluate variable expressions with integers.			
translate English phrases to algebraic expressions.			
use integers in applications.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Visualize Fractions

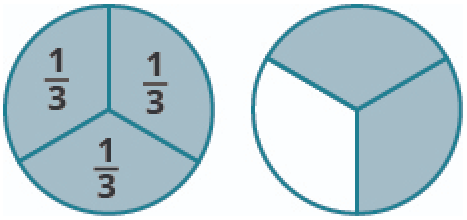
By the end of this section, you will be able to:

- Find equivalent fractions
- Simplify fractions
- Multiply fractions
- Divide fractions
- Simplify expressions written with a fraction bar
- Translate phrases to expressions with fractions

Note:
A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **Fractions**.

Find Equivalent Fractions

Fractions are a way to represent parts of a whole. The fraction $\frac{1}{3}$ means that one whole has been divided into 3 equal parts and each part is one of the three equal parts. See [link]. The fraction $\frac{2}{3}$ represents two of three equal parts. In the fraction $\frac{2}{3}$, the 2 is called the **numerator** and the 3 is called the **denominator**.



The circle on the left has been divided into 3 equal parts. Each part is $\frac{1}{3}$ of the 3 equal parts. In the circle on the right, $\frac{2}{3}$ of the circle is shaded (2 of the 3 equal parts).

Note: Doing the Manipulative Mathematics activity “Model Fractions” will help you develop a better understanding of fractions, their numerators and denominators.

Note:

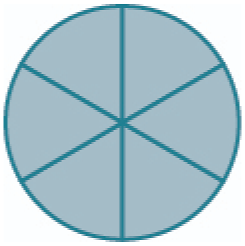
Fraction

A **fraction** is written $\frac{a}{b}$, where $b \neq 0$ and

- a is the **numerator** and b is the **denominator**.

A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

If a whole pie has been cut into 6 pieces and we eat all 6 pieces, we ate $\frac{6}{6}$ pieces, or, in other words, one whole pie.



So $\frac{6}{6} = 1$. This leads us to the property of one that tells us that any number, except zero, divided by itself is 1.

Note:
Property of One
Equation:

$$\frac{a}{a} = 1 \qquad (a \neq 0)$$

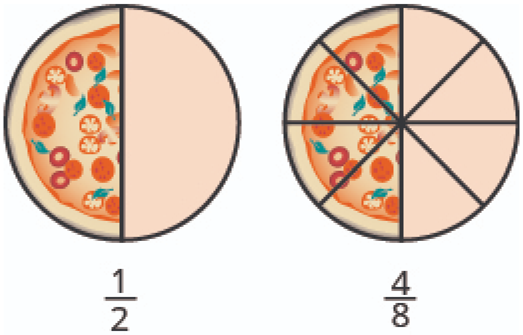
Any number, except zero, divided by itself is one.

Note: Doing the Manipulative Mathematics activity “Fractions Equivalent to One” will help you develop a better understanding of fractions that are equivalent to one.

If a pie was cut in 6 pieces and we ate all 6, we ate $\frac{6}{6}$ pieces, or, in other words, one whole pie. If the pie was cut into 8 pieces and we ate all 8, we ate $\frac{8}{8}$ pieces, or one whole pie. We ate the same amount—one whole pie.

The fractions $\frac{6}{6}$ and $\frac{8}{8}$ have the same value, 1, and so they are called equivalent fractions. **Equivalent fractions** are fractions that have the same value.

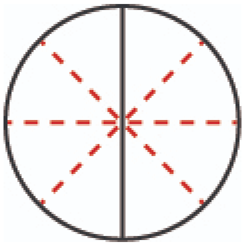
Let’s think of pizzas this time. [link] shows two images: a single pizza on the left, cut into two equal pieces, and a second pizza of the same size, cut into eight pieces on the right. This is a way to show that $\frac{1}{2}$ is equivalent to $\frac{4}{8}$. In other words, they are equivalent fractions.



Since the same amount is of each pizza is shaded, we see that $\frac{1}{2}$ is equivalent to $\frac{4}{8}$. They are equivalent fractions.

Note:
Equivalent Fractions
Equivalent fractions are fractions that have the same value.

How can we use mathematics to change $\frac{1}{2}$ into $\frac{4}{8}$? How could we take a pizza that is cut into 2 pieces and cut it into 8 pieces? We could cut each of the 2 larger pieces into 4 smaller pieces! The whole pizza would then be cut into 8 pieces instead of just 2. Mathematically, what we've described could be written like this as $\frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$. See [link].



Cutting each half of the pizza into 4 pieces, gives us pizza cut into 8 pieces: $\frac{1 \cdot 4}{2 \cdot 4} = \frac{4}{8}$.

This model leads to the following property:

Note:
Equivalent Fractions Property
If a, b, c are numbers where $b \neq 0, c \neq 0$, then
Equation:

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$$

If we had cut the pizza differently, we could get

$$\frac{1 \cdot 2}{2 \cdot 2} = \frac{2}{4} \quad \text{so} \quad \frac{1}{2} = \frac{2}{4}$$

$$\frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6} \quad \text{so} \quad \frac{1}{2} = \frac{3}{6}$$

$$\frac{1 \cdot 10}{2 \cdot 10} = \frac{10}{20} \quad \text{so} \quad \frac{1}{2} = \frac{10}{20}$$

So, we say $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, and $\frac{10}{20}$ are equivalent fractions.

Note: Doing the Manipulative Mathematics activity “Equivalent Fractions” will help you develop a better understanding of what it means when two fractions are equivalent.

Example:
Exercise:

Problem: Find three fractions equivalent to $\frac{2}{5}$.

Solution:
Solution

To find a fraction equivalent to $\frac{2}{5}$, we multiply the numerator and denominator by the same number. We can choose any number, except for zero. Let’s multiply them by 2, 3, and then 5.

$$\frac{2 \cdot 2}{5 \cdot 2} = \frac{4}{10} \qquad \frac{2 \cdot 3}{5 \cdot 3} = \frac{6}{15} \qquad \frac{2 \cdot 5}{5 \cdot 5} = \frac{10}{25}$$

So, $\frac{4}{10}$, $\frac{6}{15}$, and $\frac{10}{25}$ are equivalent to $\frac{2}{5}$.

Note:
Exercise:

Problem: Find three fractions equivalent to $\frac{3}{5}$.

Solution:

$\frac{6}{10}$, $\frac{9}{15}$, $\frac{12}{20}$; answers may vary

Note:
Exercise:

Problem: Find three fractions equivalent to $\frac{4}{5}$.

Solution:

$\frac{8}{10}$, $\frac{12}{15}$, $\frac{16}{20}$; answers may vary

Simplify Fractions

A fraction is considered **simplified** if there are no common factors, other than 1, in its numerator and denominator.

For example,

- $\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.
- $\frac{10}{15}$ is not simplified because 5 is a common factor of 10 and 15.

Note:

Simplified Fraction

A fraction is considered **simplified** if there are no common factors in its numerator and denominator.

The phrase *reduce a fraction* means to simplify the fraction. We simplify, or reduce, a fraction by removing the common factors of the numerator and denominator. A fraction is not simplified until all common factors have been removed. If an expression has fractions, it is not completely simplified until the fractions are simplified.

In [link], we used the equivalent fractions property to find equivalent fractions. Now we'll use the equivalent fractions property in reverse to simplify fractions. We can rewrite the property to show both forms together.

Note:

Equivalent Fractions Property

If a,b,c are numbers where $b\ne 0,c\ne 0,$

Equation:

$$\text{then}\quad \frac{a}{b}=\frac{a\cdot c}{b\cdot c}\quad \text{and}\quad \frac{a\cdot c}{b\cdot c}=\frac{a}{b}$$

Example:

Exercise:

Problem: Simplify: $-\frac{32}{56}$.

Solution:

Solution

	$-\frac{32}{56}$
Rewrite the numerator and denominator showing the common factors.	$-\frac{4\cdot 8}{7\cdot 8}$
Simplify using the equivalent fractions property.	$-\frac{4}{7}$

Notice that the fraction $-\frac{4}{7}$ is simplified because there are no more common factors.

Note:
Exercise:

Problem: Simplify: $-\frac{42}{54}$.

Solution:

$-\frac{7}{9}$

Note:
Exercise:

Problem: Simplify: $-\frac{45}{81}$.

Solution:

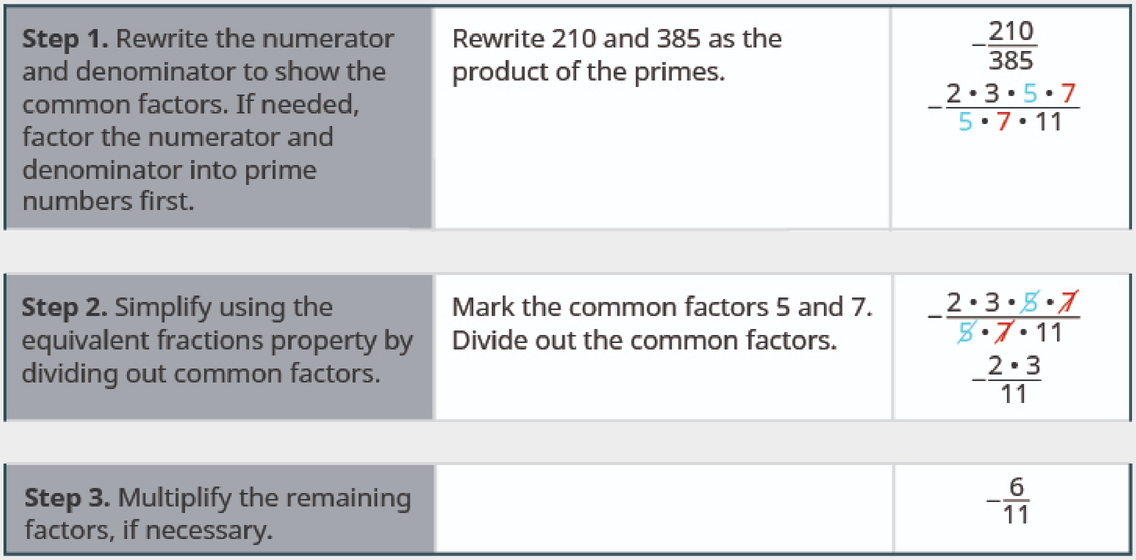
$-\frac{5}{9}$

Sometimes it may not be easy to find common factors of the numerator and denominator. When this happens, a good idea is to factor the numerator and the denominator into prime numbers. Then divide out the common factors using the equivalent fractions property.

Example:
How to Simplify a Fraction
Exercise:

Problem: Simplify: $-\frac{210}{385}$.

Solution:
Solution



Step 1. Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers first.	Rewrite 210 and 385 as the product of the primes.	$-\frac{210}{385}$ $-\frac{2\cdot3\cdot5\cdot7}{5\cdot7\cdot11}$
Step 2. Simplify using the equivalent fractions property by dividing out common factors.	Mark the common factors 5 and 7. Divide out the common factors.	$-\frac{2\cdot3\cdot\cancel{5}\cdot\cancel{7}}{\cancel{5}\cdot\cancel{7}\cdot11}$ $-\frac{2\cdot3}{11}$
Step 3. Multiply the remaining factors, if necessary.		$-\frac{6}{11}$

Note:
Exercise:

Problem: Simplify: $-\frac{69}{120}$.

Solution:

$-\frac{23}{40}$

Note:
Exercise:

Problem: Simplify: $-\frac{120}{192}$.

Solution:

$-\frac{5}{8}$

We now summarize the steps you should follow to simplify fractions.

Note:
Simplify a Fraction.

Rewrite the numerator and denominator to show the common factors.	If needed, factor the numerator and denominator into prime numbers first.
Simplify using the equivalent fractions property by dividing out common factors.	
Multiply any remaining factors, if needed.	

Example:
Exercise:

Problem: Simplify: $\frac{5x}{5y}$.

Solution:
Solution

	$\frac{5x}{5y}$
Rewrite showing the common factors, then divide out the common factors.	$\frac{\cancel{5}\cdot x}{\cancel{5}\cdot y}$
Simplify.	$\frac{x}{y}$

Note:
Exercise:

Problem: Simplify: $\frac{7x}{7y}$.

Solution:

$\frac{x}{y}$

Note:

Exercise:

Problem: Simplify: $\frac{3a}{3b}$.

Solution:

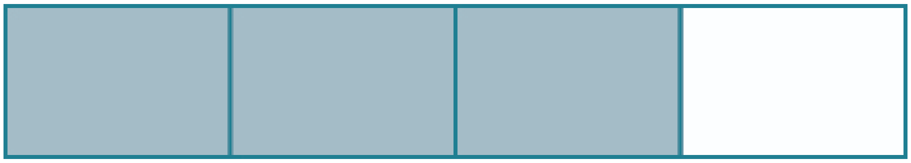
$\frac{a}{b}$

Multiply Fractions

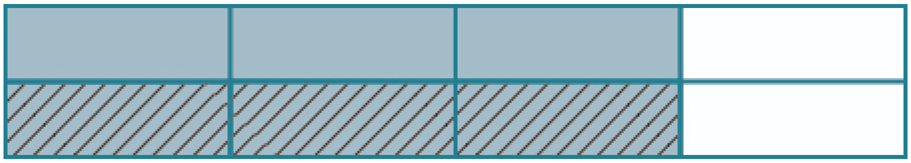
Many people find multiplying and dividing fractions easier than adding and subtracting fractions. So we will start with fraction multiplication.

Note: Doing the Manipulative Mathematics activity “Model Fraction Multiplication” will help you develop a better understanding of multiplying fractions.

We’ll use a model to show you how to multiply two fractions and to help you remember the procedure. Let’s start with $\frac{3}{4}$.



Now we’ll take $\frac{1}{2}$ of $\frac{3}{4}$.



Notice that now, the whole is divided into 8 equal parts. So $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$.

To multiply fractions, we multiply the numerators and multiply the denominators.

Note:
Fraction Multiplication
If a, b, c and d are numbers where $b \neq 0$ and $d \neq 0$, then
Equation:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

To multiply fractions, multiply the numerators and multiply the denominators.

When multiplying fractions, the properties of positive and negative numbers still apply, of course. It is a good idea to determine the sign of the product as the first step. In [link], we will multiply negative and a positive, so the product will be negative.

Example:
Exercise:

Problem: Multiply: $-\frac{11}{12} \cdot \frac{5}{7}$.

Solution:
Solution

The first step is to find the sign of the product. Since the signs are the different, the product is negative.

	$-\frac{11}{12}\cdot\frac{5}{7}$
Determine the sign of the product; multiply.	$-\frac{11\cdot5}{12\cdot7}$
Are there any common factors in the numerator and the denominator? No	$-\frac{55}{84}$

Note:
Exercise:

Problem: Multiply: $-\frac{10}{28}\cdot\frac{8}{15}$.

Solution:

$-\frac{4}{21}$

Note:
Exercise:

Problem: Multiply: $-\frac{9}{20}\cdot\frac{5}{12}$.

Solution:

$-\frac{3}{16}$

When multiplying a fraction by an integer, it may be helpful to write the integer as a fraction. Any integer, a, can be written as $\frac{a}{1}$. So, for example, $3=\frac{3}{1}$.

Example:
Exercise:

Problem: Multiply: $-\frac{12}{5}(-20x)$.

Solution:
Solution

Determine the sign of the product. The signs are the same, so the product is positive.

	$-\frac{12}{5}(-20x)$
Write $20x$ as a fraction.	$\frac{12}{5}\left(\frac{20x}{1}\right)$
Multiply.	
Rewrite 20 to show the common factor 5 and divide it out.	$\frac{12 \cdot 4 \cdot \cancel{5}x}{\cancel{5} \cdot 1}$
Simplify.	$48x$

Note:
Exercise:

Problem: Multiply: $\frac{11}{3}(-9a)$.

Solution:

$-33a$

Note:
Exercise:

Problem: Multiply: $\frac{13}{7}(-14b)$.

Solution:

$-26b$

Divide Fractions

Now that we know how to multiply fractions, we are almost ready to divide. Before we can do that, that we need some vocabulary.

The **reciprocal** of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

Notice that $\frac{2}{3} \cdot \frac{3}{2} = 1.$ A number and its reciprocal multiply to 1.

To get a product of positive 1 when multiplying two numbers, the numbers must have the same sign. So reciprocals must have the same sign.

The reciprocal of $-\frac{10}{7}$ is $-\frac{7}{10}$, since $-\frac{10}{7}\left(-\frac{7}{10}\right) = 1.$

Note:

Reciprocal

The **reciprocal** of $\frac{a}{b}$ is $\frac{b}{a}$.

A number and its reciprocal multiply to one $\frac{a}{b} \cdot \frac{b}{a} = 1$.

Note: Doing the Manipulative Mathematics activity “Model Fraction Division” will help you develop a better understanding of dividing fractions.

To divide fractions, we multiply the first fraction by the reciprocal of the second.

Note:

Fraction Division

If a, b, c and d are numbers where $b \neq 0, c \neq 0$ and $d \neq 0$, then

Equation:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

To divide fractions, we multiply the first fraction by the reciprocal of the second.

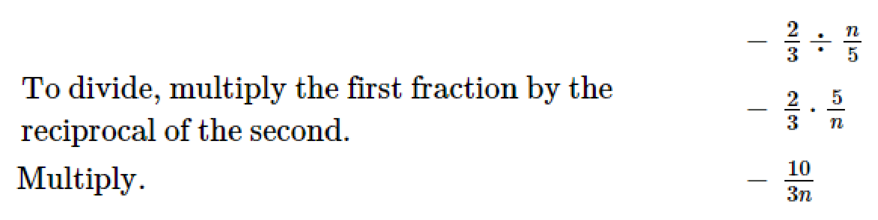
We need to say $b \neq 0, c \neq 0$ and $d \neq 0$ to be sure we don’t divide by zero!

Example:

Exercise:

Problem: Divide: $-\frac{2}{3} \div \frac{n}{5}$.

Solution:
Solution



	$-\frac{2}{3} \div \frac{n}{5}$
To divide, multiply the first fraction by the reciprocal of the second.	$-\frac{2}{3} \cdot \frac{5}{n}$
Multiply.	$-\frac{10}{3n}$

Note:
Exercise:

Problem: Divide: $-\frac{3}{5} \div \frac{p}{7}$.

Solution:

$-\frac{21}{5p}$

Note:
Exercise:

Problem: Divide: $-\frac{5}{8} \div \frac{q}{3}$.

Solution:

$-\frac{15}{8q}$

Example:
Exercise:

Problem: Find the quotient: $-\frac{7}{8} \div \left(-\frac{14}{27}\right)$.

Solution:
Solution

	$-\frac{7}{18} \div \left(-\frac{14}{27}\right)$
To divide, multiply the first fraction by the reciprocal of the second.	$-\frac{7}{18} \cdot -\frac{27}{14}$
Determine the sign of the product, and then multiply..	$\frac{7\cdot27}{18\cdot14}$
Rewrite showing common factors.	$\frac{\cancel{7}\cdot\cancel{9}\cdot3}{\cancel{9}\cdot2\cdot\cancel{7}\cdot2}$
Remove common factors.	$\frac{3}{2\cdot2}$
Simplify.	$\frac{3}{4}$

Note:
Exercise:

Problem: Find the quotient: $-\frac{7}{27} \div \left(-\frac{35}{36}\right)$.

Solution:

$\frac{4}{15}$

Note:
Exercise:

Problem: Find the quotient: $-\frac{5}{14} \div \left(-\frac{15}{28}\right)$.

Solution:

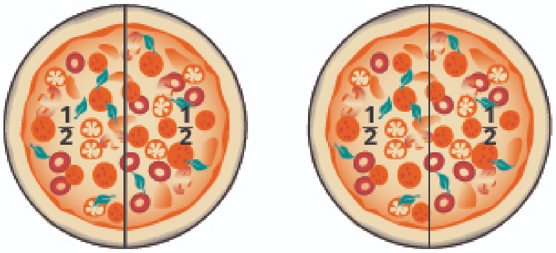
$\frac{2}{3}$

There are several ways to remember which steps to take to multiply or divide fractions. One way is to repeat the instructions to yourself. If you do this each time you do an exercise, you will have the steps memorized.

- “To multiply fractions, multiply the numerators and multiply the denominators.”
- “To divide fractions, multiply the first fraction by the reciprocal of the second.”

Another way is to keep two examples in mind:

One fourth of two pizzas is one half of a pizza.



$$2 \cdot \frac{1}{4}$$

$$\frac{2}{1} \cdot \frac{1}{4}$$

$$\frac{2}{4}$$

$$\frac{1}{2}$$

There are eight quarters in $2.00.

$$2 \div \frac{1}{4}$$

$$\frac{2}{1} \div \frac{1}{4}$$

$$\frac{2}{1} \cdot \frac{4}{1}$$

$$8$$

The numerators or denominators of some fractions contain fractions themselves. A fraction in which the numerator or the denominator is a fraction is called a **complex fraction**.

Note:
Complex Fraction
A **complex fraction** is a fraction in which the numerator or the denominator contains a fraction.

Some examples of complex fractions are:
Equation:

$$\frac{\frac{6}{7}}{3} \quad \frac{\frac{3}{4}}{\frac{5}{8}} \quad \frac{\frac{x}{2}}{\frac{5}{6}}$$

To simplify a complex fraction, we remember that the fraction bar means division. For example, the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ means $\frac{3}{4} \div \frac{5}{8}$.

Example:
Exercise:

Problem: Simplify: $\frac{\frac{3}{4}}{\frac{5}{8}}$.

Solution:
Solution

	$\frac{\frac{3}{4}}{\frac{5}{8}}$
Rewrite as division.	$\frac{3}{4} \div \frac{5}{8}$
Multiply the first fraction by the reciprocal of the second.	$\frac{3}{4} \cdot \frac{8}{5}$
Multiply.	$\frac{3 \cdot 8}{4 \cdot 5}$
Look for common factors.	$\frac{3 \cdot \cancel{4} \cdot 2}{\cancel{4} \cdot 5}$
Divide out common factors and simplify.	$\frac{6}{5}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{2}{3}}{\frac{5}{6}}$.

Solution:

$\frac{4}{5}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{3}{7}}{\frac{6}{11}}$.

Solution:

$\frac{11}{14}$

Example:
Exercise:

Problem: Simplify: $\frac{\frac{x}{2}}{\frac{xy}{6}}$.

Solution:
Solution

	$\frac{\frac{x}{2}}{\frac{xy}{6}}$
Rewrite as division.	$\frac{x}{2} \div \frac{xy}{6}$
Multiply the first fraction by the reciprocal of the second.	$\frac{x}{2} \cdot \frac{6}{xy}$
Multiply.	$\frac{x\cdot 6}{2\cdot xy}$
Look for common factors.	$\frac{\cancel{x}\cdot 3\cdot \cancel{2}}{\cancel{2}\cdot \cancel{x}\cdot y}$
Divide out common factors and simplify.	$\frac{3}{y}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{a}{8}}{\frac{ab}{6}}$.

Solution:

$\frac{3}{4b}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{p}{2}}{\frac{pq}{8}}$.

Solution:

$\frac{4}{q}$

Simplify Expressions with a Fraction Bar

The line that separates the numerator from the denominator in a fraction is called a fraction bar. A fraction bar acts as grouping symbol. The order of operations then tells us to simplify the numerator and then the denominator. Then we divide.

To simplify the expression $\frac{5-3}{7+1}$, we first simplify the numerator and the denominator separately. Then we divide.

Equation:

$$\frac{5-3}{7+1}$$

Equation:

$$\frac{2}{8}$$

Equation:

$$\frac{1}{4}$$

Note:

Simplify an Expression with a Fraction Bar.

Simplify the expression in the numerator. Simplify the expression in the denominator.
Simplify the fraction.

Example:
Exercise:

Problem: Simplify: $\frac{4-2(3)}{2^2+2}$.

Solution:
Solution

	$\frac{4-2(3)}{2^2+2}$
Use the order of operations to simplify the numerator and the denominator.	$\frac{4-6}{4+2}$
Simplify the numerator and the denominator.	$\frac{-2}{6}$
Simplify. A negative divided by a positive is negative.	$-\frac{1}{3}$

Note:
Exercise:

Problem: Simplify: $\frac{6-3(5)}{3^2+3}$.

Solution:

$-\frac{3}{4}$

Note:
Exercise:

Problem: Simplify: $\frac{4-4(6)}{3^2+3}$.

Solution:

$-\frac{5}{3}$

Where does the negative sign go in a fraction? Usually the negative sign is in front of the fraction, but you will sometimes see a fraction with a negative numerator, or sometimes with a negative denominator. Remember that fractions represent division. When the numerator and denominator have different signs, the quotient is negative.

Equation:

$$\begin{array}{ll} \frac{-1}{3} = -\frac{1}{3} & \frac{\text{negative}}{\text{positive}} = \text{negative} \\ \frac{1}{-3} = -\frac{1}{3} & \frac{\text{positive}}{\text{negative}} = \text{negative} \end{array}$$

Note:
Placement of Negative Sign in a Fraction
For any positive numbers *a* and *b*,
Equation:

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

Example:
Exercise:

Problem: Simplify: $\frac{4(-3)+6(-2)}{-3(2)-2}$.

Solution:
Solution

The fraction bar acts like a grouping symbol. So completely simplify the numerator and the denominator separately.

	$\frac{4(-3)+6(-2)}{-3(2)-2}$
Multiply.	$\frac{-12+(-12)}{-6-2}$
Simplify.	$\frac{-24}{-8}$
Divide.	3

Note:
Exercise:

Problem: Simplify: $\frac{8(-2)+4(-3)}{-5(2)+3}$.

Solution:

4

Note:
Exercise:

Problem: Simplify: $\frac{7(-1)+9(-3)}{-5(3)-2}$.

Solution:

2

Translate Phrases to Expressions with Fractions

Now that we have done some work with fractions, we are ready to translate phrases that would result in expressions with fractions.

The English words quotient and ratio are often used to describe fractions. Remember that "quotient" means division. The quotient of a and b is the result we get from dividing a by b, or $\frac{a}{b}$.

Example:
Exercise:

Problem:

Translate the English phrase into an algebraic expression: the quotient of the difference of *m* and *n*, and *p*.

Solution:
Solution

We are looking for the *quotient of* the difference of *m* and *n*, *and p*. This means we want to divide the difference of m and n by p.
Equation:

$$\frac{m-n}{p}$$

Note:
Exercise:

Problem:

Translate the English phrase into an algebraic expression: the quotient of the difference of a and b, and cd.

Solution:

$\frac{a-b}{cd}$

Note:
Exercise:

Problem:

Translate the English phrase into an algebraic expression: the quotient of the sum of p and q, and r

Solution:

$\frac{p+q}{r}$

Key Concepts

- **Equivalent Fractions Property:** If a, b, c are numbers where $b \neq 0, c \neq 0$, then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.

- **Fraction Division:** If a, b, c and d are numbers where $b \neq 0, c \neq 0,$ and $d \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$. To divide fractions, multiply the first fraction by the reciprocal of the second.
- **Fraction Multiplication:** If a, b, c and d are numbers where $b \neq 0,$ and $d \neq 0$, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. To multiply fractions, multiply the numerators and multiply the denominators.
- **Placement of Negative Sign in a Fraction:** For any positive numbers a and b, $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$.
- **Property of One:** $\frac{a}{a} = 1$; Any number, except zero, divided by itself is one.
- **Simplify a Fraction**

  Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers first.
  Simplify using the equivalent fractions property by dividing out common factors.
  Multiply any remaining factors.

- **Simplify an Expression with a Fraction Bar**

  Simplify the expression in the numerator. Simplify the expression in the denominator.
  Simplify the fraction.

Practice Makes Perfect

Find Equivalent Fractions

In the following exercises, find three fractions equivalent to the given fraction. Show your work, using figures or algebra.
Exercise:

Problem: $\frac{3}{8}$

Solution:

$\frac{6}{16}, \frac{9}{24}, \frac{12}{32}$ answers may vary

Exercise:

Problem: $\frac{5}{8}$

Exercise:

Problem: $\frac{5}{9}$

Solution:

$\frac{10}{18}, \frac{15}{27}, \frac{20}{36}$ answers may vary

Exercise:

Problem: $\frac{1}{8}$

Simplify Fractions

In the following exercises, simplify.

Exercise:

Problem: $-\frac{40}{88}$

Solution:

$-\frac{5}{11}$

Exercise:

Problem: $-\frac{63}{99}$

Exercise:

Problem: $-\frac{108}{63}$

Solution:

$-\frac{12}{7}$

Exercise:

Problem: $-\frac{104}{48}$

Exercise:

Problem: $\frac{120}{252}$

Solution:

$\frac{10}{21}$

Exercise:

Problem: $\frac{182}{294}$

Exercise:

Problem: $-\frac{3x}{12y}$

Solution:

$-\frac{x}{4y}$

Exercise:

Problem: $-\frac{4x}{32y}$

Exercise:

Problem: $\frac{14x^2}{21y}$

Solution:

$\frac{2x^2}{3y}$

Exercise:

Problem: $\frac{24a}{32b^2}$

Multiply Fractions

In the following exercises, multiply.

Exercise:

Problem: $\frac{3}{4} \cdot \frac{9}{10}$

Solution:

$\frac{27}{40}$

Exercise:

Problem: $\frac{4}{5} \cdot \frac{2}{7}$

Exercise:

Problem: $-\frac{2}{3}\left(-\frac{3}{8}\right)$

Solution:

$\frac{1}{4}$

Exercise:

Problem: $-\frac{3}{4}\left(-\frac{4}{9}\right)$

Exercise:

Problem: $-\frac{5}{9}\cdot\frac{3}{10}$

Solution:

$-\frac{1}{6}$

Exercise:

Problem: $-\frac{3}{8}\cdot\frac{4}{15}$

Exercise:

Problem: $\left(-\frac{14}{15}\right)\left(\frac{9}{20}\right)$

Solution:

$-\frac{21}{50}$

Exercise:

Problem: $\left(-\frac{9}{10}\right)\left(\frac{25}{33}\right)$

Exercise:

Problem: $\left(-\frac{63}{84}\right)\left(-\frac{44}{90}\right)$

Solution:

$\frac{11}{30}$

Exercise:

Problem: $\left(-\frac{63}{60}\right)\left(-\frac{40}{88}\right)$

Exercise:

Problem: $4 \cdot \frac{5}{11}$

Solution:

$\frac{20}{11}$

Exercise:

Problem: $5 \cdot \frac{8}{3}$

Exercise:

Problem: $\frac{3}{7} \cdot 21n$

Solution:

$9n$

Exercise:

Problem: $\frac{5}{6} \cdot 30m$

Exercise:

Problem: $-8\left(\frac{17}{4}\right)$

Solution:

-34

Exercise:

Problem: $(-1)\left(-\frac{6}{7}\right)$

Divide Fractions

In the following exercises, divide.

Exercise:

Problem: $\frac{3}{4} \div \frac{2}{3}$

Solution:

$\frac{9}{8}$

Exercise:

Problem: $\frac{4}{5} \div \frac{3}{4}$

Exercise:

Problem: $-\frac{7}{9} \div \left(-\frac{7}{4}\right)$

Solution:

$\frac{4}{9}$

Exercise:

Problem: $-\frac{5}{6} \div \left(-\frac{5}{6}\right)$

Exercise:

Problem: $\frac{3}{4} \div \frac{x}{11}$

Solution:

$\frac{33}{4x}$

Exercise:

Problem: $\frac{2}{5} \div \frac{y}{9}$

Exercise:

Problem: $\frac{5}{18} \div \left(-\frac{15}{24}\right)$

Solution:

$-\frac{4}{9}$

Exercise:

Problem: $\frac{7}{18} \div \left(-\frac{14}{27}\right)$

Exercise:

Problem: $\frac{8u}{15} \div \frac{12v}{25}$

Solution:

$\frac{10u}{9v}$

Exercise:

Problem: $\frac{12r}{25} \div \frac{18s}{35}$

Exercise:

Problem: $-5 \div \frac{1}{2}$

Solution:

-10

Exercise:

Problem: $-3 \div \frac{1}{4}$

Exercise:

Problem: $\frac{3}{4} \div (-12)$

Solution:

$-\frac{1}{16}$

Exercise:

Problem: $-15 \div \left(-\frac{5}{3}\right)$

In the following exercises, simplify.

Exercise:

Problem: $\frac{-\frac{8}{21}}{\frac{12}{35}}$

Solution:

$-\frac{10}{9}$

Exercise:

Problem: $\frac{-\frac{9}{16}}{\frac{33}{40}}$

Exercise:

Problem: $\frac{-\frac{4}{5}}{2}$

Solution:

$-\frac{2}{5}$

Exercise:

Problem: $\frac{5}{\frac{3}{10}}$

Exercise:

Problem: $\frac{\frac{m}{3}}{\frac{n}{2}}$

Solution:

$\frac{2m}{3n}$

Exercise:

Problem: $\frac{-\frac{3}{8}}{-\frac{y}{12}}$

Simplify Expressions Written with a Fraction Bar

In the following exercises, simplify.

Exercise:

Problem: $\frac{22+3}{10}$

Solution:

$\frac{5}{2}$

Exercise:

Problem: $\frac{19-4}{6}$

Exercise:

Problem: $\frac{48}{24-15}$

Solution:

$\frac{16}{3}$

Exercise:

Problem: $\frac{46}{4+4}$

Exercise:

Problem: $\frac{-6+6}{8+4}$

Solution:

0

Exercise:

Problem: $\frac{-6+3}{17-8}$

Exercise:

Problem: $\frac{4\cdot3}{6\cdot6}$

Solution:

$\frac{1}{3}$

Exercise:

Problem: $\frac{6\cdot6}{9\cdot2}$

Exercise:

Problem: $\frac{4^2-1}{25}$

Solution:

$\frac{3}{5}$

Exercise:

Problem: $\frac{7^2+1}{60}$

Exercise:

Problem: $\frac{8 \cdot 3+2 \cdot 9}{14+3}$

Solution:

$\frac{42}{17}$

Exercise:

Problem: $\frac{9 \cdot 6-4 \cdot 7}{22+3}$

Exercise:

Problem: $\frac{5 \cdot 6-3 \cdot 4}{4 \cdot 5-2 \cdot 3}$

Solution:

$\frac{9}{7}$

Exercise:

Problem: $\frac{8 \cdot 9-7 \cdot 6}{5 \cdot 6-9 \cdot 2}$

Exercise:

Problem: $\frac{5^2-3^2}{3-5}$

Solution:

$\frac{5}{2}$

Exercise:

Problem: $\frac{6^2-4^2}{4-6}$

Exercise:

Problem: $\frac{7\cdot4-2(8-5)}{9\cdot3-3\cdot5}$

Solution:

$\frac{11}{6}$

Exercise:

Problem: $\frac{9\cdot7-3(12-8)}{8\cdot7-6\cdot6}$

Exercise:

Problem: $\frac{9(8-2)-3(15-7)}{6(7-1)-3(17-9)}$

Solution:

$\frac{5}{2}$

Exercise:

Problem: $\frac{8(9-2)-4(14-9)}{7(8-3)-3(16-9)}$

Translate Phrases to Expressions with Fractions

In the following exercises, translate each English phrase into an algebraic expression.

Exercise:

Problem: the quotient of r and the sum of s and 10

Solution:

$\frac{r}{s+10}$

Exercise:

Problem: the quotient of A and the difference of 3 and B

Exercise:

Problem: the quotient of the difference of x and $y,$ and -3

Solution:

$\frac{x-y}{-3}$

Exercise:

Problem: the quotient of the sum of m and $n,$ and $4q$

Everyday Math

Exercise:

Problem:

Baking. A recipe for chocolate chip cookies calls for $\frac{3}{4}$ cup brown sugar. Imelda wants to double the recipe. ⓐ How much brown sugar will Imelda need? Show your calculation. ⓑ Measuring cups usually come in sets of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, and 1 cup. Draw a diagram to show two different ways that Imelda could measure the brown sugar needed to double the cookie recipe.

Solution:

ⓐ $1\frac{1}{2}$ cups ⓑ answers will vary

Exercise:

Problem:

Baking. Nina is making 4 pans of fudge to serve after a music recital. For each pan, she needs $\frac{2}{3}$ cup of condensed milk. ⓐ How much condensed milk will Nina need? Show your calculation. ⓑ Measuring cups usually come in sets of $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, and 1 cup. Draw a diagram to show two different ways that Nina could measure the condensed milk needed for 4 pans of fudge.

Exercise:

Problem:

Portions Don purchased a bulk package of candy that weighs 5 pounds. He wants to sell the candy in little bags that hold $\frac{1}{4}$ pound. How many little bags of candy can he fill from the bulk package?

Solution:

20 bags

Exercise:

Problem:

Portions Kristen has $\frac{3}{4}$ yards of ribbon that she wants to cut into 6 equal parts to make hair ribbons for her daughter’s 6 dolls. How long will each doll’s hair ribbon be?

Writing Exercises

Exercise:

Problem:

Rafael wanted to order half a medium pizza at a restaurant. The waiter told him that a medium pizza could be cut into 6 or 8 slices. Would he prefer 3 out of 6 slices or 4 out of 8 slices? Rafael replied that since he wasn’t very hungry, he would prefer 3 out of 6 slices. Explain what is wrong with Rafael’s reasoning.

Solution:

Answers may vary

Exercise:

Problem:

Give an example from everyday life that demonstrates how $\frac{1}{2} \cdot \frac{2}{3}$ is $\frac{1}{3}$.

Exercise:

Problem: Explain how you find the reciprocal of a fraction.

Solution:

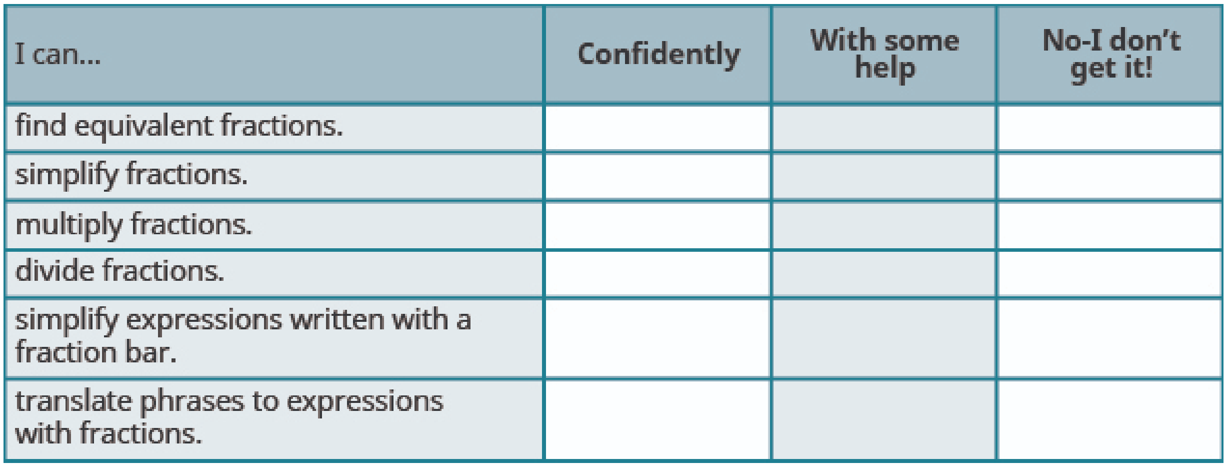
Answers may vary

Exercise:

Problem: Explain how you find the reciprocal of a negative number.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
find equivalent fractions.			
simplify fractions.			
multiply fractions.			
divide fractions.			
simplify expressions written with a fraction bar.			
translate phrases to expressions with fractions.			

ⓑ After looking at the checklist, do you think you are well prepared for the next section? Why or why not?

Glossary

complex fraction

A complex fraction is a fraction in which the numerator or the denominator contains a fraction.

denominator

The denominator is the value on the bottom part of the fraction that indicates the number of equal parts into which the whole has been divided.

equivalent fractions

Equivalent fractions are fractions that have the same value.

fraction

A fraction is written $\frac{a}{b}$, where $b \neq 0$ a is the numerator and b is the denominator. A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

numerator

The numerator is the value on the top part of the fraction that indicates how many parts of the whole are included.

reciprocal

The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$. A number and its reciprocal multiply to one: $\frac{a}{b} \cdot \frac{b}{a} = 1$.

simplified fraction

A fraction is considered simplified if there are no common factors in its numerator and denominator.

Add and Subtract Fractions
By the end of this section, you will be able to:

- Add or subtract fractions with a common denominator
- Add or subtract fractions with different denominators
- Use the order of operations to simplify complex fractions
- Evaluate variable expressions with fractions

Note:
A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **Fractions**.

Add or Subtract Fractions with a Common Denominator

When we multiplied fractions, we just multiplied the numerators and multiplied the denominators right straight across. To add or subtract fractions, they must have a common denominator.

Note:
Fraction Addition and Subtraction
If $a, b,$ and c are numbers where $c \neq 0$, then
Equation:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \quad \text{and} \quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

Note: Doing the Manipulative Mathematics activities “Model Fraction Addition” and “Model Fraction Subtraction” will help you develop a better understanding of adding and subtracting fractions.

Example:
Exercise:

Problem: Find the sum: $\frac{x}{3} + \frac{2}{3}$.

Solution:
Solution

	$\frac{x}{3} + \frac{2}{3}$
Add the numerators and place the sum over the common denominator.	$\frac{x+2}{3}$

Note:
Exercise:

Problem: Find the sum: $\frac{x}{4} + \frac{3}{4}$.

Solution:

$\frac{x+3}{4}$

Note:
Exercise:

Problem: Find the sum: $\frac{y}{8} + \frac{5}{8}$.

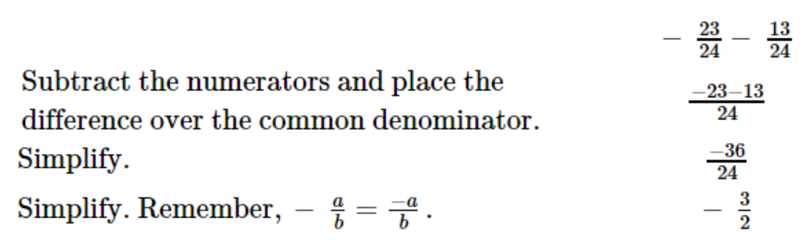
Solution:

$\frac{y+5}{8}$

Example:
Exercise:

Problem: Find the difference: $-\frac{23}{24}-\frac{13}{24}$.

Solution:
Solution



	$-\frac{23}{24}-\frac{13}{24}$
Subtract the numerators and place the difference over the common denominator.	$\frac{-23-13}{24}$
Simplify.	$\frac{-36}{24}$
Simplify. Remember, $-\frac{a}{b}=\frac{-a}{b}$.	$-\frac{3}{2}$

Note:
Exercise:

Problem: Find the difference: $-\frac{19}{28}-\frac{7}{28}$.

Solution:

$-\frac{26}{28}$

Note:
Exercise:

Problem: Find the difference: $-\frac{27}{32}-\frac{1}{32}$.

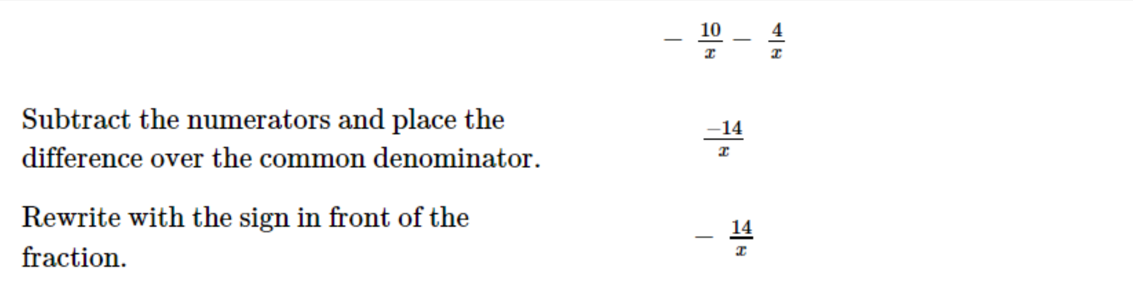
Solution:

$-\frac{7}{8}$

Example:
Exercise:

Problem: Simplify: $-\frac{10}{x}-\frac{4}{x}$.

Solution:
Solution



	$-\frac{10}{x}-\frac{4}{x}$
Subtract the numerators and place the difference over the common denominator.	$\frac{-14}{x}$
Rewrite with the sign in front of the fraction.	$-\frac{14}{x}$

Note:
Exercise:

Problem: Find the difference: $-\frac{9}{x}-\frac{7}{x}$.

Solution:

$-\frac{16}{x}$

Note:
Exercise:

Problem: Find the difference: $-\frac{17}{a}-\frac{5}{a}$.

Solution:

$-\frac{22}{a}$

Now we will do an example that has both addition and subtraction.

Example:
Exercise:

Problem: Simplify: $\frac{3}{8}+\left(-\frac{5}{8}\right)-\frac{1}{8}$.

Solution:
Solution

Add and subtract fractions—do they have a common denominator? Yes.	$\frac{3}{8}+\left(-\frac{5}{8}\right)-\frac{1}{8}$
Add and subtract the numerators and place the result over the common denominator.	$\frac{3+(-5)-1}{8}$
Simplify left to right.	$\frac{-2-1}{8}$
Simplify.	$-\frac{3}{8}$

Note:
Exercise:

Problem: Simplify: $\frac{2}{9}+\left(-\frac{4}{9}\right)-\frac{7}{9}$.

Solution:

-1

Note:
Exercise:

Problem: Simplify: $\frac{5}{9}+\left(-\frac{4}{9}\right)-\frac{7}{9}$.

Solution:

$-\frac{2}{3}$

Add or Subtract Fractions with Different Denominators

As we have seen, to add or subtract fractions, their denominators must be the same. The **least common denominator** (LCD) of two fractions is the smallest number that can be used as a common denominator of the fractions. The LCD of the two fractions is the least common multiple (LCM) of their denominators.

Note:
Least Common Denominator
The least common denominator (LCD) of two fractions is the least common multiple (LCM) of their denominators.

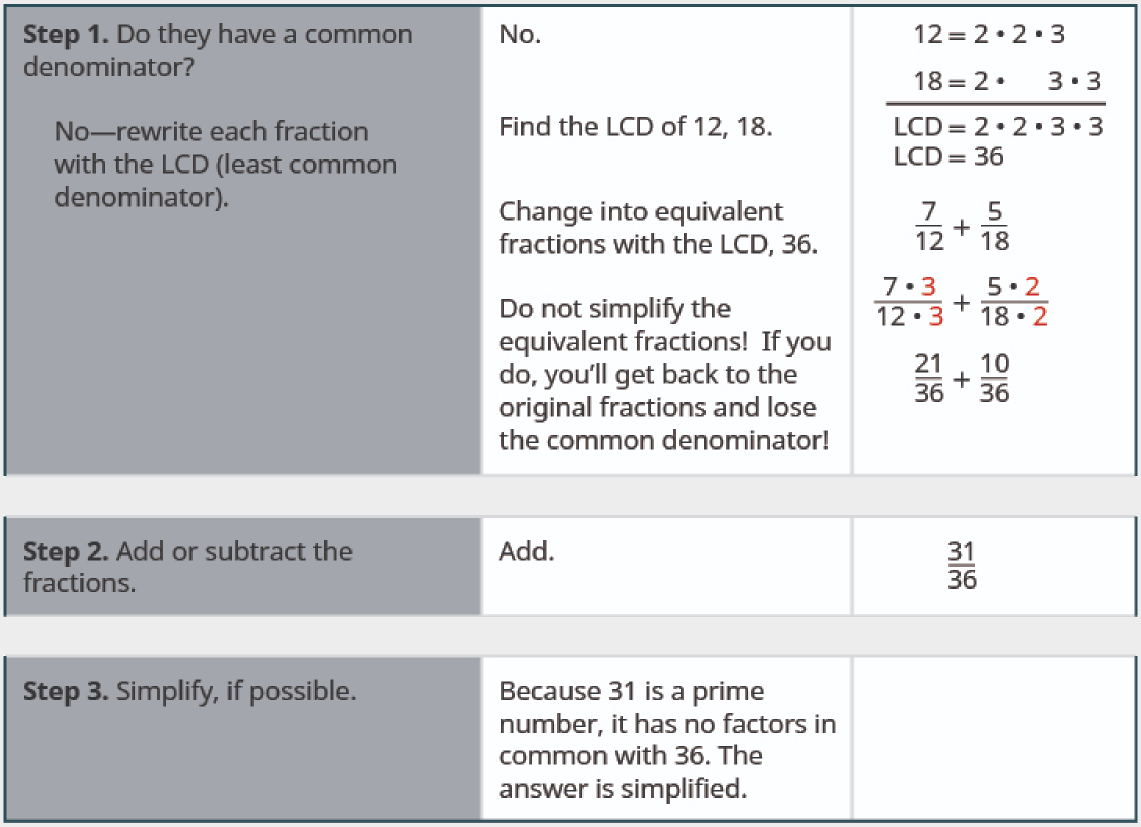
Note: Doing the Manipulative Mathematics activity “Finding the Least Common Denominator” will help you develop a better understanding of the LCD.

After we find the least common denominator of two fractions, we convert the fractions to equivalent fractions with the LCD. Putting these steps together allows us to add and subtract fractions because their denominators will be the same!

Example:
How to Add or Subtract Fractions
Exercise:

Problem: Add: $\frac{7}{12} + \frac{5}{18}$.

Solution:



Step 1. Do they have a common denominator? No—rewrite each fraction with the LCD (least common denominator).	No. Find the LCD of 12, 18. Change into equivalent fractions with the LCD, 36. Do not simplify the equivalent fractions! If you do, you'll get back to the original fractions and lose the common denominator!	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$ $\text{LCD} = 2 \cdot 2 \cdot 3 \cdot 3$ $\text{LCD} = 36$ $\frac{7}{12} + \frac{5}{18}$ $\frac{7 \cdot 3}{12 \cdot 3} + \frac{5 \cdot 2}{18 \cdot 2}$ $\frac{21}{36} + \frac{10}{36}$
Step 2. Add or subtract the fractions.	Add.	$\frac{31}{36}$
Step 3. Simplify, if possible.	Because 31 is a prime number, it has no factors in common with 36. The answer is simplified.	

Note:
Exercise:

Problem: Add: $\frac{7}{12} + \frac{11}{15}$.

Solution:

$\frac{79}{60}$

Note:
Exercise:

Problem: Add: $\frac{13}{15} + \frac{17}{20}$.

Solution:

$\frac{103}{60}$

Note:
Add or Subtract Fractions.

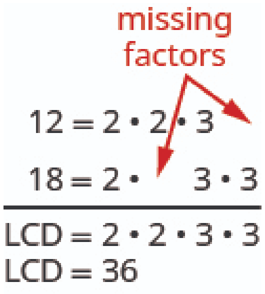
Do they have a common denominator?

- Yes—go to step 2.
- No—rewrite each fraction with the LCD (least common denominator). Find the LCD. Change each fraction into an equivalent fraction with the LCD as its denominator.

Add or subtract the fractions.
Simplify, if possible.

When finding the equivalent fractions needed to create the common denominators, there is a quick way to find the number we need to multiply both the numerator and denominator. This method works if we found the LCD by factoring into primes.

Look at the factors of the LCD and then at each column above those factors. The “missing” factors of each denominator are the numbers we need.



In [link], the LCD, 36, has two factors of 2 and two factors of 3.

The numerator 12 has two factors of 2 but only one of 3—so it is “missing” one 3—we multiply the numerator and denominator by 3.

The numerator 18 is missing one factor of 2—so we multiply the numerator and denominator by 2.

We will apply this method as we subtract the fractions in [link].

Example:
Exercise:

Problem: Subtract: $\frac{7}{15}-\frac{19}{24}$.

Solution:
Solution

Do the fractions have a common denominator? No, so we need to find the LCD.

Find the LCD.	

$\frac{7}{15} - \frac{19}{24}$ $15 = 3 \cdot 5$ $24 = 2 \cdot 2 \cdot 2 \cdot 3$ $\text{LCD} = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$ $\text{LCD} = 120$	
Notice, 15 is “missing” three factors of 2 and 24 is “missing” the 5 from the factors of the LCD. So we multiply 8 in the first fraction and 5 in the second fraction to get the LCD.	
Rewrite as equivalent fractions with the LCD.	$\frac{7 \cdot 8}{15 \cdot 8} - \frac{19 \cdot 5}{24 \cdot 5}$
Simplify.	$\frac{56}{120} - \frac{95}{120}$
Subtract.	$-\frac{39}{120}$
Check to see if the answer can be simplified.	$-\frac{13 \cdot 3}{40 \cdot 3}$
Both 39 and 120 have a factor of 3.	
Simplify.	$-\frac{13}{40}$

Do not simplify the equivalent fractions! If you do, you’ll get back to the original fractions and lose the common denominator!

Note:
Exercise:

Problem: Subtract: $\frac{13}{24} - \frac{17}{32}$.

Solution:

$\frac{1}{96}$

Note:
Exercise:

Problem: Subtract: $\frac{21}{32} - \frac{9}{28}$.

Solution:

$\frac{75}{224}$

In the next example, one of the fractions has a variable in its numerator. Notice that we do the same steps as when both numerators are numbers.

Example:
Exercise:

Problem: Add: $\frac{3}{5} + \frac{x}{8}$.

Solution:
Solution

The fractions have different denominators.

		$\frac{3}{5}+\frac{x}{8}$
Find the LCD. $\begin{array}{rl} 5 = & 5 \\ 8 = & 2\cdot 2\cdot 2 \\ \hline \text{LCD} = & 2\cdot 2\cdot 2\cdot 5 \\ \text{LCD} = & 40 \end{array}$		
Rewrite as equivalent fractions with the LCD.		$\frac{3\cdot 8}{5\cdot 8}+\frac{x\cdot 5}{8\cdot 5}$
Simplify.		$\frac{24}{40}+\frac{5x}{40}$
Add.		$\frac{24+5x}{40}$

Remember, we can only add like terms: 24 and $5x$ are not like terms.

Note:
Exercise:

Problem: Add: $\frac{y}{6}+\frac{7}{9}$.

Solution:

$\frac{9y+42}{54}$

Note:
Exercise:

Problem: Add: $\frac{x}{6}+\frac{7}{15}$.

Solution:

$\frac{5x+14}{30}$

We now have all four operations for fractions. [link] summarizes fraction operations.

Fraction Multiplication	**Fraction Division**
$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$ Multiply the numerators and multiply the denominators	$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$ Multiply the first fraction by the reciprocal of the second.
Fraction Addition	**Fraction Subtraction**
$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$ Add the numerators and place the sum over the common denominator.	$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$ Subtract the numerators and place the difference over the common denominator.

To multiply or divide fractions, an LCD is NOT needed.
To add or subtract fractions, an LCD is needed.

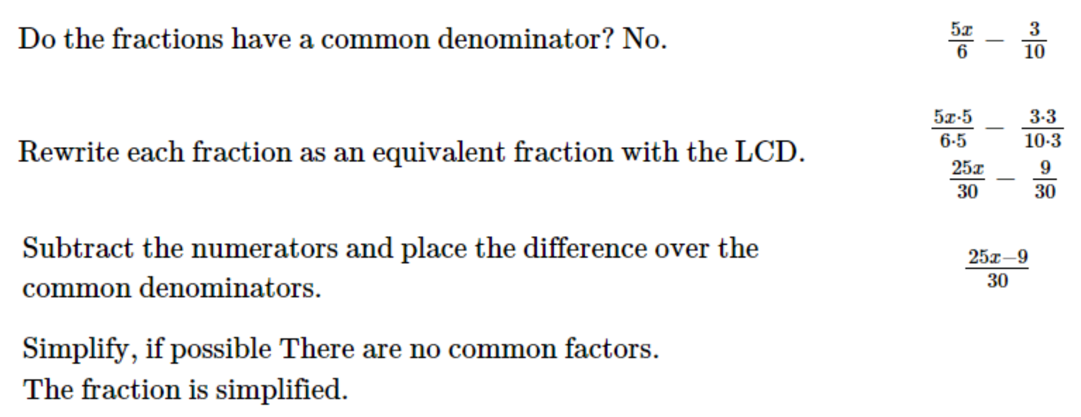
Example:
Exercise:

Problem: Simplify: ⓐ $\frac{5x}{6} - \frac{3}{10}$ ⓑ $\frac{5x}{6} \cdot \frac{3}{10}$.

Solution:
Solution

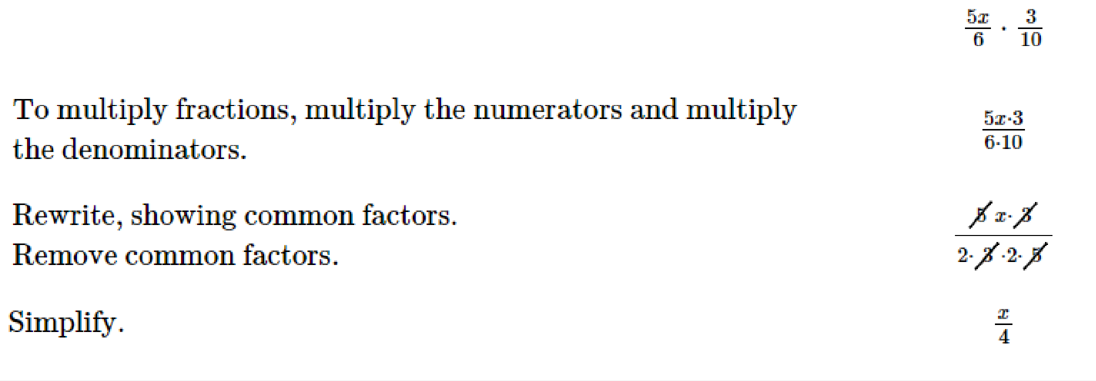
First ask, “What is the operation?” Once we identify the operation that will determine whether we need a common denominator. Remember, we need a common denominator to add or subtract, but not to multiply or divide.

ⓐ What is the operation? The operation is subtraction.



Do the fractions have a common denominator? No.	$\frac{5x}{6} - \frac{3}{10}$
Rewrite each fraction as an equivalent fraction with the LCD.	$\frac{5x\cdot 5}{6\cdot 5} - \frac{3\cdot 3}{10\cdot 3}$ $\frac{25x}{30} - \frac{9}{30}$
Subtract the numerators and place the difference over the common denominators.	$\frac{25x-9}{30}$
Simplify, if possible There are no common factors. The fraction is simplified.	

ⓑ What is the operation? Multiplication.



	$\frac{5x}{6} \cdot \frac{3}{10}$
To multiply fractions, multiply the numerators and multiply the denominators.	$\frac{5x \cdot 3}{6 \cdot 10}$
Rewrite, showing common factors. Remove common factors.	$\frac{\cancel{5} x \cdot \cancel{3}}{2 \cdot \cancel{3} \cdot 2 \cdot \cancel{5}}$
Simplify.	$\frac{x}{4}$

Notice we needed an LCD to add $\frac{5x}{6} - \frac{3}{10}$, but not to multiply $\frac{5x}{6} \cdot \frac{3}{10}$.

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{3a}{4} - \frac{8}{9}$ ⓑ $\frac{3a}{4} \cdot \frac{8}{9}$.

Solution:

ⓐ $\frac{27a-32}{36}$ ⓑ $\frac{2a}{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{4k}{5} - \frac{1}{6}$ ⓑ $\frac{4k}{5} \cdot \frac{1}{6}$.

Solution:

ⓐ $\frac{24k-5}{30}$ ⓑ $\frac{2k}{15}$

Use the Order of Operations to Simplify Complex Fractions

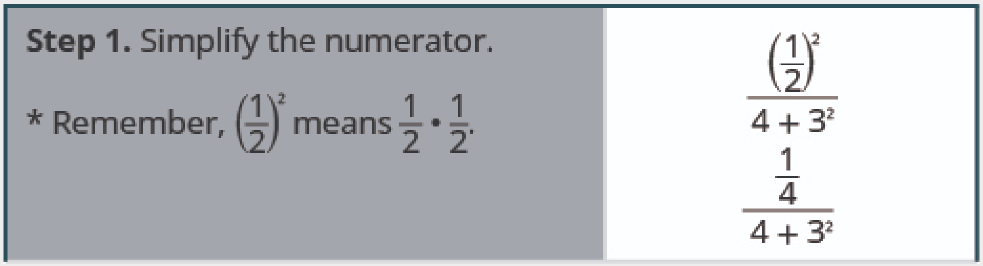
We have seen that a complex fraction is a fraction in which the numerator or denominator contains a fraction. The fraction bar indicates division. We simplified the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ by dividing $\frac{3}{4}$ by $\frac{5}{8}$.

Now we'll look at complex fractions where the numerator or denominator contains an expression that can be simplified. So we first must completely simplify the numerator and denominator separately using the order of operations. Then we divide the numerator by the denominator.

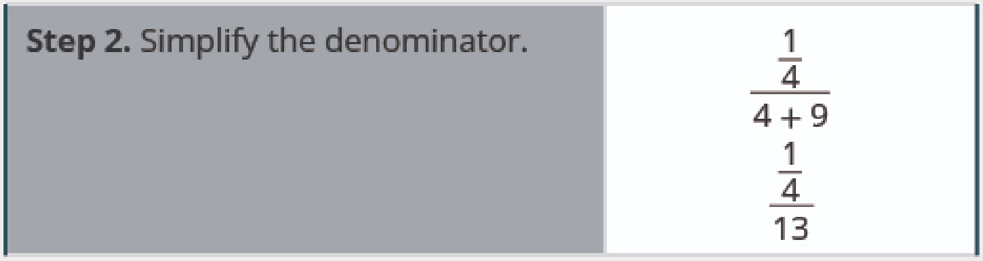
Example:
How to Simplify Complex Fractions
Exercise:

Problem: Simplify: $\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$.

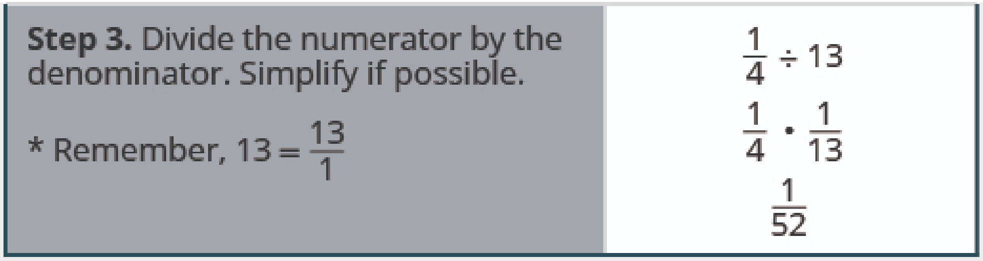
Solution:
Solution



Step 1. Simplify the numerator. * Remember, $\left(\frac{1}{2}\right)^2$ means $\frac{1}{2}\cdot\frac{1}{2}$.	$\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$ $\frac{\frac{1}{4}}{4+3^2}$



Step 2. Simplify the denominator.	$\frac{\frac{1}{4}}{4+9}$ $\frac{\frac{1}{4}}{13}$



Step 3. Divide the numerator by the denominator. Simplify if possible. * Remember, $13 = \frac{13}{1}$	$\frac{1}{4} \div 13$ $\frac{1}{4} \cdot \frac{1}{13}$ $\frac{1}{52}$

Note:
Exercise:

Problem: Simplify: $\frac{\left(\frac{1}{3}\right)^2}{2^3+2}$.

Solution:

$\frac{1}{90}$

Note:
Exercise:

Problem: Simplify: $\frac{1+4^2}{\left(\frac{1}{4}\right)^2}$.

Solution:

272

Note:
Simplify Complex Fractions.

Simplify the numerator.

Simplify the denominator.
Divide the numerator by the denominator. Simplify if possible.

Example:
Exercise:

Problem: Simplify: $\frac{\frac{1}{2}+\frac{2}{3}}{\frac{3}{4}-\frac{1}{6}}$.

Solution:
Solution

It may help to put parentheses around the numerator and the denominator.

	$\frac{\left(\frac{1}{2}+\frac{2}{3}\right)}{\left(\frac{3}{4}-\frac{1}{6}\right)}$
Simplify the numerator $(\text{LCD} = 6)$ and simplify the denominator $(\text{LCD} = 12)$.	$\frac{\left(\frac{3}{6}+\frac{4}{6}\right)}{\left(\frac{9}{12}-\frac{2}{12}\right)}$
Simplify.	$\frac{\left(\frac{7}{6}\right)}{\left(\frac{7}{12}\right)}$
Divide the numerator by the denominator.	$\frac{7}{6} \div \frac{7}{12}$
Simplify.	$\frac{7}{6} \cdot \frac{12}{7}$
Divide out common factors.	$\frac{7 \cdot 6 \cdot 2}{6 \cdot 7}$
Simplify.	2

Note:
Exercise:

Problem: Simplify: $\frac{\frac{1}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{1}{3}}$.

Solution:

2

Note:
Exercise:

Problem: Simplify: $\frac{\frac{2}{3}-\frac{1}{2}}{\frac{1}{4}+\frac{1}{3}}$.

Solution:

$\frac{2}{7}$

Evaluate Variable Expressions with Fractions

We have evaluated expressions before, but now we can evaluate expressions with fractions. Remember, to evaluate an expression, we substitute the value of the variable into the expression and then simplify.

Example:
Exercise:

Problem: Evaluate $x+\frac{1}{3}$ when ⓐ $x=-\frac{1}{3}$ ⓑ $x=-\frac{3}{4}$.

Solution:

ⓐ To evaluate $x+\frac{1}{3}$ when $x=-\frac{1}{3}$, substitute $-\frac{1}{3}$ for x in the expression.

	$x+\frac{1}{3}$
Substitute $-\frac{1}{3}$ for x.	$-\frac{1}{3}+\frac{1}{3}$
Simplify.	0

ⓑ To evaluate $x+\frac{1}{3}$ when $x=-\frac{3}{4}$, we substitute $-\frac{3}{4}$ for x in the expression.

	$x+\frac{1}{3}$
Substitute $-\frac{3}{4}$ for x.	$-\frac{3}{4}+\frac{1}{3}$
Rewrite as equivalent fractions with the LCD,	

12.	$-\frac{3\cdot 3}{4\cdot 3}+\frac{1\cdot 4}{3\cdot 4}$
Simplify.	$-\frac{9}{12}+\frac{4}{12}$
Add.	$-\frac{5}{12}$

Note:
Exercise:

Problem: Evaluate $x+\frac{3}{4}$ when ⓐ $x=-\frac{7}{4}$ ⓑ $x=-\frac{5}{4}$.

Solution:

ⓐ -1 ⓑ $-\frac{1}{2}$

Note:
Exercise:

Problem: Evaluate $y+\frac{1}{2}$ when ⓐ $y=\frac{2}{3}$ ⓑ $y=-\frac{3}{4}$.

Solution:

ⓐ $\frac{7}{6}$ ⓑ $-\frac{1}{4}$

Example:
Exercise:

Problem: Evaluate $-\frac{5}{6}-y$ when $y=-\frac{2}{3}$.

Solution:
Solution

	$-\frac{5}{6}-y$
Substitute $-\frac{2}{3}$ for y.	$-\frac{5}{6}-\left(-\frac{2}{3}\right)$
Rewrite as equivalent fractions with the LCD, 6.	$-\frac{5}{6}-\left(-\frac{4}{6}\right)$
Subtract.	$\frac{-5-(-4)}{6}$
Simplify.	$-\frac{1}{6}$

Note:
Exercise:

Problem: Evaluate $-\frac{1}{2}-y$ when $y=-\frac{1}{4}$.

Solution:

$-\frac{1}{4}$

Note:
Exercise:

Problem: Evaluate $-\frac{3}{8}-y$ when $y=-\frac{5}{2}$.

Solution:

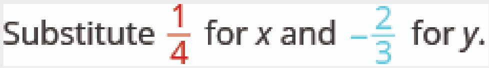
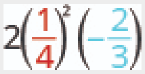
$\frac{17}{8}$

Example:
Exercise:

Problem: Evaluate $2x^2y$ when $x=\frac{1}{4}$ and $y=-\frac{2}{3}$.

Solution:
Solution

Substitute the values into the expression.

	$2x^2y$
 Substitute $\frac{1}{4}$ for x and $-\frac{2}{3}$ for y.	 $2\left(\frac{1}{4}\right)^2\left(-\frac{2}{3}\right)$
Simplify exponents first.	$2\left(\frac{1}{16}\right)\left(-\frac{2}{3}\right)$

Multiply. Divide out the common factors. Notice we write 16 as $2 \cdot 2 \cdot 4$ to make it easy to remove common factors.	$-\frac{\cancel{2}\cdot 1\cdot \cancel{2}}{\cancel{2}\cdot \cancel{2}\cdot 4\cdot 3}$
Simplify.	$-\frac{1}{12}$

Note:
Exercise:

Problem: Evaluate $3ab^2$ when $a = -\frac{2}{3}$ and $b = -\frac{1}{2}$.

Solution:

$-\frac{1}{2}$

Note:
Exercise:

Problem: Evaluate $4c^3d$ when $c = -\frac{1}{2}$ and $d = -\frac{4}{3}$.

Solution:

$\frac{2}{3}$

The next example will have only variables, no constants.

Example:

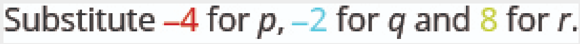
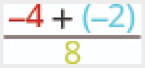
Exercise:

Problem: Evaluate $\frac{p+q}{r}$ when $p=-4, q=-2,$ and $r=8.$

Solution:

Solution

To evaluate $\frac{p+q}{r}$ when $p=-4, q=-2,$ and $r=8,$ we substitute the values into the expression.

	$\frac{p+q}{r}$
 Substitute –4 for *p*, –2 for *q* and 8 for *r*.	 $\frac{-4+(-2)}{8}$
Add in the numerator first.	$\frac{-6}{8}$
Simplify.	$-\frac{3}{4}$

Note:

Exercise:

Problem: Evaluate $\frac{a+b}{c}$ when $a=-8, b=-7,$ and $c=6.$

Solution:

$-\frac{5}{2}$

Note:
Exercise:

Problem: Evaluate $\frac{x+y}{z}$ when $x=9, y=-18,$ and $z=-6.$

Solution:

$\frac{3}{2}$

Key Concepts

- **Fraction Addition and Subtraction:** If $a, b,$ and c are numbers where $c \neq 0$, then
$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$ and $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$.

  To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.
- **Strategy for Adding or Subtracting Fractions**

Do they have a common denominator?	Yes—go to step 2.	No—Rewrite each fraction with the LCD (Least Common Denominator). Find the LCD. Change each fraction into an equivalent fraction with the LCD as its denominator.

  Add or subtract the fractions by adding or subtracting the numerators and placing that sum or difference over the common denominator.
  Simplify, if possible. To multiply or divide fractions, an LCD IS NOT needed. To add or subtract fractions, an LCD IS needed.

- **Simplify Complex Fractions**

  Simplify the numerator.
  Simplify the denominator.
  Divide the numerator by the denominator. Simplify if possible.

Practice Makes Perfect

Add and Subtract Fractions with a Common Denominator

In the following exercises, add.

Exercise:

Problem: $\frac{6}{13}+\frac{5}{13}$

Solution:

$\frac{11}{13}$

Exercise:

Problem: $\frac{4}{15}+\frac{7}{15}$

Exercise:

Problem: $\frac{x}{4}+\frac{3}{4}$

Solution:

$\frac{x+3}{4}$

Exercise:

Problem: $\frac{8}{q}+\frac{6}{q}$

Exercise:

Problem: $-\frac{3}{16}+\left(-\frac{7}{16}\right)$

Solution:

$-\frac{5}{8}$

Exercise:

Problem: $-\frac{5}{16}+\left(-\frac{9}{16}\right)$

Exercise:

Problem: $-\frac{8}{17}+\frac{15}{17}$

Solution:

$\frac{7}{17}$

Exercise:

Problem: $-\frac{9}{19}+\frac{17}{19}$

Exercise:

Problem: $\frac{6}{13}+\left(-\frac{10}{13}\right)+\left(-\frac{12}{13}\right)$

Solution:

$-\frac{16}{13}$

Exercise:

Problem: $\frac{5}{12}+\left(-\frac{7}{12}\right)+\left(-\frac{11}{12}\right)$

In the following exercises, subtract.

Exercise:

Problem: $\frac{11}{15}-\frac{7}{15}$

Solution:

$\frac{4}{15}$

Exercise:

Problem: $\frac{9}{13}-\frac{4}{13}$

Exercise:

Problem: $\frac{11}{12}-\frac{5}{12}$

Solution:

$\frac{1}{2}$

Exercise:

Problem: $\frac{7}{12}-\frac{5}{12}$

Exercise:

Problem: $\frac{19}{21}-\frac{4}{21}$

Solution:

$\frac{5}{7}$

Exercise:

Problem: $\frac{17}{21}-\frac{8}{21}$

Exercise:

Problem: $\frac{5y}{8}-\frac{7}{8}$

Solution:

$\frac{5y-7}{8}$

Exercise:

Problem: $\frac{11z}{13}-\frac{8}{13}$

Exercise:

Problem: $-\frac{23}{u}-\frac{15}{u}$

Solution:

$-\frac{38}{u}$

Exercise:

Problem: $-\frac{29}{v}-\frac{26}{v}$

Exercise:

Problem: $-\frac{3}{5}-\left(-\frac{4}{5}\right)$

Solution:

$\frac{1}{5}$

Exercise:

Problem: $-\frac{3}{7}-\left(-\frac{5}{7}\right)$

Exercise:

Problem: $-\frac{7}{9}-\left(-\frac{5}{9}\right)$

Solution:

$-\frac{2}{9}$

Exercise:

Problem: $-\frac{8}{11}-\left(-\frac{5}{11}\right)$

Mixed Practice

In the following exercises, simplify.

Exercise:

Problem: $-\frac{5}{18} \cdot \frac{9}{10}$

Solution:

$-\frac{1}{4}$

Exercise:

Problem: $-\frac{3}{14} \cdot \frac{7}{12}$

Exercise:

Problem: $\frac{n}{5} - \frac{4}{5}$

Solution:

$\frac{n-4}{5}$

Exercise:

Problem: $\frac{6}{11} - \frac{s}{11}$

Exercise:

Problem: $-\frac{7}{24} + \frac{2}{24}$

Solution:

$-\frac{5}{24}$

Exercise:

Problem: $-\frac{5}{18} + \frac{1}{18}$

Exercise:

Problem: $\frac{8}{15} \div \frac{12}{5}$

Solution:

$\frac{2}{9}$

Exercise:

Problem: $\frac{7}{12} \div \frac{9}{28}$

Add or Subtract Fractions with Different Denominators

In the following exercises, add or subtract.

Exercise:

Problem: $\frac{1}{2} + \frac{1}{7}$

Solution:

$\frac{9}{14}$

Exercise:

Problem: $\frac{1}{3} + \frac{1}{8}$

Exercise:

Problem: $\frac{1}{3} - \left(-\frac{1}{9}\right)$

Solution:

$\frac{4}{9}$

Exercise:

Problem: $\frac{1}{4} - \left(-\frac{1}{8}\right)$

Exercise:

Problem: $\frac{7}{12}+\frac{5}{8}$

Solution:

$\frac{29}{24}$

Exercise:

Problem: $\frac{5}{12}+\frac{3}{8}$

Exercise:

Problem: $\frac{7}{12}-\frac{9}{16}$

Solution:

$\frac{1}{48}$

Exercise:

Problem: $\frac{7}{16}-\frac{5}{12}$

Exercise:

Problem: $\frac{2}{3}-\frac{3}{8}$

Solution:

$\frac{7}{24}$

Exercise:

Problem: $\frac{5}{6}-\frac{3}{4}$

Exercise:

Problem: $-\frac{11}{30}+\frac{27}{40}$

Solution:

$\frac{37}{120}$

Exercise:

Problem: $-\frac{9}{20}+\frac{17}{30}$

Exercise:

Problem: $-\frac{13}{30}+\frac{25}{42}$

Solution:

$\frac{17}{105}$

Exercise:

Problem: $-\frac{23}{30}+\frac{5}{48}$

Exercise:

Problem: $-\frac{39}{56}-\frac{22}{35}$

Solution:

$-\frac{53}{40}$

Exercise:

Problem: $-\frac{33}{49}-\frac{18}{35}$

Exercise:

Problem: $-\frac{2}{3}-\left(-\frac{3}{4}\right)$

Solution:

$\frac{1}{12}$

Exercise:

Problem: $-\frac{3}{4} - \left(-\frac{4}{5}\right)$

Exercise:

Problem: $1 + \frac{7}{8}$

Solution:

$\frac{15}{8}$

Exercise:

Problem: $1 - \frac{3}{10}$

Exercise:

Problem: $\frac{x}{3} + \frac{1}{4}$

Solution:

$\frac{4x+3}{12}$

Exercise:

Problem: $\frac{y}{2} + \frac{2}{3}$

Exercise:

Problem: $\frac{y}{4} - \frac{3}{5}$

Solution:

$\frac{5y-12}{20}$

Exercise:

Problem: $\frac{x}{5} - \frac{1}{4}$

Mixed Practice

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\frac{2}{3} + \frac{1}{6}$ ⓑ $\frac{2}{3} \div \frac{1}{6}$

Solution:

ⓐ $\frac{5}{6}$ ⓑ 4

Exercise:

Problem: ⓐ $-\frac{2}{5} - \frac{1}{8}$ ⓑ $-\frac{2}{5} \cdot \frac{1}{8}$

Exercise:

Problem: ⓐ $\frac{5n}{6} \div \frac{8}{15}$ ⓑ $\frac{5n}{6} - \frac{8}{15}$

Solution:

ⓐ $\frac{25n}{16}$ ⓑ $\frac{25n-16}{30}$

Exercise:

Problem: ⓐ $\frac{3a}{8} \div \frac{7}{12}$ ⓑ $\frac{3a}{8} - \frac{7}{12}$

Exercise:

Problem: $-\frac{3}{8} \div \left(-\frac{3}{10}\right)$

Solution:

$\frac{5}{4}$

Exercise:

Problem: $-\frac{5}{12} \div \left(-\frac{5}{9}\right)$

Exercise:

Problem: $-\frac{3}{8} + \frac{5}{12}$

Solution:

$\frac{1}{24}$

Exercise:

Problem: $-\frac{1}{8} + \frac{7}{12}$

Exercise:

Problem: $\frac{5}{6} - \frac{1}{9}$

Solution:

$\frac{13}{18}$

Exercise:

Problem: $\frac{5}{9} - \frac{1}{6}$

Exercise:

Problem: $-\frac{7}{15} - \frac{y}{4}$

Solution:

$\frac{-28-15y}{60}$

Exercise:

Problem: $-\frac{3}{8} - \frac{x}{11}$

Exercise:

Problem: $\frac{11}{12a} \cdot \frac{9a}{16}$

Solution:

$\frac{33}{64}$

Exercise:

Problem: $\frac{10y}{13} \cdot \frac{8}{15y}$

Use the Order of Operations to Simplify Complex Fractions

In the following exercises, simplify.

Exercise:

Problem: $\frac{2^3+4^2}{\left(\frac{2}{3}\right)^2}$

Solution:

54

Exercise:

Problem: $\frac{3^3-3^2}{\left(\frac{3}{4}\right)^2}$

Exercise:

Problem: $\frac{\left(\frac{3}{5}\right)^2}{\left(\frac{3}{7}\right)^2}$

Solution:

$\frac{49}{25}$

Exercise:

Problem: $\frac{\left(\frac{3}{4}\right)^2}{\left(\frac{5}{8}\right)^2}$

Exercise:

Problem: $\frac{2}{\frac{1}{3}+\frac{1}{5}}$

Solution:

$\frac{15}{4}$

Exercise:

Problem: $\frac{5}{\frac{1}{4}+\frac{1}{3}}$

Exercise:

Problem: $\frac{\frac{7}{8}-\frac{2}{3}}{\frac{1}{2}+\frac{3}{8}}$

Solution:

$\frac{5}{21}$

Exercise:

Problem: $\frac{\frac{3}{4}-\frac{3}{5}}{\frac{1}{4}+\frac{2}{5}}$

Exercise:

Problem: $\frac{1}{2}+\frac{2}{3}\cdot\frac{5}{12}$

Solution:

$\frac{7}{9}$

Exercise:

Problem: $\frac{1}{3} + \frac{2}{5} \cdot \frac{3}{4}$

Exercise:

Problem: $1 - \frac{3}{5} \div \frac{1}{10}$

Solution:

-5

Exercise:

Problem: $1 - \frac{5}{6} \div \frac{1}{12}$

Exercise:

Problem: $\frac{2}{3} + \frac{1}{6} + \frac{3}{4}$

Solution:

$\frac{19}{12}$

Exercise:

Problem: $\frac{2}{3} + \frac{1}{4} + \frac{3}{5}$

Exercise:

Problem: $\frac{3}{8} - \frac{1}{6} + \frac{3}{4}$

Solution:

$\frac{23}{24}$

Exercise:

Problem: $\frac{2}{5}+\frac{5}{8}-\frac{3}{4}$

Exercise:

Problem: $12\left(\frac{9}{20}-\frac{4}{15}\right)$

Solution:

$\frac{11}{5}$

Exercise:

Problem: $8\left(\frac{15}{16}-\frac{5}{6}\right)$

Exercise:

Problem: $\frac{\frac{5}{8}+\frac{1}{6}}{\frac{19}{24}}$

Solution:

1

Exercise:

Problem: $\frac{\frac{1}{6}+\frac{3}{10}}{\frac{14}{30}}$

Exercise:

Problem: $\left(\frac{5}{9}+\frac{1}{6}\right)\div\left(\frac{2}{3}-\frac{1}{2}\right)$

Solution:

$\frac{13}{3}$

Exercise:

Problem: $\left(\frac{3}{4}+\frac{1}{6}\right)\div\left(\frac{5}{8}-\frac{1}{3}\right)$

Evaluate Variable Expressions with Fractions

In the following exercises, evaluate.

Exercise:

Problem: $x+\left(-\frac{5}{6}\right)$ when
ⓐ $x=\frac{1}{3}$
ⓑ $x=-\frac{1}{6}$

Solution:

ⓐ $-\frac{1}{2}$ ⓑ -1

Exercise:

Problem: $x+\left(-\frac{11}{12}\right)$ when
ⓐ $x=\frac{11}{12}$
ⓑ $x=\frac{3}{4}$

Exercise:

Problem: $x-\frac{2}{5}$ when
ⓐ $x=\frac{3}{5}$
ⓑ $x=-\frac{3}{5}$

Solution:

ⓐ $\frac{1}{5}$ ⓑ -1

Exercise:

Problem: $x-\frac{1}{3}$ when
ⓐ $x=\frac{2}{3}$
ⓑ $x=-\frac{2}{3}$

Exercise:

Problem: $\frac{7}{10}-w$ when
ⓐ $w=\frac{1}{2}$
ⓑ $w=-\frac{1}{2}$

Solution:

ⓐ $\frac{1}{5}$ ⓑ $\frac{6}{5}$

Exercise:

Problem: $\frac{5}{12}-w$ when
ⓐ $w=\frac{1}{4}$
ⓑ $w=-\frac{1}{4}$

Exercise:

Problem: $2x^2y^3$ when $x=-\frac{2}{3}$ and $y=-\frac{1}{2}$

Solution:

$-\frac{1}{9}$

Exercise:

Problem: $8u^2v^3$ when $u=-\frac{3}{4}$ and $v=-\frac{1}{2}$

Exercise:

Problem: $\frac{a+b}{a-b}$ when $a=-3, b=8$

Solution:

$-\frac{5}{11}$

Exercise:

Problem: $\frac{r-s}{r+s}$ when $r=10, s=-5$

Everyday Math

Exercise:

Problem:

Decorating Laronda is making covers for the throw pillows on her sofa. For each pillow cover, she needs $\frac{1}{2}$ yard of print fabric and $\frac{3}{8}$ yard of solid fabric. What is the total amount of fabric Laronda needs for each pillow cover?

Solution:

$\frac{7}{8}$ yard

Exercise:

Problem:

Baking Vanessa is baking chocolate chip cookies and oatmeal cookies. She needs $\frac{1}{2}$ cup of sugar for the chocolate chip cookies and $\frac{1}{4}$ of sugar for the oatmeal cookies. How much sugar does she need altogether?

Writing Exercises

Exercise:

Problem:

Why do you need a common denominator to add or subtract fractions? Explain.

Solution:

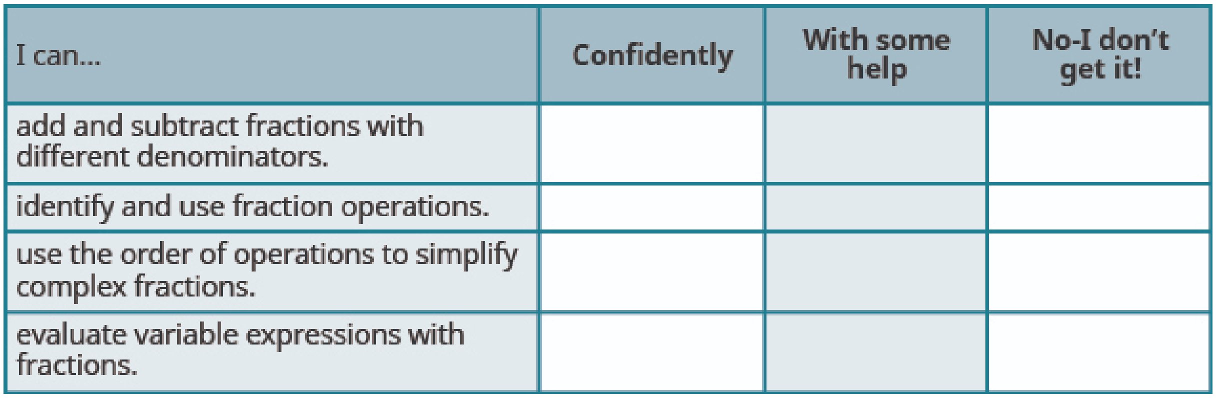
Answers may vary

Exercise:

Problem: How do you find the LCD of 2 fractions?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
add and subtract fractions with different denominators.			
identify and use fraction operations.			
use the order of operations to simplify complex fractions.			
evaluate variable expressions with fractions.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next chapter? Why or why not?

Glossary

least common denominator
: The least common denominator (LCD) of two fractions is the least common multiple (LCM) of their denominators.

Decimals

By the end of this section, you will be able to:

- Name and write decimals
- Round decimals
- Add and subtract decimals
- Multiply and divide decimals
- Convert decimals, fractions, and percents

Note:
A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **Decimals**.

Name and Write Decimals

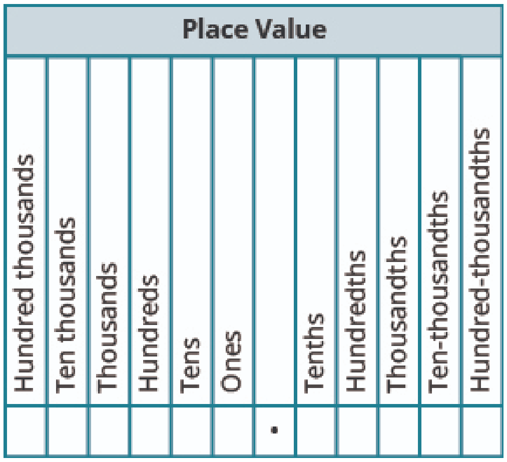
Decimals are another way of writing fractions whose denominators are powers of 10.

Equation:

$$\begin{array}{rcll} 0.1 & = & \frac{1}{10} & \qquad 0.1 \text{ is "one tenth"} \\ 0.01 & = & \frac{1}{100} & \qquad 0.01 \text{ is "one hundredth"} \\ 0.001 & = & \frac{1}{1,000} & \qquad 0.001 \text{ is "one thousandth"} \\ 0.0001 & = & \frac{1}{10,000} & \qquad 0.0001 \text{ is "one ten-thousandth"} \end{array}$$

Notice that “ten thousand” is a number larger than one, but “one ten-thousand**th**” is a number smaller than one. The “th” at the end of the name tells you that the number is smaller than one.

When we name a whole number, the name corresponds to the place value based on the powers of ten. We read 10,000 as “ten thousand” and 10,000,000 as “ten million.” Likewise, the names of the decimal places correspond to their fraction values. [link] shows the names of the place values to the left and right of the decimal point.



Place Value

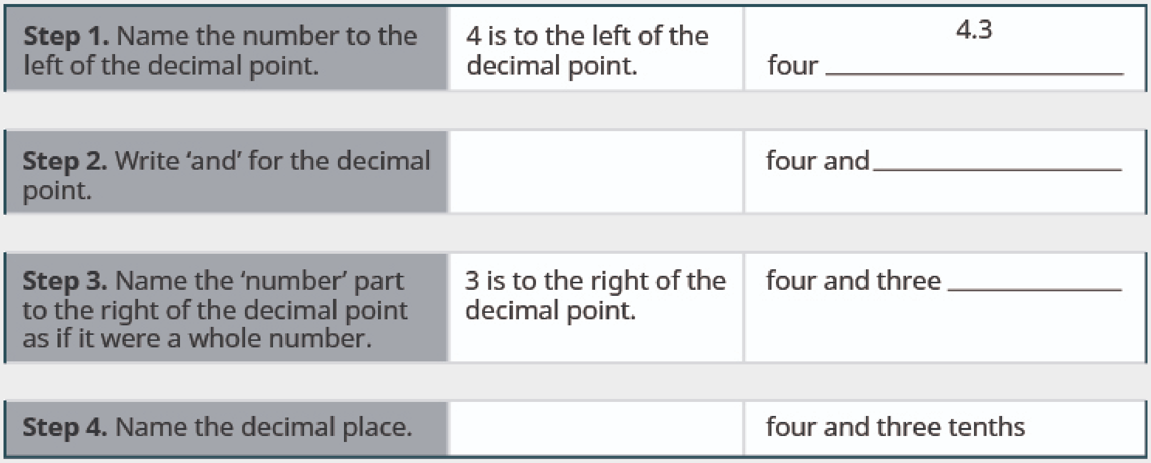
Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones		Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths
						.					

Place value of decimal numbers are shown to the left and right of the decimal point.

Example:
How to Name Decimals
Exercise:

Problem: Name the decimal: 4.3

Solution:
Solution



Step 1. Name the number to the left of the decimal point.	4 is to the left of the decimal point.	4.3 four ________________
Step 2. Write 'and' for the decimal point.		four and ______________
Step 3. Name the 'number' part to the right of the decimal point as if it were a whole number.	3 is to the right of the decimal point.	four and three __________
Step 4. Name the decimal place.		four and three tenths

Note:
Exercise:

Problem: Name the decimal: 6.7

Solution:

six and seven tenths

Note:
Exercise:

Problem: Name the decimal: 5.8

Solution:

five and eight tenths

We summarize the steps needed to name a decimal below.

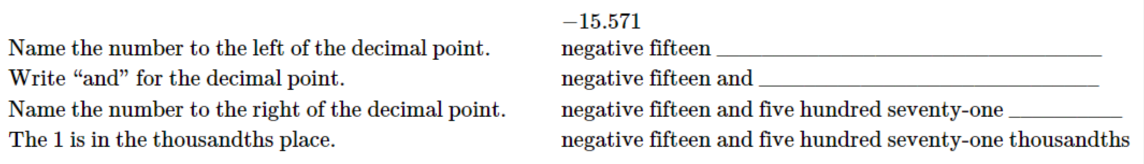
Note:
Name a Decimal.

Name the number to the left of the decimal point.
Write “and” for the decimal point.
Name the “number” part to the right of the decimal point as if it were a whole number.
Name the decimal place of the last digit.

Example:
Exercise:

Problem: Name the decimal: -15.571.

Solution:
Solution



	-15.571
Name the number to the left of the decimal point.	negative fifteen ______________________________
Write “and” for the decimal point.	negative fifteen and ___________________________
Name the number to the right of the decimal point.	negative fifteen and five hundred seventy-one _________
The 1 is in the thousandths place.	negative fifteen and five hundred seventy-one thousandths

Note:
Exercise:

Problem: Name the decimal: -13.461

Solution:

negative thirteen and four hundred sixty-one thousandths

Note:
Exercise:

Problem: Name the decimal: -2.053

Solution:

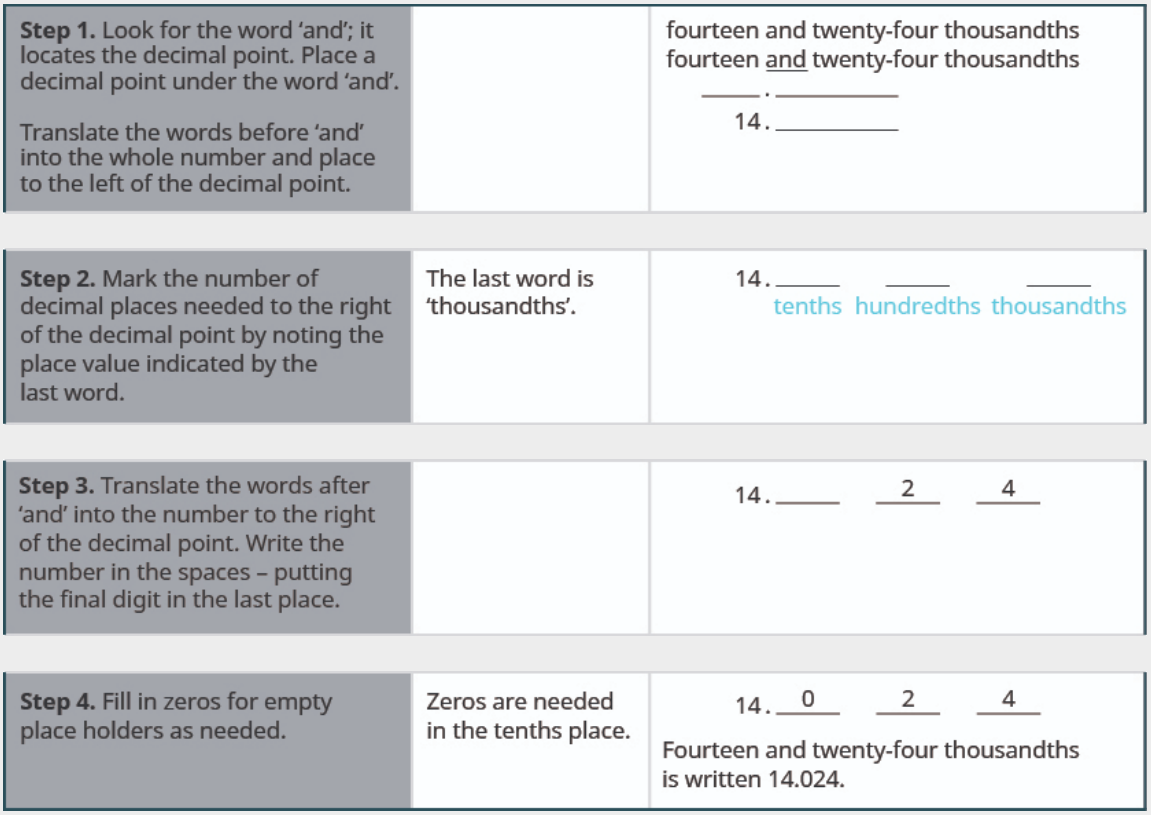
negative two and fifty-three thousandths

When we write a check we write both the numerals and the name of the number. Let’s see how to write the decimal from the name.

Example:
How to Write Decimals
Exercise:

Problem: Write “fourteen and twenty-four thousandths” as a decimal.

Solution:
Solution



Step 1. Look for the word ‘and’; it locates the decimal point. Place a decimal point under the word ‘and’. Translate the words before ‘and’ into the whole number and place to the left of the decimal point.		fourteen and twenty-four thousandths fourteen and twenty-four thousandths ___ . _____ 14 . _____
Step 2. Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.	The last word is ‘thousandths’.	14 . ___ ___ ___ tenths hundredths thousandths
Step 3. Translate the words after ‘and’ into the number to the right of the decimal point. Write the number in the spaces – putting the final digit in the last place.		14 . ___ 2 4
Step 4. Fill in zeros for empty place holders as needed.	Zeros are needed in the tenths place.	14 . 0 2 4 Fourteen and twenty-four thousandths is written 14.024.

Note:
Exercise:

Problem: Write as a decimal: thirteen and sixty-eight thousandths.

Solution:

13.068

Note:

Exercise:

Problem: Write as a decimal: five and ninety-four thousandths.

Solution:

5.094

We summarize the steps to writing a decimal.

Note:
Write a decimal.

Look for the word “and”—it locates the decimal point.

- Place a decimal point under the word “and.” Translate the words before “and” into the whole number and place it to the left of the decimal point.
- If there is no “and,” write a “0” with a decimal point to its right.

Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.
Translate the words after “and” into the number to the right of the decimal point. Write the number in the spaces—putting the final digit in the last place.
Fill in zeros for place holders as needed.

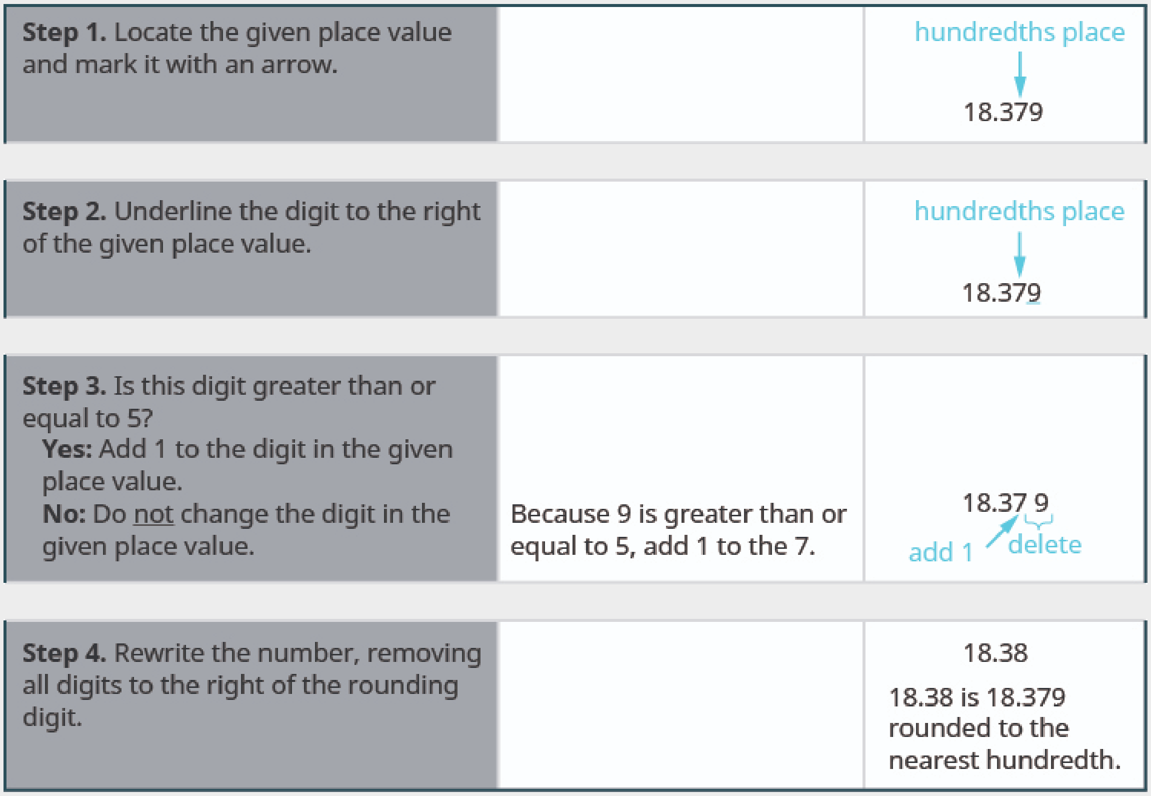
Round Decimals

Rounding decimals is very much like rounding whole numbers. We will round decimals with a method based on the one we used to round whole numbers.

Example:
How to Round Decimals
Exercise:

Problem: Round 18.379 to the nearest hundredth.

Solution:



Step 1. Locate the given place value and mark it with an arrow.		hundredths place ↓ 18.379
Step 2. Underline the digit to the right of the given place value.		hundredths place ↓ $18.37\underline{9}$
Step 3. Is this digit greater than or equal to 5? **Yes:** Add 1 to the digit in the given place value. **No:** Do not change the digit in the given place value.	Because 9 is greater than or equal to 5, add 1 to the 7.	$18.3\underset{\text{add 1}}{7}\underbrace{9}_{\text{delete}}$
Step 4. Rewrite the number, removing all digits to the right of the rounding digit.		18.38 18.38 is 18.379 rounded to the nearest hundredth.

Note:
Exercise:

Problem: Round to the nearest hundredth: 1.047

Solution:

1.05

Note:
Exercise:

Problem: Round to the nearest hundredth: 9.173

Solution:

9.17

We summarize the steps for rounding a decimal here.

Note:
Round Decimals.

Locate the given place value and mark it with an arrow.
Underline the digit to the right of the place value.
Is this digit greater than or equal to 5?

- Yes—add 1 to the digit in the given place value.
- No—do not change the digit in the given place value.

Rewrite the number, deleting all digits to the right of the rounding digit.

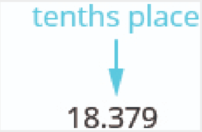
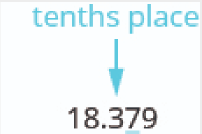
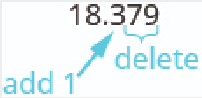
Example:
Exercise:

Problem: Round 18.379 to the nearest ⓐ tenth ⓑ whole number.

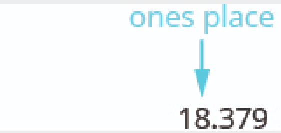
Solution:
Solution

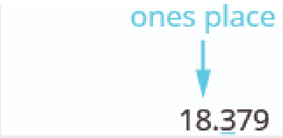
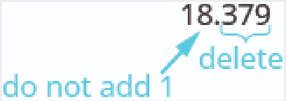
Round 18.379

ⓐ to the nearest tenth

Locate the tenths place with an arrow.	
Underline the digit to the right of the given place value.	
Because 7 is greater than or equal to 5, add 1 to the 3.	
Rewrite the number, deleting all digits to the right of the rounding digit.	18.4
Notice that the deleted digits were NOT replaced with zeros.	So, 18.379 rounded to the nearest tenth is 18.4.

ⓑ to the nearest whole number

Locate the ones place with an arrow.	
Underline the digit to the right of the given place value.	

	
Since 3 is not greater than or equal to 5, do not add 1 to the 8.	
Rewrite the number, deleting all digits to the right of the rounding digit.	18
	So, 18.379 rounded to the nearest whole number is 18.

Note:
Exercise:

Problem: Round 6.582 to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

Solution:

ⓐ 6.58 ⓑ 6.6 ⓒ 7

Note:
Exercise:

Problem: Round 15.2175 to the nearest ⓐ thousandth ⓑ hundredth ⓒ tenth.

Solution:

ⓐ 15.218 ⓑ 15.22 ⓒ 15.2

Add and Subtract Decimals

To add or subtract decimals, we line up the decimal points. By lining up the decimal points this way, we can add or subtract the corresponding place values. We then add or subtract the numbers as if they were whole numbers and then place the decimal point in the sum.

Note:
Add or Subtract Decimals.

Write the numbers so the decimal points line up vertically.
Use zeros as place holders, as needed.
Add or subtract the numbers as if they were whole numbers. Then place the decimal point in the answer under the decimal points in the given numbers.

Example:
Exercise:

Problem: Add: $23.5 + 41.38$

Solution:
Solution

Write the numbers so the decimal points line up vertically.	$\begin{array}{r} 23.5 \\ \underline{+41.38} \end{array}$
Put 0 as a placeholder after the 5 in 23.5. Remember, $\frac{5}{10} = \frac{50}{100}$ so $0.5 = 0.50$.	$\begin{array}{r} 23.50 \\ \underline{+41.38} \end{array}$
Add the numbers as if they were whole numbers. Then place the decimal point in the sum.	$\begin{array}{r} 23.50 \\ \underline{+41.38} \\ 64.88 \end{array}$

Note:

Exercise:

Problem: Add: $4.8 + 11.69$

Solution:

16.49

Note:
Exercise:

Problem: Add: $5.123 + 18.47$

Solution:

23.593

Example:
Exercise:

Problem: Subtract: $20 - 14.65$

Solution:
Solution

	$20 - 14.65$
Write the numbers so the decimal points line up vertically. Remember, 20 is a whole number, so place the decimal point after the 0.	$\begin{array}{r} 20. \\ \underline{-14.65} \end{array}$
Put in zeros to the right as placeholders.	$\begin{array}{r} 20.00 \\ \underline{-14.65} \end{array}$
Subtract and place the decimal point in the answer.	$\begin{array}{r} \overset{1}{\cancel{2}}\ \overset{\overset{9}{\cancel{10}}}{\cancel{0}}\ .\ \overset{\overset{9}{\cancel{10}}}{\cancel{0}}\ \overset{\cancel{10}}{\cancel{0}} \\ -1\,4\ .\ 6\ 5 \\ \hline 5\ .\ 3\ 5 \end{array}$

Note:
Exercise:

Problem: Subtract: 10 − 9.58

Solution:

0.42

Note:
Exercise:

Problem: Subtract: 50 − 37.42

Solution:

12.58

Multiply and Divide Decimals

Multiplying decimals is very much like multiplying whole numbers—we just have to determine where to place the decimal point. The procedure for multiplying decimals will make sense if we first convert them to fractions and then multiply.

So let’s see what we would get as the product of decimals by converting them to fractions first. We will do two examples side-by-side. Look for a pattern!

	$\underbrace{(0.3)}_{1\text{ place}}\ \underbrace{(0.7)}_{1\text{ place}}$ $\qquad$ $\underbrace{(0.2)}_{1\text{ place}}\ \underbrace{(0.46)}_{2\text{ places}}$
Convert to fractions.	$\frac{3}{10}\cdot\frac{7}{10}$ $\qquad$ $\frac{2}{10}\cdot\frac{46}{100}$
Multiply.	$\frac{21}{100}$ $\qquad$ $\frac{92}{1000}$
Convert to decimals.	$0.\underbrace{21}_{2\text{ places}}$ $\qquad$ $0.\underbrace{092}_{3\text{ places}}$

Notice, in the first example, we multiplied two numbers that each had one digit after the decimal point and the product had two decimal places. In the second example, we multiplied a number with one decimal place by a number with two decimal places and the product had three decimal places.

We multiply the numbers just as we do whole numbers, temporarily ignoring the decimal point. We then count the number of decimal points in the factors and that sum tells us the number of decimal places in the product.

The rules for multiplying positive and negative numbers apply to decimals, too, of course!

When *multiplying* two numbers,

- if their signs are the *same* the product is *positive*.
- if their signs are *different* the product is *negative*.

When we multiply signed decimals, first we determine the sign of the product and then multiply as if the numbers were both positive. Finally, we write the product with the appropriate sign.

Note:
Multiply Decimals.

Determine the sign of the product.
Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors.
Write the product with the appropriate sign.

Example:
Exercise:

Problem: Multiply: $(-3.9)\,(4.075)$

Solution:
Solution

	$(-3.9)\,(4.075)$
The signs are different. The product will	

be negative.	
Write in vertical format, lining up the numbers on the right.	$\begin{array}{r} 4.075 \\ \underline{\times\ 3.9} \end{array}$
Multiply.	$\begin{array}{r} 4.075 \\ \underline{\times\ 3.9} \\ 36675 \\ \underline{12225} \\ 158925 \end{array}$
Add the number of decimal places in the factors (1 + 3). $(-3.\underbrace{9}_{\text{1 place}})\quad(4.\underbrace{075}_{\text{3 places}})$ Place the decimal point 4 places from the right.	$\begin{array}{r} 4.075 \\ \underline{\times\ 3.9} \\ 36675 \\ \underline{12225} \\ 15.\underbrace{8925}_{\text{4 places}} \end{array}$
The signs are different, so the product is negative.	$(-3.9)\,(4.075) = -15.8925$

Note:
Exercise:

Problem: Multiply: $-4.5\,(6.107)$

Solution:

-27.4815

Note:

Exercise:

Problem: Multiply: $-10.79\,(8.12)$

Solution:

-87.6148

In many of your other classes, especially in the sciences, you will multiply decimals by powers of 10 (10, 100, 1000, etc.). If you multiply a few products on paper, you may notice a pattern relating the number of zeros in the power of 10 to number of decimal places we move the decimal point to the right to get the product.

Note:
Multiply a Decimal by a Power of Ten.

Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
Add zeros at the end of the number as needed.

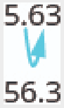
Example:
Exercise:

Problem: Multiply 5.63 ⓐ by 10 ⓑ by 100 ⓒ by $1,000$.

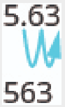
Solution:

By looking at the number of zeros in the multiple of ten, we see the number of places we need to move the decimal to the right.

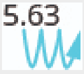
ⓐ

	$5.63(10)$
There is 1 zero in 10, so move the decimal point 1 place to the right.	 5.63 56.3

ⓑ

	$5.63(100)$
There are 2 zeros in 100, so move the decimal point 2 places to the right.	 5.63 563

ⓒ

	$5.63(1{,}000)$
There are 3 zeros in 1,000, so move the decimal point 3 places to the right.	 5.63
A zero must be added at the end.	5,630

Note:
Exercise:

Problem: Multiply 2.58 ⓐ by 10 ⓑ by 100 ⓒ by $1,000$.

Solution:

ⓐ 25.8 ⓑ 258 ⓒ $2,580$

Note:
Exercise:

Problem: Multiply 14.2 ⓐ by 10 ⓑ by 100 ⓒ by $1,000$.

Solution:

ⓐ 142 ⓑ $1,420$ ⓒ $14,200$

Just as with multiplication, division of decimals is very much like dividing whole numbers. We just have to figure out where the decimal point must be placed.

To divide decimals, determine what power of 10 to multiply the denominator by to make it a whole number. Then multiply the numerator by that same power of 10 Because of the equivalent fractions property, we haven't changed the value of the fraction! The effect is to move the decimal points in the numerator and denominator the same number of places to the right. For example:
Equation:

$$\frac{0.8}{0.4}$$
$$\frac{0.8(10)}{0.4(10)}$$
$$\frac{8}{4}$$

We use the rules for dividing positive and negative numbers with decimals, too. When dividing signed decimals, first determine the sign of the quotient and then divide as if the numbers were both positive. Finally, write the quotient with the appropriate sign.

We review the notation and vocabulary for division:
Equation:

$$\underset{\text{dividend}}{a} \div \underset{\text{divisor}}{b} = \underset{\text{quotient}}{\text{c}} \qquad\qquad \begin{array}{r} \overset{\text{c}}{\text{quotient}} \\ \underset{\text{divisor}}{b}\,\overline{\big)\,\underset{\text{dividend}}{a}} \end{array}$$

We’ll write the steps to take when dividing decimals, for easy reference.

Note:
Divide Decimals.

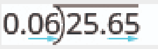
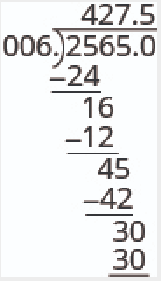
Determine the sign of the quotient.
Make the divisor a whole number by “moving” the decimal point all the way to the right. “Move” the decimal point in the dividend the same number of places—adding zeros as needed.
Divide. Place the decimal point in the quotient above the decimal point in the dividend.
Write the quotient with the appropriate sign.

Example:
Exercise:

Problem: Divide: $-25.56 \div (-0.06)$

Solution:
Solution

Remember, you can “move” the decimals in the divisor and dividend because of the Equivalent Fractions Property.

	$-25.65 \div (-0.06)$
The signs are the same.	The quotient is positive.
Make the divisor a whole number by “moving” the decimal point all the way to the right.	
“Move” the decimal point in the dividend the same number of places.	
Divide. Place the decimal point in the quotient above the decimal point in the dividend.	 427.5 006.)2565.0 −24 16 −12 45 −42 30 30
Write the quotient with the appropriate sign.	$-25.65 \div (-0.06) = 427.5$

Note:
Exercise:

Problem: Divide: $-23.492 \div (-0.04)$

Solution:

587.3

Note:
Exercise:

Problem: Divide: $-4.11 \div (-0.12)$

Solution:

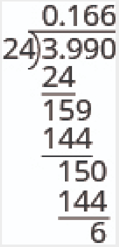
34.25

A common application of dividing whole numbers into decimals is when we want to find the price of one item that is sold as part of a multi-pack. For example, suppose a case of 24 water bottles costs \$3.99. To find the price of one water bottle, we would divide \$3.99 by 24. We show this division in [link]. In calculations with money, we will round the answer to the nearest cent (hundredth).

Example:
Exercise:

Problem: Divide: $\$3.99 \div 24$

Solution:
Solution

	$\$3.99 \div 24$
Place the decimal point in the quotient above the decimal point in the dividend.	
Divide as usual. When do we stop? Since this division involves money, we round it to the nearest cent (hundredth.) To do this, we must carry the division to the thousandths place.	

Round to the nearest cent.	$\$0.166 \approx \$0.17$ $\$3.99 \div 24 \approx \$0.17$

Note:
Exercise:

Problem: Divide: $\$6.99 \div 36$

Solution:

$\$0.19$

Note:
Exercise:

Problem: Divide: $\$4.99 \div 12$

Solution:

$\$0.42$

Convert Decimals, Fractions, and Percents

We convert decimals into fractions by identifying the place value of the last (farthest right) digit. In the decimal 0.03 the 3 is in the hundredths place, so 100 is the denominator of the fraction equivalent to 0.03.

Equation:

$$0.03 = \frac{3}{100}$$

Notice, when the number to the left of the decimal is zero, we get a fraction whose numerator is less than its denominator. Fractions like this are called proper fractions.

The steps to take to convert a decimal to a fraction are summarized in the procedure box.

Note:
Convert a Decimal to a Proper Fraction.

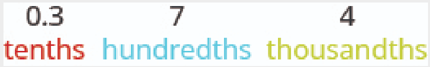
Determine the place value of the final digit.
Write the fraction.

- numerator—the “numbers” to the right of the decimal point
- denominator—the place value corresponding to the final digit

Example:
Exercise:

Problem: Write 0.374 as a fraction.

Solution:
Solution

	0.374
Determine the place value of the final digit.	
Write the fraction for 0.374: • The numerator is 374. • The denominator is 1,000.	$\frac{374}{1000}$

Simplify the fraction.	$\frac{2\cdot187}{2\cdot500}$
Divide out the common factors.	$\frac{187}{500}$ so, $0.374=\frac{187}{500}$

Did you notice that the number of zeros in the denominator of $\frac{374}{1,000}$ is the same as the number of decimal places in 0.374?

Note:
Exercise:

Problem: Write 0.234 as a fraction.

Solution:

$\frac{117}{500}$

Note:
Exercise:

Problem: Write 0.024 as a fraction.

Solution:

$\frac{3}{125}$

We’ve learned to convert decimals to fractions. Now we will do the reverse—convert fractions to decimals. Remember that the fraction bar means division. So $\frac{4}{5}$ can be written $4\div5$ or $5\overline{)4}$. This leads to the following method for converting a fraction to a decimal.

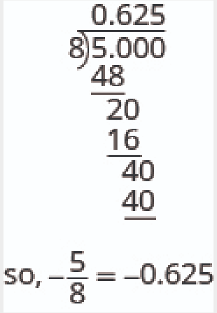
Note:
Convert a Fraction to a Decimal.
To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.

Example:
Exercise:

Problem: Write $-\frac{5}{8}$ as a decimal.

Solution:
Solution

Since a fraction bar means division, we begin by writing $\frac{5}{8}$ as $8\overline{)5}$. Now divide.



so, $-\frac{5}{8} = -0.625$

Note:
Exercise:

Problem: Write $-\frac{7}{8}$ as a decimal.

Solution:

-0.875

Note:

Exercise:

Problem: Write $-\frac{3}{8}$ as a decimal.

Solution:

-0.375

When we divide, we will not always get a zero remainder. Sometimes the quotient ends up with a decimal that repeats. A **repeating decimal** is a decimal in which the last digit or group of digits repeats endlessly. A bar is placed over the repeating block of digits to indicate it repeats.

Note:
Repeating Decimal
A **repeating decimal** is a decimal in which the last digit or group of digits repeats endlessly.

A bar is placed over the repeating block of digits to indicate it repeats.

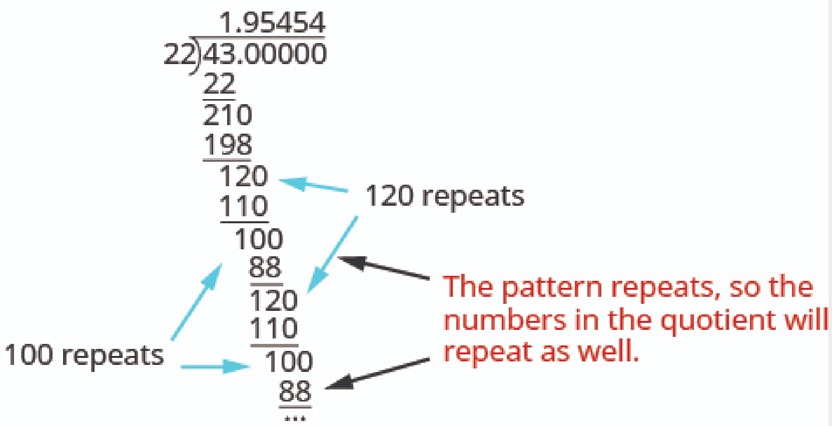
Example:
Exercise:

Problem: Write $\frac{43}{22}$ as a decimal.

Solution:
Solution

$\frac{43}{22}$

Divide 43 by 22.



so, $\frac{43}{22} = 1.9\overline{54}$

Note:
Exercise:

Problem: Write $\frac{27}{11}$ as a decimal.

Solution:

$2.\overline{45}$

Note:
Exercise:

Problem: Write $\frac{51}{22}$ as a decimal.

Solution:

$2.3\overline{18}$

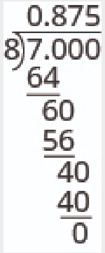
Sometimes we may have to simplify expressions with fractions and decimals together.

Example:
Exercise:

Problem: Simplify: $\frac{7}{8} + 6.4$

Solution:
Solution

First we must change one number so both numbers are in the same form. We can change the fraction to a decimal, or change the decimal to a fraction. Usually it is easier to change the fraction to a decimal.

		$\frac{7}{8} + 6.4$
Change $\frac{7}{8}$ to a decimal.		
Add.		$0.875 + 6.4$
		7.275
		So, $\frac{7}{8} + 6.4 = 7.275$

Note:
Exercise:

Problem: Simplify: $\frac{3}{8} + 4.9$

Solution:

5.275

Note:
Exercise:

Problem: Simplify: $5.7 + \frac{13}{20}$

Solution:

6.35

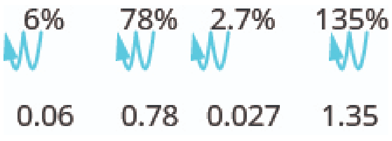
A **percent** is a ratio whose denominator is 100. Percent means per hundred. We use the percent symbol, %, to show percent.

Note:
Percent
A **percent** is a ratio whose denominator is 100.

Since a percent is a ratio, it can easily be expressed as a fraction. Percent means per 100, so the denominator of the fraction is 100. We then change the fraction to a decimal by dividing the numerator by the denominator.
Equation:

	6%	78%	135%
Write as a ratio with denominator 100.	$\frac{6}{100}$	$\frac{78}{100}$	$\frac{135}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.06	0.78	1.35

Do you see the pattern? *To convert a percent number to a decimal number, we move the decimal point two places to the left.*

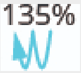


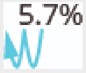
Example:
Exercise:

Problem: Convert each percent to a decimal: ⓐ 62% ⓑ 135% ⓒ 35.7%.

Solution:
Solution

ⓐ	
Move the decimal point two places to the left.	0.62

ⓑ	
	
Move the decimal point two places to the left.	1.35

ⓒ	
	
Move the decimal point two places to the left.	0.057

Note:
Exercise:

Problem: Convert each percent to a decimal: ⓐ 9% ⓑ 87% ⓒ 3.9%.

Solution:

ⓐ 0.09 ⓑ 0.87 ⓒ 0.039

Note:
Exercise:

Problem: Convert each percent to a decimal: ⓐ 3% ⓑ 91% ⓒ 8.3%.

Solution:

ⓐ 0.03 ⓑ 0.91 ⓒ 0.083

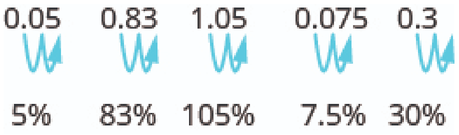
Converting a decimal to a percent makes sense if we remember the definition of percent and keep place value in mind.

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent.

Equation:

	0.83	1.05	0.075
Write as a fraction.	$\frac{83}{100}$	$1\frac{5}{100}$	$\frac{75}{1000}$
The denominator is 100.		$\frac{105}{100}$	$\frac{7.5}{100}$
Write the ratio as a percent.	83%	105%	7.5%

Recognize the pattern? *To convert a decimal to a percent, we move the decimal point two places to the right and then add the percent sign.*



Example:
Exercise:

Problem: Convert each decimal to a percent: ⓐ 0.51 ⓑ 1.25 ⓒ 0.093

Solution:
Solution

ⓐ	
	0.51
Move the decimal point two places to the right.	51%

ⓑ	
	1.25
Move the decimal point two places to the right.	125%

ⓒ	
	0.093
Move the decimal point two places to the right.	9.3%

Note:
Exercise:

Problem: Convert each decimal to a percent: ⓐ 0.17 ⓑ 1.75 ⓒ 0.0825.

Solution:

ⓐ 17% ⓑ 175% ⓒ 8.25%

Note:
Exercise:

Problem: Convert each decimal to a percent: ⓐ 0.41 ⓑ 2.25 ⓒ 0.0925

Solution:

ⓐ 41% ⓑ 225% ⓒ 9.25%

Key Concepts

- **Name a Decimal**

  Name the number to the left of the decimal point.
  Write ”and” for the decimal point.
  Name the “number” part to the right of the decimal point as if it were a whole number.
  Name the decimal place of the last digit.

- **Write a Decimal**

  Look for the word ‘and’—it locates the decimal point. Place a decimal point under the word ‘and.’ Translate the words before ‘and’ into the whole number and place it to the left of the decimal point. If there is no “and,” write a “0” with a decimal point to its right.
  Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.
  Translate the words after ‘and’ into the number to the right of the decimal point.
  Write the number in the spaces—putting the final digit in the last place.
  Fill in zeros for place holders as needed.

- **Round a Decimal**

  Locate the given place value and mark it with an arrow.
  Underline the digit to the right of the place value.
  Is this digit greater than or equal to 5? Yes—add 1 to the digit in the given place value. No—do not change the digit in the given place value.
  Rewrite the number, deleting all digits to the right of the rounding digit.

- **Add or Subtract Decimals**

  Write the numbers so the decimal points line up vertically.
  Use zeros as place holders, as needed.
  Add or subtract the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.

- **Multiply Decimals**

  Determine the sign of the product.
  Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
  Place the decimal point. The number of decimal places in the product is the sum of the decimal places in the factors.
  Write the product with the appropriate sign.

- **Multiply a Decimal by a Power of Ten**

  Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
  Add zeros at the end of the number as needed.

- **Divide Decimals**

  Determine the sign of the quotient.
  Make the divisor a whole number by “moving” the decimal point all the way to the right. “Move” the decimal point in the dividend the same number of places - adding zeros as needed.
  Divide. Place the decimal point in the quotient above the decimal point in the dividend.
  Write the quotient with the appropriate sign.

- **Convert a Decimal to a Proper Fraction**

  Determine the place value of the final digit.
  Write the fraction: numerator—the ‘numbers’ to the right of the decimal point;

denominator—the place value corresponding to the final digit.

- **Convert a Fraction to a Decimal** Divide the numerator of the fraction by the denominator.

Practice Makes Perfect

Name and Write Decimals

In the following exercises, write as a decimal.

Exercise:

Problem: Twenty-nine and eighty-one hundredths

Solution:

29.81

Exercise:

Problem: Sixty-one and seventy-four hundredths

Exercise:

Problem: Seven tenths

Solution:

0.7

Exercise:

Problem: Six tenths

Exercise:

Problem: Twenty-nine thousandths

Solution:

0.029

Exercise:

Problem: Thirty-five thousandths

Exercise:

Problem: Negative eleven and nine ten-thousandths

Solution:

-11.0009

Exercise:

Problem: Negative fifty-nine and two ten-thousandths

In the following exercises, name each decimal.

Exercise:

Problem: 5.5

Solution:

five and five tenths

Exercise:

Problem: 14.02

Exercise:

Problem: 8.71

Solution:

eight and seventy-one hundredths

Exercise:

Problem: 2.64

Exercise:

Problem: 0.002

Solution:

two thousandths

Exercise:

Problem: 0.479

Exercise:

Problem: -17.9

Solution:

negative seventeen and nine tenths

Exercise:

Problem: -31.4

Round Decimals

In the following exercises, round each number to the nearest tenth.

Exercise:

Problem: 0.67

Solution:

0.7

Exercise:

Problem: 0.49

Exercise:

Problem: 2.84

Solution:

2.8

Exercise:

Problem: 4.63

In the following exercises, round each number to the nearest hundredth.

Exercise:

Problem: 0.845

Solution:

0.85

Exercise:

Problem: 0.761

Exercise:

Problem: 0.299

Solution:

0.30

Exercise:

Problem: 0.697

Exercise:

Problem: 4.098

Solution:

4.10

Exercise:

Problem: 7.096

In the following exercises, round each number to the nearest ⓐ hundredth ⓑ tenth ⓒ whole number.

Exercise:

Problem: 5.781

Solution:

ⓐ 5.78 ⓑ 5.8 ⓒ 6

Exercise:

Problem: 1.6381

Exercise:

Problem: 63.479

Solution:

ⓐ 63.48 ⓑ 63.5 ⓒ 63

Exercise:

Problem: 84.281

Add and Subtract Decimals

In the following exercises, add or subtract.

Exercise:

Problem: $16.92 + 7.56$

Solution:

24.48

Exercise:

Problem: $248.25 - 91.29$

Exercise:

Problem: $21.76 - 30.99$

Solution:

-9.23

Exercise:

Problem: $38.6 + 13.67$

Exercise:

Problem: $-16.53 - 24.38$

Solution:

-40.91

Exercise:

Problem: $-19.47 - 32.58$

Exercise:

Problem: $-38.69 + 31.47$

Solution:

-7.22

Exercise:

Problem: $29.83 + 19.76$

Exercise:

Problem: $72.5 - 100$

Solution:

-27.5

Exercise:

Problem: $86.2 - 100$

Exercise:

Problem: $15 + 0.73$

Solution:

15.73

Exercise:

Problem: $27 + 0.87$

Exercise:

Problem: $91.95 - (-10.462)$

Solution:

102.412

Exercise:

Problem: $94.69 - (-12.678)$

Exercise:

Problem: $55.01 - 3.7$

Solution:

51.31

Exercise:

Problem: $59.08 - 4.6$

Exercise:

Problem: $2.51 - 7.4$

Solution:

-4.89

Exercise:

Problem: $3.84 - 6.1$

Multiply and Divide Decimals

In the following exercises, multiply.

Exercise:

Problem: $(0.24)\,(0.6)$

Solution:

0.144

Exercise:

Problem: $(0.81)\,(0.3)$

Exercise:

Problem: $(5.9)\,(7.12)$

Solution:

42.008

Exercise:

Problem: $(2.3)\,(9.41)$

Exercise:

Problem: $(-4.3)\,(2.71)$

Solution:

-11.653

Exercise:

Problem: $(-8.5)\,(1.69)$

Exercise:

Problem: $(-5.18)\,(-65.23)$

Solution:

337.8914

Exercise:

Problem: $(-9.16)\,(-68.34)$

Exercise:

Problem: $(0.06)\,(21.75)$

Solution:

1.305

Exercise:

Problem: $(0.08)\,(52.45)$

Exercise:

Problem: $(9.24)\,(10)$

Solution:

92.4

Exercise:

Problem: $(6.531)\,(10)$

Exercise:

Problem: $(55.2)\,(1000)$

Solution:

55,200

Exercise:

Problem: $(99.4)\,(1000)$

In the following exercises, divide.

Exercise:

Problem: $4.75 \div 25$

Solution:

0.19

Exercise:

Problem: $12.04 \div 43$

Exercise:

Problem: $\$117.25 \div 48$

Solution:

$\$2.44$

Exercise:

Problem: $\$109.24 \div 36$

Exercise:

Problem: $0.6 \div 0.2$

Solution:

3

Exercise:

Problem: $0.8 \div 0.4$

Exercise:

Problem: $1.44 \div (-0.3)$

Solution:

-4.8

Exercise:

Problem: $1.25 \div (-0.5)$

Exercise:

Problem: $-1.75 \div (-0.05)$

Solution:

35

Exercise:

Problem: $-1.15 \div (-0.05)$

Exercise:

Problem: $5.2 \div 2.5$

Solution:

2.08

Exercise:

Problem: $6.5 \div 3.25$

Exercise:

Problem: $11 \div 0.55$

Solution:

20

Exercise:

Problem: $14 \div 0.35$

Convert Decimals, Fractions and Percents

In the following exercises, write each decimal as a fraction.

Exercise:

Problem: 0.04

Solution:

$\frac{1}{25}$

Exercise:

Problem: 0.19

Exercise:

Problem: 0.52

Solution:

$\frac{13}{25}$

Exercise:

Problem: 0.78

Exercise:

Problem: 1.25

Solution:

$\frac{5}{4}$

Exercise:

Problem: 1.35

Exercise:

Problem: 0.375

Solution:

$\frac{3}{8}$

Exercise:

Problem: 0.464

Exercise:

Problem: 0.095

Solution:

$\frac{19}{200}$

Exercise:

Problem: 0.085

In the following exercises, convert each fraction to a decimal.

Exercise:

Problem: $\frac{17}{20}$

Solution:

0.85

Exercise:

Problem: $\frac{13}{20}$

Exercise:

Problem: $\frac{11}{4}$

Solution:

2.75

Exercise:

Problem: $\frac{17}{4}$

Exercise:

Problem: $-\frac{310}{25}$

Solution:

-12.4

Exercise:

Problem: $-\frac{284}{25}$

Exercise:

Problem: $\frac{15}{11}$

Solution:

$1.\overline{36}$

Exercise:

Problem: $\frac{18}{11}$

Exercise:

Problem: $\frac{15}{111}$

Solution:

$0.1\overline{35}$

Exercise:

Problem: $\frac{25}{111}$

Exercise:

Problem: $2.4+\frac{5}{8}$

Solution:

3.025

Exercise:

Problem: $3.9 + \frac{9}{20}$

In the following exercises, convert each percent to a decimal.

Exercise:

Problem: 1%

Solution:

0.01

Exercise:

Problem: 2%

Exercise:

Problem: 63%

Solution:

0.63

Exercise:

Problem: 71%

Exercise:

Problem: 150%

Solution:

1.5

Exercise:

Problem: 250%

Exercise:

Problem: 21.4%

Solution:

0.214

Exercise:

Problem: 39.3%

Exercise:

Problem: 7.8%

Solution:

0.078

Exercise:

Problem: 6.4%

In the following exercises, convert each decimal to a percent.

Exercise:

Problem: 0.01

Solution:

1%

Exercise:

Problem: 0.03

Exercise:

Problem: 1.35

Solution:

135%

Exercise:

Problem: 1.56

Exercise:

Problem: 3

Solution:

300%

Exercise:

Problem: 4

Exercise:

Problem: 0.0875

Solution:

8.75%

Exercise:

Problem: 0.0625

Exercise:

Problem: 2.254

Solution:

225.4%

Exercise:

Problem: 2.317

Everyday Math

Exercise:

Problem:

Salary Increase Danny got a raise and now makes $58,965.95 a year. Round this number to the nearest
ⓐ dollar
ⓑ thousand dollars
ⓒ ten thousand dollars.

Solution:

ⓐ $58,966 ⓑ $59,000 ⓒ $60,000

Exercise:

Problem:

New Car Purchase Selena’s new car cost $23,795.95. Round this number to the nearest
ⓐ dollar
ⓑ thousand dollars
ⓒ ten thousand dollars.

Exercise:

Problem:

Sales Tax Hyo Jin lives in San Diego. She bought a refrigerator for $1,624.99 and when the clerk calculated the sales tax it came out to exactly $142.186625. Round the sales tax to the nearest
ⓐ penny and
ⓑ dollar.

Solution:

ⓐ $142.19; ⓑ $142

Exercise:

Problem:

Sales Tax Jennifer bought a $1,038.99 dining room set for her home in Cincinnati. She calculated the sales tax to be exactly $67.53435. Round the sales tax to the nearest
ⓐ penny and
ⓑ dollar.

Exercise:

Problem:

Paycheck Annie has two jobs. She gets paid $14.04 per hour for tutoring at City College and $8.75 per hour at a coffee shop. Last week she tutored for 8 hours and worked at the coffee shop for 15 hours.
ⓐ How much did she earn?
ⓑ If she had worked all 23 hours as a tutor instead of working both jobs, how much more would she have earned?

Solution:

ⓐ $243.57 ⓑ $79.35

Exercise:

Problem:

Paycheck Jake has two jobs. He gets paid $7.95 per hour at the college cafeteria and $20.25 at the art gallery. Last week he worked 12 hours at the cafeteria and 5 hours at the art gallery.
ⓐ How much did he earn?
ⓑ If he had worked all 17 hours at the art gallery instead of working both jobs, how much more would he have earned?

Writing Exercises

Exercise:

Problem: How does knowing about US money help you learn about decimals?

Solution:

Answers may vary

Exercise:

Problem: Explain how you write “three and nine hundredths” as a decimal.

Exercise:

Problem:

Without solving the problem “44 is 80% of what number” think about what the solution might be. Should it be a number that is greater than 44 or less than 44? Explain your reasoning.

Solution:

Answers may vary

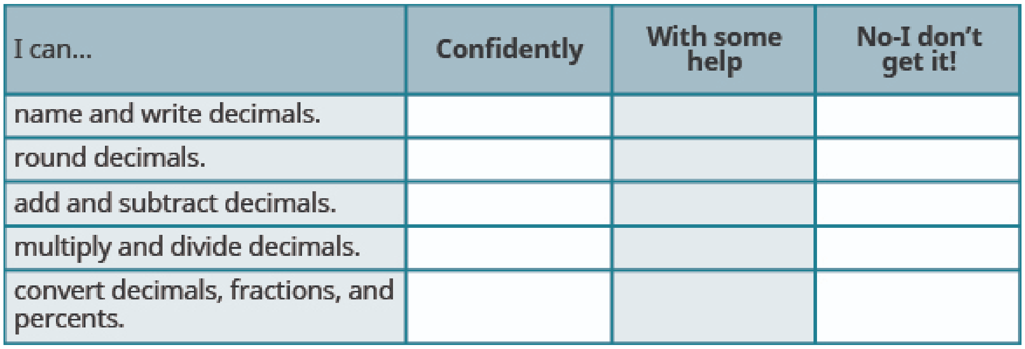
Exercise:

Problem:

When the Szetos sold their home, the selling price was 500% of what they had paid for the house 30 years ago. Explain what 500% means in this context.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don’t get it!
name and write decimals.			
round decimals.			
add and subtract decimals.			
multiply and divide decimals.			
convert decimals, fractions, and percents.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

decimal
A decimal is another way of writing a fraction whose denominator is a power of ten.

percent

A percent is a ratio whose denominator is 100.

repeating decimal

A repeating decimal is a decimal in which the last digit or group of digits repeats endlessly.

Properties of Real Numbers

By the end of this section, you will be able to:

- Use the commutative and associative properties
- Use the identity and inverse properties of addition and multiplication
- Use the properties of zero
- Simplify expressions using the distributive property

Note:
A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapter, **The Properties of Real Numbers**.

Use the Commutative and Associative Properties

Think about adding two numbers, say 5 and 3. The order we add them doesn't affect the result, does it?

Equation:

$$\begin{array}{cc} 5+3 & 3+5 \\ 8 & 8 \end{array}$$

Equation:

$$5+3=3+5$$

The results are the same.

As we can see, the order in which we add does not matter!

What about multiplying $5 \text{ and } 3$?

Equation:

$$\begin{array}{cc} 5\cdot 3 & 3\cdot 5 \\ 15 & 15 \end{array}$$

Equation:

$$5\cdot 3=3\cdot 5$$

Again, the results are the same!

The order in which we multiply does not matter!

These examples illustrate the **commutative property**. When adding or multiplying, changing the *order* gives the same result.

Note:
Commutative Property
Equation:

$$\begin{array}{lll} \textbf{of Addition} & \text{If } a, b \text{ are real numbers, then} & a+b = b+a \\ \textbf{of Multiplication} & \text{If } a, b \text{ are real numbers, then} & a \cdot b = b \cdot a \end{array}$$

When adding or multiplying, changing the *order* gives the same result.

The commutative property has to do with order. If you change the order of the numbers when adding or multiplying, the result is the same.

What about subtraction? Does order matter when we subtract numbers? Does $7-3$ give the same result as $3-7$?
Equation:

$$\begin{array}{cc} 7-3 & 3-7 \\ 4 & -4 \end{array}$$

$$\begin{array}{c} 4 \neq -4 \\ 7-3 \neq 3-7 \end{array}$$

The results are not the same.

Since changing the order of the subtraction did not give the same result, we know that *subtraction is not commutative*.

Let’s see what happens when we divide two numbers. Is division commutative?
Equation:

$$\begin{array}{cc} 12 \div 4 & 4 \div 12 \\ \frac{12}{4} & \frac{4}{12} \\ 3 & \frac{1}{3} \\ \multicolumn{2}{c}{3 \neq \frac{1}{3}} \\ \multicolumn{2}{c}{12 \div 4 \neq 4 \div 12} \end{array}$$

The results are not the same.

Since changing the order of the division did not give the same result, *division is not commutative*. The commutative properties only apply to addition and multiplication!

- Addition and multiplication *are* commutative.
- Subtraction and Division *are not* commutative.

If you were asked to simplify this expression, how would you do it and what would your answer be?
Equation:

$$7+8+2$$

Some people would think $7+8$ is 15 and then $15+2$ is 17. Others might start with $8+2$ makes 10 and then $7+10$ makes 17.

Either way gives the same result. Remember, we use parentheses as grouping symbols to indicate which operation should be done first.
Equation:

$$\begin{array}{ll} & (7+8)+2 \\ \text{Add } 7+8. & 15+2 \\ \text{Add.} & 17 \end{array}$$

$$\begin{array}{ll} & 7+(8+2) \\ \text{Add } 8+2. & 7+10 \\ \text{Add.} & 17 \end{array}$$

$$(7+8)+2=7+(8+2)$$

When adding three numbers, changing the grouping of the numbers gives the same result.

This is true for multiplication, too.
Equation:

$$\begin{array}{ll} & \left(5\cdot\frac{1}{3}\right)\cdot 3 \\ \text{Multiply.}\quad 5\cdot\frac{1}{3} & \frac{5}{3}\cdot 3 \\ \text{Multiply.} & 5 \end{array}$$

$$\begin{array}{ll} & 5\cdot\left(\frac{1}{3}\cdot 3\right) \\ \text{Multiply.}\quad \frac{1}{3}\cdot 3 & 5\cdot 1 \\ \text{Multiply.} & 5 \end{array}$$

$$\left(5\cdot\frac{1}{3}\right)\cdot 3=5\cdot\left(\frac{1}{3}\cdot 3\right)$$

When multiplying three numbers, changing the grouping of the numbers gives the same result.

You probably know this, but the terminology may be new to you. These examples illustrate the **associative property**.

Note:
Associative Property
Equation:

of Addition	If a, b, c are real numbers, then $(a+b)+c=a+(b+c)$
of Multiplication	If a, b, c are real numbers, then $(a\cdot b)\cdot c=a\cdot(b\cdot c)$

When adding or multiplying, changing the *grouping* gives the same result.

Let’s think again about multiplying $5\cdot\frac{1}{3}\cdot 3$. We got the same result both ways, but which way was easier? Multiplying $\frac{1}{3}$ and 3 first, as shown above on the right side, eliminates the fraction in the first step. Using the associative property can make the math easier!

The associative property has to do with grouping. If we change how the numbers are grouped, the result will be the same. Notice it is the same three numbers in the same order—the only difference is the grouping.

We saw that subtraction and division were not commutative. They are not associative either.

When simplifying an expression, it is always a good idea to plan what the steps will be. In order to combine like terms in the next example, we will use the commutative property of addition to write the like terms together.

Example:
Exercise:

Problem: Simplify: $18p+6q+15p+5q$.

Solution:
Solution

	$18p+6q+15p+5q$
Use the commutative property of addition to re-order so that like terms are together.	$18p+15p+6q+5q$
Add like terms.	$33p+11q$

Note:
Exercise:

Problem: Simplify: $23r+14s+9r+15s$.

Solution:

$32r + 29s$

Note:
Exercise:

Problem: Simplify: $37m + 21n + 4m - 15n$.

Solution:

$41m + 6n$

When we have to simplify algebraic expressions, we can often make the work easier by applying the commutative or associative property first, instead of automatically following the order of operations. When adding or subtracting fractions, combine those with a common denominator first.

Example:
Exercise:

Problem: Simplify: $\left(\frac{5}{13} + \frac{3}{4}\right) + \frac{1}{4}$.

Solution:
Solution

	$\left(\frac{5}{13} + \frac{3}{4}\right) + \frac{1}{4}$
Notice that the last 2 terms have a common denominator, so change the grouping.	$\frac{5}{13} + \left(\frac{3}{4} + \frac{1}{4}\right)$
Add in parentheses first.	$\frac{5}{13} + \left(\frac{4}{4}\right)$
Simplify the fraction.	$\frac{5}{13} + 1$
Add.	$1\frac{5}{13}$
Convert to an improper fraction.	$\frac{18}{13}$

Note:
Exercise:

Problem: Simplify: $\left(\frac{7}{15}+\frac{5}{8}\right)+\frac{3}{8}$.

Solution:

$\frac{22}{15}$

Note:
Exercise:

Problem: Simplify: $\left(\frac{2}{9}+\frac{7}{12}\right)+\frac{5}{12}$.

Solution:

$\frac{11}{9}$

Example:
Exercise:

Problem: Use the associative property to simplify $6\left(3x\right)$.

Solution:
Solution

Use the associative property of multiplication, $\left(a\cdot b\right)\cdot c=a\cdot\left(b\cdot c\right)$, to change the grouping.

	$6\left(3x\right)$
Change the grouping.	$(6\cdot 3)x$
Multiply in the parentheses.	$18x$

Notice that we can multiply $6\cdot 3$ but we could not multiply 3*x* without having a value for *x*.

Note:
Exercise:

Problem: Use the associative property to simplify $8\left(4x\right)$.

Solution:

$32x$

Note:
Exercise:

Problem: Use the associative property to simplify $-9\,(7y)$.

Solution:

$-63y$

Use the Identity and Inverse Properties of Addition and Multiplication

What happens when we add 0 to any number? Adding 0 doesn't change the value. For this reason, we call 0 the **additive identity**.

For example,
Equation:

$$\begin{array}{ccc} 13+0 & -14+0 & 0+(-8) \\ 13 & -14 & -8 \end{array}$$

These examples illustrate the **Identity Property of Addition** that states that for any real number a, $a+0=a$ and $0+a=a$.

What happens when we multiply any number by one? Multiplying by 1 doesn't change the value. So we call 1 the **multiplicative identity**.

For example,
Equation:

$$\begin{array}{ccc} 43\cdot 1 & -27\cdot 1 & 1\cdot \frac{3}{5} \\ 43 & -27 & \frac{3}{5} \end{array}$$

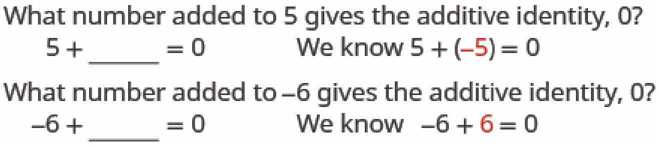
These examples illustrate the **Identity Property of Multiplication** that states that for any real number a, $a\cdot 1=a$ and $1\cdot a=a$.

We summarize the Identity Properties below.

Note:
Identity Property
Equation:

of addition For any real number a: $a+0=a \qquad 0+a=a$
0 is the **additive identity**

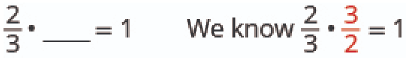
of multiplication For any real number a: $a\cdot 1=a \qquad 1\cdot a=a$
1 is the **multiplicative identity**


What number added to 5 gives the additive identity, 0?
$5 + \_\_\_\_\_ = 0$ We know $5 + (-5) = 0$
What number added to –6 gives the additive identity, 0?
$-6 + \_\_\_\_\_ = 0$ We know $-6 + 6 = 0$

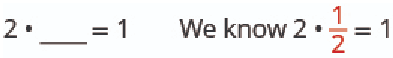
Notice that in each case, the missing number was the opposite of the number!

A number and its opposite add to zero, which is the additive identity. For this reason, we call *the opposite of a number its additive inverse.* For example, we call -5 the **additive inverse** of 5 since $-5+5=0$. This leads to the **Inverse Property of Addition** that states for any real number $a, a+(-a)=0$.

What number multiplied by $\frac{2}{3}$ gives the multiplicative identity, 1? In other words, $\frac{2}{3}$ times what results in 1?


$\frac{2}{3} \cdot \_\_\_\_ = 1$ We know $\frac{2}{3} \cdot \frac{3}{2} = 1$

What number multiplied by 2 gives the multiplicative identity, 1? In other words 2 times what results in 1?


$2 \cdot \_\_\_\_ = 1$ We know $2 \cdot \frac{1}{2} = 1$

Notice that in each case, the missing number was the reciprocal of the number!

We call $\frac{1}{a}$ the **multiplicative inverse** of a. *The reciprocal of a number is its multiplicative inverse.* A number and its reciprocal multiply to one, which is the multiplicative identity. This leads to the **Inverse Property of Multiplication** that states that for any real number $a, a\neq 0, a\cdot\frac{1}{a}=1$.

We'll formally state the inverse properties here:

Note:
Inverse Property
Equation:

of addition	For any real number a, $-a.$ is the **additive inverse** of a. A number and its opposite add to zero.	$a+(-a)=0$
of multiplication	For any real number $a, a \neq 0$ $\frac{1}{a}$ is the **multiplicative inverse** of a. A number and its reciprocal multiply to one.	$a \cdot \frac{1}{a}=1$

Example:
Exercise:

Problem: Find the additive inverse of ⓐ $\frac{5}{8}$ ⓑ 0.6 ⓒ -8 ⓓ $-\frac{4}{3}$.

Solution:
Solution

To find the additive inverse, we find its opposite.

ⓐ The additive inverse of $\frac{5}{8}$ is the opposite of $\frac{5}{8}$. The additive inverse of $\frac{5}{8}$ is $-\frac{5}{8}$.

ⓑ The additive inverse of 0.6 is the opposite of 0.6. The additive inverse of 0.6 is -0.6.

ⓒ The additive inverse of -8 is the opposite of -8. We write the opposite of -8 as $-(-8)$, and then simplify it to 8. Therefore, the additive inverse of -8 is 8.

ⓓ The additive inverse of $-\frac{4}{3}$ is the opposite of $-\frac{4}{3}$. We write this as $-\left(-\frac{4}{3}\right)$, and then simplify to $\frac{4}{3}$. Thus, the additive inverse of $-\frac{4}{3}$ is $\frac{4}{3}$.

Note:
Exercise:

Problem: Find the additive inverse of: ⓐ $\frac{7}{9}$ ⓑ 1.2 ⓒ -14 ⓓ $-\frac{9}{4}$.

Solution:

ⓐ $-\frac{7}{9}$ ⓑ -1.2 ⓒ 14 ⓓ $\frac{9}{4}$

Note:
Exercise:

Problem: Find the additive inverse of: ⓐ $\frac{7}{13}$ ⓑ 8.4 ⓒ -46 ⓓ $-\frac{5}{2}$.

Solution:

ⓐ $-\frac{7}{13}$ ⓑ -8.4 ⓒ 46 ⓓ $\frac{5}{2}$

Example:
Exercise:

Problem: Find the multiplicative inverse of ⓐ 9 ⓑ $-\frac{1}{9}$ ⓒ 0.9.

Solution:
Solution

To find the multiplicative inverse, we find the reciprocal.

ⓐ The multiplicative inverse of 9 is the reciprocal of 9, which is $\frac{1}{9}$. Therefore, the multiplicative inverse of 9 is $\frac{1}{9}$.
ⓑ The multiplicative inverse of $-\frac{1}{9}$ is the reciprocal of $-\frac{1}{9}$, which is -9. Thus, the multiplicative inverse of $-\frac{1}{9}$ is -9.
ⓒ To find the multiplicative inverse of 0.9, we first convert 0.9 to a fraction, $\frac{9}{10}$. Then we find the reciprocal of the fraction. The reciprocal of $\frac{9}{10}$ is $\frac{10}{9}$. So the multiplicative inverse of 0.9 is $\frac{10}{9}$.

Note:
Exercise:

Problem: Find the multiplicative inverse of ⓐ 4 ⓑ $-\frac{1}{7}$ ⓒ 0.3

Solution:

ⓐ $\frac{1}{4}$ ⓑ -7 ⓒ $\frac{10}{3}$

Note:
Exercise:

Problem: Find the multiplicative inverse of ⓐ 18 ⓑ $-\frac{4}{5}$ ⓒ 0.6.

Solution:

ⓐ $\frac{1}{18}$ ⓑ $-\frac{5}{4}$ ⓒ $\frac{5}{3}$

Use the Properties of Zero

The identity property of addition says that when we add 0 to any number, the result is that same number. What happens when we multiply a number by 0? Multiplying by 0 makes the product equal zero.

Note:
Multiplication by Zero
For any real number *a*.
Equation:

$$a \cdot 0 = 0 \qquad\qquad 0 \cdot a = 0$$

The product of any real number and 0 is 0.

What about division involving zero? What is $0 \div 3$? Think about a real example: If there are no cookies in the cookie jar and 3 people are to share them, how many cookies does each person get? There are no cookies to share, so each person gets 0 cookies. So,
Equation:

$$0 \div 3 = 0$$

We can check division with the related multiplication fact.
Equation:

$$12 \div 6 = 2 \text{ because } 2 \cdot 6 = 12.$$

So we know $0 \div 3 = 0$ because $0 \cdot 3 = 0$.

Note:
Division of Zero
For any real number *a*, except 0, $\frac{0}{a} = 0$ and $0 \div a = 0$.
Zero divided by any real number except zero is zero.

Now think about dividing *by* zero. What is the result of dividing 4 by 0? Think about the related multiplication fact: $4 \div 0 = ?$ means $? \cdot 0 = 4$. Is there a number that multiplied by 0 gives 4? Since any real number multiplied by 0 gives 0, there is no real number that can be multiplied by 0 to obtain 4.

We conclude that there is no answer to $4 \div 0$ and so we say that division by 0 is undefined.

Note:
Division by Zero
For any real number a, except 0, $\frac{a}{0}$ and $a \div 0$ are undefined.
Division by zero is undefined.

We summarize the properties of zero below.

Note:
Properties of Zero
Multiplication by Zero: For any real number a,

$$a \cdot 0 = 0 \quad 0 \cdot a = 0 \qquad \text{The product of any number and 0 is 0.}$$

Division of Zero, Division by Zero: For any real number a, $a \neq 0$

$\frac{0}{a} = 0$	Zero divided by any real number, except itself is zero.
$\frac{a}{0}$ is undefined	Division by zero is undefined.

Example:
Exercise:

Problem: Simplify: ⓐ $-8 \cdot 0$ ⓑ $\frac{0}{-2}$ ⓒ $\frac{-32}{0}$.

Solution:
Solution

ⓐ

	$-8 \cdot 0$
The product of any real number and 0 is 0.	0

ⓑ

	$\frac{0}{-2}$
Zero divided by any real number, except itself, is 0.	0

ⓒ

	$\frac{-32}{0}$
Division by 0 is undefined.	Undefined

Note:
Exercise:

Problem: Simplify: ⓐ $-14 \cdot 0$ ⓑ $\frac{0}{-6}$ ⓒ $\frac{-2}{0}$.

Solution:

ⓐ 0 ⓑ 0 ⓒ undefined

Note:
Exercise:

Problem: Simplify: ⓐ $0\left(-17\right)$ ⓑ $\frac{0}{-10}$ ⓒ $\frac{-5}{0}$.

Solution:

ⓐ 0 ⓑ 0 ⓒ undefined

We will now practice using the properties of identities, inverses, and zero to simplify expressions.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{0}{n+5}$, where $n \ne -5$ ⓑ $\frac{10-3p}{0}$, where $10-3p \ne 0$.

Solution:
Solution

ⓐ

	$\frac{0}{n+5}$
Zero divided by any real number except itself is 0.	0

ⓑ

	$\frac{10-3p}{0}$
Division by 0 is undefined.	Undefined

Example:
Exercise:

Problem: Simplify: $-84n+(-73n)+84n$.

Solution:
Solution

	$-84n+(-73n)+84n$
Notice that the first and third terms are opposites; use the commutative property of addition to re-order the terms.	$-84n+84n+(-73n)$
Add left to right.	$0+(-73)$
Add.	$-73n$

Note:
Exercise:

Problem: Simplify: $-27a+(-48a)+27a$.

Solution:

$-48a$

Note:
Exercise:

Problem: Simplify: $39x+(-92x)+(-39x)$.

Solution:

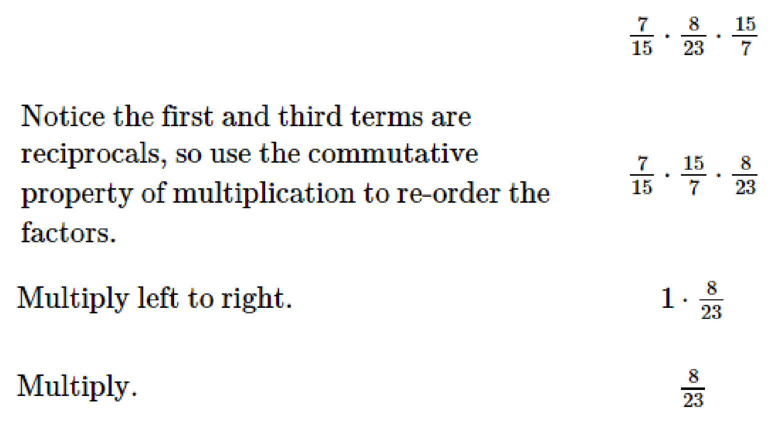
$-92x$

Now we will see how recognizing reciprocals is helpful. Before multiplying left to right, look for reciprocals—their product is 1.

Example:
Exercise:

Problem: Simplify: $\frac{7}{15} \cdot \frac{8}{23} \cdot \frac{15}{7}$.

Solution:
Solution



	$\frac{7}{15} \cdot \frac{8}{23} \cdot \frac{15}{7}$
Notice the first and third terms are reciprocals, so use the commutative property of multiplication to re-order the factors.	$\frac{7}{15} \cdot \frac{15}{7} \cdot \frac{8}{23}$
Multiply left to right.	$1 \cdot \frac{8}{23}$
Multiply.	$\frac{8}{23}$

Note:
Exercise:

Problem: Simplify: $\frac{9}{16} \cdot \frac{5}{49} \cdot \frac{16}{9}$.

Solution:

$\frac{5}{49}$

Note:
Exercise:

Problem: Simplify: $\frac{6}{17} \cdot \frac{11}{25} \cdot \frac{17}{6}$.

Solution:

$\frac{11}{25}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{0}{m+7}$, where $m \neq -7$ ⓑ $\frac{18-6c}{0}$, where $18 - 6c \neq 0$.

Solution:

ⓐ 0 ⓑ undefined

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{0}{d-4}$, where $d \neq 4$ ⓑ $\frac{15-4q}{0}$, where $15 - 4q \neq 0$.

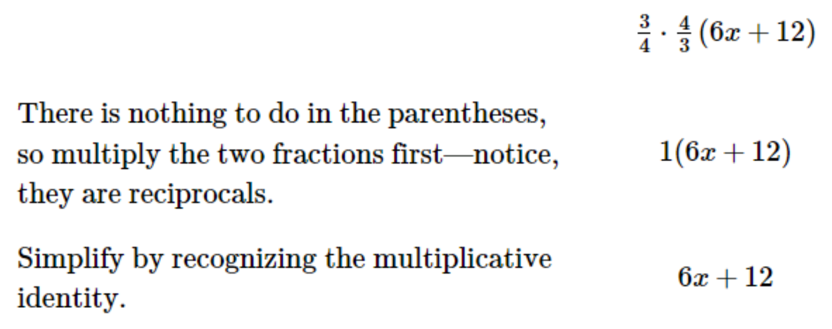
Solution:

ⓐ 0 ⓑ undefined

Example:
Exercise:

Problem: Simplify: $\frac{3}{4} \cdot \frac{4}{3}(6x + 12)$.

Solution:
Solution



	$\frac{3}{4} \cdot \frac{4}{3}(6x + 12)$
There is nothing to do in the parentheses, so multiply the two fractions first—notice, they are reciprocals.	$1(6x + 12)$
Simplify by recognizing the multiplicative identity.	$6x + 12$

Note:
Exercise:

Problem: Simplify: $\frac{2}{5} \cdot \frac{5}{2}(20y + 50)$.

Solution:

$20y + 50$

Note:
Exercise:

Problem: Simplify: $\frac{3}{8} \cdot \frac{8}{3}(12z + 16)$.

Solution:

$12z + 16$

Simplify Expressions Using the Distributive Property

Suppose that three friends are going to the movies. They each need $9.25—that's 9 dollars and 1 quarter—to pay for their tickets. How much money do they need all together?

You can think about the dollars separately from the quarters. They need 3 times $9 so $27, and 3 times 1 quarter, so 75 cents. In total, they need $27.75. If you think about doing the math in this way, you are using the **distributive property**.

Note:
Distributive Property
Equation:

$$\begin{array}{rl} \text{If } a, b, c \text{ are real numbers, then} & a(b+c) = ab + ac \\ \text{Also,} & (b+c)a = ba + ca \\ & a(b-c) = ab - ac \\ & (b-c)a = ba - ca \end{array}$$

Back to our friends at the movies, we could find the total amount of money they need like this:
Equation:

$$\begin{array}{ccc} & 3(9.25) & \\ 3(9 & + & 0.25) \\ 3(9) & + & 3(0.25) \\ 27 & + & 0.75 \\ & 27.75 & \end{array}$$

In algebra, we use the **distributive property** to remove parentheses as we simplify expressions.

For example, if we are asked to simplify the expression $3\left(x+4\right)$, the order of operations says to work in the parentheses first. But we cannot add *x* and 4, since they are not like terms. So we use the distributive property, as shown in [link].

Example:
Exercise:

Problem: Simplify: $3\left(x+4\right)$.

Solution:
Solution

	$3\left(x+4\right)$
Distribute.	$3\cdot x+3\cdot 4$
Multiply.	$3x+12$

Note:
Exercise:

Problem: Simplify: $4\left(x+2\right)$.

Solution:

$4x+8$

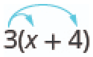
Note:
Exercise:

Problem: Simplify: $6\left(x+7\right)$.

Solution:

$6x+42$

Some students find it helpful to draw in arrows to remind them how to use the distributive property. Then the first step in [link] would look like this:

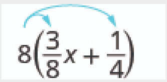


$3(x+4)$

Example:
Exercise:

Problem: Simplify: $8\left(\frac{3}{8}x+\frac{1}{4}\right)$.

Solution:
Solution

	 $8\left(\frac{3}{8}x+\frac{1}{4}\right)$
Distribute.	$8\cdot\frac{3}{8}x+8\cdot\frac{1}{4}$
Multiply.	$3x+2$

Note:
Exercise:

Problem: Simplify: $6\left(\frac{5}{6}y+\frac{1}{2}\right)$.

Solution:

$5y+3$

Note:
Exercise:

Problem: Simplify: $12\left(\frac{1}{3}n+\frac{3}{4}\right)$.

Solution:

$4n+9$

Using the distributive property as shown in [link] will be very useful when we solve money applications in later chapters.

Example:
Exercise:

Problem: Simplify: $100\left(0.3+0.25q\right)$.

Solution:
Solution

	$100(0.3+0.25q)$
Distribute.	$100(0.3)+100(0.25q)$
Multiply.	$30+25q$

Note:
Exercise:

Problem: Simplify: $100\left(0.7+0.15p\right)$.

Solution:

$70+15p$

Note:
Exercise:

Problem: Simplify: $100\left(0.04+0.35d\right)$.

Solution:

$4+35d$

When we distribute a negative number, we need to be extra careful to get the signs correct!

Example:
Exercise:

Problem: Simplify: $-2\left(4y+1\right)$.

Solution:
Solution

	$-2(4y+1)$
Distribute.	$-2\cdot 4y+(-2)\cdot 1$
Multiply.	$-8y-2$

Note:
Exercise:

Problem: Simplify: $-3\,(6m+5)$.

Solution:

$-18m-15$

Note:
Exercise:

Problem: Simplify: $-6\,(8n+11)$.

Solution:

$-48n-66$

Example:
Exercise:

Problem: Simplify: $-11\,(4-3a)$.

Solution:
Solution

Distribute.	$-11(4-3a)$
Multiply.	$-11\cdot 4-(-11)\cdot 3a$ $-44-(-33a)$
Simplify.	$-44+33a$

Notice that you could also write the result as $33a-44$. Do you know why?

Note:
Exercise:

Problem: Simplify: $-5\left(2-3a\right)$.

Solution:

$-10+15a$

Note:
Exercise:

Problem: Simplify: $-7\left(8-15y\right)$.

Solution:

$-56+105y$

[link] will show how to use the distributive property to find the opposite of an expression.

Example:
Exercise:

Problem: Simplify: $-\left(y+5\right)$.

Solution:
Solution

	$-\left(y+5\right)$
Multiplying by -1 results in the opposite.	$-1\left(y+5\right)$
Distribute.	$-1\cdot y+\left(-1\right)\cdot 5$
Simplify.	$-y+\left(-5\right)$
	$-y-5$

Note:
Exercise:

Problem: Simplify: $-(z-11)$.

Solution:

$-z+11$

Note:
Exercise:

Problem: Simplify: $-(x-4)$.

Solution:

$-x+4$

There will be times when we'll need to use the distributive property as part of the order of operations. Start by looking at the parentheses. If the expression inside the parentheses cannot be simplified, the next step would be multiply using the distributive property, which removes the parentheses. The next two examples will illustrate this.

Example:
Exercise:

Problem: Simplify: $8-2(x+3)$.

Be sure to follow the order of operations. Multiplication comes before subtraction, so we will distribute the 2 first and then subtract.

Solution:
Solution

	$8-2(x+3)$
Distribute.	$8-2\cdot x-2\cdot 3$
Multiply.	$8-2x-6$
Combine like terms.	$-2x+2$

Note:
Exercise:

Problem: Simplify: $9-3\left(x+2\right)$.

Solution:

$3-3x$

Note:
Exercise:

Problem: Simplify: $7x-5\left(x+4\right)$.

Solution:

$2x-20$

Example:
Exercise:

Problem: Simplify: $4\left(x-8\right)-\left(x+3\right)$.

Solution:
Solution

	$4\left(x-8\right)-\left(x+3\right)$
Distribute.	$4x-32-x-3$
Combine like terms.	$3x-35$

Note:
Exercise:

Problem: Simplify: $6\left(x-9\right)-\left(x+12\right)$.

Solution:

$5x-66$

Note:
Exercise:

Problem: Simplify: $8\left(x-1\right)-\left(x+5\right)$.

Solution:

$7x-13$

All the properties of real numbers we have used in this chapter are summarized in [link].

Commutative Property	
of addition If a,b are real numbers, then **of multiplication** If a,b are real numbers, then	$a+b=b+a$ $a\cdot b=b\cdot a$
Associative Property	
of addition If a,b,c are real numbers, then **of multiplication** If a,b,c are real numbers, then	$\left(a+b\right)+c=a+\left(b+c\right)$ $\left(a\cdot b\right)\cdot c=a\cdot\left(b\cdot c\right)$
Distributive Property	
If a,b,c are real numbers, then	$a\left(b+c\right)=ab+ac$
Identity Property	
of addition For any real number a: 0 is the **additive identity** **of multiplication** For any real number a: **1** is the **multiplicative identity**	$a+0=a$ $0+a=a$ $a\cdot 1=a$ $1\cdot a=a$
Inverse Property	
of addition For any real number a, $-a$ is the **additive inverse** of a	$a+\left(-a\right)=0$ $a\cdot\frac{1}{a}=1$

of multiplication For any real number $a, a \neq 0$ $\frac{1}{a}$ is the **multiplicative inverse** of a.	
Properties of Zero	
For any real number a, For any real number $a, a \neq 0$ For any real number $a, a \neq 0$	$a \cdot 0 = 0$ $0 \cdot a = 0$ $\frac{0}{a} = 0$ $\frac{a}{0}$ is undefined

Key Concepts

- **Commutative Property of**
  - **Addition:** If a, b are real numbers, then $a + b = b + a$.
  - **Multiplication:** If a, b are real numbers, then $a \cdot b = b \cdot a$. When adding or multiplying, changing the *order* gives the same result.
- **Associative Property of**
  - **Addition:** If a, b, c are real numbers, then $(a + b) + c = a + (b + c)$.
  - **Multiplication:** If a, b, c are real numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. When adding or multiplying, changing the *grouping* gives the same result.
- **Distributive Property:** If a, b, c are real numbers, then
  - $a\left(b + c\right) = ab + ac$
  - $(b + c)a = ba + ca$
  - $a\left(b - c\right) = ab - ac$
  - $(b - c)a = ba - ca$
- **Identity Property**
  - **of Addition:** For any real number a: $a + 0 = a \quad 0 + a = a$
    0 is the **additive identity**
  - **of Multiplication:** For any real number a: $a \cdot 1 = a \quad 1 \cdot a = a$
    1 is the **multiplicative identity**
- **Inverse Property**
  - **of Addition:** For any real number a, $a + (-a) = 0$. A number and its *opposite* add to zero.
    $-a$ is the **additive inverse** of a.
  - **of Multiplication:** For any real number a, $(a \neq 0)$ $a \cdot \frac{1}{a} = 1$. A number and its *reciprocal* multiply to one. $\frac{1}{a}$ is the **multiplicative inverse** of a.
- **Properties of Zero**

- For any real number a,
  $a \cdot 0 = 0 \;\; 0 \cdot a = 0$ – The product of any real number and 0 is 0.
- $\frac{0}{a} = 0$ for $a \neq 0$ – Zero divided by any real number except zero is zero.
- $\frac{a}{0}$ is undefined – Division by zero is undefined.

Practice Makes Perfect

Use the Commutative and Associative Properties

In the following exercises, use the associative property to simplify.

Exercise:

Problem: $3(4x)$

Solution:

$12x$

Exercise:

Problem: $4(7m)$

Exercise:

Problem: $(y + 12) + 28$

Solution:

$y + 40$

Exercise:

Problem: $(n + 17) + 33$

In the following exercises, simplify.

Exercise:

Problem: $\frac{1}{2} + \frac{7}{8} + \left(-\frac{1}{2}\right)$

Solution:

$\frac{7}{8}$

Exercise:

Problem: $\frac{2}{5} + \frac{5}{12} + \left(-\frac{2}{5}\right)$

Exercise:

Problem: $\frac{3}{20} \cdot \frac{49}{11} \cdot \frac{20}{3}$

Solution:

$\frac{49}{11}$

Exercise:

Problem: $\frac{13}{18} \cdot \frac{25}{7} \cdot \frac{18}{13}$

Exercise:

Problem: $-24 \cdot 7 \cdot \frac{3}{8}$

Solution:

-63

Exercise:

Problem: $-36 \cdot 11 \cdot \frac{4}{9}$

Exercise:

Problem: $\left(\frac{5}{6} + \frac{8}{15}\right) + \frac{7}{15}$

Solution:

$1\frac{5}{6}$

Exercise:

Problem: $\left(\frac{11}{12} + \frac{4}{9}\right) + \frac{5}{9}$

Exercise:

Problem: 17(0.25)(4)

Solution:

17

Exercise:

Problem: 36(0.2)(5)

Exercise:

Problem: [2.48(12)](0.5)

Solution:

14.88

Exercise:

Problem: [9.731(4)](0.75)

Exercise:

Problem: $7(4a)$

Solution:

$28a$

Exercise:

Problem: $9(8w)$

Exercise:

Problem: $-15\left(5m\right)$

Solution:

$-75m$

Exercise:

Problem: $-23\left(2n\right)$

Exercise:

Problem: $12\left(\frac{5}{6}p\right)$

Solution:

$10p$

Exercise:

Problem: $20\left(\frac{3}{5}q\right)$

Exercise:

Problem: $43m+\left(-12n\right)+\left(-16m\right)+\left(-9n\right)$

Solution:

$27m+\left(-21n\right)$

Exercise:

Problem: $-22p+17q+(-35p)+(-27q)$

Exercise:

Problem: $\frac{3}{8}g+\frac{1}{12}h+\frac{7}{8}g+\frac{5}{12}h$

Solution:

$\frac{5}{4}g+\frac{1}{2}h$

Exercise:

Problem: $\frac{5}{6}a+\frac{3}{10}b+\frac{1}{6}a+\frac{9}{10}b$

Exercise:

Problem: $6.8p+9.14q+(-4.37p)+(-0.88q)$

Solution:

$2.43p+8.26q$

Exercise:

Problem: $9.6m+7.22n+(-2.19m)+(-0.65n)$

Use the Identity and Inverse Properties of Addition and Multiplication

In the following exercises, find the additive inverse of each number.

Exercise:

Problem:
ⓐ $\frac{2}{5}$
ⓑ 4.3
ⓒ -8
ⓓ $-\frac{10}{3}$

Solution:

ⓐ $-\frac{2}{5}$ ⓑ -4.3 ⓒ 8 ⓓ $\frac{10}{3}$

Exercise:

Problem: ⓐ $\frac{5}{9}$
ⓑ 2.1
ⓒ -3
ⓓ $-\frac{9}{5}$

Exercise:

Problem: ⓐ $-\frac{7}{6}$
ⓑ -0.075
ⓒ 23
ⓓ $\frac{1}{4}$

Solution:

ⓐ $\frac{7}{6}$ ⓑ 0.075 ⓒ -23 ⓓ $-\frac{1}{4}$

Exercise:

Problem: ⓐ $-\frac{8}{3}$
ⓑ -0.019
ⓒ 52
ⓓ $\frac{5}{6}$

In the following exercises, find the multiplicative inverse of each number.

Exercise:

Problem: ⓐ 6 ⓑ $-\frac{3}{4}$ ⓒ 0.7

Solution:

ⓐ $\frac{1}{6}$ ⓑ $-\frac{4}{3}$ ⓒ $\frac{10}{7}$

Exercise:

Problem: ⓐ 12 ⓑ $-\frac{9}{2}$ ⓒ 0.13

Exercise:

Problem: ⓐ $\frac{11}{12}$ ⓑ -1.1 ⓒ -4

Solution:

ⓐ $\frac{12}{11}$ ⓑ $-\frac{10}{11}$ ⓒ $-\frac{1}{4}$

Exercise:

Problem: ⓐ $\frac{17}{20}$ ⓑ -1.5 ⓒ -3

Use the Properties of Zero

In the following exercises, simplify.

Exercise:

Problem: $\frac{6}{0}$

Solution:

undefined

Exercise:

Problem: $\frac{3}{0}$

Exercise:

Problem: $0 \div \frac{11}{12}$

Solution:

0

Exercise:

Problem: $\frac{0}{8}$

Exercise:

Problem: $\frac{0}{3}$

Solution:

0

Exercise:

Problem: $0 \cdot \frac{8}{15}$

Exercise:

Problem: $\left(-3.14\right)\left(0\right)$

Solution:

0

Exercise:

Problem: $\frac{\frac{1}{10}}{0}$

Mixed Practice

In the following exercises, simplify.

Exercise:

Problem: $19a+44-19a$

Solution:

44

Exercise:

Problem: $27c+16-27c$

Exercise:

Problem: $10(0.1d)$

Solution:

d

Exercise:

Problem: $100\left(0.01p\right)$

Exercise:

Problem: $\frac{0}{u-4.99}$, where $u\ne 4.99$

Solution:

0

Exercise:

Problem: $\frac{0}{v-65.1}$, where $v\ne 65.1$

Exercise:

Problem: $0\div \left(x-\ \frac{1}{2}\right)$, where $x\ne \frac{1}{2}$

Solution:

0

Exercise:

Problem: $0 \div \left(y - \frac{1}{6}\right)$, where $x \neq \frac{1}{6}$

Exercise:

Problem: $\frac{32-5a}{0}$, where $32 - 5a \neq 0$

Solution:

undefined

Exercise:

Problem: $\frac{28-9b}{0}$, where $28 - 9b \neq 0$

Exercise:

Problem: $\left(\frac{3}{4} + \frac{9}{10}m\right) \div 0$ where $\frac{3}{4} + \frac{9}{10}m \neq 0$

Solution:

undefined

Exercise:

Problem: $\left(\frac{5}{16}n - \frac{3}{7}\right) \div 0$ where $\frac{5}{16}n - \frac{3}{7} \neq 0$

Exercise:

Problem: $15 \cdot \frac{3}{5}(4d + 10)$

Solution:

$36d + 90$

Exercise:

Problem: $18 \cdot \frac{5}{6}(15h + 24)$

Simplify Expressions Using the Distributive Property

In the following exercises, simplify using the distributive property.

Exercise:

Problem: $8\left(4y+9\right)$

Solution:

$32y+72$

Exercise:

Problem: $9\left(3w+7\right)$

Exercise:

Problem: $6\left(c-13\right)$

Solution:

$6c-78$

Exercise:

Problem: $7\left(y-13\right)$

Exercise:

Problem: $\frac{1}{4}(3q+12)$

Solution:

$\frac{3}{4}q+3$

Exercise:

Problem: $\frac{1}{5}(4m+20)$

Exercise:

Problem: $9\left(\frac{5}{9}y-\ \frac{1}{3}\right)$

Solution:

$5y-3$

Exercise:

Problem: $10\left(\frac{3}{10}x-\ \frac{2}{5}\right)$

Exercise:

Problem: $12\left(\frac{1}{4}+\frac{2}{3}r\right)$

Solution:

$3 + 8r$

Exercise:

Problem: $12\left(\frac{1}{6} + \frac{3}{4}s\right)$

Exercise:

Problem: $2\left(s - 18\right)$

Solution:

$2s - 36$

Exercise:

Problem: $3\left(v - 10\right)$

Exercise:

Problem: $(y + 4)5$

Solution:

$5y + 20$

Exercise:

Problem: $(a + 7)4$

Exercise:

Problem: $-7\left(4p + 1\right)$

Solution:

$-28p - 7$

Exercise:

Problem: $-9\left(9a + 4\right)$

Exercise:

Problem: $-3\left(x - 6\right)$

Solution:

$-3x + 18$

Exercise:

Problem: $-4\left(q-7\right)$

Exercise:

Problem: $-\left(3x-7\right)$

Solution:

$-3x+7$

Exercise:

Problem: $-\left(5p-4\right)$

Exercise:

Problem: $16-3\left(y+8\right)$

Solution:

$-3y-8$

Exercise:

Problem: $18-4\left(x+2\right)$

Exercise:

Problem: $4-11\left(3c-2\right)$

Solution:

$-33c+26$

Exercise:

Problem: $9-6\left(7n-5\right)$

Exercise:

Problem: $22-\left(a+3\right)$

Solution:

$-a+19$

Exercise:

Problem: $8-\left(r-7\right)$

Exercise:

Problem: $(5m-3)-(m+7)$

Solution:

$4m-10$

Exercise:

Problem: $(4y-1)-(y-2)$

Exercise:

Problem: $5\,(2n+9)+12\,(n-3)$

Solution:

$22n+9$

Exercise:

Problem: $9\,(5u+8)+2\,(u-6)$

Exercise:

Problem: $9\,(8x-3)-(-2)$

Solution:

$72x-25$

Exercise:

Problem: $4\,(6x-1)-(-8)$

Exercise:

Problem: $14\,(c-1)-8\,(c-6)$

Solution:

$6c+34$

Exercise:

Problem: $11\,(n-7)-5\,(n-1)$

Exercise:

Problem: $6\,(7y+8)-(30y-15)$

Solution:

$12y+63$

Exercise:

Problem: $7\left(3n+9\right)-\left(4n-13\right)$

Everyday Math

Exercise:

Problem:

Insurance copayment Carrie had to have 5 fillings done. Each filling cost $80. Her dental insurance required her to pay 20% of the cost as a copay. Calculate Carrie's copay:

ⓐ First, by multiplying 0.20 by 80 to find her copay for each filling and then multiplying your answer by 5 to find her total copay for 5 fillings.
ⓑ Next, by multiplying [5(0.20)](80)
ⓒ Which of the properties of real numbers says that your answers to parts (a), where you multiplied 5[(0.20)(80)] and (b), where you multiplied [5(0.20)](80), should be equal?

Solution:

ⓐ $80 ⓑ $80 ⓒ answers will vary

Exercise:

Problem:

Cooking time Helen bought a 24-pound turkey for her family's Thanksgiving dinner and wants to know what time to put the turkey in to the oven. She wants to allow 20 minutes per pound cooking time. Calculate the length of time needed to roast the turkey:

ⓐ First, by multiplying $24\cdot 20$ to find the total number of minutes and then multiplying the answer by $\frac{1}{60}$ to convert minutes into hours.
ⓑ Next, by multiplying $24\left(20\cdot \frac{1}{60}\right)$.
ⓒ Which of the properties of real numbers says that your answers to parts (a), where you multiplied $\left(24\cdot 20\right)\frac{1}{60}$, and (b), where you multiplied $24\left(20\cdot \frac{1}{60}\right)$, should be equal?

Exercise:

Problem:

Buying by the case Trader Joe's grocery stores sold a bottle of wine they called "Two Buck Chuck" for $1.99. They sold a case of 12 bottles for $23.88. To find the cost of 12 bottles at $1.99, notice that 1.99 is $2-0.01$.

ⓐ Multiply 12(1.99) by using the distributive property to multiply $12\left(2-0.01\right)$.
ⓑ Was it a bargain to buy “Two Buck Chuck” by the case?

Solution:

ⓐ $23.88 ⓑ no, the price is the same

Exercise:

Problem:

Multi-pack purchase Adele’s shampoo sells for $3.99 per bottle at the grocery store. At the warehouse store, the same shampoo is sold as a 3 pack for $10.49. To find the cost of 3 bottles at $3.99, notice that 3.99 is $4-0.01$.

ⓐ Multiply 3(3.99) by using the distributive property to multiply $3\left(4-0.01\right)$.
ⓑ How much would Adele save by buying 3 bottles at the warehouse store instead of at the grocery store?

Writing Exercises

Exercise:

Problem: In your own words, state the commutative property of addition.

Solution:

Answers may vary

Exercise:

Problem:

What is the difference between the additive inverse and the multiplicative inverse of a number?

Exercise:

Problem: Simplify $8\left(x-\frac{1}{4}\right)$ using the distributive property and explain each step.

Solution:

Explanations may vary, but the expression simplifies to $8x-2$

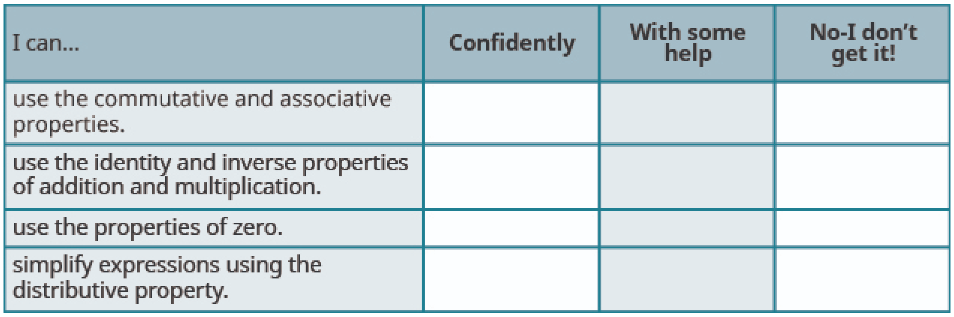
Exercise:

Problem:

Explain how you can multiply 4($5.97) without paper or calculator by thinking of $5.97 as $6-0.03$ and then using the distributive property.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don’t get it!
use the commutative and associative properties.			
use the identity and inverse properties of addition and multiplication.			
use the properties of zero.			
simplify expressions using the distributive property.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

additive identity

The additive identity is the number 0; adding 0 to any number does not change its value.

additive inverse

The opposite of a number is its additive inverse. A number and it additive inverse add to 0.

multiplicative identity

The multiplicative identity is the number 1; multiplying 1 by any number does not change the value of the number.

multiplicative inverse

The reciprocal of a number is its multiplicative inverse. A number and its multiplicative inverse multiply to one.

The Real Numbers

By the end of this section, you will be able to:

- Simplify expressions with square roots
- Identify integers, rational numbers, irrational numbers, and real numbers
- Locate fractions on the number line
- Locate decimals on the number line

Note:
A more thorough introduction to the topics covered in this section can be found in the *Prealgebra* chapters, **Decimals** and **Properties of Real Numbers**.

Simplify Expressions with Square Roots

Remember that when a number *n* is multiplied by itself, we write n^2 and read it “n squared.” The result is called the **square** of *n*. For example,

Equation:

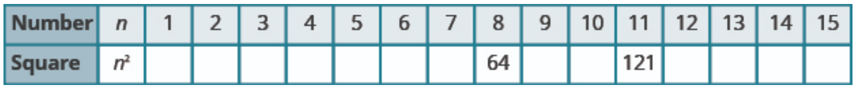
$$\begin{array}{ll} 8^2 & \text{read ‘8 squared’} \\ 64 & \text{64 is called the } \textit{square} \text{ of 8.} \end{array}$$

Similarly, 121 is the square of 11, because 11^2 is 121.

Note:
Square of a Number
If $n^2 = m$, then *m* is the **square** of *n*.

Note: Doing the Manipulative Mathematics activity “Square Numbers” will help you develop a better understanding of perfect square numbers.

Complete the following table to show the squares of the counting numbers 1 through 15.



Number	n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Square	n^2								64			121				

The numbers in the second row are called perfect square numbers. It will be helpful to learn to recognize the perfect square numbers.

The squares of the counting numbers are positive numbers. What about the squares of negative numbers? We know that when the signs of two numbers are the same, their product is positive. So the square of any negative number is also positive.

Equation:

$$(-3)^2 = 9 \qquad (-8)^2 = 64 \qquad (-11)^2 = 121 \qquad (-15)^2 = 225$$

Did you notice that these squares are the same as the squares of the positive numbers?

Sometimes we will need to look at the relationship between numbers and their squares in reverse. Because $10^2 = 100$, we say 100 is the square of 10. We also say that 10 is a *square root* of 100. A number whose square is m is called a **square root** of *m*.

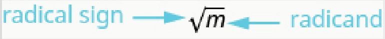
Note:
Square Root of a Number
If $n^2 = m$, then *n* is a **square root** of *m*.

Notice $(-10)^2 = 100$ also, so -10 is also a square root of 100. Therefore, both 10 and -10 are square roots of 100.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? The radical sign, $\sqrt{m}$, denotes the positive square root. The positive square root is called the principal square root. When we use the radical sign that always means we want the principal square root.

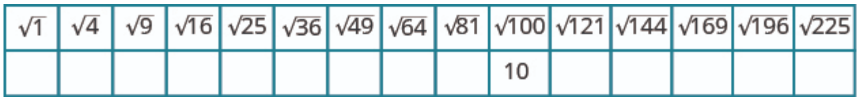
We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Note:
Square Root Notation
$\sqrt{m}$ is read "the square root of *m*"



If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.
The square root of *m*, $\sqrt{m}$, is the positive number whose square is *m*.

Since 10 is the principal square root of 100, we write $\sqrt{100} = 10$. You may want to complete the following table to help you recognize square roots.



$\sqrt{1}$	$\sqrt{4}$	$\sqrt{9}$	$\sqrt{16}$	$\sqrt{25}$	$\sqrt{36}$	$\sqrt{49}$	$\sqrt{64}$	$\sqrt{81}$	$\sqrt{100}$	$\sqrt{121}$	$\sqrt{144}$	$\sqrt{169}$	$\sqrt{196}$	$\sqrt{225}$
									10					

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{25}$ ⓑ $\sqrt{121}$.

Solution:
Solution

ⓐ

	$\sqrt{25}$
Since $5^2 = 25$	5

ⓑ

	$\sqrt{121}$
Since $11^2 = 121$	11

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{36}$ ⓑ $\sqrt{169}$.

Solution:

ⓐ 6 ⓑ 13

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{16}$ ⓑ $\sqrt{196}$.

Solution:

ⓐ 4 ⓑ 14

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{100} = 10$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{100} = -10$. We read $-\sqrt{100}$ as "the opposite of the square root of 10."

Example:
Exercise:

Problem: Simplify: ⓐ $-\sqrt{9}$ ⓑ $-\sqrt{144}$.

Solution:

ⓐ

	$-\sqrt{9}$
The negative is in front of the radical sign.	-3

ⓑ

	$-\sqrt{144}$
The negative is in front of the radical sign.	-12

Note:
Exercise:

Problem: Simplify: ⓐ $-\sqrt{4}$ ⓑ $-\sqrt{225}$.

Solution:

ⓐ -2 ⓑ -15

Note:
Exercise:

Problem: Simplify: ⓐ $-\sqrt{81}$ ⓑ $-\sqrt{100}$.

Solution:

ⓐ -9 ⓑ -10

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

We have already described numbers as *counting numbers*, *whole numbers*, and *integers*. What is the difference between these types of numbers?
Equation:

$$\begin{array}{ll} \text{Counting numbers} & 1, 2, 3, 4, \ldots \\ \text{Whole numbers} & 0, 1, 2, 3, 4, \ldots \\ \text{Integers} & \ldots -3, -2, -1, 0, 1, 2, 3, \ldots \end{array}$$

What type of numbers would we get if we started with all the integers and then included all the fractions? The numbers we would have form the set of rational numbers. A **rational number** is a number that can be written as a ratio of two integers, the second of which cannot be zero.

Note:
Rational Number
A **rational number** is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
A rational number can be written as the ratio of two integers, the second of which cannot be zero.

All signed fractions, such as $\frac{4}{5}, -\frac{7}{8}, \frac{13}{4}, -\frac{20}{3}$ are rational numbers. Each numerator and each denominator is an integer.

Are integers rational numbers? To decide if an integer is a rational number, we try to write it as a ratio of two integers. Each integer can be written as a ratio of integers in many ways. For example, 3 is equivalent to $\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \frac{15}{5} \ldots$

An easy way to write an integer as a ratio of integers is to write it as a fraction with denominator one.
Equation:

$$3 = \frac{3}{1} \qquad -8 = -\frac{8}{1} \qquad 0 = \frac{0}{1}$$

Since any integer can be written as the ratio of two integers, *all integers are rational numbers*! Remember that the counting numbers and the whole numbers are also integers, and so they, too, are rational.

What about decimals? Are they rational? Let's look at a few to see if we can write each of them as the ratio of two integers.

We've already seen that integers are rational numbers. The integer -8 could be written as the decimal -8.0. So, clearly, some decimals are rational.

Think about the decimal 7.3. Can we write it as a ratio of two integers? Because 7.3 means $7\frac{3}{10}$, we can write it as an improper fraction, $\frac{73}{10}$. So 7.3 is the ratio of the integers 73 and 10. It is a rational number.

In general, any decimal that ends after a number of digits (such as 7.3 or -1.2684) is a rational number. We can use the place value of the last digit as the denominator when writing the decimal as a fraction.

Example:
Exercise:

Problem: Write as the ratio of two integers: ⓐ -27 ⓑ 7.31.

Solution:

ⓐ

	-27
Write it as a fraction with denominator 1.	$\frac{-27}{1}$

ⓑ

	7.31
Write is as a mixed number. Remember. 7 is the whole number and the decimal part, 0.31, indicates hundredths.	$7\frac{31}{100}$
Convert to an improper fraction.	$\frac{731}{100}$

So we see that -27 and 7.31 are both rational numbers, since they can be written as the ratio of two integers.

Note:
Exercise:

Problem: Write as the ratio of two integers: ⓐ -24 ⓑ 3.57.

Solution:

ⓐ $\frac{-24}{1}$ ⓑ $\frac{357}{100}$

Note:
Exercise:

Problem: Write as the ratio of two integers: ⓐ -19 ⓑ 8.41.

Solution:

ⓐ $\frac{-19}{1}$ ⓑ $\frac{841}{100}$

Let's look at the decimal form of the numbers we know are rational.

We have seen that *every integer is a rational number*, since $a = \frac{a}{1}$ for any integer, a. We can also change any integer to a decimal by adding a decimal point and a zero.

Equation:

$$\begin{array}{lllllll} \text{Integer} & -2 & -1 & 0 & 1 & 2 & 3 \\ \text{Decimal form} & -2.0 & -1.0 & 0.0 & 1.0 & 2.0 & 3.0 \end{array}$$

Equation:

$$\text{These decimal numbers stop.}$$

We have also seen that *every fraction is a rational number*. Look at the decimal form of the fractions we considered above.

Equation:

$$\begin{array}{lllll} \text{Ratio of integers} & \frac{4}{5} & -\frac{7}{8} & \frac{13}{4} & -\frac{20}{3} \\ \text{The decimal form} & 0.8 & -0.875 & 3.25 & \begin{array}{l} -6.666\ldots \\ -6.\overline{6} \end{array} \end{array}$$

Equation:

$$\text{These decimals either stop or repeat.}$$

What do these examples tell us?

Every rational number can be written both as a ratio of integers, $(\frac{p}{q}$*, where p and q are integers and* $q \neq 0)$*, and as a decimal that either stops or repeats.*

Here are the numbers we looked at above expressed as a ratio of integers and as a decimal:

Fractions					**Integers**					
Number	$\frac{4}{5}$	$-\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$	-2	-1	0	1	2	3
Ratio of Integers	$\frac{4}{5}$	$-\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$	$-\frac{2}{1}$	$-\frac{1}{1}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$
Decimal Form	0.8	-0.875	3.25	$-6.\overline{6}$	-2.0	-1.0	0.0	1.0	2.0	3.0

Note:
Rational Number
A **rational number** is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
Its decimal form stops or repeats.

Are there any decimals that do not stop or repeat? Yes!

The number π (the Greek letter *pi*, pronounced "pie"), which is very important in describing circles, has a decimal form that does not stop or repeat.
Equation:

$$\pi = 3.141592654...$$

We can even create a decimal pattern that does not stop or repeat, such as
Equation:

$$2.01001000100001...$$

Numbers whose decimal form does not stop or repeat cannot be written as a fraction of integers. We call these numbers irrational.

Note:
Irrational Number
An **irrational number** is a number that cannot be written as the ratio of two integers.
Its decimal form does not stop and does not repeat.

Let's summarize a method we can use to determine whether a number is rational or irrational.

Note:
Rational or Irrational?
If the decimal form of a number

- *repeats or stops*, the number is **rational**.
- *does not repeat and does not stop*, the number is **irrational**.

Example:
Exercise:

Problem:

Given the numbers $0.58\overline{3}, 0.47, 3.605551275\ldots$ list the ⓐ rational numbers ⓑ irrational numbers.

Solution:
Solution

ⓐ

Look for decimals that repeat or stop.

The 3 repeats in $0.58\overline{3}$.
The decimal 0.47 stops after the 7.
So $0.58\overline{3}$ and 0.47 are rational.

ⓑ

Look for decimals that neither stop nor repeat.

$3.605551275\ldots$ has no repeating block of digits and it does not stop.
So $3.605551275\ldots$ is irrational.

Note:
Exercise:

Problem:

For the given numbers list the ⓐ rational numbers ⓑ irrational numbers: $0.29, 0.81\overline{6}, 2.515115111\ldots$.

Solution:

ⓐ $0.29, 0.81\overline{6}$ ⓑ $2.515115111\ldots$

Note:
Exercise:

Problem:

For the given numbers list the ⓐ rational numbers ⓑ irrational numbers: $2.6\overline{3}, 0.125, 0.418302\ldots$

Solution:

ⓐ $2.6\overline{3}, 0.125$ ⓑ $0.418302\ldots$

Example:
Exercise:

Problem: For each number given, identify whether it is rational or irrational: ⓐ $\sqrt{36}$ ⓑ $\sqrt{44}$.

Solution:

ⓐ Recognize that 36 is a perfect square, since $6^2 = 36$. So $\sqrt{36} = 6$, therefore $\sqrt{36}$ is rational.
ⓑ If you look at the decimal form of $\sqrt{44}$, you will notice that it never repeats. If you use a calculator that can display all the digits after the decimal, you will see that it also never stops, so $\sqrt{44}$ is irrational.

Note:
Rule of Thumb
It can be difficult to know whether a decimal number will end or not, if we can't see all the digits, like when finding the square root of a number on our calculators. A good rule of thumb for determining whether the square root of a positive integer is rational or not is to check to see whether the integer is a perfect square. If the integer is not a perfect square, then the square root of that integer cannot be rational; it must be irrational.

Note:
Exercise:

Problem: For each number given, identify whether it is rational or irrational: ⓐ $\sqrt{81}$ ⓑ $\sqrt{17}$.

Solution:

ⓐ rational ⓑ irrational

Note:
Exercise:

Problem: For each number given, identify whether it is rational or irrational: ⓐ $\sqrt{116}$ ⓑ $\sqrt{121}$.

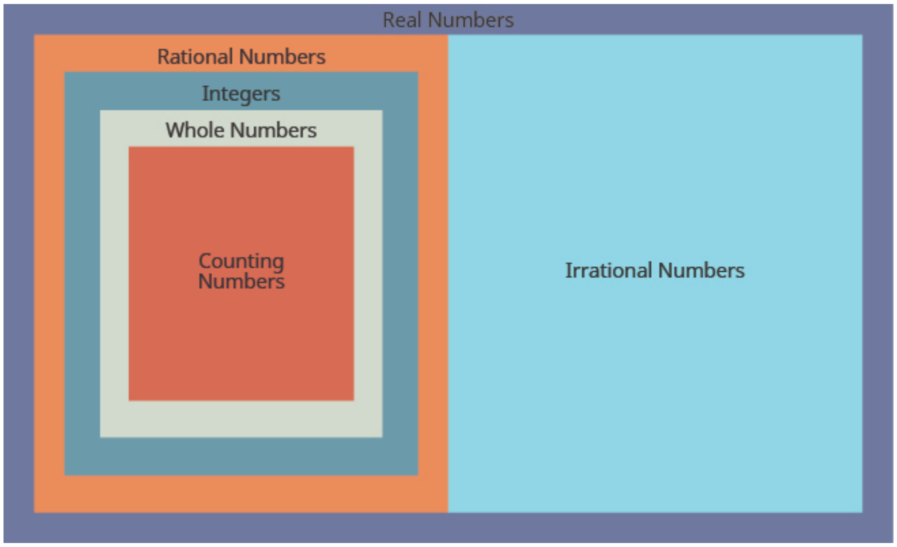
Solution:

ⓐ irrational ⓑ rational

We have seen that all counting numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. The irrational numbers are numbers whose decimal form does not stop and does not repeat. When we put together the rational numbers and the irrational numbers, we get the set of **real numbers**.

Note:
Real Number
A **real number** is a number that is either rational or irrational.

All the numbers we use in elementary algebra are real numbers. [link] illustrates how the number sets we've discussed in this section fit together.



This chart shows the number sets that make up the set of real numbers. Does the term "real numbers" seem strange to you? Are there any numbers that are not "real," and, if so, what could they be?

Can we simplify $\sqrt{-25}$? Is there a number whose square is -25?
Equation:

$$(\ \ )^2 = -25?$$

None of the numbers that we have dealt with so far has a square that is -25. Why? Any positive number squared is positive. Any negative number squared is positive. So we say there is no real number equal to $\sqrt{-25}$.

The square root of a negative number is not a real number.

Example:
Exercise:

Problem:

For each number given, identify whether it is a real number or not a real number: ⓐ $\sqrt{-169}$ ⓑ $-\sqrt{64}$.

Solution:

ⓐ There is no real number whose square is -169. Therefore, $\sqrt{-169}$ is not a real number.
ⓑ Since the negative is in front of the radical, $-\sqrt{64}$ is -8, Since -8 is a real number, $-\sqrt{64}$ is a real number.

Note:
Exercise:

Problem:

For each number given, identify whether it is a real number or not a real number: ⓐ $\sqrt{-196}$ ⓑ $-\sqrt{81}$.

Solution:

ⓐ not a real number ⓑ real number

Note:
Exercise:

Problem:

For each number given, identify whether it is a real number or not a real number: ⓐ $-\sqrt{49}$ ⓑ $\sqrt{-121}$.

Solution:

ⓐ real number ⓑ not a real number

Example:
Exercise:

Problem:

Given the numbers -7, $\frac{14}{5}$, 8, $\sqrt{5}$, 5.9, $-\sqrt{64}$, list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔ real numbers.

Solution:

ⓐ Remember, the whole numbers are 0, 1, 2, 3, … and 8 is the only whole number given.
ⓑ The integers are the whole numbers, their opposites, and 0. So the whole number 8 is an integer, and -7 is the opposite of a whole number so it is an integer, too. Also, notice that 64 is the square of 8 so $-\sqrt{64} = -8$. So the integers are $-7, 8, -\sqrt{64}$.
ⓒ Since all integers are rational, then $-7, 8, -\sqrt{64}$ are rational. Rational numbers also include fractions and decimals that repeat or stop, so $\frac{14}{5}$ and 5.9 are rational. So the list of rational numbers is -7, $\frac{14}{5}$, $8, 5.9, -\sqrt{64}$.
ⓓ Remember that 5 is not a perfect square, so $\sqrt{5}$ is irrational.
ⓔ All the numbers listed are real numbers.

Note:
Exercise:

Problem:

For the given numbers, list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔ real numbers: $-3, -\sqrt{2}, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$.

Solution:

ⓐ $4, \sqrt{49}$ ⓑ $-3, 4, \sqrt{49}$ ⓒ $-3, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$ ⓓ $-\sqrt{2}$ ⓔ $-3, -\sqrt{2}, 0.\overline{3}, \frac{9}{5}, 4, \sqrt{49}$

Note:
Exercise:

Problem:

For the given numbers, list the ⓐ whole numbers ⓑ integers ⓒ rational numbers ⓓ irrational numbers ⓔ real numbers: $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}, 2.041975\ldots$

Solution:

ⓐ $6, \sqrt{121}$ ⓑ $-\sqrt{25}, -1, 6, \sqrt{121}$ ⓒ $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}$ ⓓ $2.041975\ldots$ ⓔ $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}, 2.041975\ldots$

Locate Fractions on the Number Line

The last time we looked at the number line, it only had positive and negative integers on it. We now want to include fractions and decimals on it.

Note: Doing the Manipulative Mathematics activity “Number Line Part 3” will help you develop a better understanding of the location of fractions on the number line.

Let’s start with fractions and locate $\frac{1}{5}, -\frac{4}{5}, 3, \frac{7}{4}, -\frac{9}{2}, -5, \text{ and } \frac{8}{3}$ on the number line.

We’ll start with the whole numbers 3 and -5. because they are the easiest to plot. See [link].

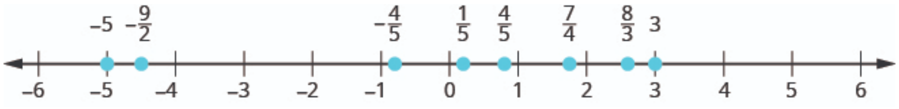
The proper fractions listed are $\frac{1}{5}$ and $-\frac{4}{5}$. We know the proper fraction $\frac{1}{5}$ has value less than one and so would be located between 0 and 1. The denominator is 5, so we divide the unit from 0 to 1 into 5 equal parts $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$. We plot $\frac{1}{5}$. See [link].

Similarly, $-\frac{4}{5}$ is between 0 and -1. After dividing the unit into 5 equal parts we plot $-\frac{4}{5}$. See [link].

Finally, look at the improper fractions $\frac{7}{4}, -\frac{9}{2}, \frac{8}{3}$. These are fractions in which the numerator is greater than the denominator. Locating these points may be easier if you change each of them to a mixed number. See [link].
Equation:

$$\frac{7}{4} = 1\frac{3}{4} \qquad -\frac{9}{2} = -4\frac{1}{2} \qquad \frac{8}{3} = 2\frac{2}{3}$$

[link] shows the number line with all the points plotted.



Example:
Exercise:

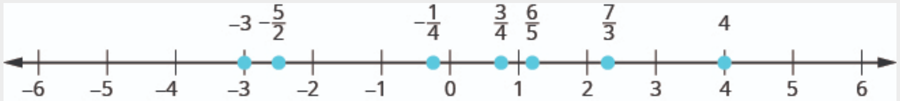
Problem: Locate and label the following on a number line: $4, \frac{3}{4}, -\frac{1}{4}, -3, \frac{6}{5}, -\frac{5}{2},$ and $\frac{7}{3}$.

Solution:

Locate and plot the integers, $4, -3$.

Locate the proper fraction $\frac{3}{4}$ first. The fraction $\frac{3}{4}$ is between 0 and 1. Divide the distance between 0 and 1 into four equal parts then, we plot $\frac{3}{4}$. Similarly plot $-\frac{1}{4}$.

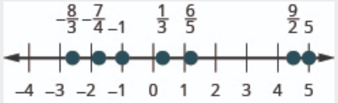
Now locate the improper fractions $\frac{6}{5}, -\frac{5}{2}, \frac{7}{3}$. It is easier to plot them if we convert them to mixed numbers and then plot them as described above: $\frac{6}{5} = 1\frac{1}{5}, -\frac{5}{2} = -2\frac{1}{2}, \frac{7}{3} = 2\frac{1}{3}$.



Note:
Exercise:

Problem: Locate and label the following on a number line: $-1, \frac{1}{3}, \frac{6}{5}, -\frac{7}{4}, \frac{9}{2}, 5, -\frac{8}{3}$.

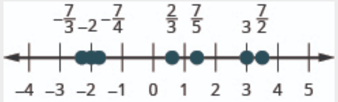
Solution:



Note:
Exercise:

Problem: Locate and label the following on a number line: $-2, \frac{2}{3}, \frac{7}{5}, -\frac{7}{4}, \frac{7}{2}, 3, -\frac{7}{3}$.

Solution:



In [link], we'll use the inequality symbols to order fractions. In previous chapters we used the number line to order numbers.

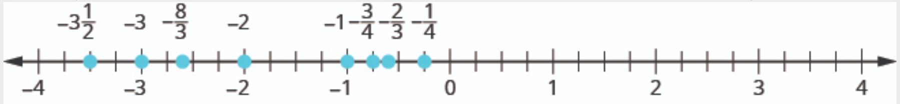
- $a < b$ "a is less than b" when a is to the left of b on the number line
- $a > b$ "a is greater than b" when a is to the right of b on the number line

As we move from left to right on a number line, the values increase.

Example:
Exercise:

Problem: Order each of the following pairs of numbers, using < or >. It may be helpful to refer [link].

ⓐ $-\frac{2}{3}$ ___ -1 ⓑ $-3\frac{1}{2}$ ___ -3 ⓒ $-\frac{3}{4}$ ___ $-\frac{1}{4}$ ⓓ -2 ___ $-\frac{8}{3}$



Solution:

Be careful when ordering negative numbers.

ⓐ

$-\frac{2}{3}$ ___ -1

$-\frac{2}{3}$ is to the right of -1 on the number line. $-\frac{2}{3} > -1$

ⓑ

$-3\frac{1}{2}$ ___ -3

$-3\frac{1}{2}$ is to the left of -3 on the number line. $-3\frac{1}{2} < -3$

ⓒ

$-\frac{3}{4}$ ___ $-\frac{1}{4}$

$-\frac{3}{4}$ is to the left of $-\frac{1}{4}$ on the number line. $-\frac{3}{4} < -\frac{1}{4}$

ⓓ

-2 ___ $-\frac{8}{3}$

-2 is to the right of $-\frac{8}{3}$ on the number line. $-2 > -\frac{8}{3}$

Note:
Exercise:

Problem: Order each of the following pairs of numbers, using < or >:

ⓐ $-\frac{1}{3}$ ___ -1 ⓑ $-1\frac{1}{2}$ ___ -2 ⓒ $-\frac{2}{3}$ ___ $-\frac{1}{3}$ ⓓ -3 ___ $-\frac{7}{3}$.

Solution:

ⓐ > ⓑ > ⓒ < ⓓ <

Note:
Exercise:

Problem: Order each of the following pairs of numbers, using < or >:

ⓐ -1 ___ $-\frac{2}{3}$ ⓑ $-2\frac{1}{4}$ ___ -2 ⓒ $-\frac{3}{5}$ ___ $-\frac{4}{5}$ ⓓ -4 ___ $-\frac{10}{3}$.

Solution:

ⓐ < ⓑ < ⓒ > ⓓ <

Locate Decimals on the Number Line

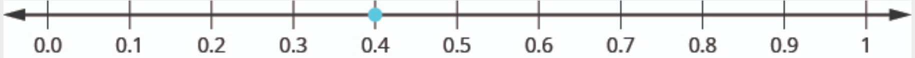
Since decimals are forms of fractions, locating decimals on the number line is similar to locating fractions on the number line.

Example:
Exercise:

Problem: Locate 0.4 on the number line.

Solution:

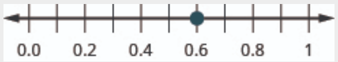
A proper fraction has value less than one. The decimal number 0.4 is equivalent to $\frac{4}{10}$, a proper fraction, so 0.4 is located between 0 and 1. On a number line, divide the interval between 0 and 1 into 10 equal parts. Now label the parts 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. We write 0 as 0.0 and 1 and 1.0, so that the numbers are consistently in tenths. Finally, mark 0.4 on the number line. See [link].



Note:
Exercise:

Problem: Locate on the number line: 0.6.

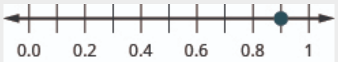
Solution:



Note:
Exercise:

Problem: Locate on the number line: 0.9.

Solution:

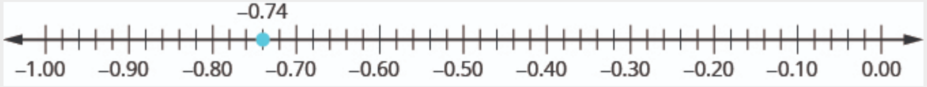


Example:
Exercise:

Problem: Locate -0.74 on the number line.

Solution:

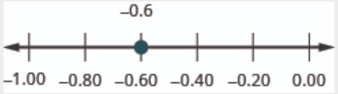
The decimal -0.74 is equivalent to $-\frac{74}{100}$, so it is located between 0 and -1. On a number line, mark off and label the hundredths in the interval between 0 and -1. See [link].



Note:
Exercise:

Problem: Locate on the number line: -0.6.

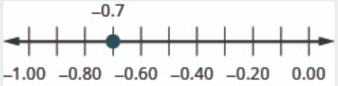
Solution:



Note:
Exercise:

Problem: Locate on the number line: -0.7.

Solution:



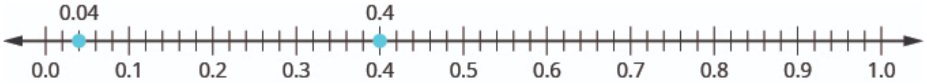
Which is larger, 0.04 or 0.40? If you think of this as money, you know that \$0.40 (forty cents) is greater than \$0.04 (four cents). So,

$0.40 > 0.04$

Again, we can use the number line to order numbers.

- $a < b$ "a is less than b" when a is to the left of b on the number line
- $a > b$ "a is greater than b" when a is to the right of b on the number line

Where are 0.04 and 0.40 located on the number line? See [link].



We see that 0.40 is to the right of 0.04 on the number line. This is another way to demonstrate that 0.40 > 0.04.

How does 0.31 compare to 0.308? This doesn't translate into money to make it easy to compare. But if we convert 0.31 and 0.308 into fractions, we can tell which is larger.

	0.31	0.308
Convert to fractions.	$\frac{31}{100}$	$\frac{308}{1000}$
We need a common denominator to compare them.	$\frac{31 \cdot 10}{100 \cdot 10}$	$\frac{308}{1000}$
	$\frac{310}{1000}$	$\frac{308}{1000}$

Because 310 > 308, we know that $\frac{310}{1000} > \frac{308}{1000}$. Therefore, 0.31 > 0.308.

Notice what we did in converting 0.31 to a fraction—we started with the fraction $\frac{31}{100}$ and ended with the equivalent fraction $\frac{310}{1000}$. Converting $\frac{310}{1000}$ back to a decimal gives 0.310. So 0.31 is equivalent to 0.310. Writing zeros at the end of a decimal does not change its value!

Equation:

$$\frac{31}{100} = \frac{310}{1000} \quad \text{and} \quad 0.31 = 0.310$$

We say 0.31 and 0.310 are **equivalent decimals**.

Note:
Equivalent Decimals
Two decimals are equivalent if they convert to equivalent fractions.

We use equivalent decimals when we order decimals.

The steps we take to order decimals are summarized here.

Note:
Order Decimals.

Write the numbers one under the other, lining up the decimal points.
Check to see if both numbers have the same number of digits. If not, write zeros at the end of the one with fewer digits to make them match.
Compare the numbers as if they were whole numbers.
Order the numbers using the appropriate inequality sign.

Example:
Exercise:

Problem: Order 0.64 ___ 0.6 using $<$ or $>$.

Solution:
Solution

Write the numbers one under the other, lining up the decimal points.	0.64 0.6
Add a zero to 0.6 to make it a decimal with 2 decimal places. Now they are both hundredths.	0.64 0.60
64 is greater than 60.	$64 > 60$
64 hundredths is greater than 60 hundredths.	$0.64 > 0.60$
	$0.64 > 0.6$

Note:
Exercise:

Problem: Order each of the following pairs of numbers, using $<$ or $>$: 0.42 ___ 0.4.

Solution:

$>$

Note:
Exercise:

Problem: Order each of the following pairs of numbers, using $<$ or $>$: 0.18 ___ 0.1.

Solution:

$>$

Example:
Exercise:

Problem: Order 0.83 ___ 0.803 using $<$ or $>$.

Solution:
Solution

	0.83 ___ 0.803
Write the numbers one under the other, lining up the decimals.	0.83 0.803
They do not have the same number of digits. Write one zero at the end of 0.83.	0.830 0.803
Since $830 > 803$, 830 thousandths is greater than 803 thousandths.	$0.830 > 0.803$
	$0.83 > 0.803$

Note:
Exercise:

Problem: Order the following pair of numbers, using $<$ or $>$: 0.76 ___ 0.706.

Solution:

$>$

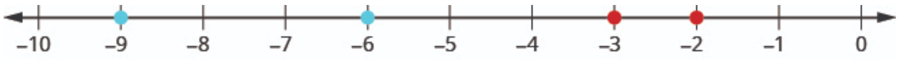
Note:
Exercise:

Problem: Order the following pair of numbers, using $<$ or $>$: 0.305 ___ 0.35.

Solution:

$<$

When we order negative decimals, it is important to remember how to order negative integers. Recall that larger numbers are to the right on the number line. For example, because -2 lies to the right of -3 on the number line, we know that $-2 > -3$. Similarly, smaller numbers lie to the left on the number line. For example, because -9 lies to the left of -6 on the number line, we know that $-9 < -6$. See [link].



If we zoomed in on the interval between 0 and -1, as shown in [link], we would see in the same way that $-0.2 > -0.3$ and $-0.9 < -0.6$.

Example:
Exercise:

Problem: Use $<$ or $>$ to order -0.1 ___ -0.8.

Solution:
Solution

	-0.1 ___ -0.8
Write the numbers one under the other, lining up the decimal points. They have the same number of digits.	-0.1 -0.8
Since $-1 > -8$, -1 tenth is greater than -8 tenths.	$-0.1 > -0.8$

Note:
Exercise:

Problem: Order the following pair of numbers, using < or >: -0.3 ___ -0.5.

Solution:

>

Note:
Exercise:

Problem: Order the following pair of numbers, using < or >: -0.6 ___ -0.7.

Solution:

>

Key Concepts

- **Square Root Notation**
  $\sqrt{m}$ is read 'the square root of m.' If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.
- **Order Decimals**

  Write the numbers one under the other, lining up the decimal points.
  Check to see if both numbers have the same number of digits. If not, write zeros at the end of the one with fewer digits to make them match.
  Compare the numbers as if they were whole numbers.
  Order the numbers using the appropriate inequality sign.

Practice Makes Perfect

Simplify Expressions with Square Roots

In the following exercises, simplify.
Exercise:

Problem: $\sqrt{36}$

Solution:

6

Exercise:

Problem: $\sqrt{4}$

Exercise:

Problem: $\sqrt{64}$

Solution:

8

Exercise:

Problem: $\sqrt{169}$

Exercise:

Problem: $\sqrt{9}$

Solution:

3

Exercise:

Problem: $\sqrt{16}$

Exercise:

Problem: $\sqrt{100}$

Solution:

10

Exercise:

Problem: $\sqrt{144}$

Exercise:

Problem: $-\sqrt{4}$

Solution:

-2

Exercise:

Problem: $-\sqrt{100}$

Exercise:

Problem: $-\sqrt{1}$

Solution:

-1

Exercise:

Problem: $-\sqrt{121}$

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

In the following exercises, write as the ratio of two integers.

Exercise:

Problem: ⓐ 5 ⓑ 3.19

Solution:

ⓐ $\frac{5}{1}$ ⓑ $\frac{319}{100}$

Exercise:

Problem: ⓐ 8 ⓑ 1.61

Exercise:

Problem: ⓐ -12 ⓑ 9.279

Solution:

ⓐ $\frac{-12}{1}$ ⓑ $\frac{9279}{1000}$

Exercise:

Problem: ⓐ -16 ⓑ 4.399

In the following exercises, list the ⓐ rational numbers, ⓑ irrational numbers

Exercise:

Problem: $0.75, 0.22\bar{3}, 1.39174$

Solution:

ⓐ $0.75, 0.22\bar{3}$ ⓑ $1.39174...$

Exercise:

Problem: $0.36, 0.94729..., 2.52\bar{8}$

Exercise:

Problem: $0.4\bar{5}, 1.919293..., 3.59$

Solution:

ⓐ $0.4\bar{5}, 3.59$ ⓑ $1.919293...$

Exercise:

Problem: $0.1\bar{3}, 0.42982..., 1.875$

In the following exercises, identify whether each number is rational or irrational.

Exercise:

Problem: ⓐ $\sqrt{25}$ ⓑ $\sqrt{30}$

Solution:

ⓐ rational ⓑ irrational

Exercise:

Problem: ⓐ $\sqrt{44}$ ⓑ $\sqrt{49}$

Exercise:

Problem: ⓐ $\sqrt{164}$ ⓑ $\sqrt{169}$

Solution:

ⓐ irrational ⓑ rational

Exercise:

Problem: ⓐ $\sqrt{225}$ ⓑ $\sqrt{216}$

In the following exercises, identify whether each number is a real number or not a real number.

Exercise:

Problem: ⓐ $-\sqrt{81}$ ⓑ $\sqrt{-121}$

Solution:

ⓐ real number ⓑ not a real number

Exercise:

Problem: ⓐ $-\sqrt{64}$ ⓑ $\sqrt{-9}$

Exercise:

Problem: ⓐ $\sqrt{-36}$ ⓑ $-\sqrt{144}$

Solution:

ⓐ not a real number ⓑ real number

Exercise:

Problem: ⓐ $\sqrt{-49}$ ⓑ $-\sqrt{144}$

In the following exercises, list the ⓐ whole numbers, ⓑ integers, ⓒ rational numbers, ⓓ irrational numbers, ⓔ real numbers for each set of numbers.

Exercise:

Problem: $-8, 0, 1.95286\ldots, \frac{12}{5}, \sqrt{36}, 9$

Solution:

ⓐ $0, \sqrt{36}, 9$ ⓑ $-8, \sqrt{36}, 9$ ⓒ $-8, 0, \frac{12}{5}, \sqrt{36}, 9$ ⓓ $1.95286\ldots$ ⓔ $-8, 0, 1.95286\ldots, \frac{12}{5}, \sqrt{36}, 9$

Exercise:

Problem: $-9, -3\frac{4}{9}, -\sqrt{9}, 0.40\overline{9}, \frac{11}{6}, 7$

Exercise:

Problem: $-\sqrt{100}, -7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$

Solution:

ⓐ none ⓑ $-\sqrt{100}, -7, -1$ ⓒ $-\sqrt{100}, -7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$ ⓓ none ⓔ $-\sqrt{100}, -7, -\frac{8}{3}, -1, 0.77, 3\frac{1}{4}$

Exercise:

Problem: $-6, -\frac{5}{2}, 0, 0.7\overline{1428}5, 2\frac{1}{5}, \sqrt{14}$

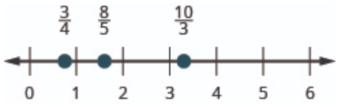
Locate Fractions on the Number Line

In the following exercises, locate the numbers on a number line.

Exercise:

Problem: $\frac{3}{4}, \frac{8}{5}, \frac{10}{3}$

Solution:



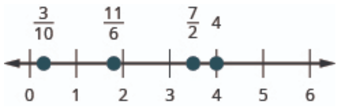
Exercise:

Problem: $\frac{1}{4}, \frac{9}{5}, \frac{11}{3}$

Exercise:

Problem: $\frac{3}{10}, \frac{7}{2}, \frac{11}{6}, 4$

Solution:



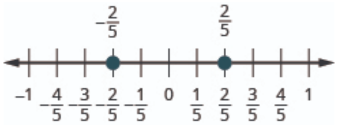
Exercise:

Problem: $\frac{7}{10}, \frac{5}{2}, \frac{13}{8}, 3$

Exercise:

Problem: $\frac{2}{5}, -\frac{2}{5}$

Solution:



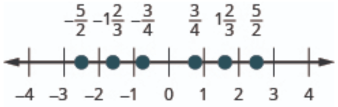
Exercise:

Problem: $\frac{3}{4}, -\frac{3}{4}$

Exercise:

Problem: $\frac{3}{4}, -\frac{3}{4}, 1\frac{2}{3}, -1\frac{2}{3}, \frac{5}{2}, -\frac{5}{2}$

Solution:



Exercise:

Problem: $\frac{1}{5}, -\frac{2}{5}, 1\frac{3}{4}, -1\frac{3}{4}, \frac{8}{3}, -\frac{8}{3}$

In the following exercises, order each of the pairs of numbers, using < or >.

Exercise:

Problem: -1 ___ $-\frac{1}{4}$

Solution:

<

Exercise:

Problem: -1 ___ $-\frac{1}{3}$

Exercise:

Problem: $-2\frac{1}{2}$ ___ -3

Solution:

>

Exercise:

Problem: $-1\frac{3}{4}$ ___ -2

Exercise:

Problem: $-\frac{5}{12}$ ___ $-\frac{7}{12}$

Solution:

$>$

Exercise:

Problem: $-\frac{9}{10}$ ___ $-\frac{3}{10}$

Exercise:

Problem: -3 ___ $-\frac{13}{5}$

Solution:

$<$

Exercise:

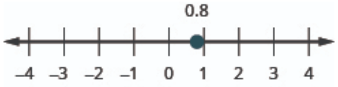
Problem: -4 ___ $-\frac{23}{6}$

Locate Decimals on the Number Line In the following exercises, locate the number on the number line.

Exercise:

Problem: 0.8

Solution:



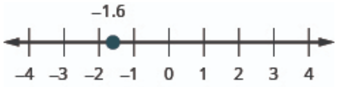
Exercise:

Problem: -0.9

Exercise:

Problem: -1.6

Solution:



Exercise:

Problem: 3.1

In the following exercises, order each pair of numbers, using < or >.

Exercise:

Problem: 0.37 ___ 0.63

Solution:

$<$

Exercise:

Problem: 0.86 ___ 0.69

Exercise:

Problem: 0.91 ___ 0.901

Solution:

$>$

Exercise:

Problem: 0.415 ___ 0.41

Exercise:

Problem: -0.5 ___ -0.3

Solution:

$<$

Exercise:

Problem: -0.1 ___ -0.4

Exercise:

Problem: -0.62 ___ -0.619

Solution:

$<$

Exercise:

Problem: -7.31 ___ -7.3

Everyday Math

Exercise:

Problem:

Field trip All the 5th graders at Lincoln Elementary School will go on a field trip to the science museum. Counting all the children, teachers, and chaperones, there will be 147 people. Each bus holds 44 people.

ⓐ How many busses will be needed?
ⓑ Why must the answer be a whole number?
ⓒ Why shouldn’t you round the answer the usual way, by choosing the whole number closest to the exact answer?

Solution:

ⓐ 4 busses ⓑ answers may vary ⓒ answers may vary

Exercise:

Problem:

Child care Serena wants to open a licensed child care center. Her state requires there be no more than 12 children for each teacher. She would like her child care center to serve 40 children.

ⓐ How many teachers will be needed?
ⓑ Why must the answer be a whole number?
ⓒ Why shouldn’t you round the answer the usual way, by choosing the whole number closest to the exact answer?

Writing Exercises

Exercise:

Problem: In your own words, explain the difference between a rational number and an irrational number.

Solution:

Answers may vary

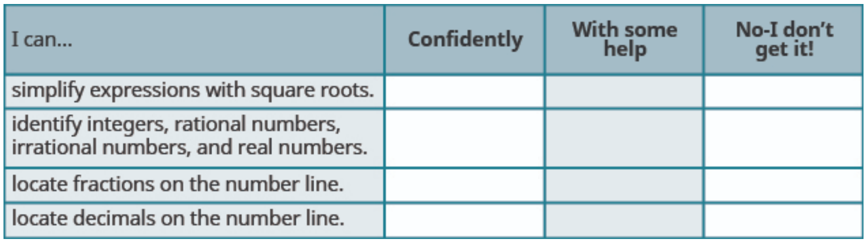
Exercise:

Problem:

Explain how the sets of numbers (counting, whole, integer, rational, irrationals, reals) are related to each other.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objective of this section.



I can…	Confidently	With some help	No-I don’t get it!
simplify expressions with square roots.			
identify integers, rational numbers, irrational numbers, and real numbers.			
locate fractions on the number line.			
locate decimals on the number line.			

ⓑ On a scale of $1-10$, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

equivalent decimals

Two decimals are equivalent if they convert to equivalent fractions.

irrational number

An irrational number is a number that cannot be written as the ratio of two integers. Its decimal form does not stop and does not repeat.

rational number

A rational number is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. A rational number can be written as the ratio of two integers. Its decimal form stops or repeats.

radical sign

A radical sign is the symbol $\sqrt{m}$ that denotes the positive square root.

real number

A real number is a number that is either rational or irrational.

square and square root

If $n^2 = m$, then m is the square of n and n is a square root of m.

Converting Units

In this modules, students will learn how to convert units, using the method of dimensional analysis.

Dimensional analysis, also known as unit analysis, is a systematic method from converting from one kind of unit of measurement to another. It is used extensively in chemistry and other science related fields. It is an extremely valuable skill to learn, especially when solving complicated application problems.

To convert from one unit of measurement to another (like inches to feet) using dimensional analysis, we use what are called **unit ratios**. Unit ratios are ratios (or fractions) which are equivalent to 1. Remember that a fraction is equivalent to 1 when the numerator and denominator are the same. With unit ratios, the numerator and denominator represent the same measurement, but with different units.

Here are some examples of unit ratios involving units of length:

$$\frac{12in}{1ft}, \frac{1ft}{12in}, \frac{1yd}{3ft}, \frac{3ft}{1yd}$$

These ratios are created from the conversion facts that there are 12 inches in a foot and 3 feet in a yard.

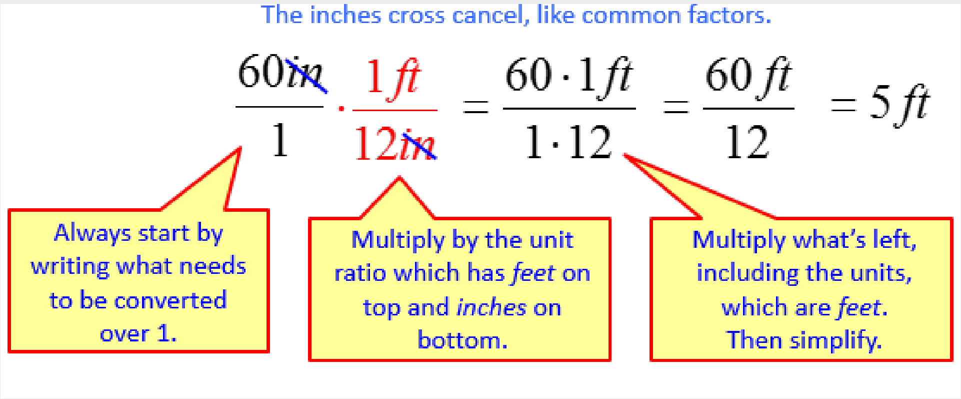
Notice that each conversion fact above gives two possible unit ratios. Deciding which one to use will depend on where units need to be placed in order to cross cancel. In dimensional analysis, we multiply ratios together and cancel common units the same way we cancel common factors when multiplying fractions.

Refer to the U.S. Standard Measuring System for conversion facts that will be used in the following examples and homework problems.

Example:
Exercise:

Problem: Convert 60 inches into feet.

Solution:



The inches cross cancel, like common factors.

$$\frac{60\cancel{in}}{1}\cdot\frac{1\,ft}{12\cancel{in}} = \frac{60\cdot 1\,ft}{1\cdot 12} = \frac{60\,ft}{12} = 5\,ft$$

Always start by writing what needs to be converted over 1.

Multiply by the unit ratio which has *feet* on top and *inches* on bottom.

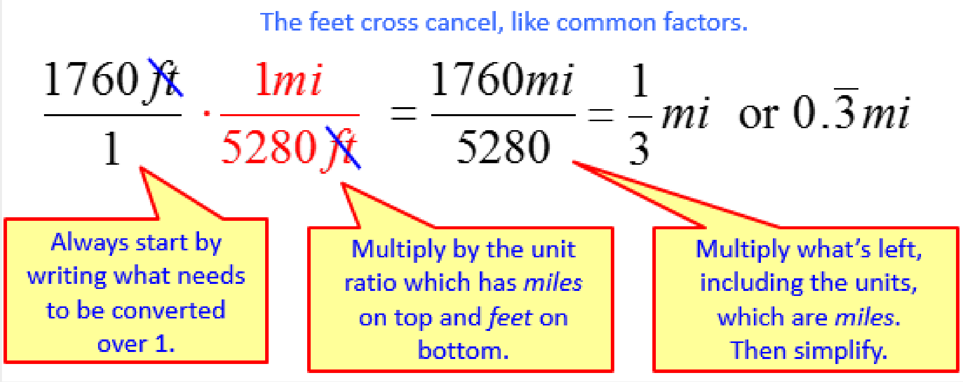
Multiply what's left, including the units, which are *feet*. Then simplify.

Notice that we start by writing what we are converting as a ratio, by placing it over a 1. This is similar to writing a whole number as a fraction when we want to multiply it times a fraction.

Example:
Exercise:

Problem: Convert 1760 feet into miles.

Solution:



The feet cross cancel, like common factors.

$$\frac{1760\,\cancel{ft}}{1}\cdot\frac{1mi}{5280\,\cancel{ft}} = \frac{1760mi}{5280} = \frac{1}{3}mi \text{ or } 0.\overline{3}mi$$

Always start by writing what needs to be converted over 1.

Multiply by the unit ratio which has *miles* on top and *feet* on bottom.

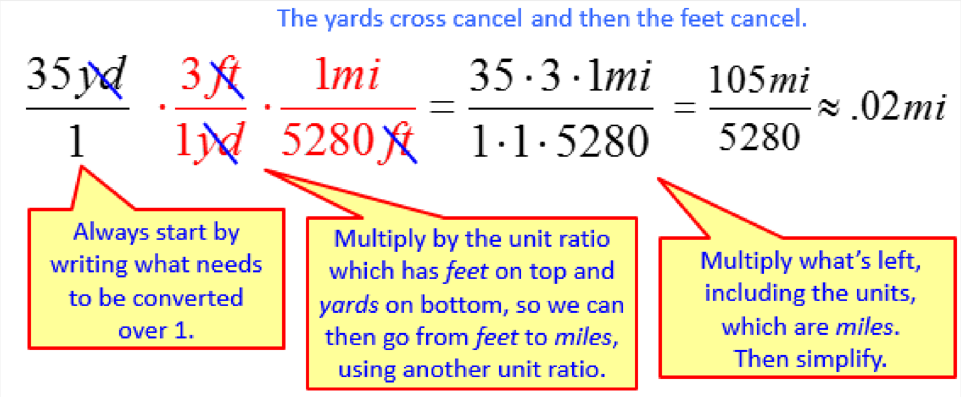
Multiply what's left, including the units, which are *miles*. Then simplify.

In the two previous examples, only one unit ratio was needed to make the conversion. However, in most cases, we need more than one unit ratio. In those situations, we have to consider the conversion facts that are available and then make a plan.

Example:
Exercise:

Problem: Convert 35 yards into miles.

Solution:



The yards cross cancel and then the feet cancel.

$$\frac{35\cancel{yd}}{1}\cdot\frac{3\cancel{ft}}{1\cancel{yd}}\cdot\frac{1mi}{5280\cancel{ft}}=\frac{35\cdot3\cdot1mi}{1\cdot1\cdot5280}=\frac{105mi}{5280}\approx .02mi$$

Always start by writing what needs to be converted over 1.

Multiply by the unit ratio which has *feet* on top and *yards* on bottom, so we can then go from *feet* to *miles*, using another unit ratio.

Multiply what's left, including the units, which are *miles*. Then simplify.

Since there are no conversion facts on our list of facts to memorize that relate yards and miles, we had to use two unit ratios. First, we converted yards to feet, using the fact that there are 3 feet in a yard. Then, we converted the feet to miles, using the fact that there are 5280 feet in a mile.

Note:
Try It
Exercise:

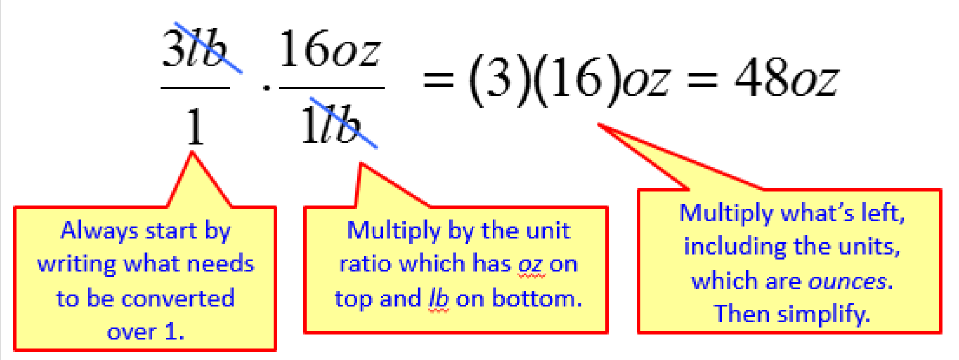
Problem: Use dimensional analysis to convert 0.75 miles into inches.

Solution:

$$\frac{0.75\,\cancel{mi}}{1}\cdot\frac{5280\,\cancel{ft}}{1\,\cancel{mi}}\cdot\frac{12in}{1\,\cancel{ft}}=47{,}520\,in$$

Example:

Convert 3 pounds into ounces.



$$\frac{3\cancel{lb}}{1}\cdot\frac{16oz}{1\cancel{lb}}=(3)(16)oz=48oz$$

Always start by writing what needs to be converted over 1.

Multiply by the unit ratio which has *oz* on top and *lb* on bottom.

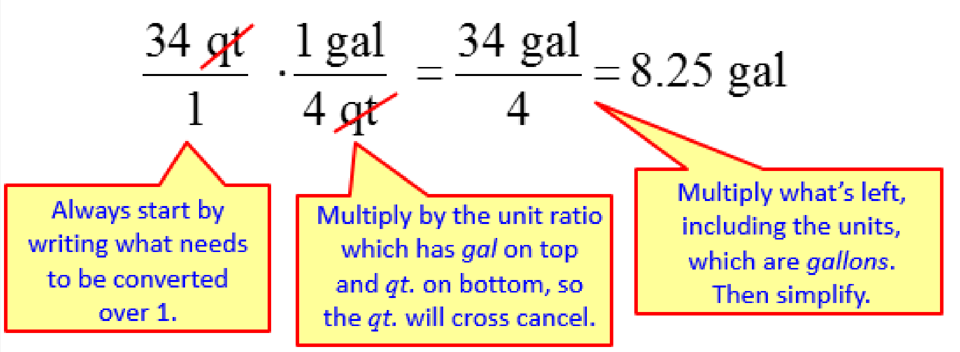
Multiply what's left, including the units, which are *ounces*. Then simplify.

Example:

Exercise:

Problem: How many gallons are in 34 quarts?

Solution:



$$\frac{34\ \cancel{qt}}{1}\cdot\frac{1\ \text{gal}}{4\ \cancel{qt}}=\frac{34\ \text{gal}}{4}=8.25\ \text{gal}$$

Always start by writing what needs to be converted over 1.

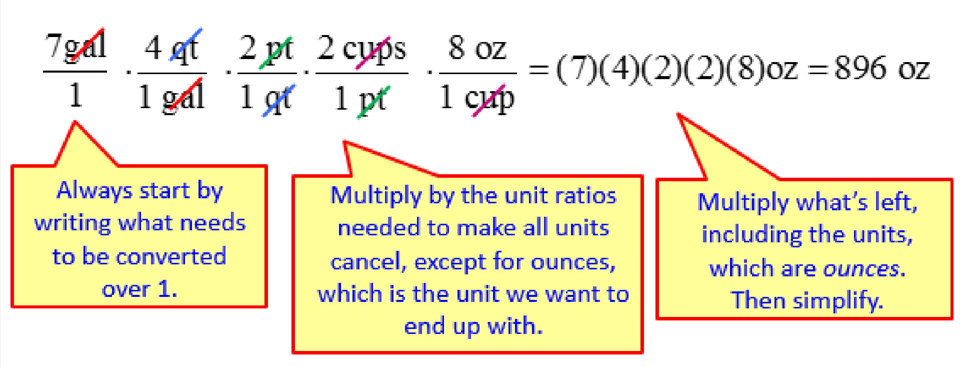
Multiply by the unit ratio which has *gal* on top and *qt.* on bottom, so the *qt.* will cross cancel.

Multiply what's left, including the units, which are *gallons*. Then simplify.

Example:
Exercise:

Problem: How many ounces are in 7 gallons?

Solution:



$$\frac{7\cancel{gal}}{1}\cdot\frac{4\cancel{qt}}{1\cancel{gal}}\cdot\frac{2\cancel{pt}}{1\cancel{qt}}\cdot\frac{2\cancel{cups}}{1\cancel{pt}}\cdot\frac{8\text{ oz}}{1\cancel{cup}}=(7)(4)(2)(2)(8)\text{oz}=896\text{ oz}$$

Always start by writing what needs to be converted over 1.

Multiply by the unit ratios needed to make all units cancel, except for ounces, which is the unit we want to end up with.

Multiply what's left, including the units, which are *ounces*. Then simplify.

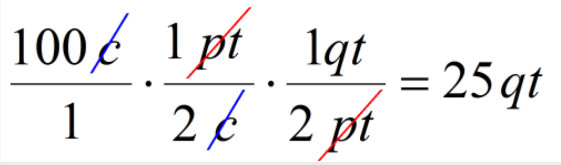
Notice that several unit ratios are needed here. The first one has gallons in the denominator so that the gallons will cross cancel. Then, a second one has quarts in the denominator so that the quarts will cross cancel. This process continues until all units cancel except for the units that you are converting to.

Note:
Try It
Exercise:

Problem:

Use dimension analysis to answer the question: How many quarts are there in 100 cups?

Solution:



$$\frac{100\cancel{c}}{1}\cdot\frac{1\cancel{pt}}{2\cancel{c}}\cdot\frac{1qt}{2\cancel{pt}}=25qt$$

Homework

Use dimensional analysis to complete the following homework exercises. Even though only final answers will be shown in the given solutions, be sure to write down the process, just as it has been shown to you.

Exercise:

Problem: Convert 22.5 yards to inches.

Solution:

810 inches

Exercise:

Problem: Convert 3 gallons to pints.

Exercise:

Problem: How many pounds are in 500 ounces?

Solution:

31.25 lb.

Exercise:

Problem: How many miles are in 26,928 ft?

Exercise:

Problem: Convert 1000 yards into miles.

Solution:

Approximately, 0.57 miles.

Exercise:

Problem: Convert 2.5 quarts to fl. oz.

Introduction class="introduction"

The rocks in this formation must remain perfectly balanced around the center for the formation to hold its shape.

If we carefully placed more rocks of equal weight on both sides of this formation, it would still balance. Similarly, the expressions in an equation remain balanced when we add the same quantity to both sides of the equation. In this chapter, we will solve equations, remembering that what we do to one side of the equation, we must also do to the other side.

Solve Equations Using the Subtraction and Addition Properties of Equality
By the end of this section, you will be able to:

- Verify a solution of an equation
- Solve equations using the Subtraction and Addition Properties of Equality
- Solve equations that require simplification
- Translate to an equation and solve
- Translate and solve applications

Note:
Before you get started, take this readiness quiz.

1. Evaluate $x+4$ when $x=-3$.
   If you missed this problem, review [link].
2. Evaluate $15-y$ when $y=-5$.
   If you missed this problem, review [link].
3. Simplify $4(4n+1)-15n$.
   If you missed this problem, review [link].
4. Translate into algebra "5 is less than x."
   If you missed this problem, review [link].

Verify a Solution of an Equation

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that make each side of the equation the same – so that we end up with a true statement. Any value of the variable that makes the equation true is called a solution to the equation. It is the answer to the puzzle!

Note:
Solution of an equation
A **solution of an equation** is a value of a variable that makes a true statement when substituted into the equation.

Note:
To determine whether a number is a solution to an equation.

Substitute the number in for the variable in the equation.
Simplify the expressions on both sides of the equation.
Determine whether the resulting equation is true (the left side is equal to the right side)

  - If it is true, the number is a solution.
  - If it is not true, the number is not a solution.

Example:
Exercise:

Problem: Determine whether $x=\frac{3}{2}$ is a solution of $4x-2=2x+1$.

Solution:
Solution

Since a solution to an equation is a value of the variable that makes the equation true, begin by substituting the value of the solution for the variable.

	$4x-2=2x+1$
Substitute $\frac{3}{2}$ for x.	$4\left(\frac{3}{2}\right)-3\overset{?}{=}2\left(\frac{3}{2}\right)+1$
Multiply.	$6-2\overset{?}{=}3+1$
Subtract.	$4=4\checkmark$

Since $x=\frac{3}{2}$ results in a true equation (4 is in fact equal to 4), $\frac{3}{2}$ is a solution to the equation $4x-2=2x+1$.

Note:
Exercise:

Problem: Is $y=\frac{4}{3}$ a solution of $9y+2=6y+3$?

Solution:

no

Note:
Exercise:

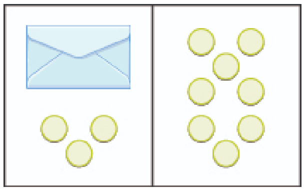
Problem: Is $y=\frac{7}{5}$ a solution of $5y+3=10y-4$?

Solution:

yes

Solve Equations Using the Subtraction and Addition Properties of Equality

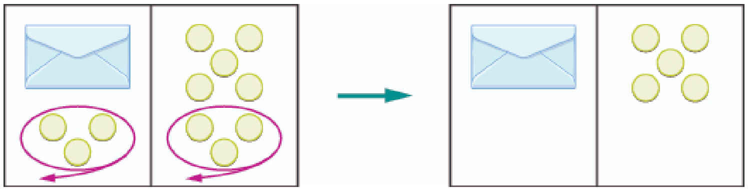
We are going to use a model to clarify the process of solving an equation. An envelope represents the variable – since its contents are unknown – and each counter represents one. We will set out one envelope and some counters on our workspace, as shown in [link]. Both sides of the workspace have the same number of counters, but some counters are “hidden” in the envelope. Can you tell how many counters are in the envelope?



The illustration shows a model of an equation with one variable. On the left side of the workspace is an unknown (envelope) and three counters, while on the right side of the workspace are eight counters.

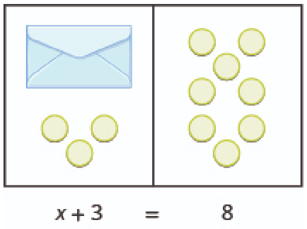
What are you thinking? What steps are you taking in your mind to figure out how many counters are in the envelope?

Perhaps you are thinking: “I need to remove the 3 counters at the bottom left to get the envelope by itself. The 3 counters on the left can be matched with 3 on the right and so I can take them away from both sides. That leaves five on the right—so there must be 5 counters in the envelope.” See [link] for an illustration of this process.



The illustration shows a model for solving an equation with one variable. On both sides of the workspace remove three counters, leaving only the unknown (envelope) and five counters on the right side. The unknown is equal to five counters.

What algebraic equation would match this situation? In [link] each side of the workspace represents an expression and the center line takes the place of the equal sign. We will call the contents of the envelope x.



The illustration shows a model for the equation $x+3=8$.

Let's write algebraically the steps we took to discover how many counters were in the envelope:

	$x+3=8$
First, we took away three from each side.	$x+3-3=8-3$
Then we were left with five.	$x=5$

Check:

Five in the envelope plus three more does equal eight!
Equation:

$$5+3=8$$

Our model has given us an idea of what we need to do to solve one kind of equation. The goal is to isolate the variable by itself on one side of the equation. To solve equations such as these mathematically, we use the **Subtraction Property of Equality**.

Note:
Subtraction Property of Equality
For any numbers *a*, *b*, and *c*,
Equation:

$$\begin{array}{lrcl}\text{If} & a & = & b, \\ \text{then} & a-c & = & b-c\end{array}$$

When you subtract the same quantity from both sides of an equation, you still have equality.

Note: Doing the Manipulative Mathematics activity “Subtraction Property of Equality” will help you develop a better understanding of how to solve equations by using the Subtraction Property of Equality.

Let’s see how to use this property to solve an equation. Remember, the goal is to isolate the variable on one side of the equation. And we check our solutions by substituting the value into the equation to make sure we have a true statement.

Example:
Exercise:

Problem: Solve: $y+37=-13$.

Solution:
Solution

To get *y* by itself, we will undo the addition of 37 by using the Subtraction Property of Equality.

	$y+37=-13$
Subtract 37 from each side to ‘undo’ the addition.	

		$y+37-37=-13-37$
Simplify.		$y=-50$
Check:	$y+37=-13$	
Substitute $y=-50$	$-50+37=-13$	
	$-13\overset{?}{=}-13$ ✓	

Since $y=-50$ makes $y+37=-13$ a true statement, we have the solution to this equation.

Note:
Exercise:

Problem: Solve: $x+19=-27$.

Solution:

$x=-46$

Note:
Exercise:

Problem: Solve: $x+16=-34$.

Solution:

$x=-50$

What happens when an equation has a number subtracted from the variable, as in the equation $x-5=8$? We use another property of equations to solve equations where a number is subtracted from the variable. We want to isolate the variable, so to ‘undo’ the subtraction we will add the number to both sides. We use the **Addition Property of Equality**.

Note:
Addition Property of Equality

For any numbers a, b, and c,

Equation:

$$\begin{array}{lrcl} \text{If} & a & = & b, \\ \text{then} & a+c & = & b+c \end{array}$$

When you add the same quantity to both sides of an equation, you still have equality.

In [link], 37 was added to the *y* and so we subtracted 37 to ‘undo’ the addition. In [link], we will need to ‘undo’ subtraction by using the Addition Property of Equality.

Example:

Exercise:

Problem: Solve: $a - 28 = -37$.

Solution:

Solution

		$a - 28 = -37$
Add 28 to each side to ‘undo’ the subtraction.		$a - 28 + 28 = -37 + 28$
Simplify.		$a = -9$
Check:	$a - 28 = -37$	
Substitute $a = -9$	$-9 - 28 = -37$	
	$-37 \overset{?}{=} -37$ ✓	
		The solution to $a - 28 = -37$ is $a = -9$.

Note:
Exercise:

Problem: Solve: $n - 61 = -75$.

Solution:

$n = -14$

Note:
Exercise:

Problem: Solve: $p - 41 = -73$.

Solution:

$p = -32$

Example:
Exercise:

Problem: Solve: $x - \frac{5}{8} = \frac{3}{4}$.

Solution:
Solution

	$x - \frac{5}{8} = \frac{3}{4}$
Use the Addition Property of Equality.	$x - \frac{5}{8} + \frac{5}{8} = \frac{3}{4} + \frac{5}{8}$
Find the LCD to add the fractions on the right.	$x - \frac{5}{8} + \frac{5}{8} = \frac{6}{8} + \frac{5}{8}$
Simplify.	$x = \frac{11}{8}$

Check:	$x - \frac{5}{8} = \frac{3}{4}$	
Substitute $x = \frac{11}{8}$.	$\frac{11}{8} - \frac{5}{8} \overset{?}{=} \frac{3}{4}$	
Subtract.	$\frac{6}{8} \overset{?}{=} \frac{3}{4}$	
Simplify.	$\frac{3}{4} = \frac{3}{4}\checkmark$	
		The solution to $x - \frac{5}{8} = \frac{3}{4}$ is $x = \frac{11}{8}$.

Note:
Exercise:

Problem: Solve: $p - \frac{2}{3} = \frac{5}{6}$.

Solution:

$p = \frac{3}{2}$

Note:
Exercise:

Problem: Solve: $q - \frac{1}{2} = \frac{5}{6}$.

Solution:

$q = \frac{4}{3}$

The next example will be an equation with decimals.

Example:
Exercise:

Problem: Solve: $n - 0.63 = -4.2$.

Solution:
Solution

		$n - 0.63 = -4.2$
Use the Addition Property of Equality.		$n - 0.63 + 0.63 = -4.2 + 0.63$
Add.		$n = -3.57$
Check:	$n = -3.57$	
Let $n = -3.57$.	$-3.57 - 0.63 \stackrel{?}{=} -4.2$	
	$-4.2 = -4.2\ ✓$	

Note:
Exercise:

Problem: Solve: $b - 0.47 = -2.1$.

Solution:

$b = -1.63$

Note:
Exercise:

Problem: Solve: $c - 0.93 = -4.6$.

Solution:

$c = -3.67$

Solve Equations That Require Simplification

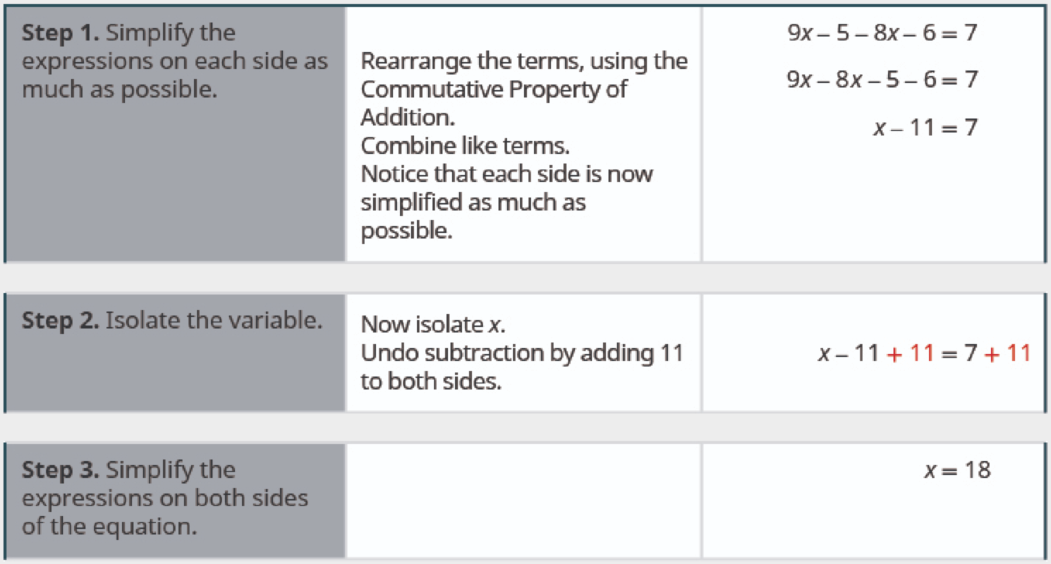
In the previous examples, we were able to isolate the variable with just one operation. Most of the equations we encounter in algebra will take more steps to solve. Usually, we will need to simplify one or both sides of an equation before using the Subtraction or Addition Properties of Equality.

You should always simplify as much as possible before you try to isolate the variable. Remember that to simplify an expression means to do all the operations in the expression. Simplify one side of the equation at a time. Note that simplification is different from the process used to solve an equation in which we apply an operation to both sides.

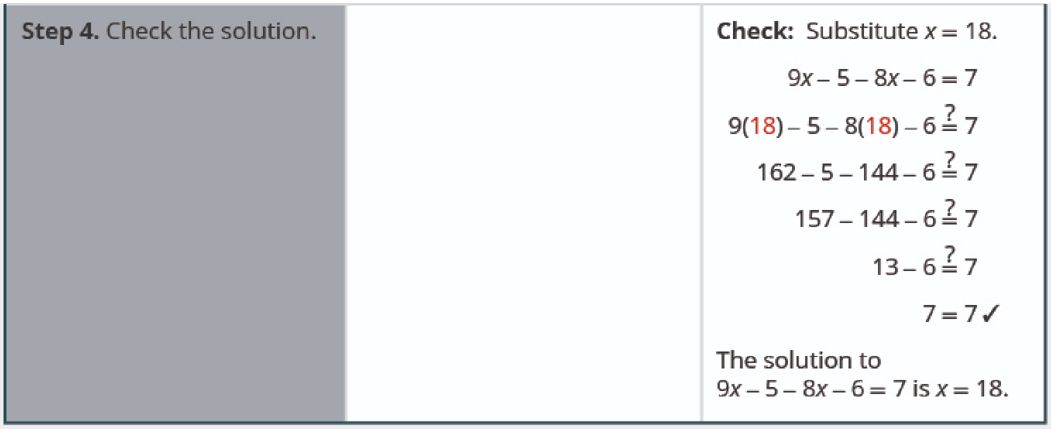
Example:
How to Solve Equations That Require Simplification
Exercise:

Problem: Solve: $9x - 5 - 8x - 6 = 7$.

Solution:
Solution



Step 1. Simplify the expressions on each side as much as possible.	Rearrange the terms, using the Commutative Property of Addition. Combine like terms. Notice that each side is now simplified as much as possible.	$9x - 5 - 8x - 6 = 7$ $9x - 8x - 5 - 6 = 7$ $x - 11 = 7$
Step 2. Isolate the variable.	Now isolate *x*. Undo subtraction by adding 11 to both sides.	$x - 11 + 11 = 7 + 11$
Step 3. Simplify the expressions on both sides of the equation.		$x = 18$



Step 4. Check the solution.		**Check:** Substitute $x = 18$. $9x - 5 - 8x - 6 = 7$ $9(18) - 5 - 8(18) - 6 \stackrel{?}{=} 7$ $162 - 5 - 144 - 6 \stackrel{?}{=} 7$ $157 - 144 - 6 \stackrel{?}{=} 7$ $13 - 6 \stackrel{?}{=} 7$ $7 = 7$ ✓ The solution to $9x - 5 - 8x - 6 = 7$ is $x = 18$.

Note:
Exercise:

Problem: Solve: $8y - 4 - 7y - 7 = 4$.

Solution:

$y = 15$

Note:
Exercise:

Problem: Solve: $6z + 5 - 5z - 4 = 3$.

Solution:

$z = 2$

Example:
Exercise:

Problem: Solve: $5(n - 4) - 4n = -8$.

Solution:
Solution

We simplify both sides of the equation as much as possible before we try to isolate the variable.

	$5(n-4)-4n=-8$
Distribute on the left.	$5n-20-4n=-8$
Use the Commutative Property to rearrange terms.	$5n-4n-20=-8$
Combine like terms.	$n-20=-8$
Each side is as simplified as possible. Next, isolate n.	
Undo subtraction by using the Addition Property of Equality.	$n-20+20=-8+20$
Add.	$n=12$
Check. Substitute $n=12$. $5(n-4)-4n=-8$ $5(12-4)-4(12)\overset{?}{=}-8z$ $5(8)-48\overset{?}{=}-8$ $40-48\overset{?}{=}-8$ $-8=-8$ ✓	
	The solution to $5(n-4)-4n=-8$ is $n=12$.

Note:
Exercise:

Problem: Solve: $5(p-3)-4p=-10$.

Solution:

$p=5$

Note:
Exercise:

Problem: Solve: $4(q+2)-3q=-8$.

Solution:

$q=-16$

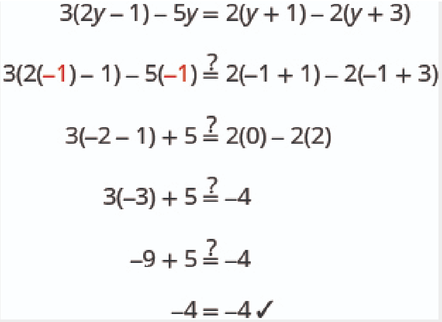
Example:
Exercise:

Problem: Solve: $3(2y-1)-5y=2\left(y+1\right)-2\left(y+3\right)$.

Solution:
Solution

We simplify both sides of the equation before we isolate the variable.

	$3(2y-1)-5y=2(y+1)-2(y+3)$
Distribute on both sides.	$6y-3-5y=2y+2-2y-6$
Use the Commutative Property of Addition.	$6y-5y-3=2y-2y+2-6$
Combine like terms.	$y-3=-4$
Each side is as simplified as possible. Next, isolate y.	
Undo subtraction by using the Addition Property of Equality.	$y-3+3=-4+3$
Add.	$y=-1$
Check. Let $y=-1$.	

 $3(2y-1)-5y=2(y+1)-2(y+3)$ $3(2(-1)-1)-5(-1)\overset{?}{=}2(-1+1)-2(-1+3)$ $3(-2-1)+5\overset{?}{=}2(0)-2(2)$ $3(-3)+5\overset{?}{=}-4$ $-9+5\overset{?}{=}-4$ $-4=-4\checkmark$	
	The solution to $3(2y-1)-5y=2\left(y+1\right)-2\left(y+3\right)$ is $y=-1.$

Note:
Exercise:

Problem: Solve: $4(2h-3)-7h=6\left(h-2\right)-6\left(h-1\right).$

Solution:

$h=6$

Note:
Exercise:

Problem: Solve: $2(5x+2)-9x=3\left(x-2\right)-3\left(x-4\right).$

Solution:

$x=2$

Translate to an Equation and Solve

To solve applications algebraically, we will begin by translating from English sentences into equations. Our first step is to look for the word (or words) that would translate to the equals sign. [link] shows us some of the words that are commonly used.

Equals =
is is equal to is the same as the result is gives was will be

The steps we use to translate a sentence into an equation are listed below.

Note:
Translate an English sentence to an algebraic equation.

Locate the “equals” word(s). Translate to an equals sign (=).
Translate the words to the left of the “equals” word(s) into an algebraic expression.
Translate the words to the right of the “equals” word(s) into an algebraic expression.

Example:
Exercise:

Problem: Translate and solve: Eleven more than x is equal to 54.

Solution:
Solution

Translate.	$\underbrace{\text{Eleven more than } x}_{x+11} \quad \underbrace{\text{is equal to}}_{=} \quad \underbrace{54}_{54}$
Subtract 11 from both sides.	$x + 11 - 11 = 54 - 11$
Simplify.	$x = 43$
Check: Is 54 eleven more than 43? $43 + 11 \overset{?}{=} 54$ $54 = 54\checkmark$	

Note:
Exercise:

Problem: Translate and solve: Ten more than *x* is equal to 41.

Solution:

$x + 10 = 41; x = 31$

Note:
Exercise:

Problem: Translate and solve: Twelve less than *x* is equal to 51.

Solution:

$y - 12 = 51; y = 63$

Example:
Exercise:

Problem: Translate and solve: The difference of $12t$ and $11t$ is -14.

Solution:
Solution

Translate.	$\underbrace{\text{The difference of } 12t \text{ and } 11t}_{12t - 11t}\ \underbrace{\text{is}}_{=}\ \underbrace{-14}_{-14}$
Simplify.	$t = -14$
Check:	

$$\begin{aligned} 12(-14)-11(-14) &\overset{?}{=} -14 \\ -168+154 &\overset{?}{=} -14 \\ -14 &= -14\checkmark \end{aligned}$$

Note:
Exercise:

Problem: Translate and solve: The difference of $4x$ and $3x$ is 14.

Solution:

$4x-3x=14; x=14$

Note:
Exercise:

Problem: Translate and solve: The difference of $7a$ and $6a$ is -8.

Solution:

$7a-6a=-8; a=-8$

Translate and Solve Applications

Most of the time a question that requires an algebraic solution comes out of a real life question. To begin with that question is asked in English (or the language of the person asking) and not in math symbols. Because of this, it is an important skill to be able to translate an everyday situation into algebraic language.

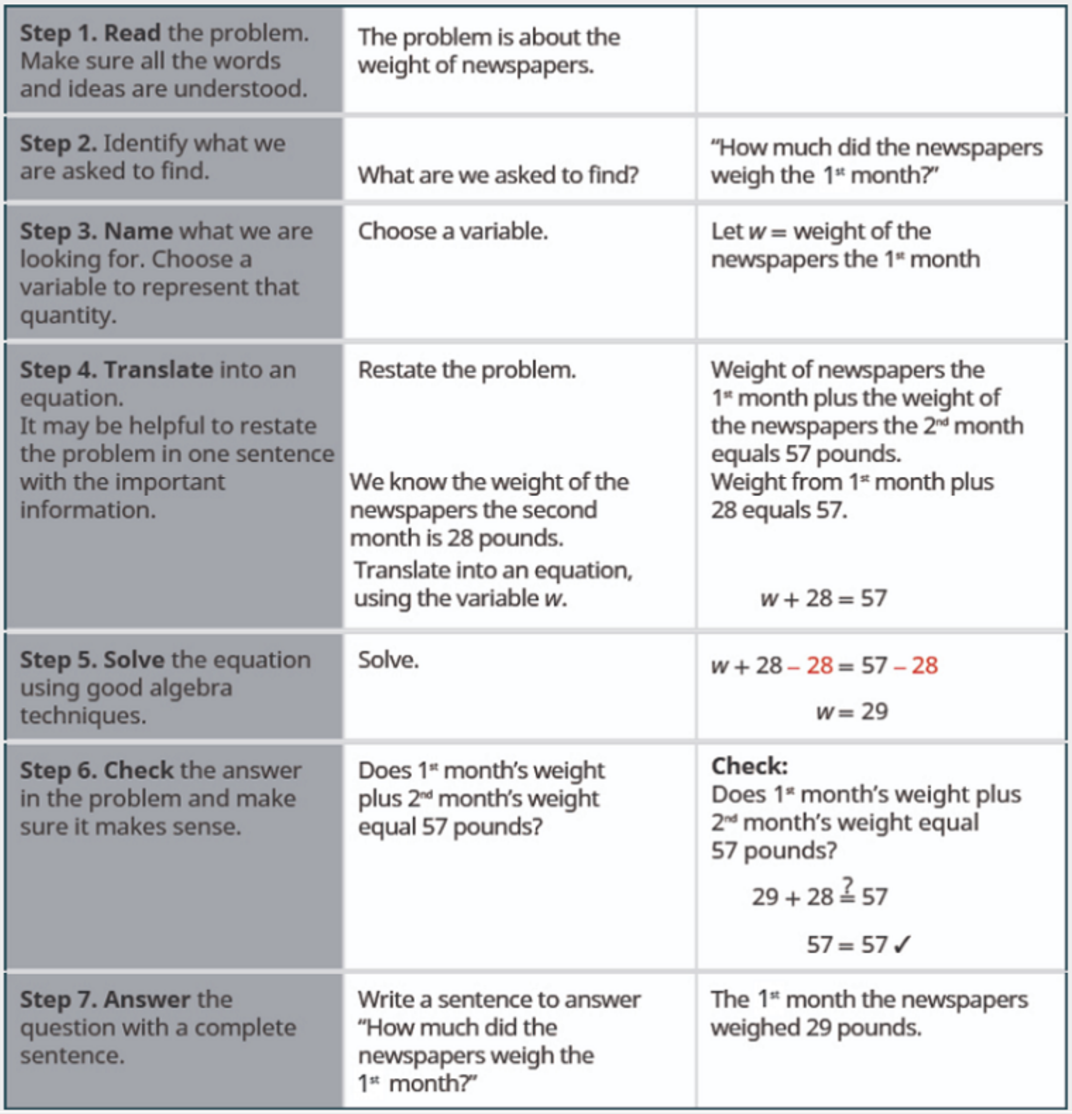
We will start by restating the problem in just one sentence, assign a variable, and then translate the sentence into an equation to solve. When assigning a variable, choose a letter that reminds you of what you are looking for. For example, you might use *q* for the number of quarters if you were solving a problem about coins.

Example:
How to Solve, Translate, and Solve Applications
Exercise:

Problem:

The MacIntyre family recycled newspapers for two months. The two months of newspapers weighed a total of 57 pounds. The second month, the newspapers weighed 28 pounds. How much did the newspapers weigh the first month?

Solution:
Solution



Step 1. Read the problem. Make sure all the words and ideas are understood.	The problem is about the weight of newspapers.	
Step 2. Identify what we are asked to find.	What are we asked to find?	"How much did the newspapers weigh the 1st month?"
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Choose a variable.	Let w = weight of the newspapers the 1st month
Step 4. Translate into an equation. It may be helpful to restate the problem in one sentence with the important information.	Restate the problem. We know the weight of the newspapers the second month is 28 pounds. Translate into an equation, using the variable w.	Weight of newspapers the 1st month plus the weight of the newspapers the 2nd month equals 57 pounds. Weight from 1st month plus 28 equals 57. $w + 28 = 57$
Step 5. Solve the equation using good algebra techniques.	Solve.	$w + 28 - 28 = 57 - 28$ $w = 29$
Step 6. Check the answer in the problem and make sure it makes sense.	Does 1st month's weight plus 2nd month's weight equal 57 pounds?	**Check:** Does 1st month's weight plus 2nd month's weight equal 57 pounds? $29 + 28 \overset{?}{=} 57$ $57 = 57$ ✓
Step 7. Answer the question with a complete sentence.	Write a sentence to answer "How much did the newspapers weigh the 1st month?"	The 1st month the newspapers weighed 29 pounds.

Note:
Exercise:

Problem: Translate into an algebraic equation and solve:

The Pappas family has two cats, Zeus and Athena. Together, they weigh 23 pounds. Zeus weighs 16 pounds. How much does Athena weigh?

Solution:

7 pounds

Note:
Exercise:

Problem: Translate into an algebraic equation and solve:

Sam and Henry are roommates. Together, they have 68 books. Sam has 26 books. How many books does Henry have?

Solution:

42 books

Note:
Solve an application.

Read the problem. Make sure all the words and ideas are understood.
Identify what we are looking for.
Name what we are looking for. Choose a variable to represent that quantity.
Translate into an equation. It may be helpful to restate the problem in one sentence with the important information.
Solve the equation using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

Example:
Exercise:

Problem:

Randell paid $28,675 for his new car. This was $875 less than the sticker price. What was the sticker price of the car?

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	"What was the sticker price of the car?"
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let $s =$ the sticker price of the car.
Step 4. Translate into an equation. Restate the problem in one sentence.	$28,675 is $875 less than the sticker price $28,675 is $875 less than s
Step 5. Solve the equation.	$\begin{aligned} 28{,}675 &= s - 875 \\ 28{,}675 + 875 &= s - 875 + 875 \\ 29{,}550 &= s \end{aligned}$
Step 6. Check the answer. Is $875 less than $29,550 equal to $28,675? $\begin{aligned} 29{,}550 - 875 &\overset{?}{=} 28{,}675 \\ 28{,}675 &= 28{,}675 \checkmark \end{aligned}$	
Step 7. Answer the question with a complete sentence.	The sticker price of the car was $29,550.

Note:
Exercise:

Problem: Translate into an algebraic equation and solve:

Eddie paid $19,875 for his new car. This was $1,025 less than the sticker price. What was the sticker price of the car?

Solution:

$20,900

Note:
Exercise:

Problem: Translate into an algebraic equation and solve:

The admission price for the movies during the day is $7.75. This is $3.25 less the price at night. How much does the movie cost at night?

Solution:

$11.00

Key Concepts

- **To Determine Whether a Number is a Solution to an Equation**

  Substitute the number in for the variable in the equation.
  Simplify the expressions on both sides of the equation.
  Determine whether the resulting statement is true.
    - If it is true, the number is a solution.
    - If it is not true, the number is not a solution.

- **Addition Property of Equality**
  - For any numbers a, b, and c, if $a = b$, then $a + c = b + c$.
- **Subtraction Property of Equality**
  - For any numbers a, b, and c, if $a = b$, then $a - c = b - c$.
- **To Translate a Sentence to an Equation**

  Locate the “equals” word(s). Translate to an equal sign (=).
  Translate the words to the left of the “equals” word(s) into an algebraic expression.
  Translate the words to the right of the “equals” word(s) into an algebraic expression.

- **To Solve an Application**

  Read the problem. Make sure all the words and ideas are understood.
  Identify what we are looking for.
  Name what we are looking for. Choose a variable to represent that quantity.
  Translate into an equation. It may be helpful to restate the problem in one sentence with the important information.
  Solve the equation using good algebra techniques.
  Check the answer in the problem and make sure it makes sense.
  Answer the question with a complete sentence.

Practice Makes Perfect

Verify a Solution of an Equation

In the following exercises, determine whether the given value is a solution to the equation.
Exercise:

Problem: Is $y = \frac{5}{3}$ a solution of $6y + 10 = 12y$?

Solution:

yes

Exercise:

Problem: Is $x = \frac{9}{4}$ a solution of $4x + 9 = 8x$?

Exercise:

Problem: Is $u = -\frac{1}{2}$ a solution of $8u - 1 = 6u$?

Solution:

no

Exercise:

Problem: Is $v = -\frac{1}{3}$ a solution of $9v - 2 = 3v$?

Solve Equations using the Subtraction and Addition Properties of Equality

In the following exercises, solve each equation using the Subtraction and Addition Properties of Equality.

Exercise:

Problem: $x + 24 = 35$

Solution:

$x = 11$

Exercise:

Problem: $x + 17 = 22$

Exercise:

Problem: $y + 45 = -66$

Solution:

$y = -111$

Exercise:

Problem: $y + 39 = -83$

Exercise:

Problem: $b + \frac{1}{4} = \frac{3}{4}$

Solution:

$b = \frac{1}{2}$

Exercise:

Problem: $a + \frac{2}{5} = \frac{4}{5}$

Exercise:

Problem: $p + 2.4 = -9.3$

Solution:

$p = -11.7$

Exercise:

Problem: $m + 7.9 = 11.6$

Exercise:

Problem: $a - 45 = 76$

Solution:

$a = 121$

Exercise:

Problem: $a - 30 = 57$

Exercise:

Problem: $m - 18 = -200$

Solution:

$m = -182$

Exercise:

Problem: $m - 12 = -12$

Exercise:

Problem: $x - \frac{1}{3} = 2$

Solution:

$x = \frac{7}{3}$

Exercise:

Problem: $x - \frac{1}{5} = 4$

Exercise:

Problem: $y - 3.8 = 10$

Solution:

$y = 13.8$

Exercise:

Problem: $y - 7.2 = 5$

Exercise:

Problem: $x - 165 = -420$

Solution:

$x = -255$

Exercise:

Problem: $z - 101 = -314$

Exercise:

Problem: $z + 0.52 = -8.5$

Solution:

$z = -9.02$

Exercise:

Problem: $x + 0.93 = -4.1$

Exercise:

Problem: $q + \frac{3}{4} = \frac{1}{2}$

Solution:

$q = -\frac{1}{4}$

Exercise:

Problem: $p + \frac{1}{3} = \frac{5}{6}$

Exercise:

Problem: $p - \frac{2}{5} = \frac{2}{3}$

Solution:

$p = \frac{16}{15}$

Exercise:

Problem: $y - \frac{3}{4} = \frac{3}{5}$

Solve Equations that Require Simplification

In the following exercises, solve each equation.

Exercise:

Problem: $c + 31 - 10 = 46$

Solution:

$c = 25$

Exercise:

Problem: $m + 16 - 28 = 5$

Exercise:

Problem: $9x + 5 - 8x + 14 = 20$

Solution:

$x = 1$

Exercise:

Problem: $6x + 8 - 5x + 16 = 32$

Exercise:

Problem: $-6x - 11 + 7x - 5 = -16$

Solution:

$x = 0$

Exercise:

Problem: $-8n - 17 + 9n - 4 = -41$

Exercise:

Problem: $5\left(y - 6\right) - 4y = -6$

Solution:

$y = 24$

Exercise:

Problem: $9\left(y - 2\right) - 8y = -16$

Exercise:

Problem: $8\left(u+1.5\right)-7u=4.9$

Solution:

$u=-7.1$

Exercise:

Problem: $5\left(w+2.2\right)-4w=9.3$

Exercise:

Problem: $6a-5\left(a-2\right)+9=-11$

Solution:

$a=-30$

Exercise:

Problem: $8c-7\left(c-3\right)+4=-16$

Exercise:

Problem: $6\left(y-2\right)-5y=4\left(y+3\right)$
$-4\left(y-1\right)$

Solution:

$y=28$

Exercise:

Problem: $9\left(x-1\right)-8x=-3\left(x+5\right)$
$+3\left(x-5\right)$

Exercise:

Problem: $3(5n-1)-14n+9$
$=10\left(n-4\right)-6n-4\left(n+1\right)$

Solution:

$n=-50$

Exercise:

Problem: $2(8m+3)-15m-4$
$=9\left(m+6\right)-2\left(m-1\right)-7m$

Exercise:

Problem: $-\left(j+2\right)+2j-1=5$

Solution:

$j=8$

Exercise:

Problem: $-\left(k+7\right)+2k+8=7$

Exercise:

Problem: $-\left(\frac{1}{4}a-\frac{3}{4}\right)+\frac{5}{4}a=-2$

Solution:

$a=-\frac{11}{4}$

Exercise:

Problem: $-\left(\frac{2}{3}d-\frac{1}{3}\right)+\frac{5}{3}d=-4$

Exercise:

Problem: $$\begin{array}{l}8(4x+5)-5(6x)-x\\ =53-6(x+1)+3(2x+2)\end{array}$$

Solution:

$x=13$

Exercise:

Problem: $$\begin{array}{l}6(9y-1)-10(5y)-3y\\ =22-4\left(2y-12\right)+8\left(y-6\right)\end{array}$$

Translate to an Equation and Solve

In the following exercises, translate to an equation and then solve it.

Exercise:

Problem: Nine more than x is equal to 52.

Solution:

$x+9=52;x=43$

Exercise:

Problem: The sum of *x* and -15 is 23.

Exercise:

Problem: Ten less than m is -14.

Solution:

$m-10=-14; m=-4$

Exercise:

Problem: Three less than y is -19.

Exercise:

Problem: The sum of y and -30 is 40.

Solution:

$y+(-30)=40; y=70$

Exercise:

Problem: Twelve more than p is equal to 67.

Exercise:

Problem: The difference of $9x$ and $8x$ is 107.

Solution:

$9x-8x=107; x=107$

Exercise:

Problem: The difference of $5c$ and $4c$ is 602.

Exercise:

Problem: The difference of n and $\frac{1}{6}$ is $\frac{1}{2}$.

Solution:

$n-\frac{1}{6}=\frac{1}{2}; n=\frac{2}{3}$

Exercise:

Problem: The difference of f and $\frac{1}{3}$ is $\frac{1}{12}$.

Exercise:

Problem: The sum of $-4n$ and $5n$ is -82.

Solution:

$-4n+5n=-82; n=-82$

Exercise:

Problem: The sum of $-9m$ and $10m$ is -95.

Translate and Solve Applications

In the following exercises, translate into an equation and solve.

Exercise:

Problem:

Distance Avril rode her bike a total of 18 miles, from home to the library and then to the beach. The distance from Avril's house to the library is 7 miles. What is the distance from the library to the beach?

Solution:

11 miles

Exercise:

Problem:

Reading Jeff read a total of 54 pages in his History and Sociology textbooks. He read 41 pages in his History textbook. How many pages did he read in his Sociology textbook?

Exercise:

Problem:

Age Eva's daughter is 15 years younger than her son. Eva's son is 22 years old. How old is her daughter?

Solution:

7 years old

Exercise:

Problem:

Age Pablo's father is 3 years older than his mother. Pablo's mother is 42 years old. How old is his father?

Exercise:

Problem:

Groceries For a family birthday dinner, Celeste bought a turkey that weighed 5 pounds less than the one she bought for Thanksgiving. The birthday turkey weighed 16 pounds. How much did the Thanksgiving turkey weigh?

Solution:

21 pounds

Exercise:

Problem:

Weight Allie weighs 8 pounds less than her twin sister Lorrie. Allie weighs 124 pounds. How much does Lorrie weigh?

Exercise:

Problem:

Health Connor’s temperature was 0.7 degrees higher this morning than it had been last night. His temperature this morning was 101.2 degrees. What was his temperature last night?

Solution:

100.5 degrees

Exercise:

Problem:

Health The nurse reported that Tricia’s daughter had gained 4.2 pounds since her last checkup and now weighs 31.6 pounds. How much did Tricia’s daughter weigh at her last checkup?

Exercise:

Problem:

Salary Ron’s paycheck this week was $17.43 less than his paycheck last week. His paycheck this week was $103.76. How much was Ron’s paycheck last week?

Solution:

$121.19

Exercise:

Problem:

Textbooks Melissa’s math book cost $22.85 less than her art book cost. Her math book cost $93.75. How much did her art book cost?

Everyday Math

Exercise:

Problem:

Construction Miguel wants to drill a hole for a $\frac{5}{8}$ inch screw. The hole should be $\frac{1}{12}$ inch smaller than the screw. Let d equal the size of the hole he should drill. Solve the equation $d-\frac{1}{12}=\frac{5}{8}$ to see what size the hole should be.

Solution:

$d=\frac{17}{24}$ inch

Exercise:

Problem:

Baking Kelsey needs $\frac{2}{3}$ cup of sugar for the cookie recipe she wants to make. She only has $\frac{3}{8}$ cup of sugar and will borrow the rest from her neighbor. Let s equal the amount of sugar she will borrow. Solve the equation $\frac{3}{8} + s = \frac{2}{3}$ to find the amount of sugar she should ask to borrow.

Writing Exercises

Exercise:

Problem: Is -8 a solution to the equation $3x = 16 - 5x$? How do you know?

Solution:

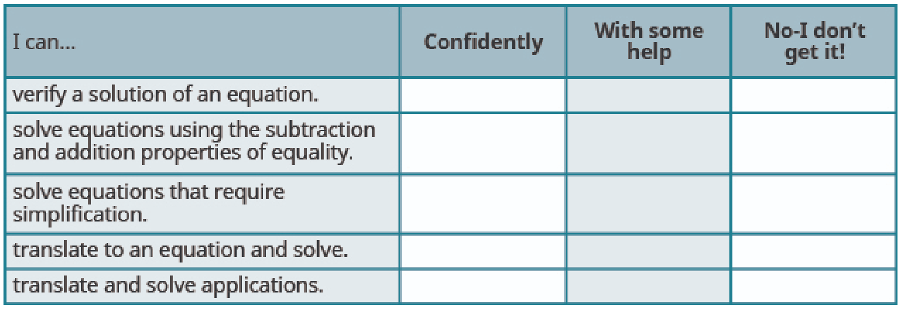
No. Justifications will vary.

Exercise:

Problem: What is the first step in your solution to the equation $10x + 2 = 4x + 26$?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
verify a solution of an equation.			
solve equations using the subtraction and addition properties of equality.			
solve equations that require simplification.			
translate to an equation and solve.			
translate and solve applications.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

solution of an equation

A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

Solve Equations using the Division and Multiplication Properties of Equality
By the end of this section, you will be able to:

- Solve equations using the Division and Multiplication Properties of Equality
- Solve equations that require simplification
- Translate to an equation and solve
- Translate and solve applications

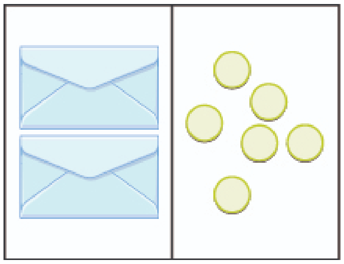
Note:
Before you get started, take this readiness quiz.

1. Simplify: $-7\left(\frac{1}{-7}\right)$.
   If you missed this problem, review [link].
2. Evaluate $9x+2$ when $x=-3$.
   If you missed this problem, review [link].

Solve Equations Using the Division and Multiplication Properties of Equality

You may have noticed that all of the equations we have solved so far have been of the form $x+a=b$ or $x-a=b$. We were able to isolate the variable by adding or subtracting the constant term on the side of the equation with the variable. Now we will see how to solve equations that have a variable multiplied by a constant and so will require division to isolate the variable.

Let’s look at our puzzle again with the envelopes and counters in [link].

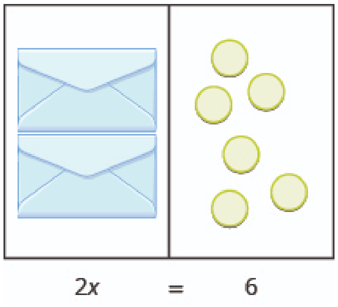


The illustration shows a model of an equation with one variable multiplied by a constant. On the left side of the workspace are two instances of the unknown (envelope), while on the right side of the workspace are six counters.

In the illustration there are two identical envelopes that contain the same number of counters. Remember, the left side of the workspace must equal the right side, but the counters on the left side are “hidden” in the envelopes. So how many counters are in each envelope?

How do we determine the number? We have to separate the counters on the right side into two groups of the same size to correspond with the two envelopes on the left side. The 6 counters divided into 2 equal groups gives 3 counters in each group (since $6 \div 2 = 3$).

What equation models the situation shown in [link]? There are two envelopes, and each contains x counters. Together, the two envelopes must contain a total of 6 counters.



The illustration shows a model of the equation $2x = 6$.

	$2x = 6$
If we divide both sides of the equation by 2, as we did with the envelopes and counters,	$\frac{2x}{2} = \frac{6}{2}$
we get:	

	$x = 3$

We found that each envelope contains 3 counters. Does this check? We know $2 \cdot 3 = 6$, so it works! Three counters in each of two envelopes does equal six!

This example leads to the **Division Property of Equality**.

Note:
The Division Property of Equality
For any numbers a, b, and c, and $c \neq 0$,
Equation:

$$\begin{array}{lccc} \text{If} & a & = & b, \\ \text{then} & \frac{a}{c} & = & \frac{b}{c} \end{array}$$

When you divide both sides of an equation by any non-zero number, you still have equality.

Note: Doing the Manipulative Mathematics activity "Division Property of Equality" will help you develop a better understanding of how to solve equations by using the Division Property of Equality.

The goal in solving an equation is to 'undo' the operation on the variable. In the next example, the variable is multiplied by 5, so we will divide both sides by 5 to 'undo' the multiplication.

Example:
Exercise:

Problem: Solve: $5x = -27$.

Solution:
Solution

To isolate x, "undo" the multiplication by 5.	$5x = -27$

Divide to ‘undo’ the multiplication.		$\frac{5x}{5} = -\frac{27}{5}$
Simplify.		$x = -\frac{27}{5}$
Check:	$5x = -27$	
Substitute $-\frac{27}{5}$ for x.	$5\left(-\frac{27}{5}\right) \overset{?}{=} -27$	
	$-27 = -27$ ✓	
		Since this is a true statement, $x = -\frac{27}{5}$ is the solution to $5x = -27$.

Note:
Exercise:

Problem: Solve: $3y = -41$.

Solution:

$y = \frac{-41}{3}$

Note:
Exercise:

Problem: Solve: $4z = -55$.

Solution:

$z = \frac{-55}{4}$

Consider the equation $\frac{x}{4} = 3$. We want to know what number divided by 4 gives 3. So to “undo” the division, we will need to multiply by 4. The **Multiplication Property of Equality** will allow us to do this. This property says that if we start with two equal quantities and multiply both by the same number, the results are equal.

Note:
The Multiplication Property of Equality
For any numbers *a*, *b*, and *c*,
Equation:

$$\begin{array}{lrcl} \text{If} & a & = & b, \\ \text{then} & ac & = & bc \end{array}$$

If you multiply both sides of an equation by the same number, you still have equality.

Example:
Exercise:

Problem: Solve: $\frac{y}{-7} = -14$.

Solution:
Solution

Here y is divided by -7. We must multiply by -7 to isolate y.

	$\frac{y}{-7} = -14$
Multiply both sides by -7.	$-7\left(\frac{y}{-7}\right) = -7(-14)$
Multiply.	$\frac{-7y}{-7} = 98$
Simplify.	

		$y = 98$
Check: $\frac{y}{-7} = -14$		
Substitute $y = 98$.	$\frac{98}{-7} \overset{?}{=} -14$	
Divide.	$-14 = -14$ ✓	

Note:
Exercise:

Problem: Solve: $\frac{a}{-7} = -42$.

Solution:

$a = 294$

Note:
Exercise:

Problem: Solve: $\frac{b}{-6} = -24$.

Solution:

$b = 144$

Example:
Exercise:

Problem: Solve: $-n = 9$.

Solution:
Solution

		$-n=9$
Remember $-n$ is equivalent to $-1n$.		$-1n=9$
Divide both sides by -1.		$\frac{-1n}{-1}=\frac{9}{-1}$
Divide.		$n=-9$
Notice that there are two other ways to solve $-n=9$. We can also solve this equation by multiplying both sides by -1 and also by taking the opposite of both sides.		
Check:	$-n=9$	
Substitute $n=-9$.	$-(-9)\overset{?}{=}9$	
Simplify.	$9=9\checkmark$	

Note:
Exercise:

Problem: Solve: $-k=8$.

Solution:

$k=-8$

Note:
Exercise:

Problem: Solve: $-g=3$.

Solution:

$g = -3$

Example:
Exercise:

Problem: Solve: $\frac{3}{4}x = 12$.

Solution:
Solution

Since the product of a number and its reciprocal is 1, our strategy will be to isolate x by multiplying by the reciprocal of $\frac{3}{4}$.

<table>
<tr><td colspan="2"></td><td>$\frac{3}{4}x = 12$</td></tr>
<tr><td colspan="2">Multiply by the reciprocal of $\frac{3}{4}$.</td><td>$\frac{4}{3} \cdot \frac{3}{4}x = \frac{4}{3} \cdot 12$</td></tr>
<tr><td colspan="2">Reciprocals multiply to 1.</td><td>$1x = \frac{4}{3} \cdot \frac{12}{1}$</td></tr>
<tr><td colspan="2">Multiply.</td><td>$x = 16$</td></tr>
<tr><td colspan="3">Notice that we could have divided both sides of the equation $\frac{3}{4}x = 12$ by $\frac{3}{4}$ to isolate x. While this would work, most people would find multiplying by the reciprocal easier.</td></tr>
<tr><td>Check:</td><td>$\frac{3}{4}x = 12$</td><td></td></tr>
<tr><td>Substitute $x = 16$.</td><td></td><td></td></tr>
</table>

	$\frac{3}{4}\cdot 16 \overset{?}{=} 12$	
	$12 = 12\checkmark$	

Note:
Exercise:

Problem: Solve: $\frac{2}{5}n = 14$.

Solution:

$n = 35$

Note:
Exercise:

Problem: Solve: $\frac{5}{6}y = 15$.

Solution:

$y = 18$

In the next example, all the variable terms are on the right side of the equation. As always, our goal in solving the equation is to isolate the variable.

Example:
Exercise:

Problem: Solve: $\frac{8}{15} = -\frac{4}{5}x$.

Solution:
Solution

		$\frac{8}{15} = -\frac{4}{5}x$
Multiply by the reciprocal of $-\frac{4}{5}$.		$\left(-\frac{5}{4}\right)\left(\frac{8}{15}\right) = \left(-\frac{5}{4}\right)\left(-\frac{4}{5}x\right)$
Reciprocals multiply to 1.		$-\frac{\cancel{5}\cdot\cancel{4}\cdot 2}{\cancel{4}\cdot 3\cdot\cancel{5}} = 1x$
Multiply.		$-\frac{2}{3} = x$
Check:	$\frac{8}{15} = -\frac{4}{5}x$	
Let $x = -\frac{2}{3}$.	$\frac{8}{15} = -\frac{4}{5}\left(-\frac{2}{3}\right)$	
	$\frac{8}{15} = \frac{8}{15}$ ✓	

Note:
Exercise:

Problem: Solve: $\frac{9}{25} = -\frac{4}{5}z$.

Solution:

$z = -\frac{9}{20}$

Note:
Exercise:

Problem: Solve: $\frac{5}{6} = -\frac{8}{3}\,r$.

Solution:

$r = -\frac{5}{16}$

Solve Equations That Require Simplification

Many equations start out more complicated than the ones we have been working with.

With these more complicated equations the first step is to simplify both sides of the equation as much as possible. This usually involves combining like terms or using the distributive property.

Example:
Exercise:

Problem: Solve: $14 - 23 = 12y - 4y - 5y$.

Solution:
Solution

Begin by simplifying each side of the equation.

		$14 - 23 = 12y - 4y - 5y.$
Simplify each side.		$-9 = 3y$
Divide both sides by 3 to isolate y.		$\frac{-9}{3} = \frac{3y}{3}$
Divide.		$-3 = y$
Check:		

	$14-23=12y-4y-5y$	
Substitute $y=-3$.	$14-23\overset{?}{=}12(-3)-4(-3)-5(-3)$	
	$14-23\overset{?}{=}-36+12+15$	
	$-9=-9$ ✓	

Note:
Exercise:

Problem: Solve: $18-27=15c-9c-3c$.

Solution:

$c=-3$

Note:
Exercise:

Problem: Solve:$18-22=12x-x-4x$.

Solution:

$x=-\frac{4}{7}$

Example:
Exercise:

Problem: Solve: $-4(a-3)-7=25$.

Solution:
Solution

Here we will simplify each side of the equation by using the distributive property first.

		$-4(a-3)-7=25$
Distribute.		$-4a+12-7=25$
Simplify.		$-4a+5=25$
Simplify.		$-4a=20$
Divide both sides by -4 to isolate a.		$\frac{-4a}{-4}=\frac{20}{-4}$
Divide.		$a=-5$
Check:	$-4(a-3)-7=25$	
Substitute $a=-5$.	$-4(-5-3)-7\overset{?}{=}25$	
	$-4(-8)-7\overset{?}{=}25$	
	$32-7\overset{?}{=}25$	
	$25=25$ ✓	

Note:
Exercise:

Problem: Solve: $-4(q-2)-8=24$.

Solution:

$q=-6$

Note:
Exercise:

Problem: Solve: $-6(r-2)-12=30$.

Solution:

$r=-5$

Now we have covered all four properties of equality—subtraction, addition, division, and multiplication. We'll list them all together here for easy reference.

Note:
Properties of Equality
Equation:

Subtraction Property of Equality

For any real numbers $a, b,$ and c,

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & a-c & = & b-c. \end{array}$$

Addition Property of Equality

For any real numbers $a, b,$ and c,

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & a+c & = & b+c. \end{array}$$

Division Property of Equality

For any numbers $a, b,$ and c, and $c \neq 0$,

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & \frac{a}{c} & = & \frac{b}{c}. \end{array}$$

Multiplication Property of Equality

For any numbers $a, b,$ and c,

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & ac & = & bc. \end{array}$$

When you add, subtract, multiply, or divide the same quantity from both sides of an equation, you still have equality.

Translate to an Equation and Solve

In the next few examples, we will translate sentences into equations and then solve the equations. You might want to review the translation table in the previous chapter.

Example:
Exercise:

Problem: Translate and solve: The number 143 is the product of -11 and y.

Solution:
Solution

Begin by translating the sentence into an equation.

Translate.	$\underbrace{\text{The number 143}}_{143}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{the product of }-11\text{ and y.}}_{-11y}$
Divide by -11.	$\frac{143}{-11}=\frac{-11y}{-11}$
Simplify.	$-13=y$
Check: $\begin{aligned} 143 &= -11y \\ 143 &\overset{?}{=} -11(-13) \\ 143 &= 143\checkmark \end{aligned}$	

Note:
Exercise:

Problem: Translate and solve: The number 132 is the product of −*12* and *y*.

Solution:

$132 = -12y; y = -11$

Note:
Exercise:

Problem: Translate and solve: The number 117 is the product of −*13* and *z*.

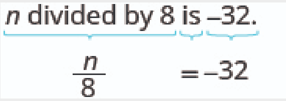
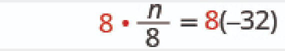
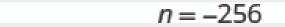
Solution:

$117 = -13z; z = -9$

Example:
Exercise:

Problem: Translate and solve: n divided by 8 is -32.

Solution:
Solution

Begin by translating the sentence into an equation. Translate.		 $\underbrace{n \text{ divided by } 8}\ \underbrace{\text{is}}\ \underbrace{-32.}$ $\frac{n}{8} = -32$
Multiple both sides by 8.		 $8 \cdot \frac{n}{8} = 8(-32)$
Simplify.		 $n = -256$
Check:	Is n divided by 8 equal to −32?	

Let $n = -256$.	Is -256 divided by 8 equal to -32?	
Translate.	$\frac{-256}{8} \overset{?}{=} -32$	
Simplify.	$-32 = -32\checkmark$	

Note:
Exercise:

Problem: Translate and solve: n divided by 7 is equal to -21.

Solution:

$\frac{n}{7} = -21; n = -147$

Note:
Exercise:

Problem: Translate and solve: n divided by 8 is equal to -56.

Solution:

$\frac{n}{8} = -56; n = -448$

Example:
Exercise:

Problem: Translate and solve: The quotient of y and -4 is 68.

Solution:
Solution

Begin by translating the sentence into an equation.

Translate.		$\underbrace{\text{The quotient of } y \text{ and } -4}_{\frac{y}{-4}}\ \underbrace{\text{is}}_{=}\ \underbrace{68.}_{68}$
Multiply both sides by -4.		$-4\left(\frac{y}{-4}\right) = -4(68)$
Simplify.		$y = -272$
Check:	Is the quotient of y and -4 equal to 68?	
Let $y = -272$.	Is the quotient of -272 and -4 equal to 68?	
Translate.	$\frac{-272}{-4} \stackrel{?}{=} 68$	
Simplify.	$68 = 68$✓	

Note:
Exercise:

Problem: Translate and solve: The quotient of q and -8 is 72.

Solution:

$\frac{q}{-8} = 72; q = -576$

Note:
Exercise:

Problem: Translate and solve: The quotient of p and -9 is 81.

Solution:

$\frac{p}{-9} = 81; p = -729$

Example:
Exercise:

Problem: Translate and solve: Three-fourths of p is 18.

Solution:
Solution

Begin by translating the sentence into an equation. Remember, “of” translates into multiplication.

Translate.		$\underbrace{\text{Three-fourths of } p}_{\frac{3}{4}p}\ \underbrace{\text{is}}_{=}\ \underbrace{18.}_{18}$
Multiply both sides by $\frac{4}{3}$.		$\frac{4}{3}\cdot\frac{3}{4}p=\frac{4}{3}\cdot 18$
Simplify.		$p=24$
Check:	Is three-fourths of p equal to 18?	
Let $p=24$.	Is three-fourths of 24 equal to 18?	
Translate.	$\frac{3}{4}\cdot 24\overset{?}{=}18$	
Simplify.	$18=18\checkmark$	

Note:
Exercise:

Problem: Translate and solve: Two-fifths of f is 16.

Solution:

$\frac{2}{5}f = 16; f = 40$

Note:
Exercise:

Problem: Translate and solve: Three-fourths of f is 21.

Solution:

$\frac{3}{4}f = 21; f = 28$

Example:
Exercise:

Problem: Translate and solve: The sum of three-eighths and x is one-half.

Solution:
Solution

Begin by translating the sentence into an equation.

Translate.	$\underbrace{\text{The sum of three – eighths and } x}_{\frac{3}{8} + x} \; \underbrace{\text{is}}_{=} \; \underbrace{\frac{1}{2}}_{\frac{1}{2}}$
Subtract $\frac{3}{8}$ from each side.	$\frac{3}{8} - \frac{3}{8} + x = \frac{1}{2} - \frac{3}{8}$
Simplify and rewrite fractions with common denominators.	$x = \frac{4}{8} - \frac{3}{8}$

Simplify.			$x = \frac{1}{8}$
Check:		Is the sum of three-eighths and x equal to one-half?	
Let $x = \frac{1}{8}$.		Is the sum of three-eighths and one-eighth equal to one-half?	
Translate.		$\frac{3}{8} + \frac{1}{8} \overset{?}{=} \frac{1}{2}$	
Simplify.		$\frac{4}{8} \overset{?}{=} \frac{1}{2}$	
Simplify.		$\frac{1}{2} = \frac{1}{2} \checkmark$	

Note:
Exercise:

Problem: Translate and solve: The sum of five-eighths and *x* is one-fourth.

Solution:

$\frac{5}{8} + x = \frac{1}{4}; x = -\frac{3}{8}$

Note:
Exercise:

Problem: Translate and solve: The sum of three-fourths and *x* is five-sixths.

Solution:

$\frac{3}{4} + x = \frac{5}{6}; x = \frac{1}{12}$

Translate and Solve Applications

To solve applications using the Division and Multiplication Properties of Equality, we will follow the same steps we used in the last section. We will state what we are trying to find, assign a variable, restate the information given in the problem in just one sentence, and then translate the sentence into an equation to solve.

Example:
Exercise:

Problem:

Denae bought 6 pounds of grapes for $10.74. What was the cost of one pound of grapes?

Solution:
Solution

What are you asked to find?	The cost of 1 pound of grapes
Assign a variable.	Let $c =$ the cost of one pound.
Write a sentence that gives the information to find it.	The cost of 6 pounds is $10.74.
Translate into an equation.	$6c = 10.74$
Solve.	$\frac{6c}{6} = \frac{10.74}{6}$ $c = 1.79$
	The grapes cost $1.79 per pound.

Check: If one pound costs $1.79, do 6 pounds cost $10.74?

$$\begin{aligned} 6\,(1.79) &\overset{?}{=} 10.74 \\ 10.74 &= 10.74\checkmark \end{aligned}$$

Note:
Exercise:

Problem: Translate and solve:

Arianna bought a 24-pack of water bottles for $9.36. What was the cost of one water bottle?

Solution:

$0.39

Note:
Exercise:

Problem: Translate and solve:

At JB's Bowling Alley, 6 people can play on one lane for $34.98. What is the cost for each person?

Solution:

$5.83

Example:
Exercise:

Problem:

Andreas bought a used car for $12,000. Because the car was 4-years old, its price was $\frac{3}{4}$ of the original price, when the car was new. What was the original price of the car?

Solution:
Solution

What are you asked to find?	The original price of the car
Assign a variable.	Let $p =$ the original price.
Write a sentence that gives the information to find it.	$12,000 is $\frac{3}{4}$ of the original price.
Translate into an equation.	$12{,}000 = \frac{3}{4}p$
Solve.	$\frac{4}{3}(12{,}000) = \frac{4}{3}\cdot\frac{3}{4}p$ $16{,}000 = p$ The original cost of the car was $16,000.

Check: Is $\frac{3}{4}$ of $16,000 equal to $12,000?

$$\begin{aligned}\frac{3}{4}\cdot 16{,}000 &\overset{?}{=} 12{,}000\\ 12{,}000 &= 12{,}000\checkmark\end{aligned}$$

Note:
Exercise:

Problem: Translate and solve:

The annual property tax on the Mehta’s house is $1,800, calculated as $\frac{15}{1,000}$ of the assessed value of the house. What is the assessed value of the Mehta’s house?

Solution:

$120,000

Note:
Exercise:

Problem: Translate and solve:

Stella planted 14 flats of flowers in $\frac{2}{3}$ of her garden. How many flats of flowers would she need to fill the whole garden?

Solution:

21 flats

Key Concepts

- **The Division Property of Equality—**For any numbers a, b, and c, and $c \neq 0$, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$.
  When you divide both sides of an equation by any non-zero number, you still have equality.
- **The Multiplication Property of Equality—**For any numbers a, b, and c, if $a = b$, then $ac = bc$.
  If you multiply both sides of an equation by the same number, you still have equality.

Practice Makes Perfect

Solve Equations Using the Division and Multiplication Properties of Equality

In the following exercises, solve each equation using the Division and Multiplication Properties of Equality and check the solution.

Exercise:

Problem: $8x = 56$

Solution:

$x = 7$

Exercise:

Problem: $7p = 63$

Exercise:

Problem: $-5c = 55$

Solution:

$c = -11$

Exercise:

Problem: $-9x = -27$

Exercise:

Problem: $-809 = 15y$

Solution:

$y = -\frac{809}{15}$

Exercise:

Problem: $-731 = 19y$

Exercise:

Problem: $-37p = -541$

Solution:

$p = -\frac{541}{37}$

Exercise:

Problem: $-19m = -586$

Exercise:

Problem: $0.25z = 3.25$

Solution:

$z = 13$

Exercise:

Problem: $0.75a = 11.25$

Exercise:

Problem: $-13x = 0$

Solution:

$x = 0$

Exercise:

Problem: $24x = 0$

Exercise:

Problem: $\frac{x}{4} = 35$

Solution:

$x = 140$

Exercise:

Problem: $\frac{z}{2} = 54$

Exercise:

Problem: $-20 = \frac{q}{-5}$

Solution:

$q = 100$

Exercise:

Problem: $\frac{c}{-3} = -12$

Exercise:

Problem: $\frac{y}{9} = -16$

Solution:

$y = -144$

Exercise:

Problem: $\frac{q}{6} = -38$

Exercise:

Problem: $\frac{m}{-12} = 45$

Solution:

$m = -540$

Exercise:

Problem: $-24 = \frac{p}{-20}$

Exercise:

Problem: $-y = 6$

Solution:

$y = -6$

Exercise:

Problem: $-u = 15$

Exercise:

Problem: $-v = -72$

Solution:

$v = 72$

Exercise:

Problem: $-x = -39$

Exercise:

Problem: $\frac{2}{3}y = 48$

Solution:

$y = 72$

Exercise:

Problem: $\frac{3}{5}r = 75$

Exercise:

Problem: $-\frac{5}{8}w = 40$

Solution:

$w = -64$

Exercise:

Problem: $24 = -\frac{3}{4}x$

Exercise:

Problem: $-\frac{2}{5} = \frac{1}{10}a$

Solution:

$a = -4$

Exercise:

Problem: $-\frac{1}{3}q = -\frac{5}{6}$

Exercise:

Problem: $-\frac{7}{10}x = -\frac{14}{3}$

Solution:

$x = \frac{20}{3}$

Exercise:

Problem: $\frac{3}{8}y = -\frac{1}{4}$

Exercise:

Problem: $\frac{7}{12} = -\frac{3}{4}p$

Solution:

$p = -\frac{7}{9}$

Exercise:

Problem: $\frac{11}{18} = -\frac{5}{6}q$

Exercise:

Problem: $-\frac{5}{18} = -\frac{10}{9}u$

Solution:

$u = \frac{1}{4}$

Exercise:

Problem: $-\frac{7}{20} = -\frac{7}{4}v$

Solve Equations That Require Simplification

In the following exercises, solve each equation requiring simplification.

Exercise:

Problem: $100 - 16 = 4p - 10p - p$

Solution:

$p = -12$

Exercise:

Problem: $-18 - 7 = 5t - 9t - 6t$

Exercise:

Problem: $\frac{7}{8}n - \frac{3}{4}n = 9 + 2$

Solution:

$n = 88$

Exercise:

Problem: $\frac{5}{12}q + \frac{1}{2}q = 25 - 3$

Exercise:

Problem: $0.25d + 0.10d = 6 - 0.75$

Solution:

$d = 15$

Exercise:

Problem: $0.05p - 0.01p = 2 + 0.24$

Exercise:

Problem: $-10(q - 4) - 57 = 93$

Solution:

$q = -11$

Exercise:

Problem: $-12(d - 5) - 29 = 43$

Exercise:

Problem: $-10(x + 4) - 19 = 85$

Solution:

$x = -\frac{72}{5}$

Exercise:

Problem: $-15(z + 9) - 11 = 75$

Mixed Practice

In the following exercises, solve each equation.

Exercise:

Problem: $\frac{9}{10}x = 90$

Solution:

$x = 100$

Exercise:

Problem: $\frac{5}{12}y = 60$

Exercise:

Problem: $y + 46 = 55$

Solution:

$y = 9$

Exercise:

Problem: $x + 33 = 41$

Exercise:

Problem: $\frac{w}{-2} = 99$

Solution:

$w = -198$

Exercise:

Problem: $\frac{s}{-3} = -60$

Exercise:

Problem: $27 = 6a$

Solution:

$a = \frac{9}{2}$

Exercise:

Problem: $-a = 7$

Exercise:

Problem: $-x = 2$

Solution:

$x = -2$

Exercise:

Problem: $z - 16 = -59$

Exercise:

Problem: $m - 41 = -14$

Solution:

$m = 27$

Exercise:

Problem: $0.04r = 52.60$

Exercise:

Problem: $63.90 = 0.03p$

Solution:

$p = 2130$

Exercise:

Problem: $-15x = -120$

Exercise:

Problem: $84 = -12z$

Solution:

$z = -7$

Exercise:

Problem: $19.36 = x - 0.2x$

Exercise:

Problem: $c - 0.3c = 35.70$

Solution:

$c = 51$

Exercise:

Problem: $-y=-9$

Exercise:

Problem: $-x=-8$

Solution:

$x=8$

Translate to an Equation and Solve

In the following exercises, translate to an equation and then solve.

Exercise:

Problem: 187 is the product of -17 and *m*.

Exercise:

Problem: 133 is the product of -19 and *n*.

Solution:

$133=-19n;n=-7$

Exercise:

Problem: -184 is the product of 23 and *p*.

Exercise:

Problem: -152 is the product of 8 and *q*.

Solution:

$-152=8q;q=-19$

Exercise:

Problem: *u* divided by 7 is equal to -49.

Exercise:

Problem: *r* divided by 12 is equal to -48.

Solution:

$\frac{r}{12}=-48;r=-576$

Exercise:

Problem: h divided by -13 is equal to -65.

Exercise:

Problem: j divided by -20 is equal to -80.

Solution:

$\frac{j}{-20} = -80; j = 1{,}600$

Exercise:

Problem: The quotient c and -19 is 38.

Exercise:

Problem: The quotient of b and -6 is 18.

Solution:

$\frac{b}{-6} = 18; b = -108$

Exercise:

Problem: The quotient of h and 26 is -52.

Exercise:

Problem: The quotient k and 22 is -66.

Solution:

$\frac{k}{22} = -66; k = -1{,}452$

Exercise:

Problem: Five-sixths of y is 15.

Exercise:

Problem: Three-tenths of x is 15.

Solution:

$\frac{3}{10}x = 15; x = 50$

Exercise:

Problem: Four-thirds of *w* is 36.

Exercise:

Problem: Five-halves of *v* is 50.

Solution:

$\frac{5}{2}v = 50; v = 20$

Exercise:

Problem: The sum of nine-tenths and *g* is two-thirds.

Exercise:

Problem: The sum of two-fifths and *f* is one-half.

Solution:

$\frac{2}{5} + f = \frac{1}{2}; f = \frac{1}{10}$

Exercise:

Problem: The difference of *p* and one-sixth is two-thirds.

Exercise:

Problem: The difference of *q* and one-eighth is three-fourths.

Solution:

$q - \frac{1}{8} = \frac{3}{4}; q = \frac{7}{8}$

Translate and Solve Applications

In the following exercises, translate into an equation and solve.

Exercise:

Problem:

Kindergarten Connie’s kindergarten class has 24 children. She wants them to get into 4 equal groups. How many children will she put in each group?

Exercise:

Problem:

Balloons Ramona bought 18 balloons for a party. She wants to make 3 equal bunches. How many balloons did she use in each bunch?

Solution:

6 balloons

Exercise:

Problem: **Tickets** Mollie paid $36.25 for 5 movie tickets. What was the price of each ticket?

Exercise:

Problem:

Shopping Serena paid $12.96 for a pack of 12 pairs of sport socks. What was the price of pair of sport socks?

Solution:

$1.08

Exercise:

Problem:

Sewing Nancy used 14 yards of fabric to make flags for one-third of the drill team. How much fabric, would Nancy need to make flags for the whole team?

Exercise:

Problem:

MPG John's SUV gets 18 miles per gallon (mpg). This is half as many mpg as his wife's hybrid car. How many miles per gallon does the hybrid car get?

Solution:

36 mpg

Exercise:

Problem: **Height** Aiden is 27 inches tall. He is $\frac{3}{8}$ as tall as his father. How tall is his father?

Exercise:

Problem:

Real estate Bea earned $11,700 commission for selling a house, calculated as $\frac{6}{100}$ of the selling price. What was the selling price of the house?

Solution:

$195,000

Everyday Math

Exercise:

Problem:

Commission Every week Perry gets paid $150 plus 12% of his total sales amount. Solve the equation $840=150+0.12(a-1250)$ for a, to find the total amount Perry must sell in order to be paid $840 one week.

Exercise:

Problem:

Stamps Travis bought $9.45 worth of 49-cent stamps and 21-cent stamps. The number of 21-cent stamps was 5 less than the number of 49-cent stamps. Solve the equation $0.49s+0.21(s-5)=9.45$ for s, to find the number of 49-cent stamps Travis bought.

Solution:

15 49-cent stamps

Writing Exercises

Exercise:

Problem:

Frida started to solve the equation $-3x=36$ by adding 3 to both sides. Explain why Frida’s method will not solve the equation.

Exercise:

Problem:

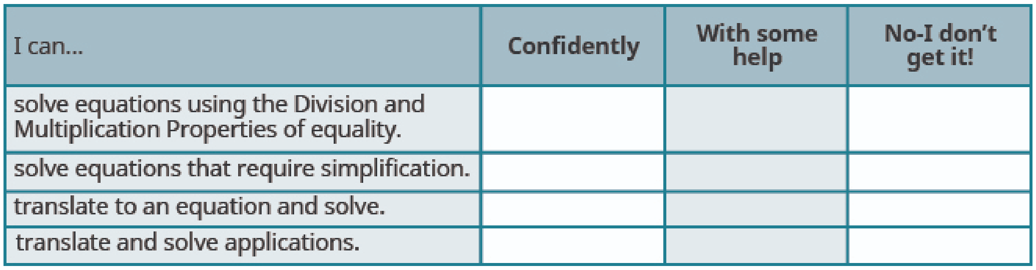
Emiliano thinks $x=40$ is the solution to the equation $\frac{1}{2}x=80$. Explain why he is wrong.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
solve equations using the Division and Multiplication Properties of equality.			
solve equations that require simplification.			
translate to an equation and solve.			
translate and solve applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Solve Equations with Variables and Constants on Both Sides
By the end of this section, you will be able to:

- Solve an equation with constants on both sides
- Solve an equation with variables on both sides
- Solve an equation with variables and constants on both sides

Note:
Before you get started, take this readiness quiz.

1. Simplify: $4y-9+9$.
   If you missed this problem, review [link].

Solve Equations with Constants on Both Sides

In all the equations we have solved so far, all the variable terms were on only one side of the equation with the constants on the other side. This does not happen all the time—so now we will learn to solve equations in which the variable terms, or constant terms, or both are on both sides of the equation.

Our strategy will involve choosing one side of the equation to be the “variable side”, and the other side of the equation to be the “constant side.” Then, we will use the Subtraction and Addition Properties of Equality to get all the variable terms together on one side of the equation and the constant terms together on the other side.

By doing this, we will transform the equation that began with variables and constants on both sides into the form $ax=b$. We already know how to solve equations of this form by using the Division or Multiplication Properties of Equality.

Example:
Exercise:

Problem: Solve: $7x + 8 = -13$.

Solution:
Solution

In this equation, the variable is found only on the left side. It makes sense to call the left side the “variable” side. Therefore, the right side will be the “constant” side. We will write the labels above the equation to help us remember what goes where.

$$\begin{array}{cc} \text{variable} & \text{constant} \\ 7x + 8 = & -13 \end{array}$$

Since the left side is the “x”, or variable side, the 8 is out of place. We must “undo” adding 8 by subtracting 8, and to keep the equality we must subtract 8 from both sides.

	$\begin{array}{cc} \text{variable} & \text{constant} \\ 7x + 8 = & -13 \end{array}$
Use the Subtraction Property of Equality.	$7x + 8 - 8 = -13 - 8$
Simplify.	$7x = -21$
Now all the variables are on the left and the constant on the right.	

The equation looks like those you learned to solve earlier.		
Use the Division Property of Equality.		$\frac{7x}{7} = \frac{-21}{7}$
Simplify.		$x = -3$
Check:	$7x + 8 = -13$	
Let $x = -3$.	$7(-3) + 8 \overset{?}{=} -13$	
	$-21 + 8 \overset{?}{=} -13$ $-13 = -13$ ✓	

Note:
Exercise:

Problem: Solve: $3x + 4 = -8$.

Solution:

$x = -4$

Note:
Exercise:

Problem: Solve: $5a+3=-37$.

Solution:

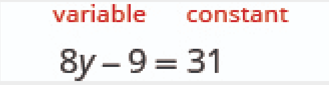
$a=-8$

Example:
Exercise:

Problem: Solve: $8y-9=31$.

Solution:
Solution

Notice, the variable is only on the left side of the equation, so we will call this side the “variable” side, and the right side will be the “constant” side. Since the left side is the “variable” side, the 9 is out of place. It is subtracted from the $8y$, so to “undo” subtraction, add 9 to both sides. Remember, whatever you do to the left, you must do to the right.

	 $8y-9=31$
Add 9 to both sides.	

		$8y-9+9=31+9$
Simplify.		$8y=40$
		The variables are now on one side and the constants on the other. We continue from here as we did earlier.
Divide both sides by 8.		$\frac{8y}{8}=\frac{40}{8}$
Simplify.		$y=5$
Check:	$8y-9=31$	
Let $y=5$.	$8\cdot 5-9\overset{?}{=}31$	
	$40-9\overset{?}{=}31$	
	$31=31$ ✓	

Note:
Exercise:

Problem: Solve: $5y - 9 = 16$.

Solution:

$y = 5$

Note:
Exercise:

Problem: Solve: $3m - 8 = 19$.

Solution:

$m = 9$

Solve Equations with Variables on Both Sides

What if there are variables on both sides of the equation? For equations like this, begin as we did above—choose a “variable” side and a “constant” side, and then use the subtraction and addition properties of equality to collect all variables on one side and all constants on the other side.

Example:
Exercise:

Problem: Solve: $9x = 8x - 6$.

Solution:
Solution

Here the variable is on both sides, but the constants only appear on the right side, so let’s make the right side the “constant” side. Then the left side will be the “variable” side.

		$\overset{\text{variable}}{9x} = \overset{\text{constant}}{8x - 6}$
We don’t want any x’s on the right, so subtract the $8x$ from both sides.		$9x - 8x = 8x - 8x - 6$
Simplify.		$x = -6$
We succeeded in getting the variables on one side and the constants on the other, and have obtained the solution.		
Check:	$9x = 8x - 6$	
Let $x = -6$.	$9(-6) \stackrel{?}{=} 8(-6) - 6$	
	$-54 \stackrel{?}{=} -48 - 6$	
	$-54 = -54\ \checkmark$	

Note:
Exercise:

Problem: Solve: $6n = 5n - 10$.

Solution:

$n = -10$

Note:
Exercise:

Problem: Solve: $-6c = -7c - 1$.

Solution:

$c = -1$

Example:
Exercise:

Problem: Solve: $5y - 9 = 8y$.

Solution:
Solution

The only constant is on the left and the y’s are on both sides. Let’s leave the constant on the left and get the variables to the right.

		$5y \overset{\text{constant}}{-9} = \overset{\text{variable}}{8y}$
Subtract $5y$ from both sides.		$5y - 5y - 9 = 8y - 5y$
Simplify.		$-9 = 3y$
We have the y’s on the right and the constants on the left. Divide both sides by 3.		$\frac{-9}{3} = \frac{3y}{3}$
Simplify.		$-3 = y$
Check:	$5y - 9 = 8y$	
Let $y = -3$.	$5(-3) - 9 \overset{?}{=} 8(-3)$	
	$-15 - 9 \overset{?}{=} -24$	
	$-24 = -24$ ✓	

Note:
Exercise:

Problem: Solve: $3p - 14 = 5p$.

Solution:

$p = -7$

Note:
Exercise:

Problem: Solve: $8m + 9 = 5m$.

Solution:

$m = -3$

Example:
Exercise:

Problem: Solve: $12x = -x + 26$.

Solution:
Solution

The only constant is on the right, so let the left side be the “variable” side.

	$\overset{\text{variable}}{12x} = \overset{\text{constant}}{-x + 26}$
Remove the $-x$ from the right side by adding x to both sides.	$12x + x = -x + x + 26$
Simplify.	$13x = 26$
All the x's are on the left and the constants are on the right. Divide both sides by 13.	$\frac{13x}{13} = \frac{26}{13}$
Simplify.	$x = 2$

Note:
Exercise:

Problem: Solve: $12j = -4j + 32$.

Solution:

$j = 2$

Note:
Exercise:

Problem: Solve: $8h = -4h + 12$.

Solution:

$h = 1$

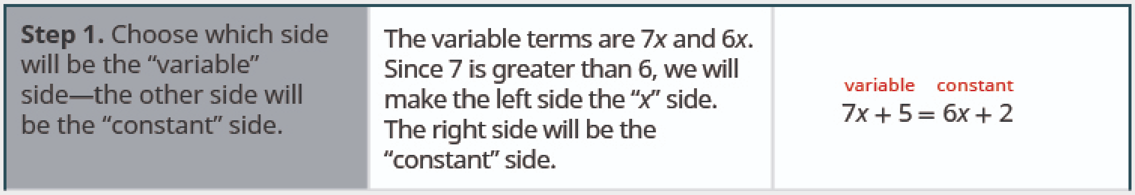
Solve Equations with Variables and Constants on Both Sides

The next example will be the first to have variables and constants on both sides of the equation. It may take several steps to solve this equation, so we need a clear and organized strategy.

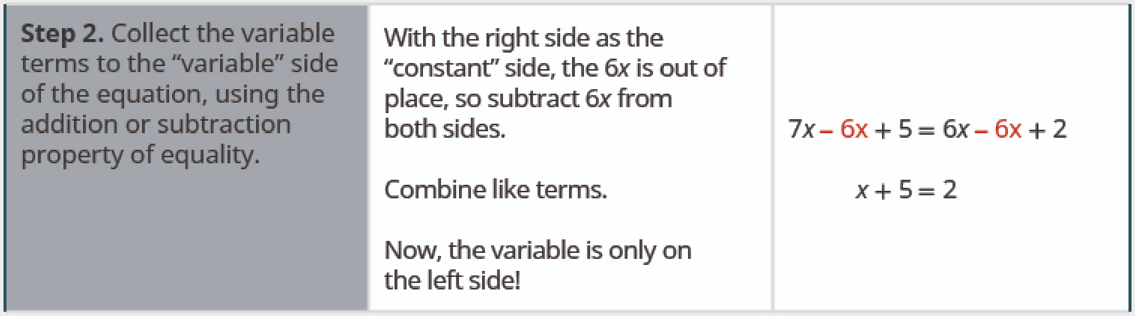
Example:
How to Solve Equations with Variables and Constants on Both Sides
Exercise:

Problem: Solve: $7x + 5 = 6x + 2$.

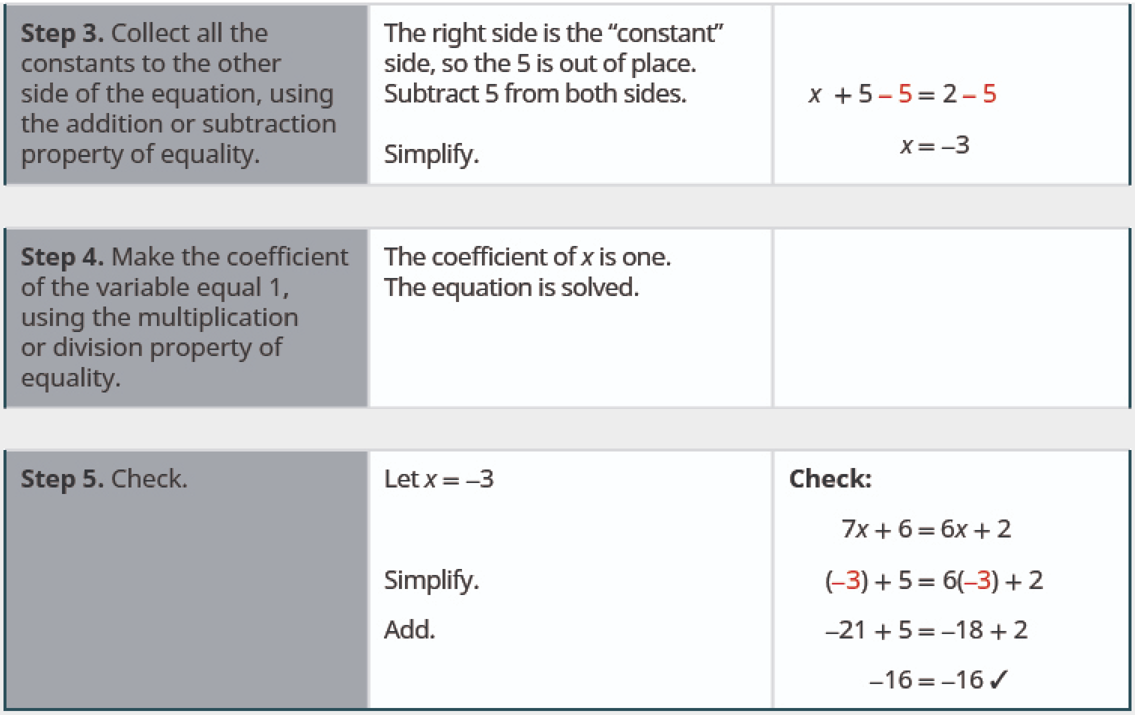
Solution:
Solution



Step 1. Choose which side will be the "variable" side—the other side will be the "constant" side.	The variable terms are 7*x* and 6*x*. Since 7 is greater than 6, we will make the left side the "*x*" side. The right side will be the "constant" side.	$\overset{\text{variable}}{7x+5} = \overset{\text{constant}}{6x+2}$



Step 2. Collect the variable terms to the "variable" side of the equation, using the addition or subtraction property of equality.	With the right side as the "constant" side, the 6*x* is out of place, so subtract 6*x* from both sides. Combine like terms. Now, the variable is only on the left side!	$7x - 6x + 5 = 6x - 6x + 2$ $x + 5 = 2$



Step 3. Collect all the constants to the other side of the equation, using the addition or subtraction property of equality.	The right side is the “constant” side, so the 5 is out of place. Subtract 5 from both sides. Simplify.	$x + 5 - 5 = 2 - 5$ $x = -3$
Step 4. Make the coefficient of the variable equal 1, using the multiplication or division property of equality.	The coefficient of *x* is one. The equation is solved.	
Step 5. Check.	Let $x = -3$ Simplify. Add.	**Check:** $7x + 6 = 6x + 2$ $(-3) + 5 = 6(-3) + 2$ $-21 + 5 = -18 + 2$ $-16 = -16$ ✓

Note:
Exercise:

Problem: Solve: $12x + 8 = 6x + 2$.

Solution:

$x = -1$

Note:
Exercise:

Problem: Solve: $9y + 4 = 7y + 12$.

Solution:

$y = 4$

We’ll list the steps below so you can easily refer to them. But we’ll call this the ‘Beginning Strategy’ because we’ll be adding some steps later in this chapter.

Note:
Beginning Strategy for Solving Equations with Variables and Constants on Both Sides of the Equation.

Choose which side will be the “variable” side—the other side will be the “constant” side.
Collect the variable terms to the “variable” side of the equation, using the Addition or Subtraction Property of Equality.
Collect all the constants to the other side of the equation, using the Addition or Subtraction Property of Equality.
Make the coefficient of the variable equal 1, using the Multiplication or Division Property of Equality.
Check the solution by substituting it into the original equation.

In Step 1, a helpful approach is to make the “variable” side the side that has the variable with the larger coefficient. This usually makes the arithmetic easier.

Example:
Exercise:

Problem: Solve: $8n - 4 = -2n + 6$.

Solution:
Solution

In the first step, choose the variable side by comparing the coefficients of the variables on each side.

Since $8 > -2$, make the left side the “variable” side.		$\overset{\text{variable}}{8n-4} = \overset{\text{constant}}{-2n+6}$
We don’t want variable terms on the right side—add $2n$ to both sides to leave only constants on the right.		$8n + 2n - 4 = -2n + 2n + 6$
Combine like terms.		$10n - 4 = 6$
We don’t want any constants on the left side, so add 4 to both sides.		$10n - 4 + 4 = 6 + 4$
Simplify.		$10n = 10$
The variable term is on the left and the constant term is on the right. To get the coefficient of n to be one, divide both sides by 10.		$\frac{10n}{10} = \frac{10}{10}$
Simplify.		$n = 1$
Check:	$8n - 4 = -2n + 6$	

Let $n = 1$.	$8 \cdot 1 - 4 \overset{?}{=} -2 \cdot 1 + 6$	
	$8 - 4 \overset{?}{=} -2 + 6$	
	$4 = 4 \checkmark$	

Note:
Exercise:

Problem: Solve: $8q - 5 = -4q + 7$.

Solution:

$q = 1$

Note:
Exercise:

Problem: Solve: $7n - 3 = n + 3$.

Solution:

$n = 1$

Example:
Exercise:

Problem: Solve: $7a - 3 = 13a + 7$.

Solution:
Solution

In the first step, choose the variable side by comparing the coefficients of the variables on each side.

Since $13 > 7$, make the right side the "variable" side and the left side the "constant" side.

	$\overset{\text{constant}}{7a-3} = \overset{\text{variable}}{13a+7}$
Subtract $7a$ from both sides to remove the variable term from the left.	$7a - 7a - 3 = 13a - 7a + 7$
Combine like terms.	$-3 = 6a + 7$
Subtract 7 from both sides to remove the constant from the right.	$-3 - 7 = 6a + 7 - 7$
Simplify.	$-10 = 6a$
Divide both sides by 6 to make 1 the coefficient of a.	$\frac{-10}{6} = \frac{6a}{6}$

Simplify.		$-\frac{5}{3} = a$
Check:	$7a - 3 = 13a + 7$	
Let $a = -\frac{5}{3}$.	$7\left(-\frac{5}{3}\right) - 3 \overset{?}{=} 13\left(-\frac{5}{3}\right) + 7$	
	$-\frac{35}{3} - \frac{9}{3} \overset{?}{=} -\frac{65}{3} + \frac{21}{3}$	
	$-\frac{54}{3} = -\frac{54}{3}$ ✓	

Note:
Exercise:

Problem: Solve: $2a - 2 = 6a + 18$.

Solution:

$a = -5$

Note:

Exercise:

Problem: Solve: $4k - 1 = 7k + 17$.

Solution:

$k = -6$

In the last example, we could have made the left side the "variable" side, but it would have led to a negative coefficient on the variable term. (Try it!) While we could work with the negative, there is less chance of errors when working with positives. The strategy outlined above helps avoid the negatives!

To solve an equation with fractions, we just follow the steps of our strategy to get the solution!

Example:
Exercise:

Problem: Solve: $\frac{5}{4}x + 6 = \frac{1}{4}x - 2$.

Solution:
Solution

Since $\frac{5}{4} > \frac{1}{4}$, make the left side the "variable" side and the right side the "constant" side.

	variable constant $\frac{5}{4}x+6=\frac{1}{4}x-2$
Subtract $\frac{1}{4}x$ from both sides.	$\frac{5}{4}x-\frac{1}{4}x+6=\frac{1}{4}x-\frac{1}{4}x-2$
Combine like terms.	$x+6=-2$
Subtract 6 from both sides.	$x+6-6=-2-6$
Simplify.	$x=-8$
Check: Let $x=-8$.	$\frac{5}{4}x+6=\frac{1}{4}x-2$ $\frac{5}{4}(-8)+6\overset{?}{=}\frac{1}{4}(-8)-2$ $-10+6\overset{?}{=}-2-2$ $-4=-4$✓

Note:
Exercise:

Problem: Solve: $\frac{7}{8}x-12=-\frac{1}{8}x-2.$

Solution:

$x=10$

Note:

Exercise:

Problem: Solve: $\frac{7}{6}y + 11 = \frac{1}{6}y + 8$.

Solution:

$y = -3$

We will use the same strategy to find the solution for an equation with decimals.

Example:
Exercise:

Problem: Solve: $7.8x + 4 = 5.4x - 8$.

Solution:
Solution

Since $7.8 > 5.4$, make the left side the “variable” side and the right side the “constant” side.

	$\overset{\text{variable side}}{7.8x + 4} = \overset{\text{constant side}}{5.4x - 8}$
Subtract $5.4x$ from both sides.	$7.8x - 5.4x + 4 = 5.4x - 5.4x - 8$

<table>
<tr><td colspan="2">Combine like terms.</td><td>$2.4x + 4 = -8$</td></tr>
<tr><td colspan="2">Subtract 4 from both sides.</td><td>$2.4x + 4 - 4 = -8 - 4$</td></tr>
<tr><td colspan="2">Simplify.</td><td>$2.4x = -12$</td></tr>
<tr><td colspan="2">Use the Division Propery of Equality.</td><td>$\frac{2.4x}{2.4} = \frac{-12}{2.4}$</td></tr>
<tr><td colspan="2">Simplify.</td><td>$x = -5$</td></tr>
<tr><td>Check:</td><td>$7.8x + 4 = 5.4x - 5$</td><td></td></tr>
<tr><td>Let $x = -5$.</td><td>$7.8(-5) + 4 = 5.4(-5) - 8$</td><td></td></tr>
<tr><td></td><td>$-39 + 4 \overset{?}{=} -27 - 8$</td><td></td></tr>
<tr><td></td><td>$-35 = -35$ ✓</td><td></td></tr>
</table>

Note:
Exercise:

Problem: Solve: $2.8x + 12 = -1.4x - 9$.

Solution:

$x = -5$

Note:
Exercise:

Problem: Solve: $3.6y + 8 = 1.2y - 4$.

Solution:

$y = -5$

Key Concepts

- **Beginning Strategy for Solving an Equation with Variables and Constants on Both Sides of the Equation**

  Choose which side will be the “variable” side—the other side will be the “constant” side.
  Collect the variable terms to the “variable” side of the equation, using the Addition or Subtraction Property of Equality.
  Collect all the constants to the other side of the equation, using the Addition or Subtraction Property of Equality.
  Make the coefficient of the variable equal 1, using the Multiplication or Division Property of Equality.
  Check the solution by substituting it into the original equation.

Practice Makes Perfect

Solve Equations with Constants on Both Sides

In the following exercises, solve the following equations with constants on both sides.

Exercise:

Problem: $9x - 3 = 60$

Exercise:

Problem: $12x - 8 = 64$

Solution:

$x = 6$

Exercise:

Problem: $14w + 5 = 117$

Exercise:

Problem: $15y + 7 = 97$

Solution:

$y = 6$

Exercise:

Problem: $2a + 8 = -28$

Exercise:

Problem: $3m + 9 = -15$

Solution:

$m = -8$

Exercise:

Problem: $-62 = 8n - 6$

Exercise:

Problem: $-77 = 9b - 5$

Solution:

$b = -8$

Exercise:

Problem: $35 = -13y + 9$

Exercise:

Problem: $60 = -21x - 24$

Solution:

$x = -4$

Exercise:

Problem: $-12p - 9 = 9$

Exercise:

Problem: $-14q - 2 = 16$

Solution:

$q = -\frac{9}{7}$

Solve Equations with Variables on Both Sides

In the following exercises, solve the following equations with variables on both sides.

Exercise:

Problem: $19z = 18z - 7$

Exercise:

Problem: $21k = 20k - 11$

Solution:

$k = -11$

Exercise:

Problem: $9x + 36 = 15x$

Exercise:

Problem: $8x + 27 = 11x$

Solution:

$x = 9$

Exercise:

Problem: $c = -3c - 20$

Exercise:

Problem: $b = -4b - 15$

Solution:

$b = -3$

Exercise:

Problem: $9q = 44 - 2q$

Exercise:

Problem: $5z = 39 - 8z$

Solution:

$z = 3$

Exercise:

Problem: $6y + \frac{1}{2} = 5y$

Exercise:

Problem: $4x + \frac{3}{4} = 3x$

Solution:

$x = -\frac{3}{4}$

Exercise:

Problem: $-18a - 8 = -22a$

Exercise:

Problem: $-11r - 8 = -7r$

Solution:

$r = -2$

Solve Equations with Variables and Constants on Both Sides

In the following exercises, solve the following equations with variables and constants on both sides.

Exercise:

Problem: $8x - 15 = 7x + 3$

Exercise:

Problem: $6x - 17 = 5x + 2$

Solution:

$x = 19$

Exercise:

Problem: $26 + 13d = 14d + 11$

Exercise:

Problem: $21 + 18f = 19f + 14$

Solution:

$f = 7$

Exercise:

Problem: $2p - 1 = 4p - 33$

Exercise:

Problem: $12q - 5 = 9q - 20$

Solution:

$q = -5$

Exercise:

Problem: $4a+5=-a-40$

Exercise:

Problem: $8c+7=-3c-37$

Solution:

$c=-4$

Exercise:

Problem: $5y-30=-5y+30$

Exercise:

Problem: $7x-17=-8x+13$

Solution:

$x=2$

Exercise:

Problem: $7s+12=5+4s$

Exercise:

Problem: $9p+14=6+4p$

Solution:

$p=-\frac{8}{5}$

Exercise:

Problem: $2z-6=23-z$

Exercise:

Problem: $3y-4=12-y$

Solution:

$y=4$

Exercise:

Problem: $\frac{5}{3}c-3=\frac{2}{3}c-16$

Exercise:

Problem: $\frac{7}{4}m-7=\frac{3}{4}m-13$

Solution:

$m=-6$

Exercise:

Problem: $8-\frac{2}{5}q=\frac{3}{5}q+6$

Exercise:

Problem: $11-\frac{1}{5}a=\frac{4}{5}a+4$

Solution:

$a=7$

Exercise:

Problem: $\frac{4}{3}n+9=\frac{1}{3}n-9$

Exercise:

Problem: $\frac{5}{4}a+15=\frac{3}{4}a-5$

Solution:

$a=-40$

Exercise:

Problem: $\frac{1}{4}y+7=\frac{3}{4}y-3$

Exercise:

Problem: $\frac{3}{5}p+2=\frac{4}{5}p-1$

Solution:

$p=15$

Exercise:

Problem: $14n+8.25=9n+19.60$

Exercise:

Problem: $13z+6.45=8z+23.75$

Solution:

$z=3.46$

Exercise:

Problem: $2.4w-100=0.8w+28$

Exercise:

Problem: $2.7w-80=1.2w+10$

Solution:

$w = 60$

Exercise:

Problem: $5.6r + 13.1 = 3.5r + 57.2$

Exercise:

Problem: $6.6x - 18.9 = 3.4x + 54.7$

Solution:

$x = 23$

Everyday Math

Exercise:

Problem:

Concert tickets At a school concert the total value of tickets sold was \$1506. Student tickets sold for \$6 and adult tickets sold for \$9. The number of adult tickets sold was 5 less than 3 times the number of student tickets. Find the number of student tickets sold, s, by solving the equation $6s + 9(3s - 5) = 1506$.

Exercise:

Problem:

Making a fence Jovani has 150 feet of fencing to make a rectangular garden in his backyard. He wants the length to be 15 feet more than the width. Find the width, w, by solving the equation $150 = 2w + 2(w + 15)$.

Solution:

30 feet

Writing Exercises

Exercise:

Problem:

Solve the equation $\frac{6}{5}y - 8 = \frac{1}{5}y + 7$ explaining all the steps of your solution as in the examples in this section.

Exercise:

Problem:

Solve the equation $10x + 14 = -2x + 38$ explaining all the steps of your solution as in the examples in this section.

Solution:

$x = 2$ Justifications will vary.

Exercise:

Problem:

When solving an equation with variables on both sides, why is it usually better to choose the side with the larger coefficient of x to be the “variable” side?

Exercise:

Problem:

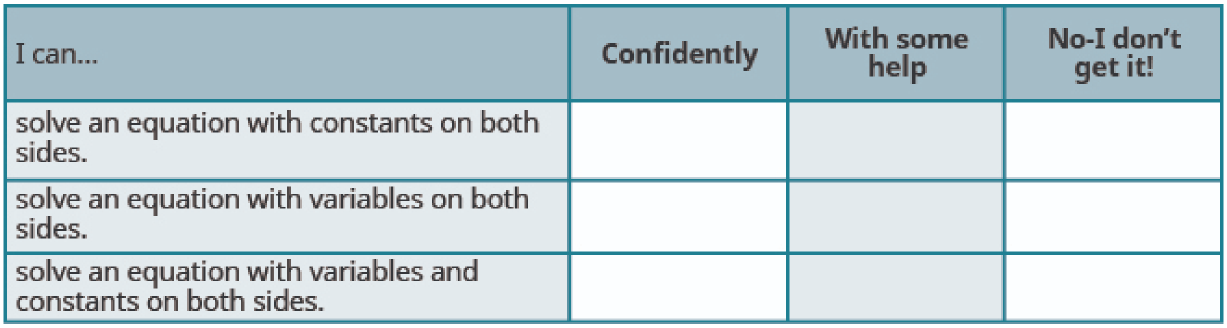
Is $x = -2$ a solution to the equation $5 - 2x = -4x + 1$? How do you know?

Solution:

Yes. Justifications will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
solve an equation with constants on both sides.			
solve an equation with variables on both sides.			
solve an equation with variables and constants on both sides.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Use a General Strategy to Solve Linear Equations
By the end of this section, you will be able to:

- Solve equations using a general strategy
- Classify equations

Note:
Before you get started, take this readiness quiz.

1. Simplify: $-(a-4)$.
   If you missed this problem, review [link].
2. Multiply: $\frac{3}{2}(12x+20)$.
   If you missed this problem, review [link].
3. Simplify: $5-2(n+1)$.
   If you missed this problem, review [link].
4. Multiply: $3(7y+9)$.
   If you missed this problem, review [link].
5. Multiply: $(2.5)\,(6.4)$.
   If you missed this problem, review [link].

Solve Equations Using the General Strategy

Until now we have dealt with solving one specific form of a linear equation. It is time now to lay out one overall strategy that can be used to solve any linear equation. Some equations we solve will not require all these steps to solve, but many will.

Beginning by simplifying each side of the equation makes the remaining steps easier.

Example:

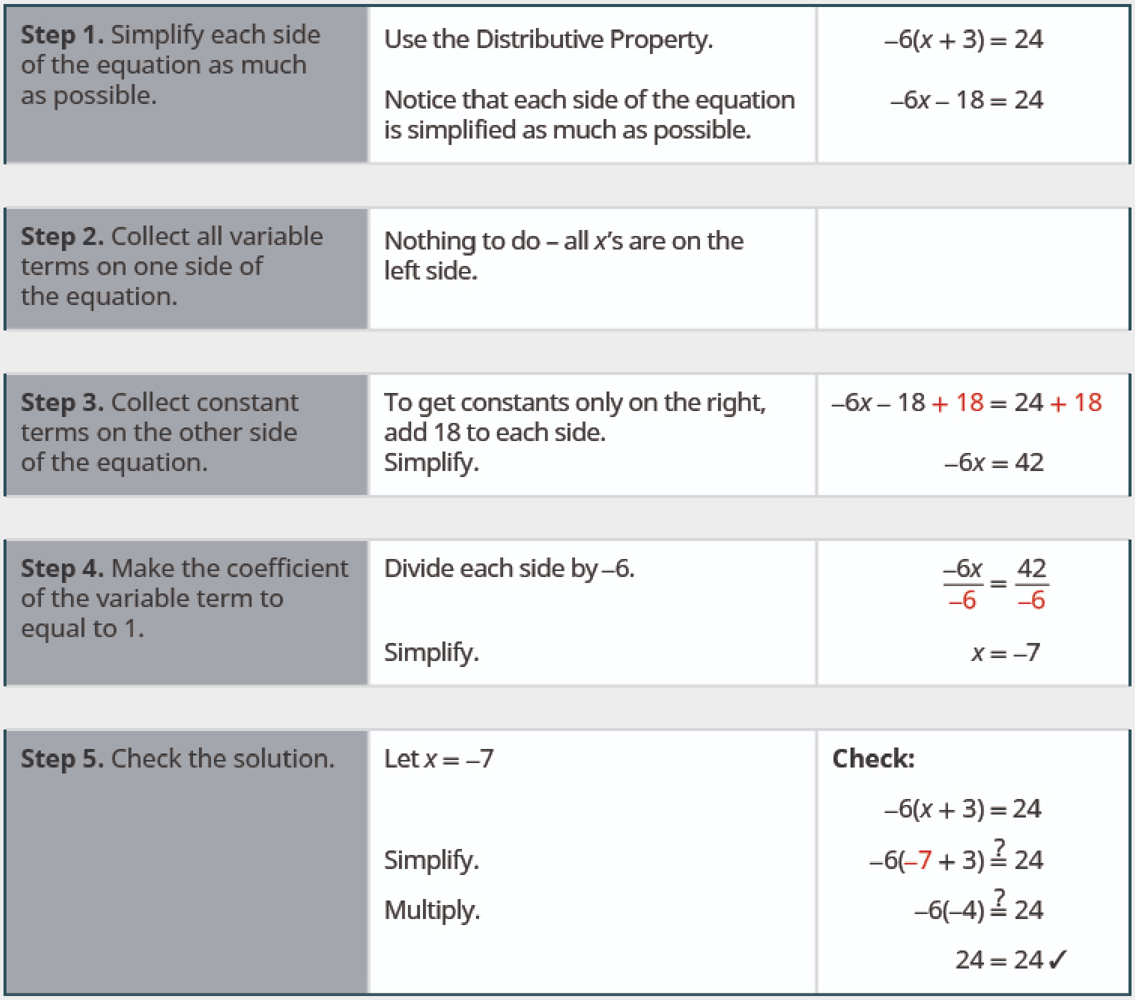
How to Solve Linear Equations Using the General Strategy

Exercise:

Problem: Solve: $-6\left(x+3\right)=24$.

Solution:

Solution



Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is simplified as much as possible.	$-6(x+3)=24$ $-6x-18=24$
Step 2. Collect all variable terms on one side of the equation.	Nothing to do – all x’s are on the left side.	
Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 18 to each side. Simplify.	$-6x-18+18=24+18$ $-6x=42$
Step 4. Make the coefficient of the variable term to equal to 1.	Divide each side by –6. Simplify.	$\frac{-6x}{-6}=\frac{42}{-6}$ $x=-7$
Step 5. Check the solution.	Let $x=-7$ Simplify. Multiply.	**Check:** $-6(x+3)=24$ $-6(-7+3)\overset{?}{=}24$ $-6(-4)\overset{?}{=}24$ $24=24$ ✓

Note:

Exercise:

Problem: Solve: $5\left(x+3\right)=35$.

Solution:

$x=4$

Note:
Exercise:

Problem: Solve: $6\left(y-4\right)=-18$.

Solution:

$y=1$

Note:
General strategy for solving linear equations.

Simplify each side of the equation as much as possible.	Use the Distributive Property to remove any parentheses.	Combine like terms.
Collect all the variable terms on one side of the equation.	Use the Addition or Subtraction Property of Equality.	
Collect all the constant terms on the other side of the equation.	Use the Addition or Subtraction Property of Equality.	
Make the coefficient of the variable term to equal to 1.	Use the Multiplication or Division Property of Equality.	State the solution to the equation.
Check the solution.	Substitute the solution into the original equation to make sure the result is a true statement.	

Example:
Exercise:

Problem: Solve: $-(y+9)=8$.

Solution:
Solution

	$-(y+9)=8$
Simplify each side of the equation as much as possible by distributing.	$-y-9=8$
The only y term is on the left side, so all variable terms are on the left side of the equation.	
Add 9 to both sides to get all constant terms on the right side of the equation.	$-y-9+9=8+9$
Simplify.	$-y=17$
Rewrite $-y$ as $-1y$.	$-1y=17$
Make the coefficient of the variable term to equal to 1 by dividing both sides by -1.	$\frac{-1y}{-1}=\frac{17}{-1}$
Simplify.	$y=-17$

Check:	$-(y+9)=8$	
Let $y=-17$.	$-(-17+9)\overset{?}{=}8$	
	$-(-8)\overset{?}{=}8$	
	$8=8$ ✓	

Note:
Exercise:

Problem: Solve: $-(y+8)=-2$.

Solution:

$y=-6$

Note:
Exercise:

Problem: Solve: $-(z+4)=-12$.

Solution:

$z = 8$

Example:
Exercise:

Problem: Solve: $5\left(a - 3\right) + 5 = -10.$

Solution:
Solution

	$5(a-3)+5=-10$
Simplify each side of the equation as much as possible.	
Distribute.	$5a-15+5=-10$
Combine like terms.	$5a-10=-10$
The only a term is on the left side, so all variable terms are on one side of the equation.	
Add 10 to both sides to get all constant terms on the other side of the equation.	$5a-10+10=-10+10$
Simplify.	$5a=0$

Make the coefficient of the variable term to equal to 1 by dividing both sides by 5.		$\frac{5a}{5}=\frac{0}{5}$
Simplify.		$a=0$
Check:	$5(a-3)+5=-10$	
Let $a=0$.	$5(0-3)+5\overset{?}{=}-10$	
	$5(-3)+5\overset{?}{=}-10$	
	$-15+5\overset{?}{=}-10$	
	$-10=-10$ ✓	

Note:
Exercise:

Problem: Solve: $2(m-4)+3=-1$.

Solution:

$m=2$

Note:
Exercise:

Problem: Solve: $7\left(n-3\right)-8=-15.$

Solution:

$n=2$

Example:
Exercise:

Problem: Solve: $\frac{2}{3}\left(6m-3\right)=8-m.$

Solution:
Solution

	$\frac{2}{3}(6m-3)=8-m$
Distribute.	$4m-2=8-m$
Add m to get the variables only to the left.	$4m+m-2=8-m+m$

Simplify.		$5m-2=8$
Add 2 to get constants only on the right.		$5m-2+2=8+2$
Simplify.		$5m=10$
Divide by 5.		$\frac{5m}{5}=\frac{10}{5}$
Simplify.		$m=2$
Check:	$\frac{2}{3}(6m-3)=8-m$	
Let $m=2$.	$\frac{2}{3}(6\cdot 2-3)\overset{?}{=}8-2$	
	$\frac{2}{3}(12-3)\overset{?}{=}6$	
	$\frac{2}{3}(9)\overset{?}{=}6$	
	$6=6$ ✓	

Note:
Exercise:

Problem: Solve: $\frac{1}{3}(6u+3)=7-u$.

Solution:

$u=2$

Note:
Exercise:

Problem: Solve: $\frac{2}{3}(9x-12)=8+2x$.

Solution:

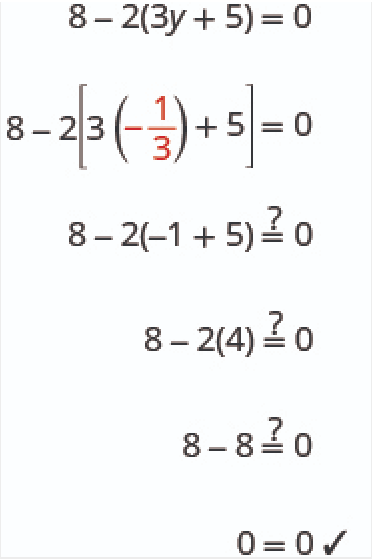
$x=4$

Example:
Exercise:

Problem: Solve: $8-2\left(3y+5\right)=0$.

Solution:
Solution

	$8 - 2(3y + 5) = 0$
Simplify—use the Distributive Property.	$8 - 6y - 10 = 0$
Combine like terms.	$-6y - 2 = 0$
Add 2 to both sides to collect constants on the right.	$-6y - 2 + 2 = 0 + 2$
Simplify.	$-6y = 2$
Divide both sides by -6.	$\frac{-6y}{-6} = \frac{2}{-6}$
Simplify.	$y = -\frac{1}{3}$
Check: Let $y = -\frac{1}{3}$.	



$$8-2(3y+5)=0$$

$$8-2\left[3\left(-\frac{1}{3}\right)+5\right]=0$$

$$8-2(-1+5)\overset{?}{=}0$$

$$8-2(4)\overset{?}{=}0$$

$$8-8\overset{?}{=}0$$

$$0=0\checkmark$$

Note:
Exercise:

Problem: Solve: $12-3\left(4j+3\right)=-17$.

Solution:

$j=\frac{5}{3}$

Note:
Exercise:

Problem: Solve: $-6-8\left(k-2\right)=-10$.

Solution:

$k=\frac{5}{2}$

Example:
Exercise:

Problem: Solve: $4\left(x-1\right)-2=5\left(2x+3\right)+6$.

Solution:
Solution

	$4(x-1)-2=5(2x+3)+6$
Distribute.	$4x-4-2=10x+15+6$
Combine like terms.	$4x-6=10x+21$
Subtract $4x$ to get the variables only on the right side since $10>4$.	$4x-4x-6=10x-4x+21$
Simplify.	$-6=6x+21$
Subtract 21 to get the constants on left.	$-6-21=6x+21-21$
Simplify.	$-27=6x$

Divide by 6.		$\frac{-27}{\color{red}6}=\frac{6x}{\color{red}6}$
Simplify.		$-\frac{9}{2}=x$
Check:	$4(x-1)-2=5(2x+3)+6$	
Let $x=-\frac{9}{2}$.	$4\left({\color{red}-\frac{9}{2}}-1\right)-2\overset{?}{=}5\left[2\left({\color{red}-\frac{9}{2}}\right)+3\right]+6$	
	$4\left(-\frac{11}{2}\right)-2\overset{?}{=}5(-9+3)+6$	
	$-22-2\overset{?}{=}5(-6)+6$	
	$-24\overset{?}{=}-30+6$	
	$-24=-24\checkmark$	

Note:
Exercise:

Problem: Solve: $6\left(p-3\right)-7=5\left(4p+3\right)-12.$

Solution:

$p=-2$

Note:
Exercise:

Problem: Solve: $8\left(q+1\right)-5=3\left(2q-4\right)-1.$

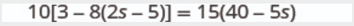
Solution:

$q=-8$

Example:
Exercise:

Problem: Solve: $10\left[3-8\left(2s-5\right)\right]=15\left(40-5s\right).$

Solution:
Solution

	$10[3-8(2s-5)]=15(40-5s)$

Simplify from the innermost parentheses first.	$10[3-16s+40]=15(40-5s)$
Combine like terms in the brackets.	$10[43-16s]=15(40-5s)$
Distribute.	$430-160s=600-75s$
Add $160s$ to get the s’s to the right.	$430-160s+160s=600-75s+160s$
Simplify.	$430=600+85s$
Subtract 600 to get the constants to the left.	$430-600=600+85s-600$
Simplify.	$-170=85s$
Divide.	$\frac{-170}{85}=\frac{85s}{85}$
Simplify.	$-2=s$
Check:	$10[3-8(2s-5)]=15(40-5s)$
Substitute $s=-2$.	$10[3-8(2(-2)-5)]\stackrel{?}{=}15(40-5(-2))$
	$10[3-8(-4-5)]\stackrel{?}{=}15(40+10)$

	$10[3-8(-9)] \overset{?}{=} 15(50)$	
	$10[3+72] \overset{?}{=} 750$	
	$10[75] \overset{?}{=} 750$	
	$750 = 750$ ✓	

Note:
Exercise:

Problem: Solve: $6\left[4-2\left(7y-1\right)\right]=8\left(13-8y\right)$.

Solution:

$y=-\frac{17}{5}$

Note:
Exercise:

Problem: Solve: $12\left[1-5\left(4z-1\right)\right]=3\left(24+11z\right)$.

Solution:

$z=0$

Example:
Exercise:

Problem: Solve: $0.36(100n+5)=0.6(30n+15)$.

Solution:
Solution

	$0.36(100n+5)=0.6(30n+15)$
Distribute.	$36n+1.8=18n+9$
Subtract $18n$ to get the variables to the left.	$36n-18n+1.8=18n-18n+9$
Simplify.	$18n+1.8=9$
Subtract 1.8 to get the constants to the right.	$18n+1.8-1.8=9-1.8$
Simplify.	$18n=7.2$
Divide.	

		$\frac{18n}{\color{red}18}=\frac{7.2}{\color{red}18}$
Simplify.		$n=0.4$
Check:	$0.36(100n+5)=0.6(30n+15)$	
Let $n=0.4$.	$0.36(100({\color{red}0.4})+5)\overset{?}{=}0.6(30({\color{red}0.4})+15)$	
	$0.36(40+5)\overset{?}{=}0.6(12+15)$	
	$0.36(45)\overset{?}{=}0.6(27)$	
	$16.2=16.2$ ✓	

Note:
Exercise:

Problem: Solve: $0.55(100n+8)=0.6(85n+14)$.

Solution:

$n=1$

Note:
Exercise:

Problem: Solve: $0.15(40m-120)=0.5(60m+12)$.

Solution:

$m=-1$

Classify Equations

Consider the equation we solved at the start of the last section, $7x+8=-13$. The solution we found was $x=-3$. This means the equation $7x+8=-13$ is true when we replace the variable, *x*, with the value -3. We showed this when we checked the solution $x=-3$ and evaluated $7x+8=-13$ for $x=-3$.

$$\begin{aligned} 7(-3)+8 &\overset{?}{=} -13 \\ -21+8 &\overset{?}{=} -13 \\ -13 &= -13\ \checkmark \end{aligned}$$

If we evaluate $7x+8$ for a different value of *x*, the left side will not be -13.

The equation $7x+8=-13$ is true when we replace the variable, *x*, with the value -3, but not true when we replace *x* with any other value. Whether or not the equation $7x+8=-13$ is true depends on the value of the variable. Equations like this are called conditional equations.

All the equations we have solved so far are conditional equations.

Note:

Conditional equation
An equation that is true for one or more values of the variable and false for all other values of the variable is a **conditional equation**.

Now let’s consider the equation $2y+6=2\left(y+3\right)$. Do you recognize that the left side and the right side are equivalent? Let’s see what happens when we solve for *y*.

	$2y+6=2(y+3)$
Distribute.	$2y+6=2y+6$
Subtract $2y$ to get the y’s to one side.	$2y-2y+6=2y-2y+6$
Simplify—the y’s are gone!	$6=6$

But $6=6$ is true.

This means that the equation $2y+6=2\left(y+3\right)$ is true for any value of *y*. We say the solution to the equation is all of the real numbers. An equation that is true for any value of the variable like this is called an identity.

Note:

Identity
An equation that is true for any value of the variable is called an **identity**.
The solution of an identity is all real numbers.

What happens when we solve the equation $5z = 5z - 1$?

	$5z = 5z - 1$
Subtract $5z$ to get the constant alone on the right.	$5z - 5z = 5z - 5z - 1$
Simplify—the z’s are gone!	$0 \neq -1$

But $0 \neq -1$.

Solving the equation $5z = 5z - 1$ led to the false statement $0 = -1$. The equation $5z = 5z - 1$ will not be true for any value of z. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a contradiction.

Note:
Contradiction
An equation that is false for all values of the variable is called a **contradiction**.
A contradiction has no solution.

Example:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$$6(2n-1)+3=2n-8+5(2n+1)$$

Solution:
Solution

	$6(2n-1)+3=2n-8+5(2n+1)$
Distribute.	$12n-6+3=2n-8+10n+5$
Combine like terms.	$12n-3=12n-3$
Subtract $12n$ to get the n’s to one side.	$12n-12n-3=12n-12n-3$
Simplify.	$-3=-3$
This is a true statement.	The equation is an identity. The solution is all real numbers.

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$4 + 9(3x - 7) = -42x - 13 + 23(3x - 2)$$

Solution:

identity; all real numbers

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$8(1 - 3x) + 15(2x + 7) = 2(x + 50) + 4(x + 3) + 1$$

Solution:

identity; all real numbers

Example:
Exercise:

Problem:

Classify as a conditional equation, an identity, or a contradiction. Then state the solution.

$10+4(p-5)=0$

Solution:
Solution

	$10+4(p-5)=0$
Distribute.	$10+4p-20=0$
Combine like terms.	$4p-10=0$
Add 10 to both sides.	$4p-10+10=0+10$
Simplify.	$4p=10$
Divide.	$\frac{4p}{4}=\frac{10}{4}$
Simplify.	

	$p = \frac{5}{2}$
The equation is true when $p = \frac{5}{2}$.	This is a conditional equation. The solution is $p = \frac{5}{2}$.

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $11(q+3)-5=19$

Solution:

conditional equation; $q = -\frac{9}{11}$

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6+14(k-8)=95$

Solution:

conditional equation; $k = \frac{201}{14}$

Example:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$5m+3\left(9+3m\right)=2\left(7m-11\right)$

Solution:
Solution

	$5m+3(9+3m)=2(7m-11)$
Distribute.	$5m+27+9m=14m-22$
Combine like terms.	$14m+27=14m-22$
Subtract $14m$ from both sides.	$14m+27-14m=14m-22-14m$
Simplify.	$27\ne-22$
But $27\ne-22.$	The equation is a contradiction. It has no solution.

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$12c + 5(5 + 3c) = 3(9c - 4)$$

Solution:

contradiction; no solution

Note:
Exercise:

Problem:

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$4(7d + 18) = 13(3d - 2) - 11d$$

Solution:

contradiction; no solution

Type of equation	What happens when you solve it?	Solution
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	**True** for any value of the variable	All real numbers
Contradiction	**False** for all values of the variable	No solution

Key Concepts

- **General Strategy for Solving Linear Equations**

Simplify each side of the equation as much as possible.	Use the Distributive Property to remove any parentheses.	Combine like terms.
Collect all the variable terms on one side of the equation.	Use the Addition or Subtraction Property of Equality.	
Collect all the constant terms on the other side of the equation.	Use the Addition or Subtraction Property of Equality.	
Make the coefficient of the variable term to equal to 1.	Use the Multiplication or Division Property of Equality.	State the solution to the equation.
Check the solution.	Substitute the solution into the original equation.	

Practice Makes Perfect

Solve Equations Using the General Strategy for Solving Linear Equations

In the following exercises, solve each linear equation.

Exercise:

Problem: $15\left(y-9\right)=-60$

Exercise:

Problem: $21\left(y-5\right)=-42$

Solution:

$y=3$

Exercise:

Problem: $-9\left(2n+1\right)=36$

Exercise:

Problem: $-16\left(3n+4\right)=32$

Solution:

$n=-2$

Exercise:

Problem: $8\left(22+11r\right)=0$

Exercise:

Problem: $5\left(8+6p\right)=0$

Solution:

$p=-\frac{4}{3}$

Exercise:

Problem: $-(w-12)=30$

Exercise:

Problem: $-(t-19)=28$

Solution:

$t=-9$

Exercise:

Problem: $9\,(6a+8)+9=81$

Exercise:

Problem: $8\,(9b-4)-12=100$

Solution:

$b=2$

Exercise:

Problem: $32+3(z+4)=41$

Exercise:

Problem: $21+2(m-4)=25$

Solution:

$m=6$

Exercise:

Problem: $51+5(4-q)=56$

Exercise:

Problem: $-6+6(5-k)=15$

Solution:

$k=\frac{3}{2}$

Exercise:

Problem: $2(9s-6)-62=16$

Exercise:

Problem: $8(6t-5)-35=-27$

Solution:

$t=1$

Exercise:

Problem: $3(10-2x)+54=0$

Exercise:

Problem: $-2(11-7x)+54=4$

Solution:

$x=-2$

Exercise:

Problem: $\frac{2}{3}(9c-3)=22$

Exercise:

Problem: $\frac{3}{5}(10x-5)=27$

Solution:

$x=5$

Exercise:

Problem: $\frac{1}{5}(15c+10)=c+7$

Exercise:

Problem: $\frac{1}{4}(20d+12)=d+7$

Solution:

$d=1$

Exercise:

Problem: $18-(9r+7)=-16$

Exercise:

Problem: $15-(3r+8)=28$

Solution:

$r=-7$

Exercise:

Problem: $5-(n-1)=19$

Exercise:

Problem: $-3-(m-1)=13$

Solution:

$m = -15$

Exercise:

Problem: $11 - 4(y - 8) = 43$

Exercise:

Problem: $18 - 2(y - 3) = 32$

Solution:

$y = -4$

Exercise:

Problem: $24 - 8(3v + 6) = 0$

Exercise:

Problem: $35 - 5(2w + 8) = -10$

Solution:

$w = \frac{1}{2}$

Exercise:

Problem: $4(a - 12) = 3\left(a + 5\right)$

Exercise:

Problem: $-2(a - 6) = 4\left(a - 3\right)$

Solution:

$a = 4$

Exercise:

Problem: $2(5 - u) = -3\left(2u + 6\right)$

Exercise:

Problem: $5(8 - r) = -2\left(2r - 16\right)$

Solution:

$r = 8$

Exercise:

Problem: $3(4n - 1) - 2 = 8n + 3$

Exercise:

Problem: $9(2m - 3) - 8 = 4m + 7$

Solution:

$m = 3$

Exercise:

Problem: $12 + 2(5 - 3y) = -9\left(y - 1\right) - 2$

Exercise:

Problem: $-15 + 4(2 - 5y) = -7\left(y - 4\right) + 4$

Solution:

$y = -3$

Exercise:

Problem: $8\left(x-4\right)-7x=14$

Exercise:

Problem: $5\left(x-4\right)-4x=14$

Solution:

$x=34$

Exercise:

Problem: $5+6(3s-5)=-3+2\left(8s-1\right)$

Exercise:

Problem: $-12+8(x-5)=-4+3\left(5x-2\right)$

Solution:

$x=-6$

Exercise:

Problem: $4\left(u-1\right)-8=6\left(3u-2\right)-7$

Exercise:

Problem: $7\left(2n-5\right)=8\left(4n-1\right)-9$

Solution:

$n=-1$

Exercise:

Problem: $4\left(p-4\right)-\left(p+7\right)=5\left(p-3\right)$

Exercise:

Problem: $3\left(a-2\right)-\left(a+6\right)=4\left(a-1\right)$

Solution:

$a=-4$

Exercise:

Problem: $-(9y+5)-(3y-7)=16-(4y-2)$

Exercise:

Problem: $-\left(7m+4\right)-\left(2m-5\right)=14-\left(5m-3\right)$

Solution:

$m=-4$

Exercise:

Problem: $4\left[5-8\left(4c-3\right)\right]=12\left(1-13c\right)-8$

Exercise:

Problem: $5\left[9-2\left(6d-1\right)\right]=11\left(4-10d\right)-139$

Solution:

$d=-3$

Exercise:

Problem: $$3\left[-9+8\left(4h-3\right)\right]$$ $$=2\left(5-12h\right)-19$$

Exercise:

Problem: $$3\left[-14+2\left(15k-6\right)\right]$$ $$=8\left(3-5k\right)-24$$

Solution:

$k=\frac{3}{5}$

Exercise:

Problem: $$5\left[2\left(m+4\right)+8\left(m-7\right)\right]$$ $$=2\left[3\left(5+m\right)-\left(21-3m\right)\right]$$

Exercise:

Problem: $$10\left[5\left(n+1\right)+4\left(n-1\right)\right]$$ $$=11[7\left(5+n\right)-\left(25-3n\right)]$$

Solution:

$n=-5$

Exercise:

Problem: $5\left(1.2u-4.8\right)=-12$

Exercise:

Problem: $4\left(2.5v-0.6\right)=7.6$

Solution:

$v=1$

Exercise:

Problem: $0.25(q-6)=0.1\,(q+18)$

Exercise:

Problem: $0.2(p-6)=0.4\,(p+14)$

Solution:

$p=-34$

Exercise:

Problem: $0.2\,(30n+50)=28$

Exercise:

Problem: $0.5\,(16m+34)=-15$

Solution:

$m=-4$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

Exercise:

Problem: $23z+19=3\,(5z-9)+8z+46$

Exercise:

Problem: $15y+32=2\,(10y-7)-5y+46$

Solution:

identity; all real numbers

Exercise:

Problem: $5\left(b-9\right)+4\left(3b+9\right)=6\left(4b-5\right)-7b+21$

Exercise:

Problem: $9\left(a-4\right)+3\left(2a+5\right)=7\left(3a-4\right)-6a+7$

Solution:

identity; all real numbers

Exercise:

Problem: $18(5j-1)+29=47$

Exercise:

Problem: $24(3d-4)+100=52$

Solution:

conditional equation; $d=\frac{2}{3}$

Exercise:

Problem: $22(3m-4)=8\left(2m+9\right)$

Exercise:

Problem: $30(2n-1)=5\left(10n+8\right)$

Solution:

conditional equation; $n=7$

Exercise:

Problem: $7v+42=11\left(3v+8\right)-2\left(13v-1\right)$

Exercise:

Problem: $18u-51=9\left(4u+5\right)-6\left(3u-10\right)$

Solution:

contradiction; no solution

Exercise:

Problem: $3\left(6q-9\right)+7\left(q+4\right)=5\left(6q+8\right)-5\left(q+1\right)$

Exercise:

Problem: $5\left(p+4\right)+8\left(2p-1\right)=9\left(3p-5\right)-6\left(p-2\right)$

Solution:

contradiction; no solution

Exercise:

Problem: $12(6h-1)=8\left(8h+5\right)-4$

Exercise:

Problem: $9(4k-7)=11\left(3k+1\right)+4$

Solution:

conditional equation; $k=26$

Exercise:

Problem: $45(3y-2)=9\left(15y-6\right)$

Exercise:

Problem: $60(2x-1)=15\,(8x+5)$

Solution:

contradiction; no solution

Exercise:

Problem: $16(6n+15)=48\,(2n+5)$

Exercise:

Problem: $36(4m+5)=12\,(12m+15)$

Solution:

identity; all real numbers

Exercise:

Problem: $9(14d+9)+4d=13\,(10d+6)+3$

Exercise:

Problem: $11(8c+5)-8c=2\,(40c+25)+5$

Solution:

identity; all real numbers

Everyday Math

Exercise:

Problem:

Fencing Micah has 44 feet of fencing to make a dog run in his yard. He wants the length to be 2.5 feet more than the width. Find the length, L, by solving the equation $2L + 2(L - 2.5) = 44$.

Exercise:

Problem:

Coins Rhonda has $1.90 in nickels and dimes. The number of dimes is one less than twice the number of nickels. Find the number of nickels, n, by solving the equation $0.05n + 0.10(2n - 1) = 1.90$.

Solution:

8 nickels

Writing Exercises

Exercise:

Problem:

Using your own words, list the steps in the general strategy for solving linear equations.

Exercise:

Problem:

Explain why you should simplify both sides of an equation as much as possible before collecting the variable terms to one side and the constant terms to the other side.

Solution:

Answers will vary.

Exercise:

Problem:

What is the first step you take when solving the equation $3 - 7(y - 4) = 38$? Why is this your first step?

Exercise:

Problem:

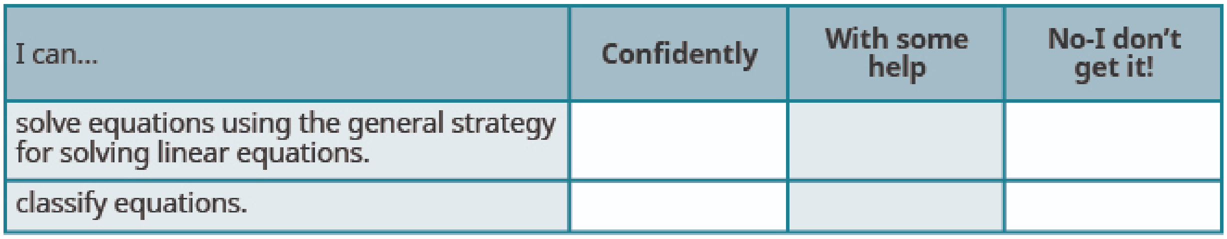
Solve the equation $\frac{1}{4}(8x + 20) = 3x - 4$ explaining all the steps of your solution as in the examples in this section.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objective of this section.



I can...	Confidently	With some help	No-I don't get it!
solve equations using the general strategy for solving linear equations.			
classify equations.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

conditional equation

An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

contradiction

An equation that is false for all values of the variable is called a contradiction. A contradiction has no solution.

identity

An equation that is true for any value of the variable is called an identity. The solution of an identity is all real numbers.

Solve Equations with Fractions or Decimals

By the end of this section, you will be able to:

- Solve equations with fraction coefficients
- Solve equations with decimal coefficients

Note:
Before you get started, take this readiness quiz.

1. Multiply: $8 \cdot \frac{3}{8}$.
   If you missed this problem, review [link].
2. Find the LCD of $\frac{5}{6}$ and $\frac{1}{4}$.
   If you missed this problem, review [link].
3. Multiply 4.78 by 100.
   If you missed this problem, review [link].

Solve Equations with Fraction Coefficients

Let's use the general strategy for solving linear equations introduced earlier to solve the equation, $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$.

	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$
To isolate the x term, subtract $\frac{1}{2}$ from both sides.	$\frac{1}{8}x + \frac{1}{2} - \frac{1}{2} = \frac{1}{4} - \frac{1}{2}$

Simplify the left side.	$\frac{1}{8}x=\frac{1}{4}-\frac{1}{2}$
Change the constants to equivalent fractions with the LCD.	$\frac{1}{8}x=\frac{1}{4}-\frac{2}{4}$
Subtract.	$\frac{1}{8}x=-\frac{1}{4}$
Multiply both sides by the reciprocal of $\frac{1}{8}$.	$\frac{8}{1}\cdot\frac{1}{8}x=\frac{8}{1}\left(-\frac{1}{4}\right)$
Simplify.	$x=-2$

This method worked fine, but many students do not feel very confident when they see all those fractions. So, we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the least common denominator of all the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but without fractions. This process is called “clearing” the equation of fractions.

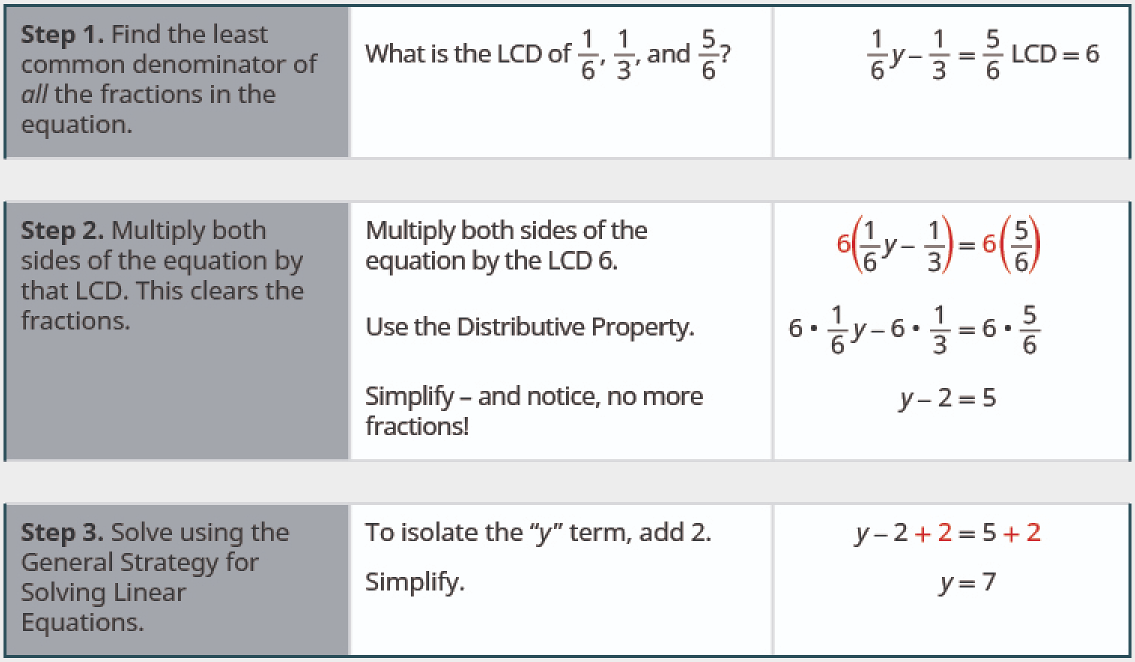
Let’s solve a similar equation, but this time use the method that eliminates the fractions.

Example:
How to Solve Equations with Fraction Coefficients

Exercise:

Problem: Solve: $\frac{1}{6}y-\frac{1}{3}=\frac{5}{6}$.

Solution:
Solution



Step 1. Find the least common denominator of *all* the fractions in the equation.	What is the LCD of $\frac{1}{6}$, $\frac{1}{3}$, and $\frac{5}{6}$?	$\frac{1}{6}y-\frac{1}{3}=\frac{5}{6}$ LCD = 6
Step 2. Multiply both sides of the equation by that LCD. This clears the fractions.	Multiply both sides of the equation by the LCD 6. Use the Distributive Property. Simplify – and notice, no more fractions!	$6\left(\frac{1}{6}y-\frac{1}{3}\right)=6\left(\frac{5}{6}\right)$ $6\cdot\frac{1}{6}y-6\cdot\frac{1}{3}=6\cdot\frac{5}{6}$ $y-2=5$
Step 3. Solve using the General Strategy for Solving Linear Equations.	To isolate the “y” term, add 2. Simplify.	$y-2+2=5+2$ $y=7$

Note:
Exercise:

Problem: Solve: $\frac{1}{4}x+\frac{1}{2}=\frac{5}{8}$.

Solution:

$x=\frac{1}{2}$

Note:
Exercise:

Problem: Solve: $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$.

Solution:

$x = -2$

Notice in [link], once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve! We then used the General Strategy for Solving Linear Equations.

Note:
Strategy to solve equations with fraction coefficients.

Find the least common denominator of*all*the fractions in the equation.
Multiply both sides of the equation by that LCD. This clears the fractions.
Solve using the General Strategy for Solving Linear Equations.

Example:
Exercise:

Problem: Solve: $6 = \frac{1}{2}v + \frac{2}{5}v - \frac{3}{4}v$.

Solution:
Solution

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the LCD of all fractions in the equation.	$6 = \frac{1}{2}v + \frac{2}{5}v - \frac{3}{4}v$
The LCD is 20.	
Multiply both sides of the equation by 20.	$20(6) = 20 \cdot \left(\frac{1}{2}v + \frac{2}{5}v - \frac{3}{4}v\right)$
Distribute.	$20(6) = 20 \cdot \frac{1}{2}v + 20 \cdot \frac{2}{5}v - 20 \cdot \frac{3}{4}v$
Simplify—notice, no more fractions!	$120 = 10v + 8v - 15v$
Combine like terms.	$120 = 3v$
Divide by 3.	$\frac{120}{3} = \frac{3v}{3}$
Simplify.	$40 = v$
Check: $6 = \frac{1}{2}v + \frac{2}{5}v - \frac{3}{4}v$	

Let $v = 40$.	$6 \overset{?}{=} \frac{1}{2}(40) + \frac{2}{5}(40) - \frac{3}{4}(40)$	
	$6 \overset{?}{=} 20 + 16 - 30$	
	$6 = 6$ ✓	

Note:
Exercise:

Problem: Solve: $7 = \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x$.

Solution:

$x = 12$

Note:
Exercise:

Problem: Solve: $-1 = \frac{1}{2}u + \frac{1}{4}u - \frac{2}{3}u$.

Solution:

$u = -12$

In the next example, we again have variables on both sides of the equation.

Example:
Exercise:

Problem: Solve: $a+\frac{3}{4}=\frac{3}{8}a-\frac{1}{2}$.

Solution:
Solution

	$a+\frac{3}{4}=\frac{3}{8}a-\frac{1}{2}$
Find the LCD of all fractions in the equation. The LCD is 8.	
Multiply both sides by the LCD.	$8\left(a+\frac{3}{4}\right)=8\left(\frac{3}{8}a-\frac{1}{2}\right)$
Distribute.	$8\cdot a+8\cdot\frac{3}{4}=8\cdot\frac{3}{8}a-8\cdot\frac{1}{2}$
Simplify—no more fractions.	$8a+6=3a-4$
Subtract $3a$ from both sides.	$8a-3a+6=3a-3a-4$

Simplify.		$5a+6=-4$
Subtract 6 from both sides.		$5a+6-6=-4-6$
Simplify.		$5a=-10$
Divide by 5.		$\frac{5a}{5}=\frac{-10}{5}$
Simplify.		$a=-2$
Check:	$a+\frac{3}{4}=\frac{3}{8}a-\frac{1}{2}$	
Let $a=-2$.	$-2+\frac{3}{4}\stackrel{?}{=}\frac{3}{8}(-2)-\frac{1}{2}$	
	$-\frac{8}{4}+\frac{3}{4}\stackrel{?}{=}-\frac{16}{8}-\frac{4}{8}$	
	$-\frac{5}{4}=-\frac{10}{8}$	
	$-\frac{5}{4}=-\frac{5}{4}\checkmark$	

Note:
Exercise:

Problem: Solve: $x+\frac{1}{3}=\frac{1}{6}x-\frac{1}{2}$.

Solution:

$x=-1$

Note:
Exercise:

Problem: Solve: $c+\frac{3}{4}=\frac{1}{2}c-\frac{1}{4}$.

Solution:

$c=-2$

In the next example, we start by using the Distributive Property. This step clears the fractions right away.

Example:
Exercise:

Problem: Solve: $-5=\frac{1}{4}(8x+4)$.

Solution:
Solution

		$-5=\frac{1}{4}(8x+4)$
Distribute.		$-5=\frac{1}{4}\cdot 8x+\frac{1}{4}\cdot 4$
Simplify. Now there are no fractions.		$-5=2x+1$
Subtract 1 from both sides.		$-5-1=2x+1-1$
Simplify.		$-6=2x$
Divide by 2.		$\frac{-6}{2}=\frac{2x}{2}$
Simplify.		$-3=x$
Check:	$-5=\frac{1}{4}(8x+4)$	

Let $x = -3$.	$-5 \overset{?}{=} \frac{1}{2}(4(-3) + 2)$	
	$-5 \overset{?}{=} \frac{1}{2}(-12 + 2)$	
	$-5 \overset{?}{=} \frac{1}{2}(-10)$	
	$-5 = -5$ ✓	

Note:
Exercise:

Problem: Solve: $-11 = \frac{1}{2}(6p + 2)$.

Solution:

$p = -4$

Note:
Exercise:

Problem: Solve: $8 = \frac{1}{3}(9q + 6)$.

Solution:

$q = 2$

In the next example, even after distributing, we still have fractions to clear.

Example:
Exercise:

Problem: Solve: $\frac{1}{2}(y-5) = \frac{1}{4}(y-1)$.

Solution:
Solution

	$\frac{1}{2}(y-5) = \frac{1}{4}(y-1)$
Distribute.	$\frac{1}{2} \cdot y - \frac{1}{2} \cdot 5 = \frac{1}{4} \cdot y - \frac{1}{4} \cdot 1$
Simplify.	$\frac{1}{2}y - \frac{5}{2} = \frac{1}{4}y - \frac{1}{4}$
Multiply by the LCD, 4.	$4\left(\frac{1}{2}y - \frac{5}{2}\right) = 4\left(\frac{1}{4}y - \frac{1}{4}\right)$

Distribute.		$4\cdot\frac{1}{2}y-4\cdot\frac{5}{2}=4\cdot\frac{1}{4}y-4\cdot\frac{1}{4}$
Simplify.		$2y-10=y-1$
Collect the variables to the left.		$2y-y-10=y-y-1$
Simplify.		$y-10=-1$
Collect the constants to the right.		$y-10+10=-1+10$
Simplify.		$y=9$
Check:	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$	
Let $y=9$.	$\frac{1}{2}(9-5)\overset{?}{=}\frac{1}{4}(9-1)$	
Finish the check on your own.		

Note:
Exercise:

Problem: Solve: $\frac{1}{5}(n+3)=\frac{1}{4}(n+2)$.

Solution:

$n=2$

Note:
Exercise:

Problem: Solve: $\frac{1}{2}(m-3)=\frac{1}{4}(m-7)$.

Solution:

$m=-1$

Example:
Exercise:

Problem: Solve: $\frac{5x-3}{4}=\frac{x}{2}$.

Solution:
Solution

	$\frac{5x-3}{4}=\frac{x}{2}$

Multiply by the LCD, 4.		$4\left(\frac{5x-3}{4}\right)=4\left(\frac{x}{2}\right)$
Simplify.		$5x-3=2x$
Collect the variables to the right.		$5x-5x-3=2x-5x$
Simplify.		$-3=-3x$
Divide.		$\frac{-3}{-3}=\frac{-3x}{-3}$
Simplify.		$1=x$
Check:	$\frac{5x-3}{4}=\frac{x}{2}$	
Let $x=1$.	$\frac{5(1)-3}{4}\overset{?}{=}\frac{1}{2}$	
	$\frac{2}{4}\overset{?}{=}\frac{1}{2}$	
	$\frac{1}{2}=\frac{1}{2}$ ✓	

Note:
Exercise:

Problem: Solve: $\frac{4y-7}{3} = \frac{y}{6}$.

Solution:

$y = 2$

Note:
Exercise:

Problem: Solve: $\frac{-2z-5}{4} = \frac{z}{8}$.

Solution:

$z = -2$

Example:
Exercise:

Problem: Solve: $\frac{a}{6} + 2 = \frac{a}{4} + 3$.

Solution:
Solution

		$\frac{a}{6}+2=\frac{a}{4}+3$
Multiply by the LCD, 12.		$12\left(\frac{a}{6}+2\right)=12\left(\frac{a}{4}+3\right)$
Distribute.		$12\cdot\frac{a}{6}+12\cdot2=12\cdot\frac{a}{4}+12\cdot3$
Simplify.		$2a+24=3a+36$
Collect the variables to the right.		$2a-2a+24=3a-2a+36$
Simplify.		$24=a+36$
Collect the constants to the left.		$24-36=a+36-36$
Simplify.		$a=-12$
Check:	$\frac{a}{6}+2=\frac{a}{4}+3$	
Let $a=-12$.	$\frac{-12}{6}+2\overset{?}{=}\frac{-12}{4}+3$	
	$-2+2\overset{?}{=}-3+3$	

	$0 = 0$ ✓	

Note:
Exercise:

Problem: Solve: $\frac{b}{10} + 2 = \frac{b}{4} + 5$.

Solution:

$b = -20$

Note:
Exercise:

Problem: Solve: $\frac{c}{6} + 3 = \frac{c}{3} + 4$.

Solution:

$c = -6$

Example:
Exercise:

Problem: Solve: $\frac{4q+3}{2} + 6 = \frac{3q+5}{4}$.

Solution:
Solution

	$\frac{4q+3}{2}+6=\frac{3q+5}{4}$
Multiply by the LCD, 4.	$4\left(\frac{4q+3}{2}+6\right)=4\left(\frac{3q+5}{4}\right)$
Distribute.	$4\left(\frac{4q+3}{2}\right)+4\cdot 6=4\cdot\left(\frac{3q+5}{4}\right)$
Simplify.	$2(4q+3)+24=3q+5$ $8q+6+24=3q+5$ $8q+30=3q+5$
Collect the variables to the left.	$8q-3q+30=3q-3q+5$
Simplify.	$5q+30=5$
Collect the constants to the right.	

		$5q+30-30=5-30$
Simplify.		$5q=-25$
Divide by 5.		$\frac{5q}{5}=\frac{-25}{5}$
Simplify.		$q=-5$
Check:	$\frac{4q+3}{2}+6=\frac{3q+5}{4}$	
Let $q=-5$.	$\frac{4(-5)+3}{2}+6\overset{?}{=}\frac{3(-5)+5}{4}$	
Finish the check on your own.		

Note:
Exercise:

Problem: Solve: $\frac{3r+5}{6}+1=\frac{4r+3}{3}$.

Solution:

$r=1$

Note:
Exercise:

Problem: Solve: $\frac{2s+3}{2}+1=\frac{3s+2}{4}$.

Solution:

$s=-8$

Solve Equations with Decimal Coefficients

Some equations have decimals in them. This kind of equation will occur when we solve problems dealing with money or percentages. But decimals can also be expressed as fractions. For example, $0.3=\frac{3}{10}$ and $0.17=\frac{17}{100}$. So, with an equation with decimals, we can use the same method we used to clear fractions—multiply both sides of the equation by the least common denominator.

Example:
Exercise:

Problem: Solve: $0.06x+0.02=0.25x-1.5$.

Solution:
Solution

Look at the decimals and think of the equivalent fractions.
Equation:

$$0.06=\frac{6}{100} \qquad 0.02=\frac{2}{100} \qquad 0.25=\frac{25}{100} \qquad 1.5=1\frac{5}{10}$$

Notice, the LCD is 100.

By multiplying by the LCD, we will clear the decimals from the equation.

	$0.06x+0.02=0.25x-1.5$
Multiply both sides by 100.	$100(0.06x+0.02)=100(0.25x-1.5)$
Distribute.	$100(0.06x)+100(0.02)=100(0.25x)-100(1.5)$
Multiply, and now we have no more decimals.	$6x+2=25x-150$
Collect the variables to the right.	$6x-6x+2=25x-6x-150$
Simplify.	$2=19x-150$
Collect the constants to the left.	$2+150=19x-150+150$
Simplify.	$152=19x$
Divide by 19.	$\frac{152}{19}=\frac{19x}{19}$

Simplify.	$8 = x$
Check: Let $x = 8$. $0.06(8) + 0.02 \overset{?}{=} 0.25(8) - 1.5$ $0.48 + 0.02 \overset{?}{=} 2.00 - 1.5$ $0.50 = 0.50$ ✓	

Note:
Exercise:

Problem: Solve: $0.14h + 0.12 = 0.35h - 2.4$.

Solution:

$h = 12$

Note:
Exercise:

Problem: Solve: $0.65k - 0.1 = 0.4k - 0.35$.

Solution:

$k = -1$

The next example uses an equation that is typical of the money applications in the next chapter. Notice that we distribute the decimal before we clear all the decimals.

Example:
Exercise:

Problem: Solve: $0.25x + 0.05(x + 3) = 2.85$.

Solution:
Solution

	$0.25x + 0.05(x + 3) = 2.85$
Distribute first.	$0.25x + 0.05x + 0.15 = 2.85$
Combine like terms.	$0.30x + 0.15 = 2.85$
To clear decimals, multiply by 100.	$100(0.30x + 0.15) = 100(2.85)$
Distribute.	$30x + 15 = 285$
Subtract 15 from both sides.	$30x + 15 - 15 = 285 - 15$

Simplify.	$30x=270$
Divide by 30.	$\frac{30x}{30}=\frac{270}{30}$
Simplify.	$x=9$
Check it yourself by substituting $x=9$ into the original equation.	

Note:
Exercise:

Problem: Solve: $0.25n+0.05(n+5)=2.95.$

Solution:

$n=9$

Note:
Exercise:

Problem: Solve: $0.10d+0.05(d-5)=2.15.$

Solution:

$d=16$

Key Concepts

- **Strategy to Solve an Equation with Fraction Coefficients**

  Find the least common denominator of all the fractions in the equation.
  Multiply both sides of the equation by that LCD. This clears the fractions.
  Solve using the General Strategy for Solving Linear Equations.

Practice Makes Perfect

Solve Equations with Fraction Coefficients

In the following exercises, solve each equation with fraction coefficients.

Exercise:

Problem: $\frac{1}{4}x - \frac{1}{2} = -\frac{3}{4}$

Exercise:

Problem: $\frac{3}{4}x - \frac{1}{2} = \frac{1}{4}$

Solution:

$x = 1$

Exercise:

Problem: $\frac{5}{6}y - \frac{2}{3} = -\frac{3}{2}$

Exercise:

Problem: $\frac{5}{6}y - \frac{1}{3} = -\frac{7}{6}$

Solution:

$y = -1$

Exercise:

Problem: $\frac{1}{2}a + \frac{3}{8} = \frac{3}{4}$

Exercise:

Problem: $\frac{5}{8}b + \frac{1}{2} = -\frac{3}{4}$

Solution:

$b = -2$

Exercise:

Problem: $2 = \frac{1}{3}x - \frac{1}{2}x + \frac{2}{3}x$

Exercise:

Problem: $2 = \frac{3}{5}x - \frac{1}{3}x + \frac{2}{5}x$

Solution:

$x = 3$

Exercise:

Problem: $\frac{1}{4}m - \frac{4}{5}m + \frac{1}{2}m = -1$

Exercise:

Problem: $\frac{5}{6}n - \frac{1}{4}n - \frac{1}{2}n = -2$

Solution:

$n = -24$

Exercise:

Problem: $x + \frac{1}{2} = \frac{2}{3}x - \frac{1}{2}$

Exercise:

Problem: $x + \frac{3}{4} = \frac{1}{2}x - \frac{5}{4}$

Solution:

$x = -4$

Exercise:

Problem: $\frac{1}{3}w + \frac{5}{4} = w - \frac{1}{4}$

Exercise:

Problem: $\frac{3}{2}z + \frac{1}{3} = z - \frac{2}{3}$

Solution:

$z = -2$

Exercise:

Problem: $\frac{1}{2}x - \frac{1}{4} = \frac{1}{12}x + \frac{1}{6}$

Exercise:

Problem: $\frac{1}{2}a - \frac{1}{4} = \frac{1}{6}a + \frac{1}{12}$

Solution:

$a = 1$

Exercise:

Problem: $\frac{1}{3}b+\frac{1}{5}=\frac{2}{5}b-\frac{3}{5}$

Exercise:

Problem: $\frac{1}{3}x+\frac{2}{5}=\frac{1}{5}x-\frac{2}{5}$

Solution:

$x=-6$

Exercise:

Problem: $1=\frac{1}{6}(12x-6)$

Exercise:

Problem: $1=\frac{1}{5}(15x-10)$

Solution:

$x=1$

Exercise:

Problem: $\frac{1}{4}(p-7)=\frac{1}{3}(p+5)$

Exercise:

Problem: $\frac{1}{5}(q+3)=\frac{1}{2}(q-3)$

Solution:

$q=7$

Exercise:

Problem: $\frac{1}{2}(x+4)=\frac{3}{4}$

Exercise:

Problem: $\frac{1}{3}(x+5)=\frac{5}{6}$

Solution:

$x=-\frac{5}{2}$

Exercise:

Problem: $\frac{5q-8}{5}=\frac{2q}{10}$

Exercise:

Problem: $\frac{4m+2}{6}=\frac{m}{3}$

Solution:

$m=-1$

Exercise:

Problem: $\frac{4n+8}{4}=\frac{n}{3}$

Exercise:

Problem: $\frac{3p+6}{3}=\frac{p}{2}$

Solution:

$p=-4$

Exercise:

Problem: $\frac{u}{3} - 4 = \frac{u}{2} - 3$

Exercise:

Problem: $\frac{v}{10} + 1 = \frac{v}{4} - 2$

Solution:

$v = 20$

Exercise:

Problem: $\frac{c}{15} + 1 = \frac{c}{10} - 1$

Exercise:

Problem: $\frac{d}{6} + 3 = \frac{d}{8} + 2$

Solution:

$d = -24$

Exercise:

Problem: $\frac{3x+4}{2} + 1 = \frac{5x+10}{8}$

Exercise:

Problem: $\frac{10y-2}{3} + 3 = \frac{10y+1}{9}$

Solution:

$y = -1$

Exercise:

Problem: $\frac{7u-1}{4} - 1 = \frac{4u+8}{5}$

Exercise:

Problem: $\frac{3v-6}{2} + 5 = \frac{11v-4}{5}$

Solution:

$v = 4$

Solve Equations with Decimal Coefficients

In the following exercises, solve each equation with decimal coefficients.

Exercise:

Problem: $0.6y + 3 = 9$

Exercise:

Problem: $0.4y - 4 = 2$

Solution:

$y = 15$

Exercise:

Problem: $3.6j - 2 = 5.2$

Exercise:

Problem: $2.1k + 3 = 7.2$

Solution:

$k = 2$

Exercise:

Problem: $0.4x + 0.6 = 0.5x - 1.2$

Exercise:

Problem: $0.7x + 0.4 = 0.6x + 2.4$

Solution:

$x = 20$

Exercise:

Problem: $0.23x + 1.47 = 0.37x - 1.05$

Exercise:

Problem: $0.48x + 1.56 = 0.58x - 0.64$

Solution:

$x = 22$

Exercise:

Problem: $0.9x - 1.25 = 0.75x + 1.75$

Exercise:

Problem: $1.2x - 0.91 = 0.8x + 2.29$

Solution:

$x = 8$

Exercise:

Problem: $0.05n + 0.10\left(n + 8\right) = 2.15$

Exercise:

Problem: $0.05n + 0.10\left(n + 7\right) = 3.55$

Solution:

$n = 19$

Exercise:

Problem: $0.10d + 0.25\left(d + 5\right) = 4.05$

Exercise:

Problem: $0.10d + 0.25\left(d + 7\right) = 5.25$

Solution:

$d = 10$

Exercise:

Problem: $0.05\left(q - 5\right) + 0.25q = 3.05$

Exercise:

Problem: $0.05\left(q - 8\right) + 0.25q = 4.10$

Solution:

$q = 15$

Everyday Math

Exercise:

Problem:

Coins Taylor has \$2.00 in dimes and pennies. The number of pennies is 2 more than the number of dimes. Solve the equation $0.10d + 0.01(d + 2) = 2$ for d, the number of dimes.

Exercise:

Problem:

Stamps Paula bought \$22.82 worth of 49-cent stamps and 21-cent stamps. The number of 21-cent stamps was 8 less than the number of 49-cent stamps. Solve the equation $0.49s + 0.21(s - 8) = 22.82$ for *s*, to find the number of 49-cent stamps Paula bought.

Solution:

$s = 35$

Writing Exercises

Exercise:

Problem:

Explain how you find the least common denominator of $\frac{3}{8}$, $\frac{1}{6}$, and $\frac{2}{3}$.

Exercise:

Problem:

If an equation has several fractions, how does multiplying both sides by the LCD make it easier to solve?

Solution:

Answers will vary.

Exercise:

Problem:

If an equation has fractions only on one side, why do you have to multiply both sides of the equation by the LCD?

Exercise:

Problem:

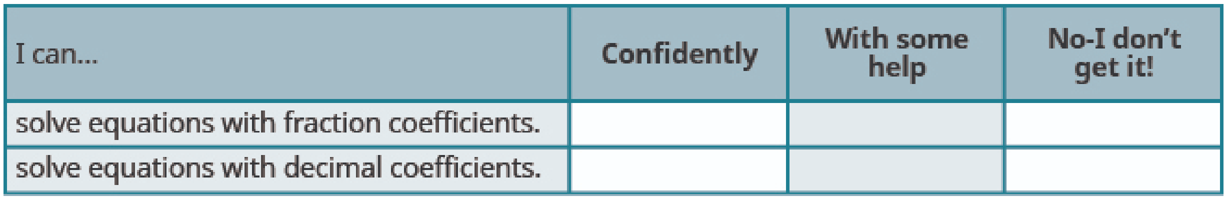
In the equation $0.35x + 2.1 = 3.85$ what is the LCD? How do you know?

Solution:

100. Justifications will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
solve equations with fraction coefficients.			
solve equations with decimal coefficients.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Solve a Formula for a Specific Variable
By the end of this section, you will be able to:

- Use the Distance, Rate, and Time formula
- Solve a formula for a specific variable

Note:
Before you get started, take this readiness quiz.

1. Solve: $15t = 120$.
   If you missed this problem, review [link].
2. Solve: $6x + 24 = 96$.
   If you missed this problem, review [link].

Use the Distance, Rate, and Time Formula

One formula you will use often in algebra and in everyday life is the formula for distance traveled by an object moving at a constant rate. Rate is an equivalent word for “speed.” The basic idea of rate may already familiar to you. Do you know what distance you travel if you drive at a steady rate of 60 miles per hour for 2 hours? (This might happen if you use your car’s cruise control while driving on the highway.) If you said 120 miles, you already know how to use this formula!

Note:
Distance, Rate, and Time
For an object moving at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula:
Equation:

$$d = rt \quad \text{where} \quad \begin{aligned} d &= \text{distance} \\ r &= \text{rate} \\ t &= \text{time} \end{aligned}$$

We will use the Strategy for Solving Applications that we used earlier in this chapter. When our problem requires a formula, we change Step 4. In place of writing a sentence, we write the appropriate formula. We write the revised steps here for reference.

Note:
Solve an application (with a formula).

Readthe problem. Make sure all the words and ideas are understood.
Identifywhat we are looking for.
Namewhat we are looking for. Choose a variable to represent that quantity.
Translateinto an equation. Write the appropriate formula for the situation.
Substitute in the given information.
Solvethe equation using good algebra techniques.
Checkthe answer in the problem and make sure it makes sense.
Answerthe question with a complete sentence.

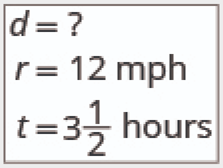
You may want to create a mini-chart to summarize the information in the problem. See the chart in this first example.

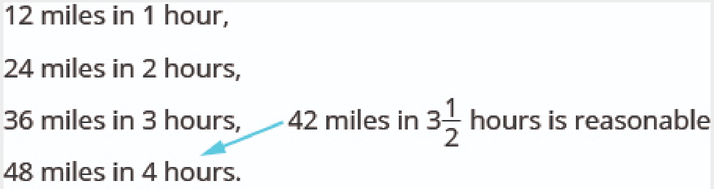
Example:
Exercise:

Problem:

Jamal rides his bike at a uniform rate of 12 miles per hour for $3\frac{1}{2}$ hours. What distance has he traveled?

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	distance traveled
Step 3. Name. Choose a variable to represent it.	Let *d* = distance.
Step 4. Translate: Write the appropriate formula.	$d = rt$
	 $d = ?$ $r = 12$ mph $t = 3\frac{1}{2}$ hours
Substitute in the given information.	$d = 12 \cdot 3\frac{1}{2}$
Step 5. Solve the equation.	$d = 42$ miles
Step 6. Check	

Does 42 miles make sense?	
Jamal rides:	
 12 miles in 1 hour, 24 miles in 2 hours, 36 miles in 3 hours, 42 miles in $3\frac{1}{2}$ hours is reasonable 48 miles in 4 hours.	
Step 7. Answer the question with a complete sentence.	Jamal rode 42 miles.

Note:
Exercise:

Problem:

Lindsay drove for $5\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?

Solution:

330 miles

Note:
Exercise:

Problem:

Trinh walked for $2\frac{1}{3}$ hours at 3 miles per hour. How far did she walk?

Solution:

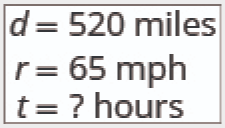
7 miles

Example:
Exercise:

Problem:

Rey is planning to drive from his house in San Diego to visit his grandmother in Sacramento, a distance of 520 miles. If he can drive at a steady rate of 65 miles per hour, how many hours will the trip take?

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	How many hours (time)
Step 3. Name. Choose a variable to represent it.	Let *t* = time.
	 $d = 520$ miles $r = 65$ mph $t = ?$ hours
Step 4. Translate.	$d = rt$

Write the appropriate formula.	
Substitute in the given information.	$520 = 65t$
Step 5. Solve the equation.	$t = 8$
Step 6. Check. Substitute the numbers into the formula and make sure the result is a true statement.	
$\begin{aligned} d &= rt \\ 520 &\overset{?}{=} 65 \cdot 8 \\ 520 &= 520\checkmark \end{aligned}$	
Step 7. Answer the question with a complete sentence. Rey’s trip will take 8 hours.	

Note:
Exercise:

Problem:

Lee wants to drive from Phoenix to his brother’s apartment in San Francisco, a distance of 770 miles. If he drives at a steady rate of 70 miles per hour, how many hours will the trip take?

Solution:

11 hours

Note:
Exercise:

Problem:

Yesenia is 168 miles from Chicago. If she needs to be in Chicago in 3 hours, at what rate does she need to drive?

Solution:

56 mph

Solve a Formula for a Specific Variable

You are probably familiar with some geometry formulas. A formula is a mathematical description of the relationship between variables. Formulas are also used in the sciences, such as chemistry, physics, and biology. In medicine they are used for calculations for dispensing medicine or determining body mass index. Spreadsheet programs rely on formulas to make calculations. It is important to be familiar with formulas and be able to manipulate them easily.

In [link] and [link], we used the formula $d = rt$. This formula gives the value of d, distance, when you substitute in the values of r and t, the rate and time. But in [link], we had to find the value of t. We substituted in values of d and r and then used algebra to solve for t. If you had to do this often, you might wonder why there is not a formula that gives the value of t when you substitute in the values of d and r. We can make a formula like this by solving the formula $d = rt$ for t.

To solve a formula for a specific variable means to isolate that variable on one side of the equals sign with a coefficient of 1. All other variables and constants are on the other side of the equals sign. To see how to solve a formula for a specific variable, we will start with the distance, rate and time formula.

Example:
Exercise:

Problem: Solve the formula $d = rt$ for t:

ⓐ when $d = 520$ and $r = 65$
ⓑ in general

Solution:
Solution

We will write the solutions side-by-side to demonstrate that solving a formula in general uses the same steps as when we have numbers to substitute.

ⓐ when $d = 520$ and $r = 65$				ⓑ in general	
Write the formula.	$d = rt$			Write the formula.	$d = rt$
Substitute.	$520 = 65t$				
Divide, to isolate t.	$\frac{520}{65} = \frac{65t}{65}$			Divide, to isolate t.	$\frac{d}{r} = \frac{rt}{r}$
Simplify.	$8 = t$			Simplify.	$\frac{d}{r} = t$

We say the formula $t=\frac{d}{r}$ is solved for t.

Note:
Exercise:

Problem: Solve the formula $d=rt$ for r:

ⓐ when $d=180$ and $t=4$ ⓑ in general

Solution:

ⓐ $r=45$ ⓑ $r=\frac{d}{t}$

Note:
Exercise:

Problem: Solve the formula $d=rt$ for r:

ⓐ when $d=780$ and $t=12$ ⓑ in general

Solution:

ⓐ $r=65$ ⓑ $r=\frac{d}{t}$

The area of a triangle is given by the formula $A=\frac{1}{2}bh$, where b is the length of the base of the triangle and h is the height of the triangle.

Example:

Exercise:

Problem: Solve the formula $A = \frac{1}{2}bh$ for h:

ⓐ when $A = 90$ and $b = 15$ ⓑ in general

Solution:
Solution

ⓐ when $A = 90$ and $b = 15$				ⓑ in general	
Write the formula.	$A = \frac{1}{2}bh$			Write the formula.	$A = \frac{1}{2}bh$
Substitute.	$90 = \frac{1}{2} \cdot 15 \cdot h$				
Clear the fractions.	$2 \cdot 90 = 2 \cdot \frac{1}{2}15h$			Clear the fractions.	$2 \cdot A = 2 \cdot \frac{1}{2}bh$
Simplify.	$180 = 15h$			Simplify.	$2A = bh$
Solve for h.	$12 = h$			Solve for h.	$\frac{2A}{b} = h$

We can now find the height of a triangle, if we know the area and the base, by using the formula $h = \frac{2A}{b}$.

Note:
Exercise:

Problem: Use the formula $A = \frac{1}{2}bh$ to solve for h:

ⓐ when $A = 170$ and $b = 17$ ⓑ in general

Solution:

ⓐ $h = 20$ ⓑ $h = \frac{2A}{b}$

Note:
Exercise:

Problem: Use the formula $A = \frac{1}{2}bh$ to solve for b:

ⓐ when $A = 62$ and $h = 31$ ⓑ in general

Solution:

ⓐ $b = 4$ ⓑ $b = \frac{2A}{h}$

The formula $I = Prt$ is used to calculate simple interest, I, for a principal, P, invested at rate, r, for t years.

Example:
Exercise:

Problem: Solve the formula $I = Prt$ to find the principal, P:

ⓐ when $I = \$5{,}600, r = 4\%, t = 7$ years ⓑ in general

Solution:
Solution

ⓐ $I = \$5{,}600$, $r = 4\%$, $t = 7$ years				ⓑ in general	
Write the formula.	$I = Prt$			Write the formula.	$I = Prt$
Substitute.	$5600 = P(0.04)(7)$				
Simplify.	$5600 = P(0.28)$			Simplify.	$I = P(rt)$
Divide, to isolate *P*.	$\frac{5600}{0.28} = \frac{P(0.28)}{0.28}$			Divide, to isolate *P*.	$\frac{I}{rt} = \frac{P(rt)}{rt}$
Simplify.	$20{,}000 = P$			Simplify.	$\frac{I}{rt} = P$
The principal is	$\$20{,}000$				$P = \frac{I}{rt}$

Note:
Exercise:

Problem: Use the formula $I = Prt$ to find the principal, P:

ⓐ when $I = \$2{,}160, r = 6\%, t = 3$ years ⓑ in general

Solution:

ⓐ $12,000 ⓑ $P = \frac{I}{rt}$

Note:
Exercise:

Problem: Use the formula $I = Prt$ to find the principal,P:

ⓐ when $I = \$5{,}400, r = 12\%, t = 5$ years ⓑ in general

Solution:

ⓐ $9,000 ⓑ $P = \frac{I}{rt}$

Later in this class, and in future algebra classes, you'll encounter equations that relate two variables, usually *x* and *y*. You might be given an equation that is solved for *y* and need to solve it for *x*, or vice versa. In the following example, we're given an equation with both *x* and *y* on the same side and we'll solve it for *y*.

Example:
Exercise:

Problem: Solve the formula $3x + 2y = 18$ for y:

ⓐ when $x = 4$ ⓑ in general

Solution:
Solution

ⓐ when $x = 4$				ⓑ in general	
	$3x + 2y = 18$				$3x + 2y = 18$
Substitute.	$3(4) + 2y = 18$				
Subtract to isolate the y-term.	$12 - 12 + 2y = 18 - 12$			Subtract to isolate the y-term.	$3x - 3x + 2y = 18 - 3x$
Divide.	$\frac{2y}{2} = \frac{6}{2}$			Divide.	$\frac{2y}{2} = \frac{18}{2} - \frac{3x}{2}$
Simplify.	$y = 3$			Simplify.	$y = -\frac{3x}{2} + 9$

Note:

Exercise:

Problem: Solve the formula $3x+4y=10$ for y:

ⓐ when $x=\frac{14}{3}$ ⓑ in general

Solution:

ⓐ $y=-1$ ⓑ $y=\frac{10-3x}{4}$

Note:
Exercise:

Problem: Solve the formula $5x+2y=18$ for y:

ⓐ when $x=4$ ⓑ in general

Solution:

ⓐ $y=-1$ ⓑ $y=\frac{18-5x}{2}$

In Examples 1.60 through 1.64 we used the numbers in part ⓐ as a guide to solving in general in part ⓑ. Now we will solve a formula in general without using numbers as a guide.

Example:
Exercise:

Problem: Solve the formula $P=a+b+c$ for a.

Solution:
Solution

We will isolate a on one side of the equation.	$P = a + b + c$
Both b and c are added to a, so we subtract them from both sides of the equation.	$P - b - c = a + b + c - b - c$
Simplify.	$P - b - c = a$ $a = P - b - c$

Note:
Exercise:

Problem: Solve the formula $P = a + b + c$ for b.

Solution:

$b = P - a - c$

Note:

Exercise:

Problem: Solve the formula $P = a + b + c$ for c.

Solution:

$c = P - a - b$

Example:
Exercise:

Problem: Solve the formula $6x + 5y = 13$ for y.

Solution:
Solution

	$6x + 5y = 13$
Subtract $6x$ from both sides to isolate the term with y.	$6x - 6x + 5y = 13 - 6x$
Simplify.	$5y = 13 - 6x$
Divide by 5 to make the coefficient 1.	$\frac{5y}{5} = \frac{13 - 6x}{5}$

Simplify.	$y = \frac{13 - 6x}{5}$

Note:
Exercise:

Problem: Solve the formula $4x + 7y = 9$ for *y*.

Solution:

$y = \frac{9-4x}{7}$

Note:
Exercise:

Problem: Solve the formula $5x + 8y = 1$ for *y*.

Solution:

$y = \frac{1-5x}{8}$

Key Concepts

- **To Solve an Application (with a formula)**

  Read the problem. Make sure all the words and ideas are understood.
  Identify what we are looking for.
  Name what we are looking for. Choose a variable to represent that

quantity.

Translate into an equation. Write the appropriate formula for the situation. Substitute in the given information.
Solve the equation using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

- **Distance, Rate and Time**
  For an object moving at a uniform (constant) rate, the distance traveled, the elapsed time, and the rate are related by the formula: $d=rt$ where d = distance, r = rate, t = time.
- **To solve a formula for a specific variable** means to get that variable by itself with a coefficient of 1 on one side of the equation and all other variables and constants on the other side.

Practice Makes Perfect

Use the Distance, Rate, and Time Formula

In the following exercises, solve.

Exercise:

Problem:

Steve drove for $8\frac{1}{2}$ hours at 72 miles per hour. How much distance did he travel?

Exercise:

Problem:

Socorro drove for $4\frac{5}{6}$ hours at 60 miles per hour. How much distance did she travel?

Solution:

290 miles

Exercise:

Problem:

Yuki walked for $1\frac{3}{4}$ hours at 4 miles per hour. How far did she walk?

Exercise:

Problem:

Francie rode her bike for $2\frac{1}{2}$ hours at 12 miles per hour. How far did she ride?

Solution:

30 miles

Exercise:

Problem:

Connor wants to drive from Tucson to the Grand Canyon, a distance of 338 miles. If he drives at a steady rate of 52 miles per hour, how many hours will the trip take?

Exercise:

Problem:

Megan is taking the bus from New York City to Montreal. The distance is 380 miles and the bus travels at a steady rate of 76 miles per hour. How long will the bus ride be?

Solution:

5 hours

Exercise:

Problem:

Aurelia is driving from Miami to Orlando at a rate of 65 miles per hour. The distance is 235 miles. To the nearest tenth of an hour, how long will the trip take?

Exercise:

Problem:

Kareem wants to ride his bike from St. Louis to Champaign, Illinois. The distance is 180 miles. If he rides at a steady rate of 16 miles per hour, how many hours will the trip take?

Solution:

11.25 hours

Exercise:

Problem:

Javier is driving to Bangor, 240 miles away. If he needs to be in Bangor in 4 hours, at what rate does he need to drive?

Exercise:

Problem:

Alejandra is driving to Cincinnati, 450 miles away. If she wants to be there in 6 hours, at what rate does she need to drive?

Solution:

75 mph

Exercise:

Problem:

Aisha took the train from Spokane to Seattle. The distance is 280 miles and the trip took 3.5 hours. What was the speed of the train?

Exercise:

Problem:

Philip got a ride with a friend from Denver to Las Vegas, a distance of 750 miles. If the trip took 10 hours, how fast was the friend driving?

Solution:

75 mph

Solve a Formula for a Specific Variable

In the following exercises, use the formula $d = rt$.

Exercise:

Solve for t
ⓐ when $d = 350$ and $r = 70$
Problem: ⓑ in general

Exercise:

Solve for t
ⓐ when $d = 240$ and $r = 60$
Problem: ⓑ in general

Solution:

ⓐ $t = 4$ ⓑ $t = \frac{d}{r}$

Exercise:

Solve for t
ⓐ when $d = 510$ and $r = 60$
Problem: ⓑ in general

Exercise:

Solve for t
ⓐ when $d = 175$ and $r = 50$
Problem: ⓑ in general

Solution:

ⓐ $t = 3.5$ ⓑ $t = \frac{d}{r}$

Exercise:

Solve for r
ⓐ when $d = 204$ and $t = 3$
Problem: ⓑ in general

Exercise:

Solve for r
ⓐ when $d = 420$ and $t = 6$
Problem: ⓑ in general

Solution:

ⓐ $r = 70$ ⓑ $r = \frac{d}{t}$

Exercise:

Solve for r
ⓐ when $d = 160$ and $t = 2.5$
Problem: ⓑ in general

Exercise:

Solve for r
ⓐ when $d = 180$ and $t = 4.5$
Problem: ⓑ in general

Solution:

ⓐ $r=40$ ⓑ $r=\frac{d}{t}$

In the following exercises, use the formula $A=\frac{1}{2}bh$.

Exercise:

Problem: Solve for b
ⓐ when $A=126$ and $h=18$
ⓑ in general

Exercise:

Problem: Solve for h
ⓐ when $A=176$ and $b=22$
ⓑ in general

Solution:

ⓐ $h=16$ ⓑ $h=\frac{2A}{b}$

Exercise:

Problem: Solve for h
ⓐ when $A=375$ and $b=25$
ⓑ in general

Exercise:

Problem: Solve for b
ⓐ when $A=65$ and $h=13$
ⓑ in general

Solution:

ⓐ $b=10$ ⓑ $b=\frac{2A}{h}$

In the following exercises, use the formula $I = Prt$.

Exercise:

Problem: Solve for the principal, P for
ⓐ $I = \$5{,}480, r = 4\%,$
$t = 7$ years
ⓑ in general

Exercise:

Problem: Solve for the principal, P for
ⓐ $I = \$3{,}950, r = 6\%,$
$t = 5$ years
ⓑ in general

Solution:

ⓐ $P = \$\ 13,166.67$ ⓑ $P = \frac{I}{rt}$

Exercise:

Problem: Solve for the time, t for
ⓐ $I = \$2{,}376, P = \$9{,}000,$
$r = 4.4\%$
ⓑ in general

Exercise:

Problem: Solve for the time, t for
ⓐ $I = \$624, P = \$6{,}000,$
$r = 5.2\%$
ⓑ in general

Solution:

ⓐ $t = 2$ years ⓑ $t = \frac{I}{\text{Pr}}$

In the following exercises, solve.

Exercise:

Solve the formula $2x + 3y = 12$ for *y*
ⓐ when $x = 3$
Problem: ⓑ in general

Exercise:

Solve the formula $5x + 2y = 10$ for *y*
ⓐ when $x = 4$
Problem: ⓑ in general

Solution:

ⓐ $y = -5$ ⓑ $y = \frac{10-5x}{2}$

Exercise:

Solve the formula $3x - y = 7$ for *y*
ⓐ when $x = -2$
Problem: ⓑ in general

Exercise:

Solve the formula $4x + y = 5$ for *y*
ⓐ when $x = -3$
Problem: ⓑ in general

Solution:

ⓐ $y = 17$ ⓑ $y = 5 - 4x$

Exercise:

Problem: Solve $a + b = 90$ for *b*.

Exercise:

Problem: Solve $a + b = 90$ for a.

Solution:

$a = 90 - b$

Exercise:

Problem: Solve $180 = a + b + c$ for a.

Exercise:

Problem: Solve $180 = a + b + c$ for c.

Solution:

$c = 180 - a - b$

Exercise:

Problem: Solve the formula $8x + y = 15$ for *y*.

Exercise:

Problem: Solve the formula $9x + y = 13$ for *y*.

Solution:

$y = 13 - 9x$

Exercise:

Problem: Solve the formula $-4x + y = -6$ for *y*.

Exercise:

Problem: Solve the formula $-5x + y = -1$ for *y*.

Solution:

$y = -1 + 5x$

Exercise:

Problem: Solve the formula $4x + 3y = 7$ for y.

Exercise:

Problem: Solve the formula $3x + 2y = 11$ for y.

Solution:

$y = \frac{11-3x}{4}$

Exercise:

Problem: Solve the formula $x - y = -4$ for y.

Exercise:

Problem: Solve the formula $x - y = -3$ for y.

Solution:

$y = 3 + x$

Exercise:

Problem: Solve the formula $P = 2L + 2W$ for L.

Exercise:

Problem: Solve the formula $P = 2L + 2W$ for W.

Solution:

$W = \frac{P - 2L}{2}$

Exercise:

Problem: Solve the formula $C = \pi d$ for d.

Exercise:

Problem: Solve the formula $C = \pi d$ for π.

Solution:

$\pi = \frac{C}{d}$

Exercise:

Problem: Solve the formula $V = LWH$ for L.

Exercise:

Problem: Solve the formula $V = LWH$ for H.

Solution:

$H = \frac{V}{LW}$

Everyday Math

Exercise:

Problem:

Converting temperature While on a tour in Greece, Tatyana saw that the temperature was 40^{o} Celsius. Solve for F in the formula $C = \frac{5}{9}(F - 32)$ to find the Fahrenheit temperature.

Exercise:

Problem:

Converting temperature Yon was visiting the United States and he saw that the temperature in Seattle one day was 50^o Fahrenheit. Solve for C in the formula $F = \frac{9}{5}C + 32$ to find the Celsius temperature.

Solution:

10°C

Writing Exercises

Exercise:

Solve the equation $2x + 3y = 6$ for y
ⓐ when $x = -3$
ⓑ in general

Problem: ⓒ Which solution is easier for you, ⓐ or ⓑ? Why?

Exercise:

Solve the equation $5x - 2y = 10$ for x
ⓐ when $y = 10$
ⓑ in general

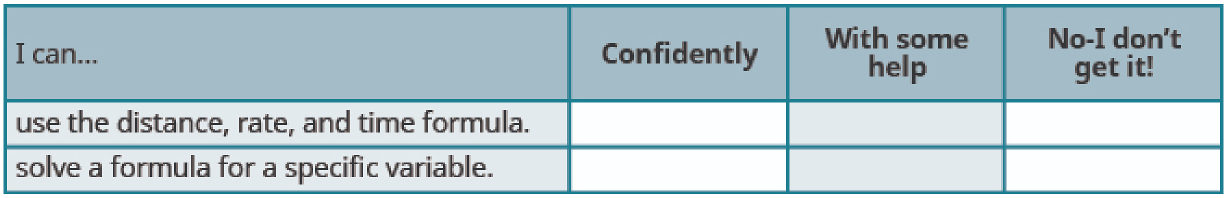
Problem: ⓒ Which solution is easier for you, ⓐ or ⓑ? Why?

Solution:

ⓐ $x = 6$
ⓑ $x = \frac{2y+10}{5}$
ⓒ Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
use the distance, rate, and time formula.			
solve a formula for a specific variable.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Solve Linear Inequalities
By the end of this section, you will be able to:

- Graph inequalities on the number line
- Solve inequalities using the Subtraction and Addition Properties of inequality
- Solve inequalities using the Division and Multiplication Properties of inequality
- Solve inequalities that require simplification
- Translate to an inequality and solve

Note:
Before you get started, take this readiness quiz.

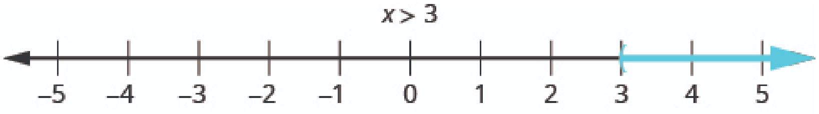
1. Translate from algebra to English: $15 > x$.
   If you missed this problem, review [link].
2. Solve: $n - 9 = -42$.
   If you missed this problem, review [link].
3. Solve: $-5p = -23$.
   If you missed this problem, review [link].
4. Solve: $3a - 12 = 7a - 20$.
   If you missed this problem, review [link].

Graph Inequalities on the Number Line

Do you remember what it means for a number to be a solution to an equation? A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

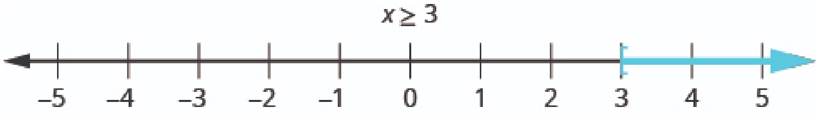
What about the solution of an inequality? What number would make the inequality $x > 3$ true? Are you thinking, ‘*x* could be 4’? That’s correct, but *x* could be 5 too, or 20, or even 3.001. Any number greater than 3 is a solution to the inequality $x > 3$.

We show the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of 3, to show that all numbers greater than 3 are solutions. Because the number 3 itself is not a solution, we put an open parenthesis at 3. The graph of $x > 3$ is shown in [link]. Please note that the following convention is used: light blue arrows point in the positive direction and dark blue arrows point in the negative direction.



The inequality $x > 3$ is graphed on this number line.

The graph of the inequality $x \geq 3$ is very much like the graph of $x > 3$, but now we need to show that 3 is a solution, too. We do that by putting a bracket at $x = 3$, as shown in [link].



The inequality $x \geq 3$ is graphed on this number line.

Notice that the open parentheses symbol, (, shows that the endpoint of the inequality is not included. The open bracket symbol, [, shows that the endpoint is included.

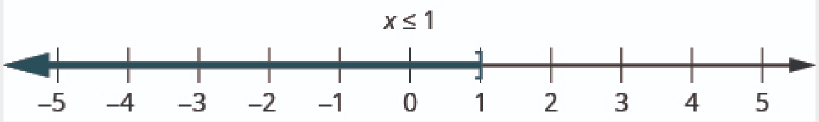
Example:
Exercise:

Problem: Graph on the number line:

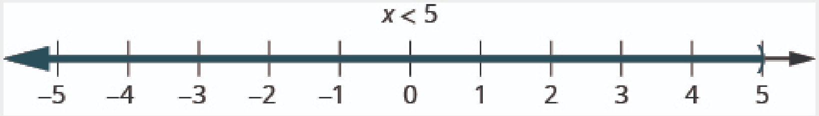
ⓐ $x \leq 1$ ⓑ $x < 5$ ⓒ $x > -1$

Solution:
Solution

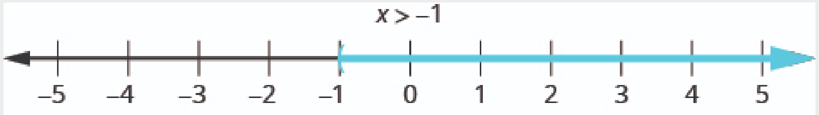
ⓐ $x \leq 1$
This means all numbers less than or equal to 1. We shade in all the numbers on the number line to the left of 1 and put a bracket at $x = 1$ to show that it is included.



ⓑ $x < 5$
This means all numbers less than 5, but not including 5. We shade in all the numbers on the number line to the left of 5 and put a parenthesis at $x = 5$ to show it is not included.



ⓒ $x > -1$
This means all numbers greater than -1, but not including -1. We shade in all the numbers on the number line to the right of -1, then put a parenthesis at $x = -1$ to show it is not included.

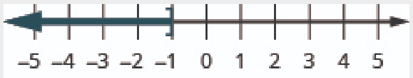


Note:
Exercise:

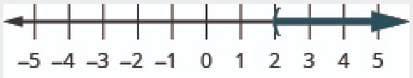
Problem: Graph on the number line: ⓐ $x \leq -1$ ⓑ $x > 2$ ⓒ $x < 3$

Solution:

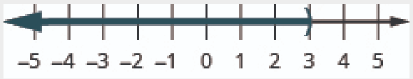
ⓐ



ⓑ



ⓒ

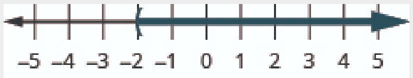


Note:
Exercise:

Problem: Graph on the number line: ⓐ $x > -2$ ⓑ $x < -3$ ⓒ $x \geq -1$

Solution:

ⓐ



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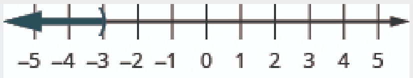
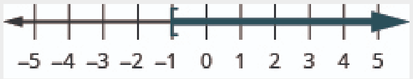
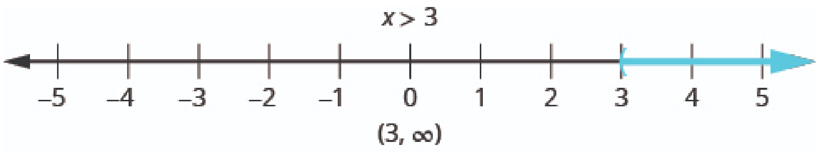


ⓒ

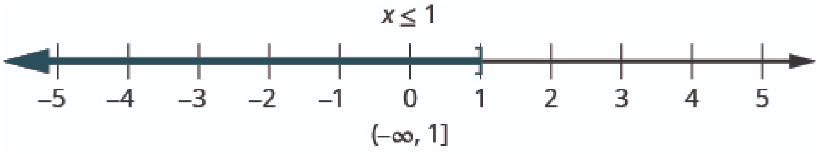


We can also represent inequalities using *interval notation.* As we saw above, the inequality $x > 3$ means all numbers greater than 3. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as ‘infinity’. It is not an actual number. [link] shows both the number line and the interval notation.



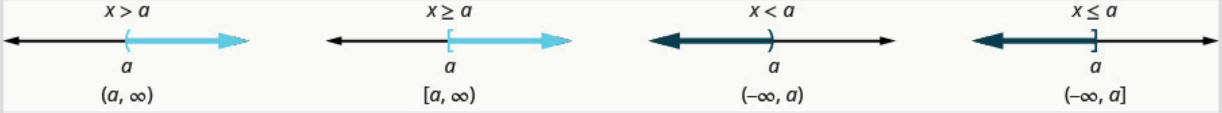
The inequality $x > 3$ is graphed on this number line and written in interval notation.

The inequality $x \leq 1$ means all numbers less than or equal to 1. There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as ‘negative infinity’. [link] shows both the number line and interval notation.

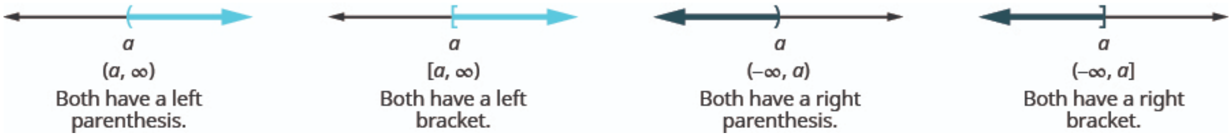


The inequality $x \leq 1$ is graphed on this number line and written in interval notation.

Note:
Inequalities, Number Lines, and Interval Notation



Did you notice how the parenthesis or bracket in the interval notation matches the symbol at the endpoint of the arrow? These relationships are shown in [link].


a
(a, ∞)
Both have a left parenthesis.
a
$[a, \infty)$
Both have a left bracket.
a
$(-\infty, a)$
Both have a right parenthesis.
a
$(-\infty, a]$
Both have a right bracket.

The notation for inequalities on a number line and in interval notation use similar symbols to express the endpoints of intervals.

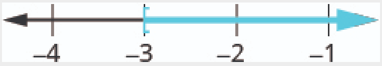
Example:
Exercise:

Problem: Graph on the number line and write in interval notation.

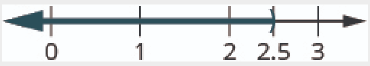
ⓐ $x \geq -3$ ⓑ $x < 2.5$ ⓒ $x \leq -\frac{3}{5}$

Solution:
Solution

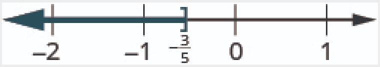
ⓐ

	$x \geq -3$
Shade to the right of -3, and put a bracket at -3.	
Write in interval notation.	$[-3, \infty)$

ⓑ

	$x < 2.5$
Shade to the left of 2.5, and put a parenthesis at 2.5.	
Write in interval notation.	$(-\infty, 2.5)$

ⓒ

	$x \le -\frac{3}{5}$
Shade to the left of $-\frac{3}{5}$, and put a bracket at $-\frac{3}{5}$.	
Write in interval notation.	$\left(-\infty, -\frac{3}{5}\right]$

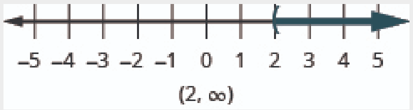
Note:
Exercise:

Problem: Graph on the number line and write in interval notation:

ⓐ $x > 2$ ⓑ $x \le -1.5$ ⓒ $x \ge \frac{3}{4}$

Solution:

ⓐ



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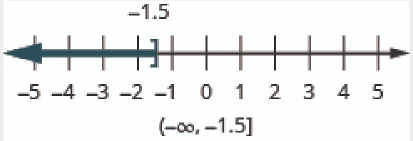

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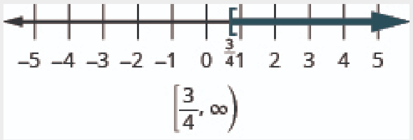

–5 –4 –3 –2 –1 0 $\frac{3}{4}$ 1 2 3 4 5
$\left[\frac{3}{4}, \infty\right)$

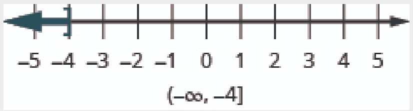
Note:
Exercise:

Problem: Graph on the number line and write in interval notation:

ⓐ $x \le -4$ ⓑ $x \ge 0.5$ ⓒ $x < -\frac{2}{3}$

Solution:

ⓐ



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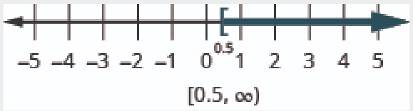
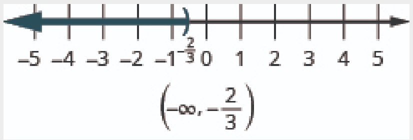


ⓒ


–5 –4 –3 –2 –1 $-\frac{2}{3}$ 0 1 2 3 4 5
$\left(-\infty, -\frac{2}{3}\right)$

Solve Inequalities using the Subtraction and Addition Properties of Inequality

The Subtraction and Addition Properties of Equality state that if two quantities are equal, when we add or subtract the same amount from both quantities, the results will be equal.

Note:
Properties of Equality
Equation:

Subtraction Property of Equality

For any numbers a, b, and c,

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & a - c & = & b - c. \end{array}$$

Addition Property of Equality

For any numbers a, b, and c,

$$\begin{array}{lrcl} \text{if} & a & = & b, \\ \text{then} & a + c & = & b + c. \end{array}$$

Similar properties hold true for inequalities.

For example, we know that −4 is less than 2.	$-4 < 2$
If we subtract 5 from both quantities, is the left side still less than the right side?	$-4 - 5 \; ? \; 2 - 5$
We get −9 on the left and −3 on the right.	$-9 \; ? \; -3$
And we know −9 is less than −3.	$-9 < -3$
	The inequality sign stayed the same.

Similarly we could show that the inequality also stays the same for addition.

This leads us to the Subtraction and Addition Properties of Inequality.

Note:
Properties of Inequality
Equation:

Subtraction Property of Inequality

For any numbers $a, b,$ and c,

$$\begin{array}{lrcl} \text{if} & a & < & b \\ \text{then} & a-c & < & b-c. \end{array}$$

$$\begin{array}{lrcl} \text{if} & a & > & b \\ \text{then} & a-c & > & b-c. \end{array}$$

Addition Property of Inequality

For any numbers $a, b,$ and c,

$$\begin{array}{lrcl} \text{if} & a & < & b \\ \text{then} & a+c & < & b+c. \end{array}$$

$$\begin{array}{lrcl} \text{if} & a & > & b \\ \text{then} & a+c & > & b+c. \end{array}$$

We use these properties to solve inequalities, taking the same steps we used to solve equations. Solving the inequality $x+5>9$, the steps would look like this:

Equation:

$$\begin{array}{lrcl} & x+5 & > & 9 \\ \text{Subtract 5 from both sides to isolate } x. & x+5-5 & > & 9-5 \\ \text{Simplify.} & x & > & 4 \end{array}$$

Any number greater than 4 is a solution to this inequality.

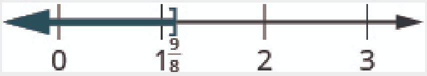
Example:
Exercise:

Problem:

Solve the inequality $n-\frac{1}{2}\leq\frac{5}{8}$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

	$n-\frac{1}{2}\leq\frac{5}{8}$
Add $\frac{1}{2}$ to both sides of the inequality.	

	$n-\frac{1}{2}+\frac{1}{2}\le\frac{5}{8}+\frac{1}{2}$
Simplify.	$n\le\frac{9}{8}$
Graph the solution on the number line.	
Write the solution in interval notation.	$\left(-\infty,\frac{9}{8}\right]$

Note:
Exercise:

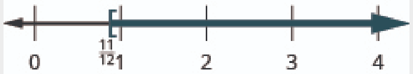
Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$p-\frac{3}{4}\ge\frac{1}{6}$

Solution:

$p\ge\frac{11}{12}$



$\left[\frac{11}{12},\infty\right)$

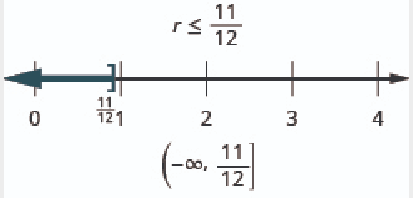
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$r-\frac{1}{3}\leq\frac{7}{12}$

Solution:



Solve Inequalities using the Division and Multiplication Properties of Inequality

The Division and Multiplication Properties of Equality state that if two quantities are equal, when we divide or multiply both quantities by the same amount, the results will also be equal (provided we don’t divide by 0).

Note:
Properties of Equality
Equation:

$$\begin{array}{ll}\textbf{Division Property of Equality} & \textbf{Multiplication Property of Equality}\\ \text{For any numbers } a,b,c,\text{and } c\neq 0, & \text{For any real numbers } a,b,c,\\ \quad\begin{array}{lrcl}\text{if} & a & = & b,\\ \text{then} & \frac{a}{c} & = & \frac{b}{c}.\end{array} & \quad\begin{array}{lrcl}\text{if} & a & = & b,\\ \text{then} & ac & = & bc.\end{array}\end{array}$$

Are there similar properties for inequalities? What happens to an inequality when we divide or multiply both sides by a constant?

Consider some numerical examples.

	$10 < 15$		$10 < 15$
Divide both sides by 5.	$\frac{10}{5} ? \frac{15}{5}$	Multiply both sides by 5.	$10(5) ? 15(5)$
Simplify.	$2 ? 3$		$50 ? 75$
Fill in the inequality signs.	$2 < 3$		$50 < 75$

Equation:

The inequality signs stayed the same.

Does the inequality stay the same when we divide or multiply by a negative number?

	$10 < 15$		$10 < 15$
Divide both sides by −5.	$\frac{10}{-5} ? \frac{15}{-5}$	Multiply both sides by −5.	$10(-5) ? 15(-5)$
Simplify.	$-2 ? -3$		$-50 ? -75$
Fill in the inequality signs.	$-2 > -3$		$-50 > -75$

Equation:

The inequality signs reversed their direction.

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses.

Here are the Division and Multiplication Properties of Inequality for easy reference.

Note:
Division and Multiplication Properties of Inequality
Equation:

$$\begin{array}{l} \text{For any real numbers } a, b, c \\ \\ \begin{array}{ll} \text{if } a < b \text{ and } c > 0, \text{then} & \frac{a}{c} < \frac{b}{c} \text{ and } ac < bc. \\ \text{if } a > b \text{ and } c > 0, \text{then} & \frac{a}{c} > \frac{b}{c} \text{ and } ac > bc. \\ \text{if } a < b \text{ and } c < 0, \text{then} & \frac{a}{c} > \frac{b}{c} \text{ and } ac > bc. \\ \text{if } a > b \text{ and } c < 0, \text{then} & \frac{a}{c} < \frac{b}{c} \text{ and } ac < bc. \end{array} \end{array}$$

When we **divide or multiply** an inequality by a:

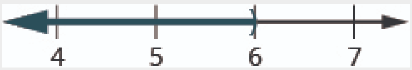
- **positive** number, the inequality stays the **same**.
- **negative** number, the inequality **reverses**.

Example:
Exercise:

Problem:

Solve the inequality $7y < 42$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

	$7y < 42$
Divide both sides of the inequality by 7. Since $7 > 0$, the inequality stays the same.	$\frac{7y}{7} < \frac{42}{7}$
Simplify.	$y < 6$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, 6)$

Note:
Exercise:

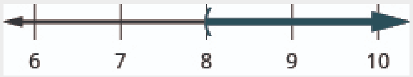
Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$9c > 72$

Solution:

$c > 8$



Note:
Exercise:

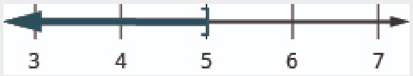
Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$12d \leq 60$

Solution:

$d \leq 5$



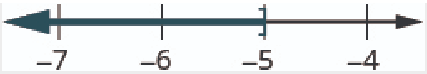
Example:
Exercise:

Problem:

Solve the inequality $-10a \geq 50$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

	$-10a \geq 50$
Divide both sides of the inequality by −10. Since $-10 < 0$, the inequality reverses.	$\frac{-10a}{-10} \leq \frac{50}{-10}$
Simplify.	$a \leq -5$
Graph the solution on the number line.	

	
Write the solution in interval notation.	(−∞, −5]

Note:
Exercise:

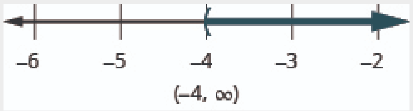
Problem:

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$-8q < 32$

Solution:

$q > -4$



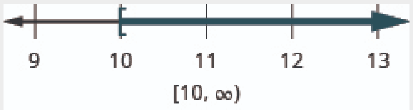
Note:
Exercise:

Problem:

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$-7r \leq -70$

Solution:



Note:
Solving Inequalities
Sometimes when solving an inequality, the variable ends up on the right. We can rewrite the inequality in reverse to get the variable to the left.
Equation:

$$x > a \text{ has the same meaning as } a < x$$

Think about it as “If Xavier is taller than Alex, then Alex is shorter than Xavier.”

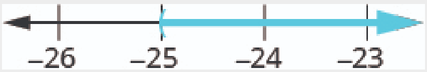
Example:
Exercise:

Problem:

Solve the inequality $-20 < \frac{4}{5}u$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

	$-20 < \frac{4}{5}u$
Multiply both sides of the inequality by $\frac{5}{4}$. Since $\frac{5}{4} > 0$, the inequality stays the same.	$\frac{5}{4}(-20) < \frac{5}{4}\left(\frac{4}{5}u\right)$
Simplify.	$-25 < u$
Rewrite the variable on the left.	$u > -25$

Graph the solution on the number line.	
Write the solution in interval notation.	(–25, ∞)

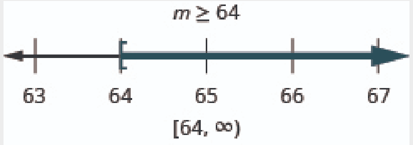
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$24 \leq \frac{3}{8}m$

Solution:



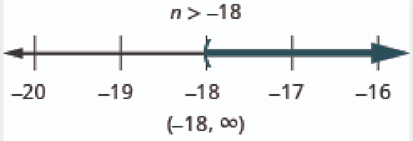
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$-24 < \frac{4}{3}n$

Solution:

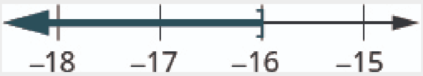


Example:
Exercise:

Problem:

Solve the inequality $\frac{t}{-2} \geq 8$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

	$\frac{t}{-2} \geq 8$
Multiply both sides of the inequality by -2. Since $-2 < 0$, the inequality reverses.	$-2\left(\frac{t}{-2}\right) \leq -2(8)$
Simplify.	$t \leq -16$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -16]$

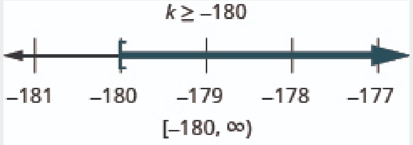
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$\frac{k}{-12} \leq 15$

Solution:



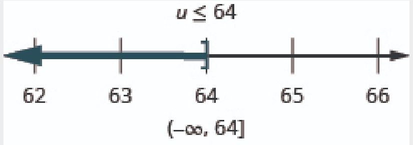
Note:
Exercise:

Problem:

Solve the inequality, graph the solution on the number line, and write the solution in interval notation.

$\frac{u}{-4} \geq -16$

Solution:



Solve Inequalities That Require Simplification

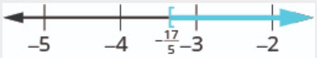
Most inequalities will take more than one step to solve. We follow the same steps we used in the general strategy for solving linear equations, but be sure to pay close attention during multiplication or division.

Example:
Exercise:

Problem:

Solve the inequality $4m \leq 9m + 17$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

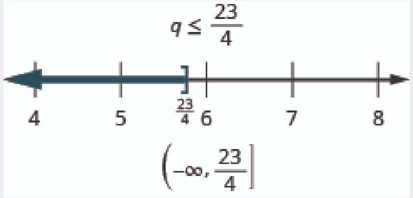
	$4m \leq 9m + 17$
Subtract $9m$ from both sides to collect the variables on the left.	$4m - 9m \leq 9m - 9m + 17$
Simplify.	$-5m \leq 17$
Divide both sides of the inequality by -5, and reverse the inequality.	$\frac{-5m}{-5} \geq \frac{17}{-5}$
Simplify.	$m \geq -\frac{17}{5}$
Graph the solution on the number line.	
Write the solution in interval notation.	$\left[-\frac{17}{5}, \infty\right)$

Note:
Exercise:

Problem:

Solve the inequality $3q \geq 7q - 23$, graph the solution on the number line, and write the solution in interval notation.

Solution:

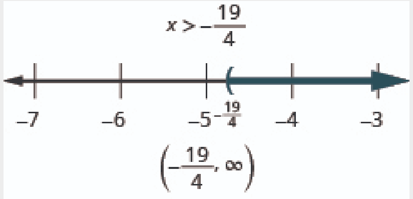


Note:
Exercise:

Problem:

Solve the inequality $6x < 10x + 19$, graph the solution on the number line, and write the solution in interval notation.

Solution:

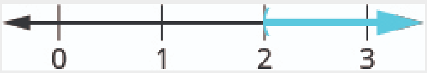


Example:
Exercise:

Problem:

Solve the inequality $8p + 3(p - 12) > 7p - 28$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

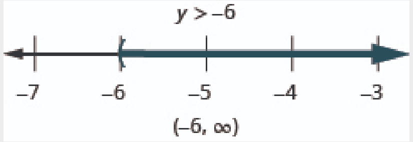
Simplify each side as much as possible.	$8p+3(p-12)>7p-28$
Distribute.	$8p+3p-36>7p-28$
Combine like terms.	$11p-36>7p-28$
Subtract $7p$ from both sides to collect the variables on the left.	$11p-36-7p>7p-28-7p$
Simplify.	$4p-36>-28$
Add 36 to both sides to collect the constants on the right.	$4p-36+36>-28+36$
Simplify.	$4p>8$
Divide both sides of the inequality by 4; the inequality stays the same.	$\frac{4p}{4}>\frac{8}{4}$
Simplify.	$p>2$
Graph the solution on the number line.	
Write the solution in interal notation.	$(2,\infty)$

Note:
Exercise:

Problem:

Solve the inequality $9y+2(y+6)>5y-24$, graph the solution on the number line, and write the solution in interval notation.

Solution:

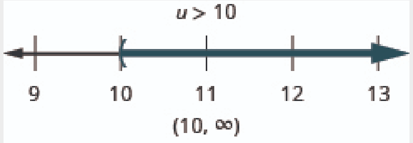


Note:
Exercise:

Problem:

Solve the inequality $6u+8(u-1)>10u+32$, graph the solution on the number line, and write the solution in interval notation.

Solution:



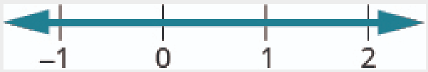
Just like some equations are identities and some are contradictions, inequalities may be identities or contradictions, too. We recognize these forms when we are left with only constants as we solve the inequality. If the result is a true statement, we have an identity. If the result is a false statement, we have a contradiction.

Example:
Exercise:

Problem:

Solve the inequality $8x-2\left(5-x\right)<4\left(x+9\right)+6x$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

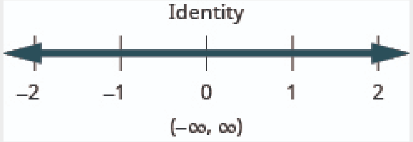
Simplify each side as much as possible.	$8x-2(5-x)<4(x+9)+6x$
Distribute.	$8x-10+2x<4x+36+6x$
Combine like terms.	$10x-10<10x+36$
Subtract $10x$ from both sides to collect the variables on the left.	$10x-10-10x<10x+36-10x$
Simplify.	$-10<36$
The x's are gone, and we have a true statement.	The inequality is an identity. The solution is all real numbers.
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty,\infty)$

Note:
Exercise:

Problem:

Solve the inequality $4b - 3(3 - b) > 5(b - 6) + 2b$, graph the solution on the number line, and write the solution in interval notation.

Solution:

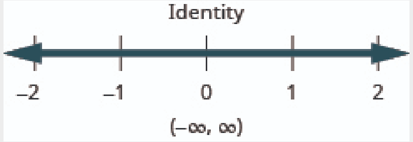


Note:
Exercise:

Problem:

Solve the inequality $9h - 7\left(2 - h\right) < 8\left(h + 11\right) + 8h$, graph the solution on the number line, and write the solution in interval notation.

Solution:

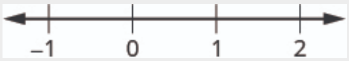


Example:
Exercise:

Problem:

Solve the inequality $\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$, graph the solution on the number line, and write the solution in interval notation.

Solution:
Solution

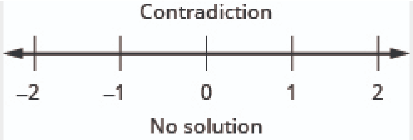
	$\frac{1}{3}a-\frac{1}{8}a>\frac{5}{24}a+\frac{3}{4}$
Multiply both sides by the LCD, 24, to clear the fractions.	$24\left(\frac{1}{3}a-\frac{1}{8}a\right)>24\left(\frac{5}{24}a+\frac{3}{4}\right)$
Simplify.	$8a-3a>5a+18$
Combine like terms.	$5a>5a+18$
Subtract $5a$ from both sides to collect the variables on the left.	$5a-5a>5a-5a+18$
Simplify.	$0>18$
The statement is false!	The inequality is a contradiction.
	There is no solution.
Graph the solution on the number line.	
Write the solution in interval notation.	There is no solution.

Note:
Exercise:

Problem:

Solve the inequality $\frac{1}{4}x-\frac{1}{12}x>\frac{1}{6}x+\frac{7}{8}$, graph the solution on the number line, and write the solution in interval notation.

Solution:

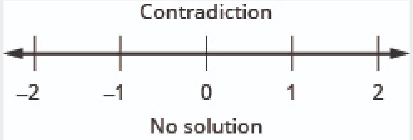


Note:
Exercise:

Problem:

Solve the inequality $\frac{2}{5}z - \frac{1}{3}z < \frac{1}{15}z - \frac{3}{5}$, graph the solution on the number line, and write the solution in interval notation.

Solution:



Translate to an Inequality and Solve

To translate English sentences into inequalities, we need to recognize the phrases that indicate the inequality. Some words are easy, like ‘more than’ and ‘less than’. But others are not as obvious.

Think about the phrase ‘at least’ – what does it mean to be ‘at least 21 years old’? It means 21 or more. The phrase ‘at least’ is the same as ‘greater than or equal to’.

[link] shows some common phrases that indicate inequalities.

$>$	$\geq$	$<$	$\leq$
is greater than	is greater than or equal to	is less than	is less than or equal to

$>$	$\geq$	$<$	$\leq$
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

Example:
Exercise:

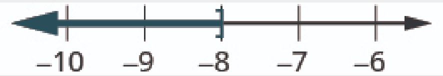
Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Twelve times *c* is no more than 96.

Solution:
Solution

Translate.	$\text{Twelve times } c \underbrace{\text{ is no more than }}_{\leq} 96$ $12c \leq 96$
Solve—divide both sides by 12.	$\frac{12c}{12} \leq \frac{96}{12}$
Simplify.	$c \leq 8$
Write in interval notation.	$(-\infty, -8]$

Graph on the number line.	

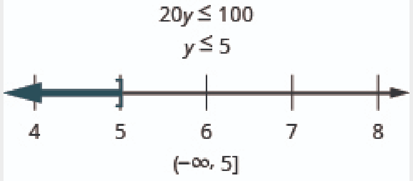
Note:
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Twenty times *y* is at most 100

Solution:



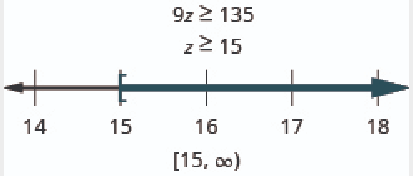
Note:
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Nine times *z* is no less than 135

Solution:



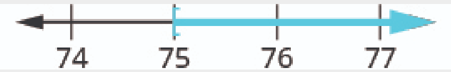
Example:
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Thirty less than *x* is at least 45.

Solution:
Solution

Translate.	Thirty less than x $\underbrace{\text{is at least}}$ 45. $x - 30 \geq 45$
Solve—add 30 to both sides.	$x - 30 + 30 \geq 45 + 30$
Simplify.	$x \geq 75$
Write in interval notation.	$[75, \infty)$
Graph on the number line.	

Note:

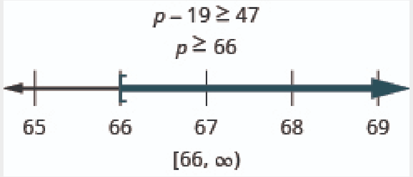
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Nineteen less than *p* is no less than 47

Solution:



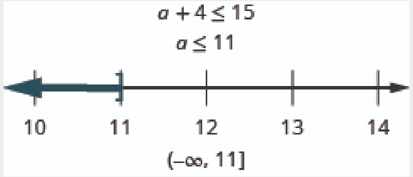
Note:
Exercise:

Problem:

Translate and solve. Then write the solution in interval notation and graph on the number line.

Four more than *a* is at most 15.

Solution:



Key Concepts

- **Subtraction Property of Inequality**
  For any numbers a, b, and c,
  if $a < b$ then $a - c < b - c$ and
  if $a > b$ then $a - c > b - c$.

- **Addition Property of Inequality**
  For any numbers a, b, and c,
  if $a < b$ then $a + c < b + c$ and
  if $a > b$ then $a + c > b + c$.
- **Division and Multiplication Properties of Inequality**
  For any numbers a, b, and c,
  if $a < b$ and $c > 0$, then $\frac{a}{c} < \frac{b}{c}$ and $ac < bc$.
  if $a > b$ and $c > 0$, then $\frac{a}{c} > \frac{b}{c}$ and $ac > bc$.
  if $a < b$ and $c < 0$, then $\frac{a}{c} > \frac{b}{c}$ and $ac > bc$.
  if $a > b$ and $c < 0$, then $\frac{a}{c} < \frac{b}{c}$ and $ac < bc$.
- When we **divide or multiply** an inequality by a:
  - **positive** number, the inequality stays the **same**.
  - **negative** number, the inequality **reverses**.

Practice Makes Perfect

Graph Inequalities on the Number Line

In the following exercises, graph each inequality on the number line.

Exercise:

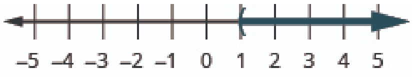
ⓐ $x \leq 2$
ⓑ $x > -1$
Problem: ⓒ $x < 0$

Exercise:

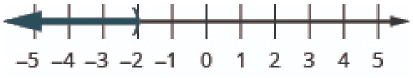
ⓐ $x > 1$
ⓑ $x < -2$
Problem: ⓒ $x \geq -3$

Solution:

ⓐ



ⓑ



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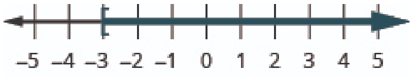


Exercise:

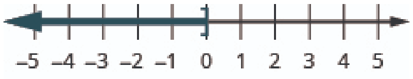
ⓐ $x \geq -3$
ⓑ $x < 4$
Problem: ⓒ $x \leq -2$

Exercise:

ⓐ $x \leq 0$
ⓑ $x > -4$
Problem: ⓒ $x \geq -1$

Solution:

ⓐ



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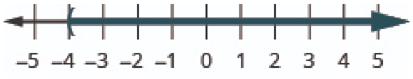

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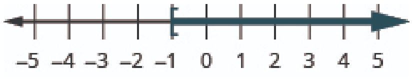


In the following exercises, graph each inequality on the number line and write in interval notation.

Exercise:

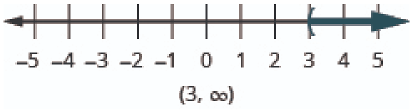
ⓐ $x < -2$
ⓑ $x \geq -3.5$
Problem: ⓒ $x \leq \frac{2}{3}$

Exercise:

ⓐ $x > 3$
ⓑ $x \leq -0.5$
Problem: ⓒ $x \geq \frac{1}{3}$

Solution:

ⓐ



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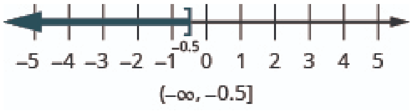
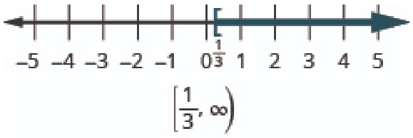


ⓒ



Exercise:

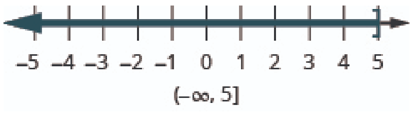
ⓐ $x \geq -4$
ⓑ $x < 2.5$
Problem: ⓒ $x > -\frac{3}{2}$

Exercise:

ⓐ $x \leq 5$
ⓑ $x \geq -1.5$
Problem: ⓒ $x < -\frac{7}{3}$

Solution:

ⓐ



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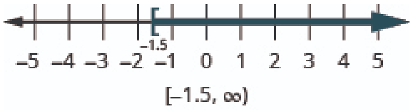

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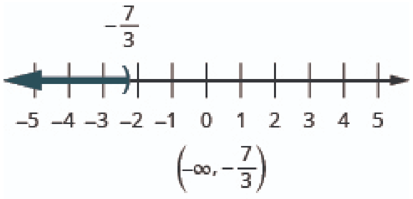


Solve Inequalities using the Subtraction and Addition Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

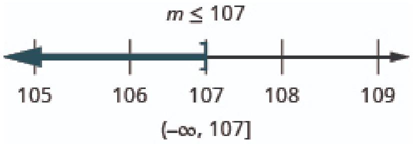
Exercise:

Problem: $n - 11 < 33$

Exercise:

Problem: $m - 45 \leq 62$

Solution:



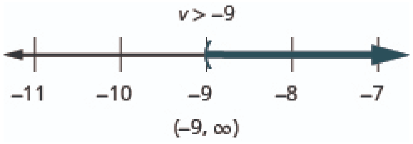
Exercise:

Problem: $u + 25 > 21$

Exercise:

Problem: $v + 12 > 3$

Solution:



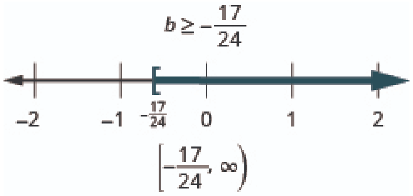
Exercise:

Problem: $a + \frac{3}{4} \geq \frac{7}{10}$

Exercise:

Problem: $b + \frac{7}{8} \geq \frac{1}{6}$

Solution:



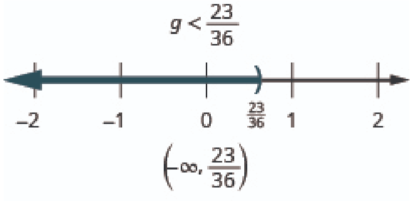
Exercise:

Problem: $f - \frac{13}{20} < -\frac{5}{12}$

Exercise:

Problem: $g - \frac{11}{12} < -\frac{5}{18}$

Solution:



Solve Inequalities using the Division and Multiplication Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

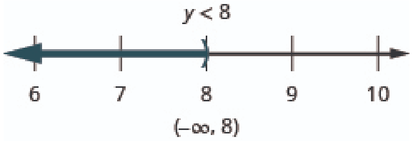
Exercise:

Problem: $8x > 72$

Exercise:

Problem: $6y < 48$

Solution:



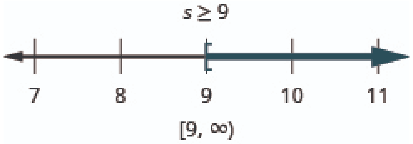
Exercise:

Problem: $7r \leq 56$

Exercise:

Problem: $9s \geq 81$

Solution:



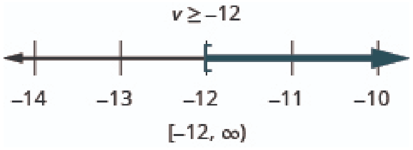
Exercise:

Problem: $-5u \geq 65$

Exercise:

Problem: $-8v \leq 96$

Solution:



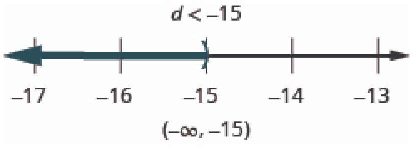
Exercise:

Problem: $-9c < 126$

Exercise:

Problem: $-7d > 105$

Solution:



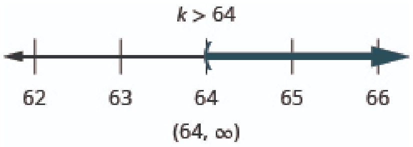
Exercise:

Problem: $20 > \frac{2}{5}h$

Exercise:

Problem: $40 < \frac{5}{8}k$

Solution:



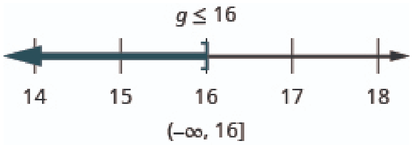
Exercise:

Problem: $\frac{7}{6}j \geq 42$

Exercise:

Problem: $\frac{9}{4}g \leq 36$

Solution:



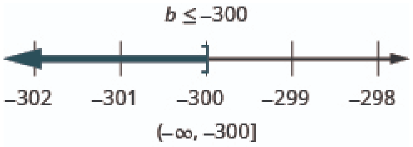
Exercise:

Problem: $\frac{a}{-3} \leq 9$

Exercise:

Problem: $\frac{b}{-10} \geq 30$

Solution:



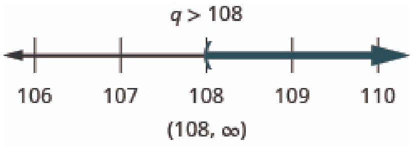
Exercise:

Problem: $-25 < \frac{p}{-5}$

Exercise:

Problem: $-18 > \frac{q}{-6}$

Solution:



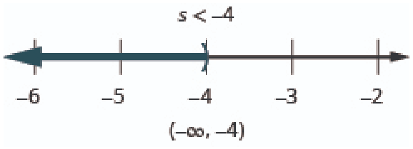
Exercise:

Problem: $9t \geq -27$

Exercise:

Problem: $7s < -28$

Solution:



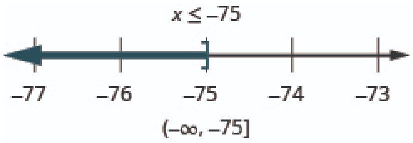
Exercise:

Problem: $\frac{2}{3}y > -36$

Exercise:

Problem: $\frac{3}{5}x \leq -45$

Solution:



Solve Inequalities That Require Simplification

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

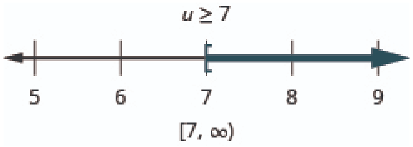
Exercise:

Problem: $4v \geq 9v - 40$

Exercise:

Problem: $5u \leq 8u - 21$

Solution:



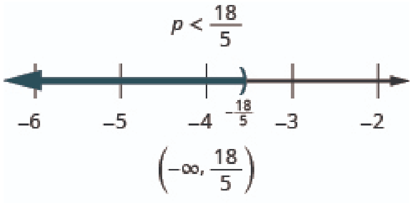
Exercise:

Problem: $13q < 7q - 29$

Exercise:

Problem: $9p > 14p - 18$

Solution:



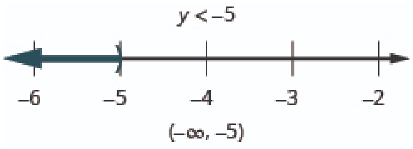
Exercise:

Problem: $12x + 3(x + 7) > 10x - 24$

Exercise:

Problem: $9y + 5(y + 3) < 4y - 35$

Solution:



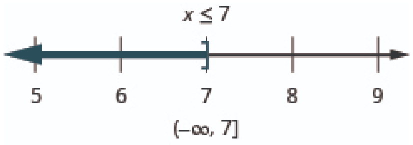
Exercise:

Problem: $6h - 4(h - 1) \leq 7h - 11$

Exercise:

Problem: $4k - (k - 2) \geq 7k - 26$

Solution:



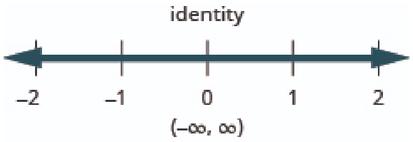
Exercise:

Problem: $8m - 2\left(14 - m\right) \geq 7\left(m - 4\right) + 3m$

Exercise:

Problem: $6n - 12\left(3 - n\right) \leq 9\left(n - 4\right) + 9n$

Solution:



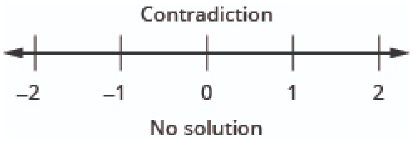
Exercise:

Problem: $\frac{3}{4}b - \frac{1}{3}b < \frac{5}{12}b - \frac{1}{2}$

Exercise:

Problem: $9u + 5(2u - 5) \geq 12(u - 1) + 7u$

Solution:



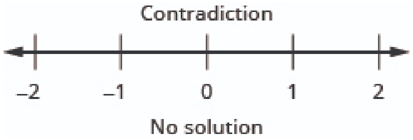
Exercise:

Problem: $\frac{2}{3}g - \frac{1}{2}(g - 14) \leq \frac{1}{6}(g + 42)$

Exercise:

Problem: $\frac{5}{6}a - \frac{1}{4}a > \frac{7}{12}a + \frac{2}{3}$

Solution:



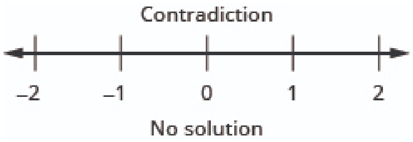
Exercise:

Problem: $\frac{4}{5}h - \frac{2}{3}(h - 9) \geq \frac{1}{15}(2h + 90)$

Exercise:

Problem: $12v + 3(4v - 1) \leq 19(v - 2) + 5v$

Solution:



Mixed practice

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

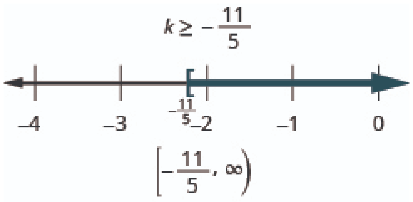
Exercise:

Problem: $15k \leq -40$

Exercise:

Problem: $35k \geq -77$

Solution:



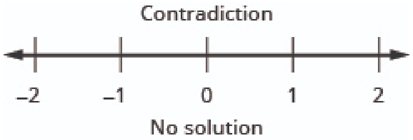
Exercise:

Problem: $23p-2(6-5p)>3(11p-4)$

Exercise:

Problem: $18q-4(10-3q)<5(6q-8)$

Solution:



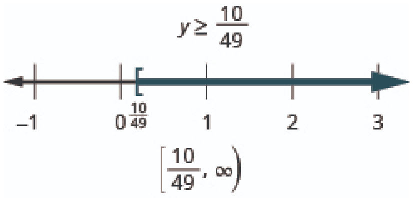
Exercise:

Problem: $-\frac{9}{4}x \ge -\frac{5}{12}$

Exercise:

Problem: $-\frac{21}{8}y \le -\frac{15}{28}$

Solution:



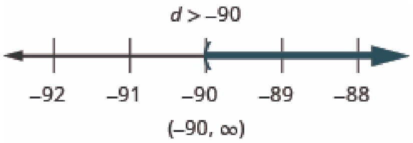
Exercise:

Problem: $c+34<-99$

Exercise:

Problem: $d+29>-61$

Solution:



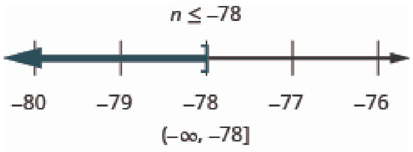
Exercise:

Problem: $\frac{m}{18} \geq -4$

Exercise:

Problem: $\frac{n}{13} \leq -6$

Solution:



Translate to an Inequality and Solve

In the following exercises, translate and solve .Then write the solution in interval notation and graph on the number line.

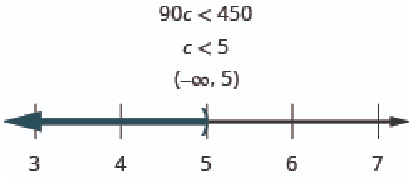
Exercise:

Problem: Fourteen times *d* is greater than 56.

Exercise:

Problem: Ninety times *c* is less than 450.

Solution:



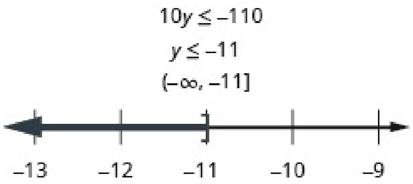
Exercise:

Problem: Eight times *z* is smaller than -40.

Exercise:

Problem: Ten times *y* is at most -110.

Solution:



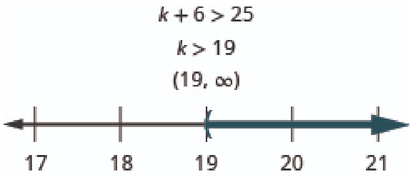
Exercise:

Problem: Three more than *h* is no less than 25.

Exercise:

Problem: Six more than *k* exceeds 25.

Solution:



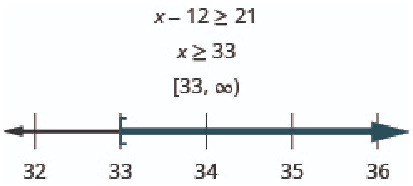
Exercise:

Problem: Ten less than *w* is at least 39.

Exercise:

Problem: Twelve less than *x* is no less than 21.

Solution:



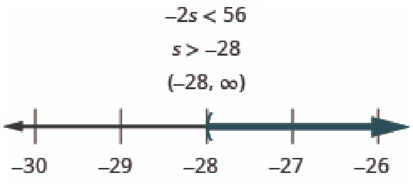
Exercise:

Problem: Negative five times *r* is no more than 95.

Exercise:

Problem: Negative two times *s* is lower than 56.

Solution:



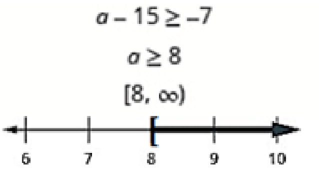
Exercise:

Problem: Nineteen less than *b* is at most -22.

Exercise:

Problem: Fifteen less than *a* is at least -7.

Solution:



Everyday Math

Exercise:

Problem:

Safety A child’s height, *h*, must be at least 57 inches for the child to safely ride in the front seat of a car. Write this as an inequality.

Exercise:

Problem:

Fighter pilots The maximum height, *h*, of a fighter pilot is 77 inches. Write this as an inequality.

Solution:

$h \leq 77$

Exercise:

Problem:

Elevators The total weight, *w*, of an elevator's passengers can be no more than 1,200 pounds. Write this as an inequality.

Exercise:

Problem:

Shopping The number of items, *n*, a shopper can have in the express check-out lane is at most 8. Write this as an inequality.

Solution:

$n \leq 8$

Writing Exercises

Exercise:

Problem: Give an example from your life using the phrase 'at least'.

Exercise:

Problem: Give an example from your life using the phrase 'at most'.

Solution:

Answers will vary.

Exercise:

Problem:

Explain why it is necessary to reverse the inequality when solving $-5x > 10$.

Exercise:

Problem:

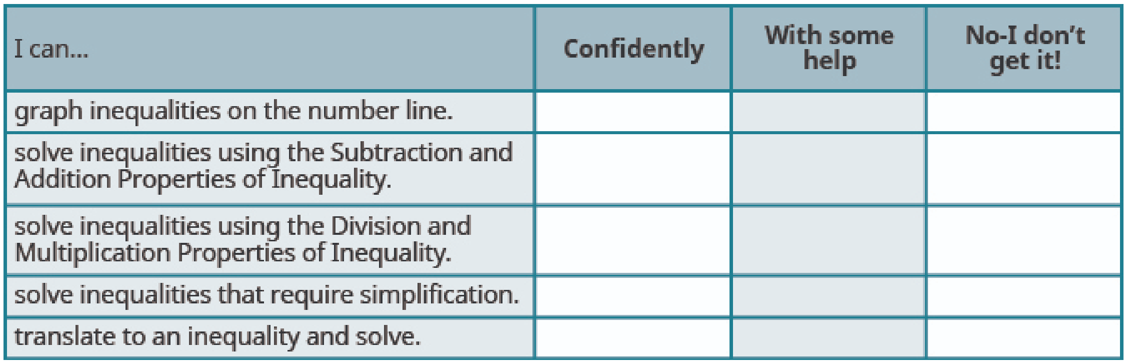
Explain why it is necessary to reverse the inequality when solving $\frac{n}{-3} < 12$.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
graph inequalities on the number line.			
solve inequalities using the Subtraction and Addition Properties of Inequality.			
solve inequalities using the Division and Multiplication Properties of Inequality.			
solve inequalities that require simplification.			
translate to an inequality and solve.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Chapter 2 Review Exercises

Solve Equations using the Subtraction and Addition Properties of Equality

Verify a Solution of an Equation

In the following exercises, determine whether each number is a solution to the equation.
Exercise:

Problem: $10x-1=5x; x=\frac{1}{5}$

Exercise:

Problem: $w+2=\frac{5}{8}; w=\frac{3}{8}$

Solution:

no

Exercise:

Problem: $-12n+5=8n; n=-\frac{5}{4}$

Exercise:

Problem: $6a-3=-7a, a=\frac{3}{13}$

Solution:

yes

Solve Equations using the Subtraction and Addition Properties of Equality

In the following exercises, solve each equation using the Subtraction Property of Equality.

Exercise:

Problem: $x+7=19$

Exercise:

Problem: $y+2=-6$

Solution:

$y=-8$

Exercise:

Problem: $a+\frac{1}{3}=\frac{5}{3}$

Exercise:

Problem: $n+3.6=5.1$

Solution:

$n=1.5$

In the following exercises, solve each equation using the Addition Property of Equality.

Exercise:

Problem: $u-7=10$

Exercise:

Problem: $x-9=-4$

Solution:

$x = 5$

Exercise:

Problem: $c - \frac{3}{11} = \frac{9}{11}$

Exercise:

Problem: $p - 4.8 = 14$

Solution:

$p = 18.8$

In the following exercises, solve each equation.

Exercise:

Problem: $n - 12 = 32$

Exercise:

Problem: $y + 16 = -9$

Solution:

$y = -25$

Exercise:

Problem: $f + \frac{2}{3} = 4$

Exercise:

Problem: $d - 3.9 = 8.2$

Solution:

$d = 12.1$

Solve Equations That Require Simplification

In the following exercises, solve each equation.

Exercise:

Problem: $y+8-15=-3$

Exercise:

Problem: $7x+10-6x+3=5$

Solution:

$x=-8$

Exercise:

Problem: $6(n-1)-5n=-14$

Exercise:

Problem: $8(3p+5)-23(p-1)=35$

Solution:

$p=-28$

Translate to an Equation and Solve

In the following exercises, translate each English sentence into an algebraic equation and then solve it.

Exercise:

Problem: The sum of -6 and m is 25.

Exercise:

Problem: Four less than n is 13.

Solution:

$n-4=13; n=17$

Translate and Solve Applications

In the following exercises, translate into an algebraic equation and solve.

Exercise:

Problem:

Rochelle's daughter is 11 years old. Her son is 3 years younger. How old is her son?

Exercise:

Problem:

Tan weighs 146 pounds. Minh weighs 15 pounds more than Tan. How much does Minh weigh?

Solution:

161 pounds

Exercise:

Problem:

Peter paid \$9.75 to go to the movies, which was \$46.25 less than he paid to go to a concert. How much did he pay for the concert?

Exercise:

Problem:

Elissa earned \$152.84 this week, which was \$21.65 more than she earned last week. How much did she earn last week?

Solution:

\$131.19

Solve Equations using the Division and Multiplication Properties of Equality

Solve Equations Using the Division and Multiplication Properties of Equality

In the following exercises, solve each equation using the division and multiplication properties of equality and check the solution.

Exercise:

Problem: $8x = 72$

Exercise:

Problem: $13a = -65$

Solution:

$a = -5$

Exercise:

Problem: $0.25p = 5.25$

Exercise:

Problem: $-y = 4$

Solution:

$y = -4$

Exercise:

Problem: $\frac{n}{6} = 18$

Exercise:

Problem: $\frac{y}{-10} = 30$

Solution:

$y = -300$

Exercise:

Problem: $36 = \frac{3}{4}x$

Exercise:

Problem: $\frac{5}{8}u = \frac{15}{16}$

Solution:

$u = \frac{3}{2}$

Exercise:

Problem: $-18m = -72$

Exercise:

Problem: $\frac{c}{9} = 36$

Solution:

$c = 324$

Exercise:

Problem: $0.45x = 6.75$

Exercise:

Problem: $\frac{11}{12} = \frac{2}{3}y$

Solution:

$y = \frac{11}{8}$

Solve Equations That Require Simplification

In the following exercises, solve each equation requiring simplification.

Exercise:

Problem: $5r - 3r + 9r = 35 - 2$

Exercise:

Problem: $24x + 8x - 11x = -7 - 14$

Solution:

$x = -1$

Exercise:

Problem: $\frac{11}{12}n - \frac{5}{6}n = 9 - 5$

Exercise:

Problem: $-9(d - 2) - 15 = -24$

Solution:

$d = 3$

Translate to an Equation and Solve

In the following exercises, translate to an equation and then solve.

Exercise:

Problem: 143 is the product of -11 and y.

Exercise:

Problem: The quotient of b and 9 is -27.

Solution:

$\frac{b}{9} = -27; b = -243$

Exercise:

Problem: The sum of q and one-fourth is one.

Exercise:

Problem: The difference of s and one-twelfth is one fourth.

Solution:

$s - \frac{1}{12} = \frac{1}{4}; s = \frac{1}{3}$

Translate and Solve Applications

In the following exercises, translate into an equation and solve.

Exercise:

Problem:

Ray paid $21 for 12 tickets at the county fair. What was the price of each ticket?

Exercise:

Problem:

Janet gets paid $24 per hour. She heard that this is $\frac{3}{4}$ of what Adam is paid. How much is Adam paid per hour?

Solution:

$32

Solve Equations with Variables and Constants on Both Sides

Solve an Equation with Constants on Both Sides

In the following exercises, solve the following equations with constants on both sides.

Exercise:

Problem: $8p + 7 = 47$

Exercise:

Problem: $10w - 5 = 65$

Solution:

$w = 7$

Exercise:

Problem: $3x + 19 = -47$

Exercise:

Problem: $32 = -4 - 9n$

Solution:

$n = -4$

Solve an Equation with Variables on Both Sides

In the following exercises, solve the following equations with variables on both sides.

Exercise:

Problem: $7y = 6y - 13$

Exercise:

Problem: $5a + 21 = 2a$

Solution:

$a = -7$

Exercise:

Problem: $k = -6k - 35$

Exercise:

Problem: $4x - \frac{3}{8} = 3x$

Solution:

$x = \frac{3}{8}$

Solve an Equation with Variables and Constants on Both Sides

In the following exercises, solve the following equations with variables and constants on both sides.

Exercise:

Problem: $12x - 9 = 3x + 45$

Exercise:

Problem: $5n - 20 = -7n - 80$

Solution:

$n = -5$

Exercise:

Problem: $4u + 16 = -19 - u$

Exercise:

Problem: $\frac{5}{8}c - 4 = \frac{3}{8}c + 4$

Solution:

$c = 32$

Use a General Strategy for Solving Linear Equations

Solve Equations Using the General Strategy for Solving Linear Equations

In the following exercises, solve each linear equation.

Exercise:

Problem: $6(x+6)=24$

Exercise:

Problem: $9(2p-5)=72$

Solution:

$p=\frac{13}{2}$

Exercise:

Problem: $-(s+4)=18$

Exercise:

Problem: $8+3(n-9)=17$

Solution:

$n=12$

Exercise:

Problem: $23-3(y-7)=8$

Exercise:

Problem: $\frac{1}{3}(6m+21)=m-7$

Solution:

$m=-14$

Exercise:

Problem: $4(3.5y+0.25)=365$

Exercise:

Problem: $0.25(q-8)=0.1(q+7)$

Solution:

$q=18$

Exercise:

Problem: $8(r-2)=6(r+10)$

Exercise:

Problem: $5+7(2-5x)=2(9x+1)-(13x-57)$

Solution:

$x=-1$

Exercise:

Problem: $(9n+5)-(3n-7)=20-(4n-2)$

Exercise:

Problem: $2[-16+5(8k-6)]=8(3-4k)-32$

Solution:

$k=\frac{3}{4}$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

Exercise:

Problem: $17y-3(4-2y)=11(y-1)+12y-1$

Exercise:

Problem: $9u+32=15(u-4)-3(2u+21)$

Solution:

contradiction; no solution

Exercise:

Problem: $-8(7m + 4) = -6(8m + 9)$

Exercise:

Problem: $21(c - 1) - 19(c + 1) = 2(c - 20)$

Solution:

identity; all real numbers

Solve Equations with Fractions and Decimals

Solve Equations with Fraction Coefficients

In the following exercises, solve each equation with fraction coefficients.

Exercise:

Problem: $\frac{2}{5}n - \frac{1}{10} = \frac{7}{10}$

Exercise:

Problem: $\frac{1}{3}x + \frac{1}{5}x = 8$

Solution:

$x = 15$

Exercise:

Problem: $\frac{3}{4}a - \frac{1}{3} = \frac{1}{2}a - \frac{5}{6}$

Exercise:

Problem: $\frac{1}{2}(k - 3) = \frac{1}{3}(k + 16)$

Solution:

$k = 41$

Exercise:

Problem: $\frac{3x-2}{5} = \frac{3x+4}{8}$

Exercise:

Problem: $\frac{5y-1}{3} + 4 = \frac{-8y+4}{6}$

Solution:

$y = -1$

Solve Equations with Decimal Coefficients

In the following exercises, solve each equation with decimal coefficients.

Exercise:

Problem: $0.8x - 0.3 = 0.7x + 0.2$

Exercise:

Problem: $0.36u + 2.55 = 0.41u + 6.8$

Solution:

$u = -85$

Exercise:

Problem: $0.6p - 1.9 = 0.78p + 1.7$

Exercise:

Problem: $0.6p - 1.9 = 0.78p + 1.7$

Solution:

$d = -20$

Solve a Formula for a Specific Variable

Use the Distance, Rate, and Time Formula

In the following exercises, solve.

Exercise:

Problem:

Natalie drove for $7\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?

Exercise:

Problem:

Mallory is taking the bus from St. Louis to Chicago. The distance is 300 miles and the bus travels at a steady rate of 60 miles per hour. How long will the bus ride be?

Solution:

5 hours

Exercise:

Problem:

Aaron’s friend drove him from Buffalo to Cleveland. The distance is 187 miles and the trip took 2.75 hours. How fast was Aaron’s friend driving?

Exercise:

Problem:

Link rode his bike at a steady rate of 15 miles per hour for $2\frac{1}{2}$ hours. How much distance did he travel?

Solution:

37.5 miles

Solve a Formula for a Specific Variable

In the following exercises, solve.

Exercise:

Problem: Use the formula. $d = rt$ to solve for t
ⓐ when $d = 510$ and $r = 60$
ⓑ in general

Exercise:

Problem: Use the formula. $d = rt$ to solve for r
ⓐ when when $d = 451$ and $t = 5.5$
ⓑ in general

Solution:

ⓐ $r = 82\,\text{mph}$; ⓑ $r = \frac{D}{t}$

Exercise:

Problem: Use the formula $A = \frac{1}{2}bh$ to solve for b
ⓐ when $A = 390$ and $h = 26$
ⓑ in general

Exercise:

Problem: Use the formula $A = \frac{1}{2}bh$ to solve for h
ⓐ when $A = 153$ and $b = 18$
ⓑ in general

Solution:

ⓐ $h = 17$ ⓑ $h = \frac{2A}{b}$

Exercise:

Problem: Use the formula $I = Prt$ to solve for the principal, P for
ⓐ $I = \$2,501, r = 4.1\%,$
$t = 5\,\text{years}$
ⓑ in general

Exercise:

Problem: Solve the formula $4x + 3y = 6$ for y
ⓐ when $x = -2$
ⓑ in general

Solution:

ⓐ $y = \frac{14}{3}$ ⓑ $y = \frac{6-4x}{3}$

Exercise:

Problem: Solve $180 = a + b + c$ for c.

Exercise:

Problem: Solve the formula $V = LWH$ for H.

Solution:

$H = \frac{V}{LW}$

Solve Linear Inequalities

Graph Inequalities on the Number Line

In the following exercises, graph each inequality on the number line.

Exercise:

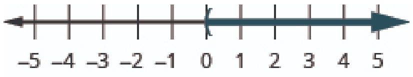
ⓐ $x \leq 4$
ⓑ $x > -2$
Problem: ⓒ $x < 1$

Exercise:

ⓐ $x > 0$
ⓑ $x < -3$
Problem: ⓒ $x \geq 1$

Solution:

ⓐ



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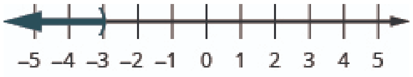

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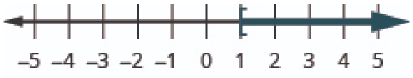


In the following exercises, graph each inequality on the number line and write in interval notation.

Exercise:

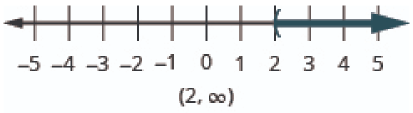
Problem: ⓐ $x < -1$
ⓑ $x \geq -2.5$
ⓒ $x \leq \frac{5}{4}$

Exercise:

Problem: ⓐ $x > 2$
ⓑ $x \leq -1.5$
ⓒ $x \geq \frac{5}{3}$

Solution:

ⓐ



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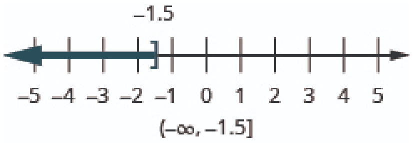

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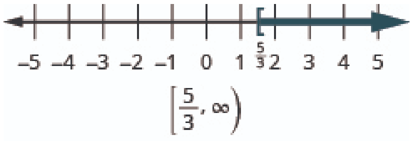


Solve Inequalities using the Subtraction and Addition Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

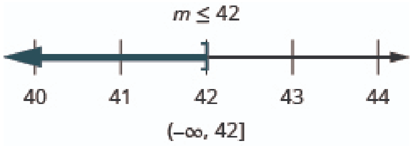
Exercise:

Problem: $n - 12 \leq 23$

Exercise:

Problem: $m + 14 \leq 56$

Solution:



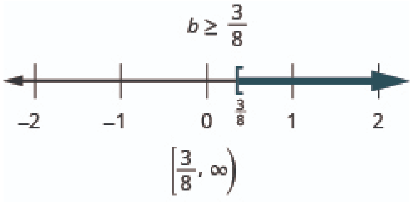
Exercise:

Problem: $a + \frac{2}{3} \geq \frac{7}{12}$

Exercise:

Problem: $b - \frac{7}{8} \geq -\frac{1}{2}$

Solution:



Solve Inequalities using the Division and Multiplication Properties of Inequality

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

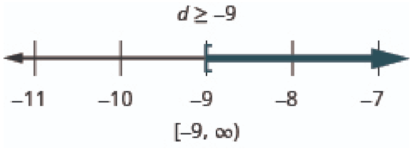
Exercise:

Problem: $9x > 54$

Exercise:

Problem: $-12d \leq 108$

Solution:



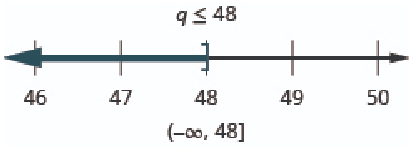
Exercise:

Problem: $\frac{5}{2}j < -60$

Exercise:

Problem: $\frac{q}{-2} \geq -24$

Solution:



Solve Inequalities That Require Simplification

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

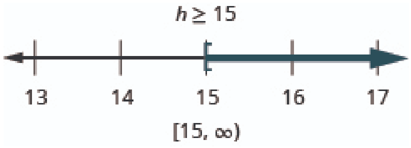
Exercise:

Problem: $6p > 15p - 30$

Exercise:

Problem: $9h - 7(h - 1) \leq 4h - 23$

Solution:



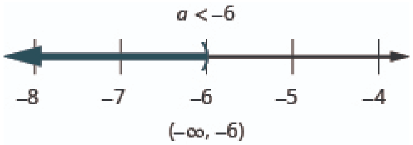
Exercise:

Problem: $5n - 15(4 - n) < 10(n - 6) + 10n$

Exercise:

Problem: $\frac{3}{8}a - \frac{1}{12}a > \frac{5}{12}a + \frac{3}{4}$

Solution:


$a < -6$

$(-\infty, -6)$

Translate to an Inequality and Solve

In the following exercises, translate and solve. Then write the solution in interval notation and graph on the number line.

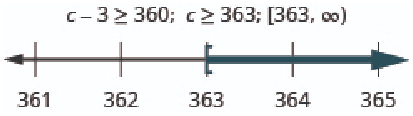
Exercise:

Problem: Five more than *z* is at most 19.

Exercise:

Problem: Three less than *c* is at least 360.

Solution:


$c - 3 \geq 360;\ c \geq 363;\ [363, \infty)$

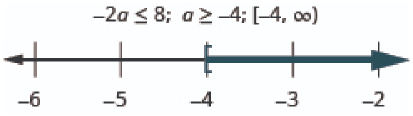
Exercise:

Problem: Nine times *n* exceeds 42.

Exercise:

Problem: Negative two times *a* is no more than 8.

Solution:


$-2a \leq 8;\ a \geq -4;\ [-4, \infty)$

Everyday Math

Exercise:

Problem:

Describe how you have used two topics from this chapter in your life outside of your math class during the past month.

Chapter 2 Practice Test

Exercise:

Problem:

Determine whether each number is a solution to the equation $6x - 3 = x + 20$.

ⓐ 5
ⓑ $\frac{23}{5}$

Solution:

ⓐ no ⓑ yes

In the following exercises, solve each equation.

Exercise:

Problem: $n - \frac{2}{3} = \frac{1}{4}$

Exercise:

Problem: $\frac{9}{2}c = 144$

Solution:

$c = 32$

Exercise:

Problem: $4y - 8 = 16$

Exercise:

Problem: $-8x - 15 + 9x - 1 = -21$

Solution:

$x = -5$

Exercise:

Problem: $-15a = 120$

Exercise:

Problem: $\frac{2}{3}x = 6$

Solution:

$x = 9$

Exercise:

Problem: $x - 3.8 = 8.2$

Exercise:

Problem: $10y = -5y - 60$

Solution:

$y = -4$

Exercise:

Problem: $8n - 2 = 6n - 12$

Exercise:

Problem: $9m - 2 - 4m - m = 42 - 8$

Solution:

$m = 9$

Exercise:

Problem: $-5(2x - 1) = 45$

Exercise:

Problem: $-(d-9)=23$

Solution:

$d=-14$

Exercise:

Problem: $\frac{1}{4}(12m-28)=6-2(3m-1)$

Exercise:

Problem: $2(6x-5)-8=-22$

Solution:

$x=-\frac{1}{3}$

Exercise:

Problem: $8(3a-5)-7(4a-3)=20-3a$

Exercise:

Problem: $\frac{1}{4}p-\frac{1}{3}=\frac{1}{2}$

Solution:

$p=\frac{10}{3}$

Exercise:

Problem: $0.1d+0.25(d+8)=4.1$

Exercise:

Problem: $14n-3(4n+5)=-9+2(n-8)$

Solution:

contradiction; no solution

Exercise:

Problem: $9(3u-2)-4[6-8(u-1)]=3(u-2)$

Exercise:

Problem: Solve the formula $x-2y=5$ for y
ⓐ when $x=-3$
ⓑ in general

Solution:

ⓐ $y=-4$ ⓑ $y=\frac{5-x}{-2}$

In the following exercises, graph on the number line and write in interval notation.

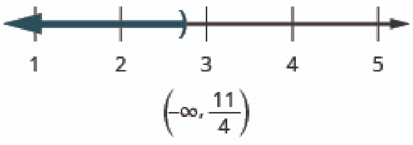
Exercise:

Problem: $x \geq -3.5$

Exercise:

Problem: $x < \frac{11}{4}$

Solution:



In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

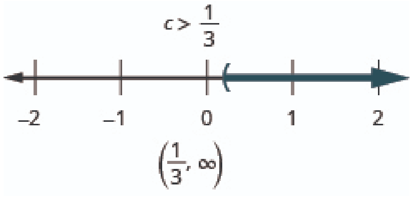
Exercise:

Problem: $8k \geq 5k-120$

Exercise:

Problem: $3c-10(c-2) < 5c+16$

Solution:



In the following exercises, translate to an equation or inequality and solve.

Exercise:

Problem: 4 less than twice *x* is 16.

Exercise:

Problem: Fifteen more than *n* is at least 48.

Solution:

$n + 15 \geq 48; n \geq 33$

Exercise:

Problem:

Samuel paid $25.82 for gas this week, which was $3.47 less than he paid last week. How much had he paid last week?

Exercise:

Problem:

Jenna bought a coat on sale for $120, which was $\frac{2}{3}$ of the original price. What was the original price of the coat?

Solution:

$120 = \frac{2}{3}p$; The original price was $180.

Exercise:

Problem:

Sean took the bus from Seattle to Boise, a distance of 506 miles. If the trip took $7\frac{2}{3}$ hours, what was the speed of the bus?

Introduction class="introduction"

Sophisticated mathematical models are used to predict traffic patterns on our nation’s highways.

Mathematical formulas model phenomena in every facet of our lives. They are used to explain events and predict outcomes in fields such as transportation, business, economics, medicine, chemistry, engineering, and many more. In this chapter, we will apply our skills in solving equations to solve problems in a variety of situations.

Use a Problem-Solving Strategy
By the end of this section, you will be able to:

- Approach word problems with a positive attitude
- Use a problem-solving strategy for word problems
- Solve number problems

Note:
Before you get started, take this readiness quiz.

1. Translate “6 less than twice *x*” into an algebraic expression.
   If you missed this problem, review [link].
2. Solve: $\frac{2}{3}x = 24$.
   If you missed this problem, review [link].
3. Solve: $3x + 8 = 14$.
   If you missed this problem, review [link].

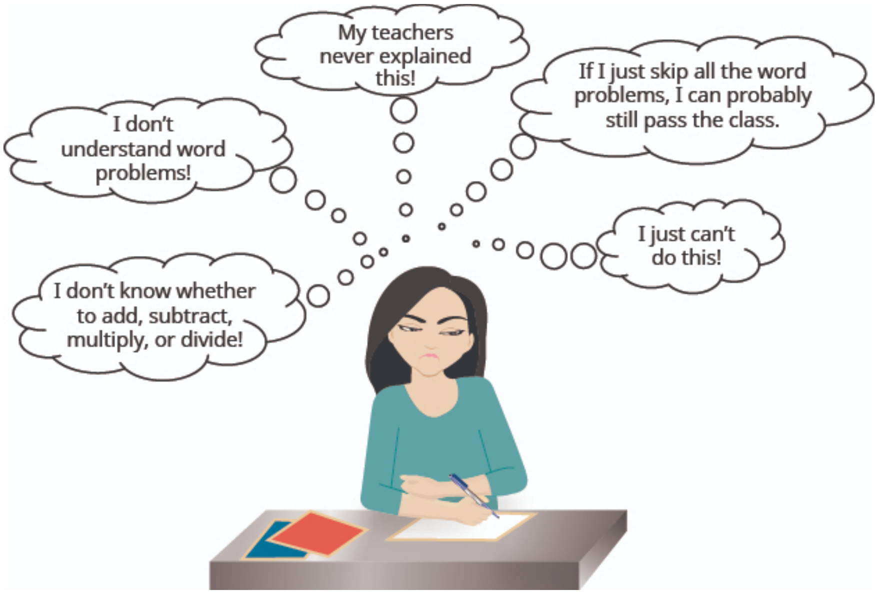
Approach Word Problems with a Positive Attitude

“If you think you can… or think you can’t… you’re right.”—Henry Ford

The world is full of word problems! Will my income qualify me to rent that apartment? How much punch do I need to make for the party? What size diamond can I afford to buy my girlfriend? Should I fly or drive to my family reunion?

How much money do I need to fill the car with gas? How much tip should I leave at a restaurant? How many socks should I pack for vacation? What size turkey do I need to buy for Thanksgiving dinner, and then what time do I need to put it in the oven? If my sister and I buy our mother a present, how much does each of us pay?

Now that we can solve equations, we are ready to apply our new skills to word problems. Do you know anyone who has had negative experiences in the past with word problems? Have you ever had thoughts like the student below?


My teachers never explained this!
If I just skip all the word problems, I can probably still pass the class.
I don't understand word problems!
I just can't do this!
I don't know whether to add, subtract, multiply, or divide!

Negative thoughts can be barriers to success.

When we feel we have no control, and continue repeating negative thoughts, we set up barriers to success. We need to calm our fears and change our negative feelings.

Start with a fresh slate and begin to think positive thoughts. If we take control and believe we can be successful, we will be able to master word problems! Read the positive thoughts in [link] and say them out loud.

I am better prepared now. I think I will begin to understand word problems.
I think I can!
I think I can!
While word problems were hard in the past, I think I can try them now.
It may take time, but I can begin to solve word problems.

Thinking positive thoughts is a first step towards success.

Think of something, outside of school, that you can do now but couldn't do 3 years ago. Is it driving a car? Snowboarding? Cooking a gourmet meal? Speaking a new language? Your past experiences with word problems happened when you were younger—now you're older and ready to succeed!

Use a Problem-Solving Strategy for Word Problems

We have reviewed translating English phrases into algebraic expressions, using some basic mathematical vocabulary and symbols. We have also translated English sentences into algebraic equations and solved some word problems. The word problems applied math to everyday situations. We restated the situation in one sentence, assigned a variable, and then wrote an equation to solve the problem. This method works as long as the situation is familiar and the math is not too complicated.

Now, we'll expand our strategy so we can use it to successfully solve any word problem. We'll list the strategy here, and then we'll use it to solve some problems. We summarize below an effective strategy for problem solving.

Note:
Use a Problem-Solving Strategy to Solve Word Problems.

Read the problem. Make sure all the words and ideas are understood.
Identify what we are looking for.
Name what we are looking for. Choose a variable to represent that quantity.
Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
Solve the equation using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

Example:
Exercise:

Problem:

Pilar bought a purse on sale for $18, which is one-half of the original price. What was the original price of the purse?

Solution:
Solution

Step 1. Read the problem. Read the problem two or more times if necessary. Look up any unfamiliar words in a dictionary or on the internet.

- *In this problem, is it clear what is being discussed? Is every word familiar?*

Step 2. Identify what you are looking for. Did you ever go into your bedroom to get something and then forget what you were looking for? It's hard to find something if you are not sure what it is! Read the problem again and look for words that tell you what you are looking for!

- *In this problem, the words "what was the original price of the purse" tell us what we need to find.*

Step 3. Name what we are looking for. Choose a variable to represent that quantity. We can use any letter for the variable, but choose one that makes it easy to remember what it represents.

- Let $p =$ the original price of the purse.

Step 4. Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Translate the English sentence into an algebraic equation.

Reread the problem carefully to see how the given information is related. Often, there is one sentence that gives this information, or it may help to write one sentence with all the important information. Look for clue words to help translate the sentence into algebra. Translate the sentence into an equation.

Restate the problem in one sentence with all the important information.	$\underbrace{18}\quad\underbrace{\text{is}}\quad\underbrace{\text{one-half the original price.}}$
Translate into an equation.	$18 \quad = \quad \frac{1}{2} \cdot p$

Step 5. Solve the equation using good algebraic techniques. Even if you know the solution right away, using good algebraic techniques here will better prepare you to solve problems that do not have obvious answers.

Solve the equation.	$18 = \frac{1}{2}p$
Multiply both sides by 2.	$2 \cdot 18 = 2 \cdot \frac{1}{2}p$
Simplify.	$36 = p$

Step 6. Check the answer in the problem to make sure it makes sense. We solved the equation and found that $p = 36$, which means “the original price” was $36.

- *Does $36 make sense in the problem? Yes, because 18 is one-half of 36, and the purse was on sale at half the original price.*

Step 7. Answer the question with a complete sentence. The problem asked “What was the original price of the purse?”

- *The answer to the question is: “The original price of the purse was $36.”*

If this were a homework exercise, our work might look like this:

Pilar bought a purse on sale for $18, which is one-half the original price. What was the original price of the purse?

	Let $p =$ the original price.
	18 is one-half the original price.
	$18 = \frac{1}{2}p$
Multiply both sides by 2.	$2 \cdot 18 = 2 \cdot \frac{1}{2}p$
Simplify.	$36 = p$
Check. Is $36 a reasonable price for a purse?	
Yes.	
Is 18 one half of 36?	
$18 \overset{?}{=} \frac{1}{2} \cdot 36$	
$18 = 18\checkmark$	
	The original price of the purse was $36.

Note:
Exercise:

Problem:

Joaquin bought a bookcase on sale for $120, which was two-thirds of the original price. What was the original price of the bookcase?

Solution:

$180

Note:
Exercise:

Problem:

Two-fifths of the songs in Mariel’s playlist are country. If there are 16 country songs, what is the total number of songs in the playlist?

Solution:

40

Let’s try this approach with another example.

Example:
Exercise:

Problem:

Ginny and her classmates formed a study group. The number of girls in the study group was three more than twice the number of boys. There were 11 girls in the study group. How many boys were in the study group?

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	How many boys were in the study group?
Step 3. Name. Choose a variable to represent the number of boys.	Let $n =$ the number of boys.
Step 4. Translate. Restate the problem in one sentence with all the important information.	$\underbrace{\text{The number of girls (11)}}\ \underbrace{\text{was}}\ \underbrace{\text{three more than twice the number of boys}}$
Translate into an equation.	$11 \quad = \quad 2b+3$

Step 5. Solve the equation.	$11 = 2b + 3$
Subtract 3 from each side.	$11 - 3 = 2b + 3 - 3$
Simplify.	$8 = 2b$
Divide each side by 2.	$\frac{8}{2} = \frac{2b}{2}$
Simplify.	$4 = b$
Step 6. Check. First, is our answer reasonable? Yes, having 4 boys in a study group seems OK. The problem says the number of girls was 3 more than twice the number of boys. If there are four boys, does that make eleven girls? Twice 4 boys is 8. Three more than 8 is 11.	
Step 7. Answer the question.	There were 4 boys in the study group.

Note:
Exercise:

Problem:

Guillermo bought textbooks and notebooks at the bookstore. The number of textbooks was 3 more than twice the number of notebooks. He bought 7 textbooks. How many notebooks did he buy?

Solution:

2

Note:
Exercise:

Problem:

Gerry worked Sudoku puzzles and crossword puzzles this week. The number of Sudoku puzzles he completed is eight more than twice the number of crossword puzzles. He completed 22 Sudoku puzzles. How many crossword puzzles did he do?

Solution:

7

Solve Number Problems

Now that we have a problem solving strategy, we will use it on several different types of word problems. The first type we will work on is “number problems.” Number problems give some clues about one or more numbers. We use these clues to write an equation. Number problems don’t usually arise on an everyday basis, but they provide a good introduction to practicing the problem solving strategy outlined above.

Example:
Exercise:

Problem: The difference of a number and six is 13. Find the number.

Solution:
Solution

Step 1. Read the problem. Are all the words familiar?	
Step 2. Identify what we are looking for.	the number
Step 3. Name. Choose a variable to represent the number.	Let $n =$ the number.
Step 4. Translate. Remember to look for clue words like "difference... of... and..."	
Restate the problem as one sentence.	$\underbrace{\text{The difference of the number and 6}}\quad\underbrace{\text{is}}\quad\underbrace{13}$
Translate into an equation.	$n-6 \quad = \quad 13$
Step 5. Solve the equation.	$n-6=13$
Simplify.	$n=19$
Step 6. Check.	
The difference of 19 and 6 is 13. It checks!	
Step 7. Answer the question.	The number is 19.

Note:
Exercise:

Problem: The difference of a number and eight is 17. Find the number.

Solution:

25

Note:
Exercise:

Problem: The difference of a number and eleven is -7. Find the number.

Solution:

4

Example:
Exercise:

Problem: The sum of twice a number and seven is 15. Find the number.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the number
Step 3. Name. Choose a variable to represent the number.	Let $n =$ the number.
Step 4. Translate.	
Restate the problem as one sentence.	$\underbrace{\text{The sum of twice a number and 7}} \quad \underbrace{\text{is}} \quad \underbrace{15}$
Translate into an equation.	$2n+7 \quad = \quad 15$
Step 5. Solve the equation.	$2n+7=15$

Subtract 7 from each side and simplify.	$2n = 8$
Divide each side by 2 and simplify.	$n = 4$
Step 6. Check.	
Is the sum of twice 4 and 7 equal to 15?	
$2 \cdot 4 + 7 \overset{?}{=} 15$ $15 = 15 \checkmark$	
Step 7. Answer the question.	The number is 4.

Did you notice that we left out some of the steps as we solved this equation? If you’re not yet ready to leave out these steps, write down as many as you need.

Note:
Exercise:

Problem: The sum of four times a number and two is 14. Find the number.

Solution:

3

Note:
Exercise:

Problem: The sum of three times a number and seven is 25. Find the number.

Solution:

6

Some number word problems ask us to find two or more numbers. It may be tempting to name them all with different variables, but so far we have only solved equations with one variable. In order to avoid using more than one variable, we will define the numbers in terms of the same variable. Be sure to read the problem carefully to discover how all the numbers relate to each other.

Example:

Exercise:

Problem: One number is five more than another. The sum of the numbers is 21. Find the numbers.

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		We are looking for two numbers.
Step 3. Name. We have two numbers to name and need a name for each.		
Choose a variable to represent the first number.		Let $n = 1^{\text{st}}$ number.
What do we know about the second number?		One number is five more than another.
		$n + 5 = 2^{\text{nd}}$ number
Step 4. Translate. Restate the problem as one sentence with all the important information.		The sum of the 1^{st} number and the 2^{nd} number is 21.
Translate into an equation.		$\underbrace{1^{\text{st}}\text{ number}} + \underbrace{2^{\text{nd}}\text{ number}} \underbrace{=} \underbrace{21}$
Substitute the variable expressions.		$n \quad + \quad n+5 \quad = \quad 21$
Step 5. Solve the equation.		$n + n + 5 = 21$
Combine like terms.		$2n + 5 = 21$
Subtract 5 from both sides and simplify.		$2n = 16$
Divide by 2 and simplify.		$n = 8 \quad 1^{\text{st}}$ number
Find the second number, too.		$n + 5 \quad 2^{\text{nd}}$ number

		8 + 5
		13
Step 6. Check.		
Do these numbers check in the problem?		
Is one number 5 more than the other?	$13 \overset{?}{=} 8 + 5$	
Is thirteen 5 more than 8? Yes.	$13 = 13\checkmark$	
Is the sum of the two numbers 21?	$8 + 13 \overset{?}{=} 21$	
	$21 = 21\checkmark$	
Step 7. Answer the question.		The numbers are 8 and 13.

Note:
Exercise:

Problem:

One number is six more than another. The sum of the numbers is twenty-four. Find the numbers.

Solution:

9, 15

Note:
Exercise:

Problem:

The sum of two numbers is fifty-eight. One number is four more than the other. Find the numbers.

Solution:

27, 31

Example:
Exercise:

Problem:

The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		We are looking for two numbers.
Step 3. Name.		
Choose a variable.		Let $n = 1^{\text{st}}$ number.
One number is 4 less than the other.		$n - 4 = 2^{\text{nd}}$ number
Step 4. Translate.		
Write as one sentence.		The sum of the 2 numbers is negative 14.
Translate into an equation.		$\underbrace{1^{\text{st}}\text{ number}} + \underbrace{2^{\text{nd}}\text{ number}} \quad \underbrace{\text{is}} \quad \underbrace{\text{negative fourteen}}$
Step 5. Solve the equation.		$n \quad + \quad n - 4 \quad = \quad -14$
Combine like terms.		$n + n - 4 = -14$
Add 4 to each side and simplify.		$2n - 4 = -14$
Simplify.		$2n = -10$
Divide by 2 and simplify.		$n = -5 \quad 1^{\text{st}}$ number
		$n - 4 \quad 2^{\text{nd}}$ number
		$-5 - 4$

		−9
Step 6. Check.		
Is −9 four less than −5?	$-5-4\overset{?}{=}-9$	
	$-9=-9\checkmark$	
Is their sum −14?	$-5+(-9)\overset{?}{=}-14$	
	$-14=-14\checkmark$	
Step 7. Answer the question.		The numbers are −5 and −9.

Note:
Exercise:

Problem:

The sum of two numbers is negative twenty-three. One number is seven less than the other. Find the numbers.

Solution:

$-15, -8$

Note:
Exercise:

Problem: The sum of two numbers is -18. One number is 40 more than the other. Find the numbers.

Solution:

$-29, 11$

Example:
Exercise:

Problem: One number is ten more than twice another. Their sum is one. Find the numbers.

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what you are looking for.		We are looking for two numbers.
Step 3. Name.		
Choose a variable.		Let $x = 1^{\text{st}}$ number.
One number is 10 more than twice another.		$2x + 10 = 2^{\text{nd}}$ number
Step 4. Translate.		
Restate as one sentence.		Their sum is one.
		The sum of the two numbers is 1.
Translate into an equation.		$x + 2x + 10 = 1$
Step 5. Solve the equation.		
Combine like terms.		$x + 2x + 10 = 1$
Subtract 10 from each side.		$3x + 10 = 1$
Divide each side by 3.		$3x = -9$
		$x = -3 \quad 1^{\text{st}}$ number
		$2x + 10 \quad 2^{\text{nd}}$ number
		$2(-3) + 10$
		4
Step 6. Check.		
Is ten more than twice −3 equal to 4?	$2(-3) + 10 \stackrel{?}{=} 4$	
	$-6 + 10 \stackrel{?}{=} 4$	

	$4=4\checkmark$	
Is their sum 1?	$-3+4\overset{?}{=}1$	
	$1=1\checkmark$	
Step 7. Answer the question.		The numbers are −3 and −4.

Note:
Exercise:

Problem: One number is eight more than twice another. Their sum is negative four. Find the numbers.

Solution:

$-4, 0$

Note:
Exercise:

Problem: One number is three more than three times another. Their sum is -5. Find the numbers.

Solution:

$-3, -2$

Some number problems involve consecutive integers. *Consecutive integers* are integers that immediately follow each other. Examples of consecutive integers are:

Equation:

$$\begin{array}{l} 1, 2, 3, 4 \\ -10, -9, -8, -7 \\ 150, 151, 152, 153 \end{array}$$

Notice that each number is one more than the number preceding it. So if we define the first integer as *n*, the next consecutive integer is $n+1$. The one after that is one more than $n+1$, so it is $n+1+1$, which is $n+2$.

Equation:

$$\begin{array}{cl} n & 1^{\text{st}} \text{ integer} \\ n+1 & 2^{\text{nd}} \text{ consecutive integer} \\ n+2 & 3^{\text{rd}} \text{ consecutive integer} \ldots \text{etc.} \end{array}$$

Example:
Exercise:

Problem: The sum of two consecutive integers is 47. Find the numbers.

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what you are looking for.		two consecutive integers
Step 3. Name each number.		Let $n = 1^{\text{st}}$ integer.
		$n + 1 =$ next consecutive integer
Step 4. Translate.		
Restate as one sentence.		The sum of the integers is 47.
Translate into an equation.		$n + n + 1 = 47$
Step 5. Solve the equation.		$n + n + 1 = 47$
Combine like terms.		$2n + 1 = 47$
Subtract 1 from each side.		$2n = 46$
Divide each side by 2.		$n = 23$ 1^{st} integer
		$n + 1$ next consecutive integer
		$23 + 1$
		24
Step 6. Check.		

$$\begin{aligned} 23+24 &\overset{?}{=} 47 \\ 47 &= 47\checkmark \end{aligned}$$		
Step 7. Answer the question.		The two consecutive integers are 23 and 24.

Note:
Exercise:

Problem: The sum of two consecutive integers is 95. Find the numbers.

Solution:

47, 48

Note:
Exercise:

Problem: The sum of two consecutive integers is -31. Find the numbers.

Solution:

-16, -15

Example:
Exercise:

Problem: Find three consecutive integers whose sum is -42.

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		three consecutive integers
Step 3. Name each of the three numbers.		Let $n = 1^{\text{st}}$ integer.

		$n+1=2^{\text{nd}}$ consecutive integer
		$n+2=3^{\text{rd}}$ consecutive integer
Step 4. Translate.		
Restate as one sentence.		The sum of the three integers is −42.
Translate into an equation.		$n+n+1+n+2=-42$
Step 5. Solve the equation.		$n+n+1+n+2=-42$
Combine like terms.		$3n+3=-42$
Subtract 3 from each side.		$3n=-45$
Divide each side by 3.		$n=-15$ 1^{st} integer
		$n+1$ 2^{nd} integer
		$-15+1$
		-14
		$n+2$ 3^{rd} integer
		$-15+2$
		-13
Step 6. Check.		
$-13+(-14)+(-15) \stackrel{?}{=} -42$ $-42 = -42$✓		
Step 7. Answer the question.		The three consecutive integers are −13, −14, and

		−15.

Note:
Exercise:

Problem: Find three consecutive integers whose sum is -96.

Solution:

$-33, -32, -31$

Note:
Exercise:

Problem: Find three consecutive integers whose sum is -36.

Solution:

$-13, -12, -11$

Now that we have worked with consecutive integers, we will expand our work to include consecutive even integers and consecutive odd integers. *Consecutive even integers* are even integers that immediately follow one another. Examples of consecutive even integers are:

Equation:

$$18, 20, 22$$
$$64, 66, 68$$
$$-12, -10, -8$$

Notice each integer is 2 more than the number preceding it. If we call the first one *n*, then the next one is $n + 2$. The next one would be $n + 2 + 2$ or $n + 4$.

Equation:

$$\begin{array}{ll} n & 1^{\text{st}} \text{ even integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive even integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive even integer} \ldots \text{etc.} \end{array}$$

Consecutive odd integers are odd integers that immediately follow one another. Consider the consecutive odd integers 77, 79, and 81.

Equation:

$$77, 79, 81$$

$$n, n+2, n+4$$

Equation:

$$\begin{array}{ll} n & 1^{\text{st}} \text{ odd integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive odd integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive odd integer} \ldots \text{etc.} \end{array}$$

Does it seem strange to add 2 (an even number) to get from one odd integer to the next? Do you get an odd number or an even number when we add 2 to 3? to 11? to 47?

Whether the problem asks for consecutive even numbers or odd numbers, you don’t have to do anything different. The pattern is still the same—to get from one odd or one even integer to the next, add 2.

Example:
Exercise:

Problem: Find three consecutive even integers whose sum is 84.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	three consecutive even integers
Step 3. Name the integers.	Let $n = 1^{\text{st}}$ even integer.
	$n + 2 = 2^{\text{nd}}$ consecutive even integer
	$n + 4 = 3^{\text{rd}}$ consecutive even integer
Step 4. Translate.	
Restate as one sentence.	The sum of the three even integers is 84.
Translate into an equation.	$n + n + 2 + n + 4 \;=\; 84$
Step 5. Solve the equation.	
Combine like terms.	$n + n + 2 + n + 4 \;=\; 84$
Subtract 6 from each side.	$3n + 6 \;=\; 84$
Divide each side by 3.	$3n \;=\; 78$
	$n \;=\; 26$ 1^{st} integer
	$\begin{array}{rcl} n & + & 2 \\ 26 & + & 2 \\ & 28 & \end{array}$ 2^{nd} integer
	$\begin{array}{rcl} n & + & 4 \\ 26 & + & 4 \\ & 30 & \end{array}$ 3^{rd} integer
Step 6. Check.	
$\begin{aligned} 26 + 28 + 30 &\overset{?}{=} 84 \\ 84 &= 84\checkmark \end{aligned}$	
Step 7. Answer the question.	The three consecutive integers are 26, 28, and 30.

Note:
Exercise:

Problem: Find three consecutive even integers whose sum is 102.

Solution:

32, 34, 36

Note:
Exercise:

Problem: Find three consecutive even integers whose sum is -24.

Solution:

$-10, -8, -6$

Example:
Exercise:

Problem:

A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		How much does the husband earn?
Step 3. Name.		
Choose a variable to represent the amount the husband earns.		Let $h =$ the amount the husband earns.
The wife earns $16,000 less than twice that.		$2h - 16{,}000$ the amount the wife earns.
Step 4. Translate.		Together the husband and wife earn $110,000.
Restate the problem in one sentence with all the important information.		$\underbrace{\text{The amount the husband earns}}\ \text{plus}\ \underbrace{\text{the amount the wife earns}}\ \text{is}\ \$110{,}000$
Translate into an equation.		$h \quad + \quad 2h - 16{,}000 \quad = \quad 110{,}000$
Step 5. Solve the equation.		h + 2h − 16,000 = 110,000
Combine like terms.		3h − 16,000 = 110,000
Add 16,000 to both sides and simplify.		3h = 126,000
Divide each side by 3.		h = 42,000
		$42,000 amount husband earns

		2h − 16,000 amount wife earns
		2(42,000) − 16,000
		84,000 − 16,000
		68,000
Step 6. Check.		
If the wife earns $68,000 and the husband earns $42,000 is the total $110,000? Yes!		
Step 7. Answer the question.		The husband earns $42,000 a year.

Note:
Exercise:

Problem:

According to the National Automobile Dealers Association, the average cost of a car in 2014 was $28,500. This was $1,500 less than 6 times the cost in 1975. What was the average cost of a car in 1975?

Solution:

$5,000

Note:
Exercise:

Problem:

U.S. Census data shows that the median price of new home in the United States in November 2014 was $280,900. This was $10,700 more than 14 times the price in November 1964. What was the median price of a new home in November 1964?

Solution:

$19,300

Key Concepts

- **Problem-Solving Strategy**

  Readthe problem. Make sure all the words and ideas are understood.

Identify what we are looking for.
Name what we are looking for. Choose a variable to represent that quantity.
Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
Solve the equation using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

- **Consecutive Integers**
  Consecutive integers are integers that immediately follow each other.
  Equation:

$$\begin{array}{cl} n & 1^{\text{st}}\text{ integer} \\ n+1 & 2^{\text{nd}}\text{ consecutive integer} \\ n+2 & 3^{\text{rd}}\text{ consecutive integer} \ldots\text{ etc.} \end{array}$$

  Consecutive even integers are even integers that immediately follow one another.
  Equation:

$$\begin{array}{cl} n & 1^{\text{st}}\text{ integer} \\ n+2 & 2^{\text{nd}}\text{ consecutive integer} \\ n+4 & 3^{\text{rd}}\text{ consecutive integer} \ldots\text{ etc.} \end{array}$$

  Consecutive odd integers are odd integers that immediately follow one another.
  Equation:

$$\begin{array}{cl} n & 1^{\text{st}}\text{ integer} \\ n+2 & 2^{\text{nd}}\text{ consecutive integer} \\ n+4 & 3^{\text{rd}}\text{ consecutive integer} \ldots\text{ etc.} \end{array}$$

Practice Makes Perfect

Approach Word Problems with a Positive Attitude

In the following exercises, prepare the lists described.

Exercise:

Problem:

List five positive thoughts you can say to yourself that will help you approach word problems with a positive attitude. You may want to copy them on a sheet of paper and put it in the front of your notebook, where you can read them often.

Solution:

Answers will vary

Exercise:

Problem:

List five negative thoughts that you have said to yourself in the past that will hinder your progress on word problems. You may want to write each one on a small piece of paper and rip it up to symbolically destroy the negative thoughts.

Use a Problem-Solving Strategy for Word Problems

In the following exercises, solve using the problem solving strategy for word problems. Remember to write a complete sentence to answer each question.

Exercise:

Problem:

Two-thirds of the children in the fourth-grade class are girls. If there are 20 girls, what is the total number of children in the class?

Solution:

30

Exercise:

Problem:

Three-fifths of the members of the school choir are women. If there are 24 women, what is the total number of choir members?

Exercise:

Problem:

Zachary has 25 country music CDs, which is one-fifth of his CD collection. How many CDs does Zachary have?

Solution:

125

Exercise:

Problem:

One-fourth of the candies in a bag of M&M’s are red. If there are 23 red candies, how many candies are in the bag?

Exercise:

Problem:

There are 16 girls in a school club. The number of girls is four more than twice the number of boys. Find the number of boys.

Solution:

6

Exercise:

Problem:

There are 18 Cub Scouts in Pack 645. The number of scouts is three more than five times the number of adult leaders. Find the number of adult leaders.

Exercise:

Problem:

Huong is organizing paperback and hardback books for her club’s used book sale. The number of paperbacks is 12 less than three times the number of hardbacks. Huong had 162 paperbacks. How many hardback books were there?

Solution:

58

Exercise:

Problem:

Jeff is lining up children’s and adult bicycles at the bike shop where he works. The number of children’s bicycles is nine less than three times the number of adult bicycles. There are 42 adult bicycles. How many children’s bicycles are there?

Exercise:

Problem:

Philip pays $1,620 in rent every month. This amount is $120 more than twice what his brother Paul pays for rent. How much does Paul pay for rent?

Solution:

$750

Exercise:

Problem:

Marc just bought an SUV for $54,000. This is $7,400 less than twice what his wife paid for her car last year. How much did his wife pay for her car?

Exercise:

Problem:

Laurie has $46,000 invested in stocks and bonds. The amount invested in stocks is $8,000 less than three times the amount invested in bonds. How much does Laurie have invested in bonds?

Solution:

$13,500

Exercise:

Problem:

Erica earned a total of $50,450 last year from her two jobs. The amount she earned from her job at the store was $1,250 more than three times the amount she earned from her job at the college. How much did she earn from her job at the college?

Solve Number Problems

In the following exercises, solve each number word problem.

Exercise:

Problem: The sum of a number and eight is 12. Find the number.

Solution:

4

Exercise:

Problem: The sum of a number and nine is 17. Find the number.

Exercise:

Problem: The difference of a number and 12 is three. Find the number.

Solution:

15

Exercise:

Problem: The difference of a number and eight is four. Find the number.

Exercise:

Problem: The sum of three times a number and eight is 23. Find the number.

Solution:

5

Exercise:

Problem: The sum of twice a number and six is 14. Find the number.

Exercise:

Problem: The difference of twice a number and seven is 17. Find the number.

Solution:

12

Exercise:

Problem: The difference of four times a number and seven is 21. Find the number.

Exercise:

Problem: Three times the sum of a number and nine is 12. Find the number.

Solution:

-5

Exercise:

Problem: Six times the sum of a number and eight is 30. Find the number.

Exercise:

Problem: One number is six more than the other. Their sum is 42. Find the numbers.

Solution:

18, 24

Exercise:

Problem: One number is five more than the other. Their sum is 33. Find the numbers.

Exercise:

Problem: The sum of two numbers is 20. One number is four less than the other. Find the numbers.

Solution:

8, 12

Exercise:

Problem: The sum of two numbers is 27. One number is seven less than the other. Find the numbers.

Exercise:

Problem: The sum of two numbers is -45. One number is nine more than the other. Find the numbers.

Solution:

$-18, -27$

Exercise:

Problem: The sum of two numbers is -61. One number is 35 more than the other. Find the numbers.

Exercise:

Problem: The sum of two numbers is -316. One number is 94 less than the other. Find the numbers.

Solution:

$-111, -205$

Exercise:

Problem: The sum of two numbers is -284. One number is 62 less than the other. Find the numbers.

Exercise:

Problem:

One number is 14 less than another. If their sum is increased by seven, the result is 85. Find the numbers.

Solution:

32, 46

Exercise:

Problem:

One number is 11 less than another. If their sum is increased by eight, the result is 71. Find the numbers.

Exercise:

Problem:

One number is five more than another. If their sum is increased by nine, the result is 60. Find the numbers.

Solution:

23, 28

Exercise:

Problem:

One number is eight more than another. If their sum is increased by 17, the result is 95. Find the numbers.

Exercise:

Problem: One number is one more than twice another. Their sum is -5. Find the numbers.

Solution:

$-2, -3$

Exercise:

Problem: One number is six more than five times another. Their sum is six. Find the numbers.

Exercise:

Problem:

The sum of two numbers is 14. One number is two less than three times the other. Find the numbers.

Solution:

4, 10

Exercise:

Problem:

The sum of two numbers is zero. One number is nine less than twice the other. Find the numbers.

Exercise:

Problem: The sum of two consecutive integers is 77. Find the integers.

Solution:

38, 39

Exercise:

Problem: The sum of two consecutive integers is 89. Find the integers.

Exercise:

Problem: The sum of two consecutive integers is $-23.$ Find the integers.

Solution:

$-11, -12$

Exercise:

Problem: The sum of two consecutive integers is $-37.$ Find the integers.

Exercise:

Problem: The sum of three consecutive integers is 78. Find the integers.

Solution:

25, 26, 27

Exercise:

Problem: The sum of three consecutive integers is 60. Find the integers.

Exercise:

Problem: Find three consecutive integers whose sum is $-36.$

Solution:

$-11, -12, -13$

Exercise:

Problem: Find three consecutive integers whose sum is $-3.$

Exercise:

Problem: Find three consecutive even integers whose sum is 258.

Solution:

84, 86, 88

Exercise:

Problem: Find three consecutive even integers whose sum is 222.

Exercise:

Problem: Find three consecutive odd integers whose sum is 171.

Solution:

55, 57, 59

Exercise:

Problem: Find three consecutive odd integers whose sum is 291.

Exercise:

Problem: Find three consecutive even integers whose sum is -36.

Solution:

$-10, -12, -14$

Exercise:

Problem: Find three consecutive even integers whose sum is -84.

Exercise:

Problem: Find three consecutive odd integers whose sum is -213.

Solution:

$-69, -71, -73$

Exercise:

Problem: Find three consecutive odd integers whose sum is -267.

Everyday Math

Exercise:

Problem:

Sale Price Patty paid $35 for a purse on sale for $10 off the original price. What was the original price of the purse?

Solution:

$45

Exercise:

Problem:

Sale Price Travis bought a pair of boots on sale for $25 off the original price. He paid $60 for the boots. What was the original price of the boots?

Exercise:

Problem:

Buying in Bulk Minh spent $6.25 on five sticker books to give his nephews. Find the cost of each sticker book.

Solution:

$1.25

Exercise:

Problem:

Buying in Bulk Alicia bought a package of eight peaches for $3.20. Find the cost of each peach.

Exercise:

Problem:

Price before Sales Tax Tom paid $1,166.40 for a new refrigerator, including $86.40 tax. What was the price of the refrigerator?

Solution:

$1080

Exercise:

Problem:

Price before Sales Tax Kenji paid $2,279 for a new living room set, including $129 tax. What was the price of the living room set?

Writing Exercises

Exercise:

Problem: What has been your past experience solving word problems?

Solution:

answers will vary

Exercise:

Problem: When you start to solve a word problem, how do you decide what to let the variable represent?

Exercise:

Problem: What are consecutive odd integers? Name three consecutive odd integers between 50 and 60.

Solution:

Consecutive odd integers are odd numbers that immediately follow each other. An example of three consecutive odd integers between 50 and 60 would be 51, 53, and 55.

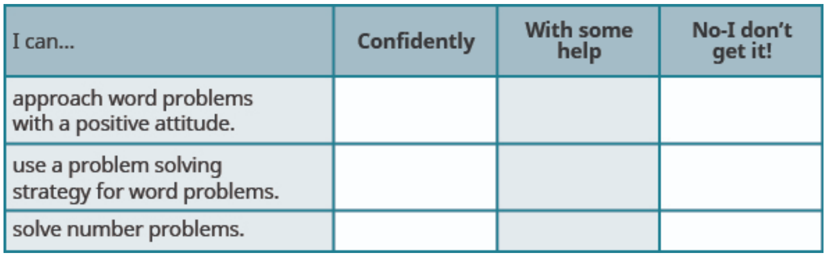
Exercise:

Problem:

What are consecutive even integers? Name three consecutive even integers between -50 and -40.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
approach word problems with a positive attitude.			
use a problem solving strategy for word problems.			
solve number problems.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential—every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

Solve Percent Applications
By the end of this section, you will be able to:

- Translate and solve basic percent equations
- Solve percent applications
- Find percent increase and percent decrease
- Solve simple interest applications
- Solve applications with discount or mark-up

Note:
Before you get started, take this readiness quiz.

1. Convert 4.5% to a decimal.
   If you missed this problem, review [link].
2. Convert 0.6 to a percent.
   If you missed this problem, review [link].
3. Round 0.875 to the nearest hundredth.
   If you missed this problem, review [link].
4. Multiply (4.5)(2.38).
   If you missed this problem, review [link].
5. Solve $3.5 = 0.7n$.
   If you missed this problem, review [link].
6. Subtract $50 - 37.45$.
   If you missed this problem, review [link].

Translate and Solve Basic Percent Equations

We will solve percent equations using the methods we used to solve equations with fractions or decimals. Without the tools of algebra, the best method available to solve percent problems was by setting them up as proportions. Now as an algebra student, you can just translate English sentences into algebraic equations and then solve the equations.

We can use any letter you like as a variable, but it is a good idea to choose a letter that will remind us of what you are looking for. We must be sure to change the given percent to a decimal when we put it in the equation.

Example:
Exercise:

Problem: Translate and solve: What number is 35% of 90?

Solution:
Solution

	$\underbrace{\text{What number}}\ \underbrace{\text{is}}\ \underbrace{35\%}\ \underbrace{\text{of}}\ \underbrace{90?}$
Translate into algebra. Let n= the number.	$n = 0.35 \cdot 90$
Remember "of" means multiply, "is" means equals.	
Multiply.	$n = 31.5$
	31.5 is 35% of 90

Note:
Exercise:

Problem: Translate and solve:

What number is 45% of 80?

Solution:

36

Note:
Exercise:

Problem: Translate and solve:

What number is 55% of 60?

Solution:

33

We must be very careful when we translate the words in the next example. The unknown quantity will not be isolated at first, like it was in [link]. We will again use direct translation to write the equation.

Example:
Exercise:

Problem: Translate and solve: 6.5% of what number is $1.17?

Solution:
Solution

	$\underbrace{6.5\%}\ \underbrace{\text{of}}\ \underbrace{\text{what number}}\ \underbrace{\text{is}}\ \underbrace{\$1.17?}$
Translate. Let $n =$ the number.	$0.065 \cdot n = 1.17$
Multiply.	$0.065n = 1.17$
Divide both sides by 0.065 and simplify.	$n = 18$
	6.5% of \$18 is \$1.17

Note:
Exercise:

Problem: Translate and solve:

7.5% of what number is $1.95?

Solution:

$26

Note:
Exercise:

Problem: Translate and solve:

8.5% of what number is \$3.06?

Solution:

\$36

In the next example, we are looking for the percent.

Example:
Exercise:

Problem: Translate and solve: 144 is what percent of 96?

Solution:
Solution

	$\underbrace{144}\quad\underbrace{\text{is}}\quad\underbrace{\text{what percent}}\quad\underbrace{\text{of}}\quad\underbrace{96?}$
Translate into algebra. Let $p =$ the percent.	$144 \quad = \quad p \quad \cdot \quad 96$
Multiply.	$144 = 96p$
Divide by 96 and simplify.	$1.5 = p$
Convert to percent.	$150\% = p$
	144 is 150% of 96

Note that we are asked to find percent, so we must have our final result in percent form.

Note:

Exercise:

Problem: Translate and solve:

110 is what percent of 88?

Solution:

125%

Note:
Exercise:

Problem: Translate and solve:

126 is what percent of 72?

Solution:

175%

Solve Applications of Percent

Many applications of percent—such as tips, sales tax, discounts, and interest—occur in our daily lives. To solve these applications we'll translate to a basic percent equation, just like those we solved in previous examples. Once we translate the sentence into a percent equation, we know how to solve it.

We will restate the problem solving strategy we used earlier for easy reference.

Note:
Use a Problem-Solving Strategy to Solve an Application.

Read the problem. Make sure all the words and ideas are understood.
Identify what we are looking for.
Name what we are looking for. Choose a variable to represent that quantity.
Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
Solve the equation using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

Now that we have the strategy to refer to, and have practiced solving basic percent equations, we are ready to solve percent applications. Be sure to ask yourself if your final answer makes sense—since many of the applications will involve everyday situations, you can rely on your own experience.

Example:
Exercise:

Problem:

Dezohn and his girlfriend enjoyed a nice dinner at a restaurant and his bill was $68.50. He wants to leave an 18% tip. If the tip will be 18% of the total bill, how much tip should he leave?

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		the amount of tip should Dezohn leave
Step 3. Name what we are looking for.		
Choose a variable to represent it.		Let *t* = amount of tip.
Step 4. Translate into an equation.		The tip is 18% of the total bill.
Write a sentence that gives the information to find it.		$\underbrace{\text{The tip}}\ \underbrace{\text{is}}\ \underbrace{18\%}\ \underbrace{\text{of}}\ \underbrace{\$68.50}$
Translate the sentence into an equation.		$t = 0.18 \cdot 68.50$
Step 5. Solve the equation. Multiply.		$t = 12.33$
Step 6. Check. Does this make sense?		
Yes, 20% of $70 is $14.		
Step 7. Answer the question with a complete sentence.		Dezohn should leave a tip of $12.33.

Notice that we used *t* to represent the unknown tip.

Note:
Exercise:

Problem:

Cierra and her sister enjoyed a dinner in a restaurant and the bill was $81.50. If she wants to leave 18% of the total bill as her tip, how much should she leave?

Solution:

$14.67

Note:
Exercise:

Problem:

Kimngoc had lunch at her favorite restaurant. She wants to leave 15% of the total bill as her tip. If her bill was $14.40, how much will she leave for the tip?

Solution:

$2.16

Example:
Exercise:

Problem:

The label on Masao’s breakfast cereal said that one serving of cereal provides 85 milligrams (mg) of potassium, which is 2% of the recommended daily amount. What is the total recommended daily amount of potassium?

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		the total amount of potassium that is

		recommended
Step 3. Name what we are looking for.		
Choose a variable to represent it.		Let a = total amount of potassium.
Step 4. Translate. Write a sentence that gives the information to find it.		$\underbrace{\text{85 mg}}\ \underbrace{\text{is}}\ \underbrace{\text{2\%}}\ \underbrace{\text{of the}}\ \underbrace{\text{total amount}}$
Translate into an equation.		$85 = 0.02 \cdot a$
Step 5. Solve the equation.		$4{,}250 = a$
Step 6. Check. Does this make sense?		
Yes, 2% is a small percent and 85 is a small part of 4,250.		
Step 7. Answer the question with a complete sentence.		The amount of potassium that is recommended is 4,250 mg.

Note:
Exercise:

Problem:

One serving of wheat square cereal has seven grams of fiber, which is 28% of the recommended daily amount. What is the total recommended daily amount of fiber?

Solution:

25 grams

Note:
Exercise:

Problem:

One serving of rice cereal has 190 mg of sodium, which is 8% of the recommended daily amount. What is the total recommended daily amount of sodium?

Solution:

2,375 mg

Example:
Exercise:

Problem:

Mitzi received some gourmet brownies as a gift. The wrapper said each brownie was 480 calories, and had 240 calories of fat. What percent of the total calories in each brownie comes from fat?

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		the percent of the total calories from fat
Step 3. Name what we are looking for.		
Choose a variable to represent it.		Let $p =$ percent of fat.
Step 4. Translate. Write a sentence that gives the information to find it.		$\underbrace{\text{What percent}}\ \underbrace{\text{of}}\ \underbrace{480}\ \underbrace{\text{is}}\ \underbrace{240?}$
Translate into an equation.		$p \cdot 480 = 240$
Step 5. Solve the equation.		$480p = 240$
Divide by 480.		$p = 0.5$
Put in a percent form.		$p = 50\%$
Step 6. Check. Does this make sense?		
Yes, 240 is half of 480, so 50% makes sense.		
Step 7. Answer the question with a complete		Of the total calories in each

sentence.		brownie, 50% is fat.

Note:
Exercise:

Problem: Solve. Round to the nearest whole percent.

Veronica is planning to make muffins from a mix. The package says each muffin will be 230 calories and 60 calories will be from fat. What percent of the total calories is from fat?

Solution:

26%

Note:
Exercise:

Problem: Solve. Round to the nearest whole percent.

The mix Ricardo plans to use to make brownies says that each brownie will be 190 calories, and 76 calories are from fat. What percent of the total calories are from fat?

Solution:

40%

Find Percent Increase and Percent Decrease

People in the media often talk about how much an amount has increased or decreased over a certain period of time. They usually express this increase or decrease as a percent.

To find the percent increase, first we find the amount of increase, the difference of the new amount and the original amount. Then we find what percent the amount of increase is of the original amount.

Note:
Find the Percent Increase.

Find the amount of increase. $\text{new amount} - \text{original amount} = \text{increase}$
Find the percent increase. The increase is what percent of the original amount?

Example:
Exercise:

Problem:

In 2011, the California governor proposed raising community college fees from \$26 a unit to \$36 a unit. Find the percent increase. (Round to the nearest tenth of a percent.)

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		the percent increase
Step 3. Name what we are looking for.		
Choose a variable to represent it.		Let $p =$ the percent.
Step 4. Translate. Write a sentence that gives the information to find it.		
First find the amount of increase.		new amount − original amount = increase
		$36 - 26 = 10$
Find the percent.		Increase is what percent of the original amount?
		$\underbrace{10}\;\underbrace{\text{is}}\;\underbrace{\text{what percent}}\;\underbrace{\text{of}}\;\underbrace{26?}$
Translate into an equation.		$10 = p \cdot 26$
Step 5. Solve the equation.		$10 = 26p$
Divide by 26.		$0.384 = p$
Change to percent form; round to the nearest tenth.		$38.4\% = p$

Step 6. Check. Does this make sense?		
Yes, 38.4% is close to $\frac{1}{3}$, and 10 is close to $\frac{1}{3}$ of 26.		
Step 7. Answer the question with a complete sentence.		The new fees represent a 38.4% increase over the old fees.

Notice that we rounded the division to the nearest thousandth in order to round the percent to the nearest tenth.

Note:
Exercise:

Problem: Find the percent increase. (Round to the nearest tenth of a percent.)

In 2011, the IRS increased the deductible mileage cost to 55.5 cents from 51 cents.

Solution:

8.8%

Note:
Exercise:

Problem: Find the percent increase.

In 1995, the standard bus fare in Chicago was $1.50. In 2008, the standard bus fare was $2.25.

Solution:

50%

Finding the percent decrease is very similar to finding the percent increase, but now the amount of decrease is the difference of the original amount and the new amount. Then we find what percent the amount of decrease is of the original amount.

Note:
Find the Percent Decrease.

Find the amount of decrease. $\text{original amount} - \text{new amount} = \text{decrease}$

Find the percent decrease.The decrease is what percent of the original amount?

Example:
Exercise:

Problem:

The average price of a gallon of gas in one city in June 2014 was \$3.71. The average price in that city in July was \$3.64. Find the percent decrease.

Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		the percent decrease
Step 3. Name what we are looking for.		
Choose a variable to represent that quantity.		Let $p =$ the percent decrease.
Step 4. Translate. Write a sentence that gives the information to find it.		
First find the amount of decrease.		$3.71 - 3.64 = 0.07$
Find the percent.		Decrease is what percent of the original amaount?
		$\underbrace{0.07}\ \underbrace{\text{is}}\ \underbrace{\text{what percent}}\ \underbrace{\text{of}}\ \underbrace{3.71?}$
Translate into an equation.		$0.07 \quad = \quad p \quad \cdot \quad 3.71$
Step 5. Solve the equation.		$0.07 = 3.71\,p$
Divide by 3.71.		$0.019 = p$
Change to percent form; round to the nearest		

tenth.		$1.9\% = p$
Step 6. Check. Does this make sense?		
Yes, if the original price was \$4, a 2% decrease would be 8 cents.		
Step 7. Answer the question with a complete sentence.		The price of gas decreased 1.9%.

Note:
Exercise:

Problem: Find the percent decrease. (Round to the nearest tenth of a percent.)

The population of North Dakota was about 672,000 in 2010. The population is projected to be about 630,000 in 2020.

Solution:

6.3%

Note:
Exercise:

Problem: Find the percent decrease.

Last year, Sheila’s salary was \$42,000. Because of furlough days, this year, her salary was \$37,800.

Solution:

10%

Solve Simple Interest Applications

Do you know that banks pay you to keep your money? The money a customer puts in the bank is called the **principal**, P, and the money the bank pays the customer is called the **interest**. The interest is computed as a certain percent of the principal; called the **rate of interest**, r. We usually express rate of interest as a percent per year, and we calculate it by using the decimal equivalent of the percent. The variable t, (for *time*) represents the number of years the money is in the account.

To find the interest we use the simple interest formula, $I = Prt$.

Note:
Simple Interest
If an amount of money, P, called the principal, is invested for a period of t years at an annual interest rate r, the amount of interest, I, earned is
Equation:

$$I = Prt \qquad \text{where} \qquad \begin{aligned} I &= \text{interest} \\ P &= \text{principal} \\ r &= \text{rate} \\ t &= \text{time} \end{aligned}$$

Interest earned according to this formula is called **simple interest**.

Interest may also be calculated another way, called compound interest. This type of interest will be covered in later math classes.

The formula we use to calculate simple interest is $I = Prt$. To use the formula, we substitute in the values the problem gives us for the variables, and then solve for the unknown variable. It may be helpful to organize the information in a chart.

Example:
Exercise:

Problem:

Nathaly deposited $12,500 in her bank account where it will earn 4% interest. How much interest will Nathaly earn in 5 years?
Equation:

$$\begin{aligned} I &= ? \\ P &= \$12{,}500 \\ r &= 4\% \\ t &= 5 \text{ years} \end{aligned}$$

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the amount of interest earned
Step 3. Name what we are looking for. Choose a variable to represent that quantity	Let $I =$ the amount of interest.
Step 4. Translate into an equation.	
Write the formula.	$I = Prt$
Substitute in the given information.	$I = (12{,}500)(.04)(5)$
Step 5. Solve the equation.	$I = 2{,}500$
Step 6. Check: Does this make sense?	
Is $2,500 is a reasonable interest on $12,500? Yes.	
Step 7. Answer the question with a complete sentence.	The interest is $2,500.

Note:
Exercise:

Problem:

Areli invested a principal of $950 in her bank account with interest rate 3%. How much interest did she earn in 5 years?

Solution:

$142.50

Note:
Exercise:

Problem:

Susana invested a principal of $36,000 in her bank account with interest rate 6.5%. How much interest did she earn in 3 years?

Solution:

$7,020

There may be times when we know the amount of interest earned on a given principal over a certain length of time, but we don't know the rate. To find the rate, we use the simple interest formula, substitute in the given values for the principal and time, and then solve for the rate.

Example:
Exercise:

Problem:

Loren loaned his brother \$3,000 to help him buy a car. In 4 years his brother paid him back the \$3,000 plus \$660 in interest. What was the rate of interest?
Equation:

$$\begin{aligned} I &= \$660 \\ P &= \$3{,}000 \\ r &= ? \\ t &= 4\text{ years} \end{aligned}$$

Solution:
Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for. — the rate of interest

Step 3. Name what we are looking for. Choose a variable to represent that quantity. — Let r = rate of interest.

Step 4. Translate into an equation.

Write the formula. — $I = Prt$

Substitute in the given information. — $660 = (3{,}000)r(4)$

Step 5. Solve the equation. — $660 = (12{,}000)r$

Divide. — $0.055 = r$

Change to percent form. — $5.5\% = r$

Step 6. Check: Does this make sense?

$$\begin{aligned} I &= Prt \\ 660 &\overset{?}{=} (3{,}000)(0.055)(4) \\ 660 &= 660\checkmark \end{aligned}$$

Step 7. Answer the question with a complete sentence. — The rate of interest was 5.5%.

Notice that in this example, Loren's brother paid Loren interest, just like a bank would have paid interest if Loren invested his money there.

Note:
Exercise:

Problem:

Jim loaned his sister $5,000 to help her buy a house. In 3 years, she paid him the $5,000, plus $900 interest. What was the rate of interest?

Solution:

6%

Note:
Exercise:

Problem:

Hang borrowed $7,500 from her parents to pay her tuition. In 5 years, she paid them $1,500 interest in addition to the $7,500 she borrowed. What was the rate of interest?

Solution:

4%

Example:
Exercise:

Problem:

Eduardo noticed that his new car loan papers stated that with a 7.5% interest rate, he would pay $6,596.25 in interest over 5 years. How much did he borrow to pay for his car?

Solution:
Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for. the amount borrowed (the principal)

Step 3. Name what we are looking for.
Choose a variable to represent that quantity. Let $P =$ principal borrowed.

Step 4. Translate into an equation.

Write the formula.
Substitute in the given information.

$$\begin{aligned} I &= Prt \\ 6{,}596.25 &= P(0.075)(5) \end{aligned}$$

Step 5. Solve the equation.

Divide.

$$\begin{aligned} 6{,}596.25 &= 0.375P \\ 17{,}590 &= P \end{aligned}$$

Step 6. Check: Does this make sense?

$$\begin{aligned} I &= Prt \\ 6{,}596.25 &\overset{?}{=} (17{,}590)(0.075)(5) \\ 6{,}596.25 &= 6{,}596.25 \checkmark \end{aligned}$$

Step 7. Answer the question with a complete sentence. The principal was \$17,590.

Note:
Exercise:

Problem:

Sean's new car loan statement said he would pay \$4,866.25 in interest from an interest rate of 8.5% over 5 years. How much did he borrow to buy his new car?

Solution:

\$11,450

Note:
Exercise:

Problem:

In 5 years, Gloria's bank account earned \$2,400 interest at 5%. How much had she deposited in the account?

Solution:

\$9,600

Solve Applications with Discount or Mark-up

Applications of discount are very common in retail settings. When you buy an item on sale, the original price has been discounted by some dollar amount. The **discount rate**, usually given as a percent, is used to determine the amount of the discount. To determine the **amount of discount**, we multiply the discount rate by the original price.

We summarize the discount model in the box below.

Note:
Discount
Equation:

$$\begin{aligned}&\text{amount of discount} = \text{discount rate} \times \text{original price}\\&\text{sale price} = \text{original price} - \text{amount of discount}\end{aligned}$$

Keep in mind that the sale price should always be less than the original price.

Example:
Exercise:

Problem:

Elise bought a dress that was discounted 35% off of the original price of $140. What was ⓐ the amount of discount and ⓑ the sale price of the dress?

Solution:
Solution

ⓐ

$$\begin{aligned}\text{Original price} &= \$140\\ \text{Discount rate} &= 35\%\\ \text{Discount} &= ?\end{aligned}$$

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the amount of discount
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let $d =$ the amount of discount.
Step 4. Translate into an equation. Write a sentence that gives the information to find it.	The discount is 35% of \$140.
Translate into an equation.	$d = 0.35(140)$
Step 5. Solve the equation.	$d = 49$
Step 6. Check: Does this make sense? Is a \$49 discount reasonable for a \$140 dress? Yes.	
Step 7. Write a complete sentence to answer the question.	The amount of discount was \$49.

ⓑ
Read the problem again.

Step 1. Identify what we are looking for.		the sale price of the dress
Step 2. Name what we are looking for.		
Choose a variable to represent that quantity.		Let $s =$ the sale price.
Step 3. Translate into an equation.		
Write a sentence that gives the information to find it.		$\underbrace{\text{The sale price}}\ \underbrace{\text{is}}\ \underbrace{\text{the \$140}}\ \underbrace{\text{minus}}\ \underbrace{\text{the \$49 discount}}$
Translate into an equation.		$s = 140 - 49$
Step 4. Solve the equation.		$s = 91$

Step 5. Check. Does this make sense?		
Is the sale price less than the original price?		
Yes, $91 is less than $140.		
Step 6. Answer the question with a complete sentence.		The sale price of the dress was $91.

Note:
Exercise:

Problem: Find ⓐ the amount of discount and ⓑ the sale price:

Sergio bought a belt that was discounted 40% from an original price of $29.

Solution:

ⓐ $11.60 ⓑ $17.40

Note:
Exercise:

Problem: Find ⓐ the amount of discount and ⓑ the sale price:

Oscar bought a barbecue that was discounted 65% from an original price of $395.

Solution:

ⓐ $256.75 ⓑ $138.25

There may be times when we know the original price and the sale price, and we want to know the discount rate. To find the discount rate, first we will find the amount of discount and then use it to compute the rate as a percent of the original price. [link] will show this case.

Example:
Exercise:

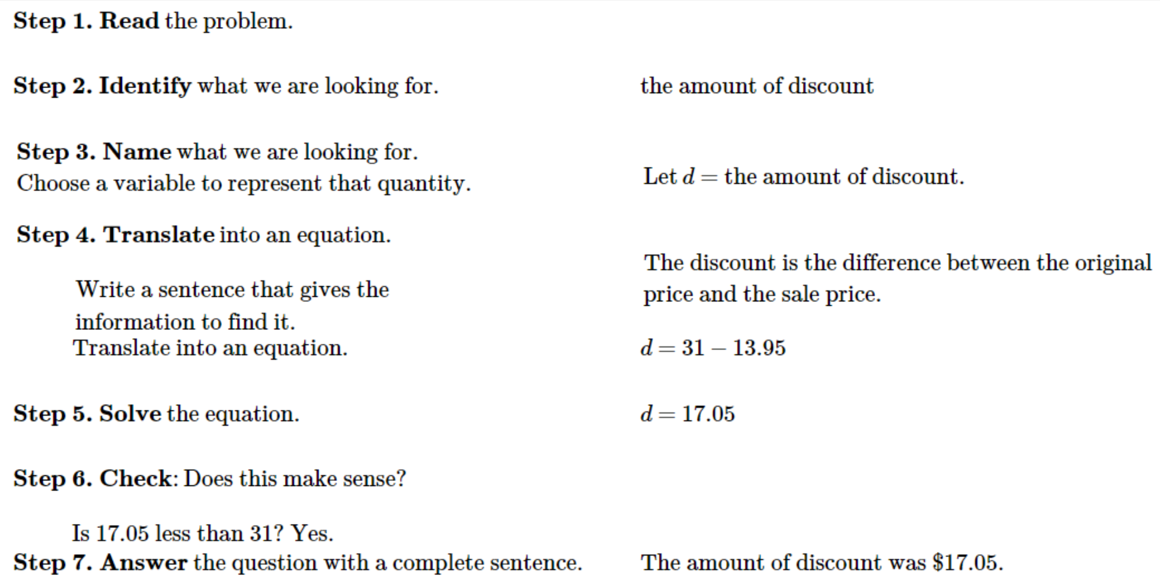
Problem:

Jeannette bought a swimsuit at a sale price of \$13.95. The original price of the swimsuit was \$31. Find the ⓐ amount of discount and ⓑ discount rate.

Solution:
Solution

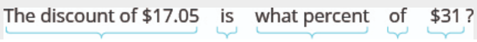
ⓐ

$$\begin{aligned} \text{Original price} &= \$31 \\ \text{Discount} &= ? \\ \text{Sale Price} &= \$13.95 \end{aligned}$$



Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the amount of discount
Step 3. Name what we are looking for. Choose a variable to represent that quantity.	Let $d =$ the amount of discount.
Step 4. Translate into an equation.	
Write a sentence that gives the information to find it.	The discount is the difference between the original price and the sale price.
Translate into an equation.	$d = 31 - 13.95$
Step 5. Solve the equation.	$d = 17.05$
Step 6. Check: Does this make sense?	
Is 17.05 less than 31? Yes.	
Step 7. Answer the question with a complete sentence.	The amount of discount was \$17.05.

ⓑ
Read the problem again.

Step 1. Identify what we are looking for.		the discount rate
Step 2. Name what we are looking for.		
Choose a variable to represent it.		Let $r =$ the discount rate.
Step 3. Translate into an equation.		
Write a sentence that gives the		 The discount of \$17.05 is what percent of \$31 ?

information to find it.		
Translate into an equation.		$17.05 = r \cdot 31$
Step 4. Solve the equation.		$17.05 = 31r$
Divide both sides by 31.		$0.55 = r$
Change to percent form.		$r = 55\%$
Step 5. Check. Does this make sense?		
Is $17.05 equal to 55% of $31?		
$17.05 \stackrel{?}{=} 0.55(31)$		
$17.05 = 17.05✓$		
Step 6. Answer the question with a complete sentence.		The rate of discount was 55%.

Note:
Exercise:

Problem: Find ⓐ the amount of discount and ⓑ the discount rate.

Lena bought a kitchen table at the sale price of $375.20. The original price of the table was $560.

Solution:

ⓐ $184.80 ⓑ 33%

Note:
Exercise:

Problem: Find ⓐ the amount of discount and ⓑ the discount rate.

Nick bought a multi-room air conditioner at a sale price of $340. The original price of the air conditioner was $400.

Solution:

ⓐ $60 ⓑ 15%

Applications of mark-up are very common in retail settings. The price a retailer pays for an item is called the **original cost**. The retailer then adds a **mark-up** to the original cost to get the **list price**, the price he sells the item for. The mark-up is usually calculated as a percent of the original cost. To determine the amount of mark-up, multiply the mark-up rate by the original cost.

We summarize the mark-up model in the box below.

Note:
Mark-Up
Equation:

$$\begin{aligned}\text{amount of mark-up} &= \text{mark-up rate } \times \text{ original cost}\\ \text{list price} &= \text{original cost} + \text{amount of mark up}\end{aligned}$$

Keep in mind that the list price should always be more than the original cost.

Example:
Exercise:

Problem:

Adam’s art gallery bought a photograph at original cost $250. Adam marked the price up 40%. Find the ⓐ amount of mark-up and ⓑ the list price of the photograph.

Solution:
Solution

ⓐ

Step 1. Read the problem.		
Step 2. Identify what we are looking for.		the amount of mark-up
Step 3. Name what we are looking for.		
Choose a variable to represent it.		Let $m =$ the amount of markup.

Step 4. Translate into an equation.		
Write a sentence that gives the information to find it.		$\underbrace{\text{The mark-up}}\ \underbrace{\text{is}}\ \underbrace{\text{40\%}}\ \underbrace{\text{of}}\ \underbrace{\text{the \$250 original cost}}$
Translate into an equation.		$m = 0.40 \cdot 250$
Step 5. Solve the equation.		$m = 100$
Step 6. Check. Does this make sense?		
Yes, 40% is less than one-half and 100 is less than half of 250.		
Step 7. Answer the question with a complete sentence.		The mark-up on the phtograph was $100.

ⓑ

Step 1. Read the problem again.		
Step 2. Identify what we are looking for.		the list price
Step 3. Name what we are looking for.		
Choose a variable to represent it.		Let $p =$ the list price.
Step 4. Translate into an equation.		
Write a sentence that gives the information to find it.		$\underbrace{\text{The list price}}\ \underbrace{\text{is}}\ \underbrace{\text{original cost}}\ \underbrace{\text{plus}}\ \underbrace{\text{the mark-up}}$
Translate into an equation.		$p = 250 + 100$
Step 5. Solve the equation.		$p = 350$
Step 6. Check. Does this make sense?		
Is the list price more than the net price?		

Is \$350 more than \$250? Yes.		
Step 7. Answer the question with a complete sentence.		The list price of the photograph was \$350.

Note:
Exercise:

Problem: Find ⓐ the amount of mark-up and ⓑ the list price.

Jim’s music store bought a guitar at original cost \$1,200. Jim marked the price up 50%.

Solution:

ⓐ \$600 ⓑ \$1,800

Note:
Exercise:

Problem: Find ⓐ the amount of mark-up and ⓑ the list price.

The Auto Resale Store bought Pablo’s Toyota for \$8,500. They marked the price up 35%.

Solution:

ⓐ \$2,975 ⓑ \$11,475

Key Concepts

- **Percent Increase** To find the percent increase:

  Find the amount of increase. $\text{increase} = \text{new amount} - \text{original amount}$
  Find the percent increase. Increase is what percent of the original amount?

- **Percent Decrease** To find the percent decrease:

  Find the amount of decrease. $\text{decrease} = \text{original amount} - \text{new amount}$
  Find the percent decrease. Decrease is what percent of the original amount?

- **Simple Interest** If an amount of money, P, called the principal, is invested for a period of t years at an annual interest rate r, the amount of interest, I earned is
  Equation:

$$\begin{aligned} & I &= \; & Prt \\ \text{where} \quad & I &= \; & \text{interest} \\ & P &= \; & \text{principal} \\ & r &= \; & \text{rate} \\ & t &= \; & \text{time} \end{aligned}$$

- **Discount**
  - amount of discount is discount rate · original price
  - sale price is original price – discount
- **Mark-up**
  - amount of mark-up is mark-up rate · original cost
  - list price is original cost + mark up

Practice Makes Perfect

Translate and Solve Basic Percent Equations

In the following exercises, translate and solve.

Exercise:

Problem: What number is 45% of 120?

Solution:

54

Exercise:

Problem: What number is 65% of 100?

Exercise:

Problem: What number is 24% of 112?

Solution:

26.88

Exercise:

Problem: What number is 36% of 124?

Exercise:

Problem: 250% of 65 is what number?

Solution:

162.5

Exercise:

Problem: 150% of 90 is what number?

Exercise:

Problem: 800% of 2250 is what number?

Solution:

18,000

Exercise:

Problem: 600% of 1740 is what number?

Exercise:

Problem: 28 is 25% of what number?

Solution:

112

Exercise:

Problem: 36 is 25% of what number?

Exercise:

Problem: 81 is 75% of what number?

Solution:

108

Exercise:

Problem: 93 is 75% of what number?

Exercise:

Problem: 8.2% of what number is $2.87?

Solution:

$35

Exercise:

Problem: 6.4% of what number is $2.88?

Exercise:

Problem: 11.5% of what number is $108.10?

Solution:

$940

Exercise:

Problem: 12.3% of what number is $92.25?

Exercise:

Problem: What percent of 260 is 78?

Solution:

30%

Exercise:

Problem: What percent of 215 is 86?

Exercise:

Problem: What percent of 1500 is 540?

Solution:

36%

Exercise:

Problem: What percent of 1800 is 846?

Exercise:

Problem: 30 is what percent of 20?

Solution:

150%

Exercise:

Problem: 50 is what percent of 40?

Exercise:

Problem: 840 is what percent of 480?

Solution:

175%

Exercise:

Problem: 790 is what percent of 395?

Solve Percent Applications

In the following exercises, solve.

Exercise:

Problem:

Geneva treated her parents to dinner at their favorite restaurant. The bill was $74.25. Geneva wants to leave 16% of the total bill as a tip. How much should the tip be?

Solution:

$11.88

Exercise:

Problem:

When Hiro and his co-workers had lunch at a restaurant near their work, the bill was $90.50. They want to leave 18% of the total bill as a tip. How much should the tip be?

Exercise:

Problem:

Trong has 12% of each paycheck automatically deposited to his savings account. His last paycheck was $2165. How much money was deposited to Trong’s savings account?

Solution:

$259.80

Exercise:

Problem:

Cherise deposits 8% of each paycheck into her retirement account. Her last paycheck was $1,485. How much did Cherise deposit into her retirement account?

Exercise:

Problem:

One serving of oatmeal has eight grams of fiber, which is 33% of the recommended daily amount. What is the total recommended daily amount of fiber?

Solution:

24.2 g

Exercise:

Problem:

One serving of trail mix has 67 grams of carbohydrates, which is 22% of the recommended daily amount. What is the total recommended daily amount of carbohydrates?

Exercise:

Problem:

A bacon cheeseburger at a popular fast food restaurant contains 2070 milligrams (mg) of sodium, which is 86% of the recommended daily amount. What is the total recommended daily amount of sodium?

Solution:

2407 mg

Exercise:

Problem:

A grilled chicken salad at a popular fast food restaurant contains 650 milligrams (mg) of sodium, which is 27% of the recommended daily amount. What is the total recommended daily amount of sodium?

Exercise:

Problem:

After 3 months on a diet, Lisa had lost 12% of her original weight. She lost 21 pounds. What was Lisa's original weight?

Solution:

175 lb.

Exercise:

Problem:

Tricia got a 6% raise on her weekly salary. The raise was $30 per week. What was her original salary?

Exercise:

Problem:

Yuki bought a dress on sale for $72. The sale price was 60% of the original price. What was the original price of the dress?

Solution:

$120

Exercise:

Problem:

Kim bought a pair of shoes on sale for \$40.50. The sale price was 45% of the original price. What was the original price of the shoes?

Exercise:

Problem: Tim left a \$9 tip for a \$50 restaurant bill. What percent tip did he leave?

Solution:

18%

Exercise:

Problem: Rashid left a \$15 tip for a \$75 restaurant bill. What percent tip did he leave?

Exercise:

Problem:

The nutrition fact sheet at a fast food restaurant says the fish sandwich has 380 calories, and 171 calories are from fat. What percent of the total calories is from fat?

Solution:

45%

Exercise:

Problem:

The nutrition fact sheet at a fast food restaurant says a small portion of chicken nuggets has 190 calories, and 114 calories are from fat. What percent of the total calories is from fat?

Exercise:

Problem:

Emma gets paid \$3,000 per month. She pays \$750 a month for rent. What percent of her monthly pay goes to rent?

Solution:

25%

Exercise:

Problem:

Dimple gets paid \$3,200 per month. She pays \$960 a month for rent. What percent of her monthly pay goes to rent?

Find Percent Increase and Percent Decrease

In the following exercises, solve.

Exercise:

Problem: Tamanika got a raise in her hourly pay, from \$15.50 to \$17.36. Find the percent increase.

Solution:

12%

Exercise:

Problem: Ayodele got a raise in her hourly pay, from \$24.50 to \$25.48. Find the percent increase.

Exercise:

Problem:

Annual student fees at the University of California rose from about \$4,000 in 2000 to about \$12,000 in 2010. Find the percent increase.

Solution:

200%

Exercise:

Problem: The price of a share of one stock rose from \$12.50 to \$50. Find the percent increase.

Exercise:

Problem:

According to *Time* magazine annual global seafood consumption rose from 22 pounds per person in the 1960s to 38 pounds per person in 2011. Find the percent increase. (Round to the nearest tenth of a percent.)

Solution:

72.7%

Exercise:

Problem:

In one month, the median home price in the Northeast rose from \$225,400 to \$241,500. Find the percent increase. (Round to the nearest tenth of a percent.)

Exercise:

Problem:

A grocery store reduced the price of a loaf of bread from \$2.80 to \$2.73. Find the percent decrease.

Solution:

2.5%

Exercise:

Problem: The price of a share of one stock fell from $8.75 to $8.54. Find the percent decrease.

Exercise:

Problem:

Hernando’s salary was $49,500 last year. This year his salary was cut to $44,055. Find the percent decrease.

Solution:

11%

Exercise:

Problem:

In 10 years, the population of Detroit fell from 950,000 to about 712,500. Find the percent decrease.

Exercise:

Problem:

In 1 month, the median home price in the West fell from $203,400 to $192,300. Find the percent decrease. (Round to the nearest tenth of a percent.)

Solution:

5.5%

Exercise:

Problem:

Sales of video games and consoles fell from $1,150 million to $1,030 million in 1 year. Find the percent decrease. (Round to the nearest tenth of a percent.)

Solve Simple Interest Applications

In the following exercises, solve.

Exercise:

Problem:

Casey deposited $1,450 in a bank account with interest rate 4%. How much interest was earned in two years?

Solution:

$116

Exercise:

Problem:

Terrence deposited $5,720 in a bank account with interest rate 6%. How much interest was earned in 4 years?

Exercise:

Problem:

Robin deposited $31,000 in a bank account with interest rate 5.2%. How much interest was earned in 3 years?

Solution:

$4,836

Exercise:

Problem:

Carleen deposited $16,400 in a bank account with interest rate 3.9%. How much interest was earned in 8 years?

Exercise:

Problem:

Hilaria borrowed $8,000 from her grandfather to pay for college. Five years later, she paid him back the $8,000, plus $1,200 interest. What was the rate of interest?

Solution:

3%

Exercise:

Problem:

Kenneth loaned his niece $1,200 to buy a computer. Two years later, she paid him back the $1,200, plus $96 interest. What was the rate of interest?

Exercise:

Problem:

Lebron loaned his daughter $20,000 to help her buy a condominium. When she sold the condominium four years later, she paid him the $20,000, plus $3,000 interest. What was the rate of interest?

Solution:

3.75%

Exercise:

Problem:

Pablo borrowed $50,000 to start a business. Three years later, he repaid the $50,000, plus $9,375 interest. What was the rate of interest?

Exercise:

Problem:

In 10 years, a bank account that paid 5.25% earned $18,375 interest. What was the principal of the account?

Solution:

$35,000

Exercise:

Problem:

In 25 years, a bond that paid 4.75% earned $2,375 interest. What was the principal of the bond?

Exercise:

Problem:

Joshua’s computer loan statement said he would pay $1,244.34 in interest for a 3-year loan at 12.4%. How much did Joshua borrow to buy the computer?

Solution:

$3,345

Exercise:

Problem:

Margaret’s car loan statement said she would pay $7,683.20 in interest for a 5-year loan at 9.8%. How much did Margaret borrow to buy the car?

Solve Applications with Discount or Mark-up

In the following exercises, find the sale price.

Exercise:

Problem:

Perla bought a cell phone that was on sale for $50 off. The original price of the cell phone was $189.

Solution:

$139

Exercise:

Problem: Sophie saw a dress she liked on sale for $15 off. The original price of the dress was $96.

Exercise:

Problem:

Rick wants to buy a tool set with original price $165. Next week the tool set will be on sale for $40 off.

Solution:

$125

Exercise:

Problem:

Angelo’s store is having a sale on televisions. One television, with original price $859, is selling for $125 off.

In the following exercises, find ⓐ the amount of discount and ⓑ the sale price.

Exercise:

Problem: Janelle bought a beach chair on sale at 60% off. The original price was $44.95.

Solution:

ⓐ $26.97 ⓑ $17.98

Exercise:

Problem: Errol bought a skateboard helmet on sale at 40% off. The original price was $49.95.

Exercise:

Problem:

Kathy wants to buy a camera that lists for $389. The camera is on sale with a 33% discount.

Solution:

ⓐ $128.37 ⓑ $260.63

Exercise:

Problem: Colleen bought a suit that was discounted 25% from an original price of $245.

Exercise:

Problem:

Erys bought a treadmill on sale at 35% off. The original price was $949.95 (round to the nearest cent.)

Solution:

ⓐ $332.48 ⓑ $617.47

Exercise:

Problem:

Jay bought a guitar on sale at 45% off. The original price was $514.75 (round to the nearest cent.)

In the following exercises, find ⓐ the amount of discount and ⓑ the discount rate. (Round to the nearest tenth of a percent if needed.)

Exercise:

Problem:

Larry and Donna bought a sofa at the sale price of $1,344. The original price of the sofa was $1,920.

Solution:

ⓐ $576 ⓑ 30%

Exercise:

Problem:

Hiroshi bought a lawnmower at the sale price of $240. The original price of the lawnmower is $300.

Exercise:

Problem:

Patty bought a baby stroller on sale for $301.75. The original price of the stroller was $355.

Solution:

ⓐ $53.25 ⓑ 15%

Exercise:

Problem: Bill found a book he wanted on sale for $20.80. The original price of the book was $32.

Exercise:

Problem:

Nikki bought a patio set on sale for $480. The original price was $850. To the nearest tenth of a percent, what was the rate of discount?

Solution:

ⓐ $370 ⓑ 43.5%

Exercise:

Problem:

Stella bought a dinette set on sale for $725. The original price was $1,299. To the nearest tenth of a percent, what was the rate of discount?

In the following exercises, find ⓐ the amount of the mark-up and ⓑ the list price.

Exercise:

Problem:

Daria bought a bracelet at original cost \$16 to sell in her handicraft store. She marked the price up 45%.

Solution:

ⓐ \$7.20 ⓑ \$23.20

Exercise:

Problem:

Regina bought a handmade quilt at original cost \$120 to sell in her quilt store. She marked the price up 55%.

Exercise:

Problem:

Tom paid \$0.60 a pound for tomatoes to sell at his produce store. He added a 33% mark-up.

Solution:

ⓐ \$0.20 ⓑ \$0.80

Exercise:

Problem:

Flora paid her supplier \$0.74 a stem for roses to sell at her flower shop. She added an 85% mark-up.

Exercise:

Problem:

Alan bought a used bicycle for \$115. After re-conditioning it, he added 225% mark-up and then advertised it for sale.

Solution:

ⓐ \$258.75 ⓑ \$373.75

Exercise:

Problem:

Michael bought a classic car for \$8,500. He restored it, then added 150% mark-up before advertising it for sale.

Everyday Math

Exercise:

Problem:

Leaving a Tip At the campus coffee cart, a medium coffee costs $1.65. MaryAnne brings $2.00 with her when she buys a cup of coffee and leaves the change as a tip. What percent tip does she leave?

Solution:

21.2%

Exercise:

Problem:

Splitting a Bill Four friends went out to lunch and the bill came to $53.75. They decided to add enough tip to make a total of $64, so that they could easily split the bill evenly among themselves. What percent tip did they leave?

Writing Exercises

Exercise:

Problem:

Without solving the problem “44 is 80% of what number” think about what the solution might be. Should it be a number that is greater than 44 or less than 44? Explain your reasoning.

Solution:

The number should be greater than 44. Since 80% equals 0.8 in decimal form, 0.8 is less than one, and we must multiply the number by 0.8 to get 44, the number must be greater than 44.

Exercise:

Problem:

Without solving the problem “What is 20% of 300?” think about what the solution might be. Should it be a number that is greater than 300 or less than 300? Explain your reasoning.

Exercise:

Problem:

After returning from vacation, Alex said he should have packed 50% fewer shorts and 200% more shirts. Explain what Alex meant.

Solution:

He meant that he should have packed half the shorts and twice the shirts.

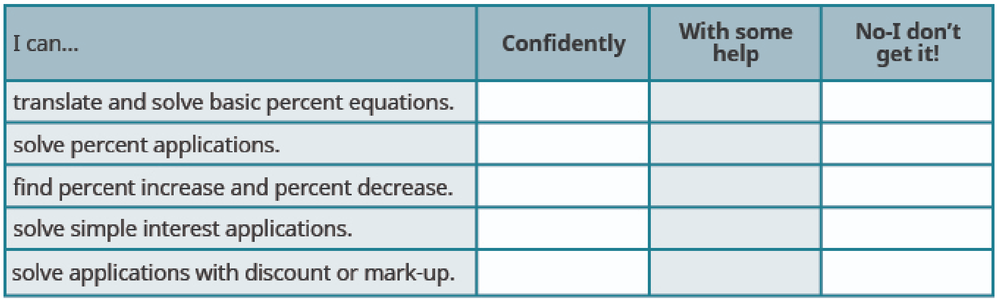
Exercise:

Problem:

Because of road construction in one city, commuters were advised to plan that their Monday morning commute would take 150% of their usual commuting time. Explain what this means.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
translate and solve basic percent equations.			
solve percent applications.			
find percent increase and percent decrease.			
solve simple interest applications.			
solve applications with discount or mark-up.			

ⓑ After reviewing this checklist, what will you do to become confident for all goals?

Glossary

amount of discount

The amount of discount is the amount resulting when a discount rate is multiplied by the original price of an item.

discount rate

The discount rate is the percent used to determine the amount of a discount, common in retail settings.

interest

Interest is the money that a bank pays its customers for keeping their money in the bank.

list price

The list price is the price a retailer sells an item for.

mark-up

A mark-up is a percentage of the original cost used to increase the price of an item.

original cost

The original cost in a retail setting, is the price that a retailer pays for an item.

principal

The principal is the original amount of money invested or borrowed for a period of time at a specific interest rate.

rate of interest

The rate of interest is a percent of the principal, usually expressed as a percent per year.

simple interest

Simple interest is the interest earned according to the formula $I = Prt$.

Solve Geometry Applications: Triangles, Rectangles, and the Pythagorean Theorem

By the end of this section, you will be able to:

- Solve applications using properties of triangles
- Use the Pythagorean Theorem
- Solve applications using rectangle properties

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\frac{1}{2}(6h)$.
   If you missed this problem, review [link].
2. The length of a rectangle is three less than the width. Let *w* represent the width. Write an expression for the length of the rectangle.
   If you missed this problem, review [link].
3. Solve: $A = \frac{1}{2}bh$ for b when $A = 260$ and $h = 52$.
   If you missed this problem, review [link].
4. Simplify: $\sqrt{144}$.
   If you missed this problem, review [link].

Solve Applications Using Properties of Triangles

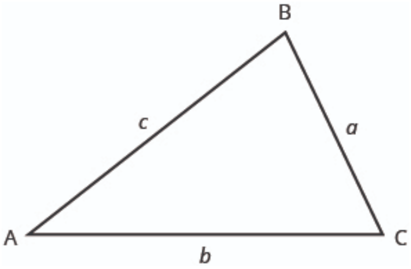
In this section we will use some common geometry formulas. We will adapt our problem-solving strategy so that we can solve geometry applications. The geometry formula will name the variables and give us the equation to solve. In addition, since these applications will all involve shapes of some sort, most people find it helpful to draw a figure and label it with the given information. We will include this in the first step of the problem solving strategy for geometry applications.

Note:
Solve Geometry Applications.

Readthe problem and make sure all the words and ideas are understood. Draw the figure and label it with the given information.
Identifywhat we are looking for.
Labelwhat we are looking for by choosing a variable to represent it.
Translateinto an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
Solvethe equation using good algebra techniques.
Checkthe answer by substituting it back into the equation solved in step 5 and by making sure it makes sense in the context of the problem.
Answerthe question with a complete sentence.

We will start geometry applications by looking at the properties of triangles. Let’s review some basic facts about triangles. Triangles have three sides and three interior angles. Usually each side is labeled with a lowercase letter to match the uppercase letter of the opposite vertex.

The plural of the word *vertex* is *vertices*. All triangles have three vertices. Triangles are named by their vertices: The triangle in [link] is called $\triangle ABC$.



Triangle ABC has vertices A, B, and C. The lengths of the sides are a, b, and c.

The three angles of a triangle are related in a special way. The sum of their measures is 180°. Note that we read $m\angle A$ as "the measure of angle A." So in $\triangle ABC$ in [link],

Equation:

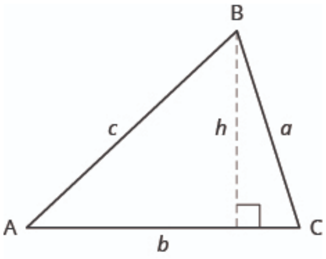
$$m\angle A + m\angle B + m\angle C = 180^\circ$$

Because the perimeter of a figure is the length of its boundary, the perimeter of $\triangle ABC$ is the sum of the lengths of its three sides.

Equation:

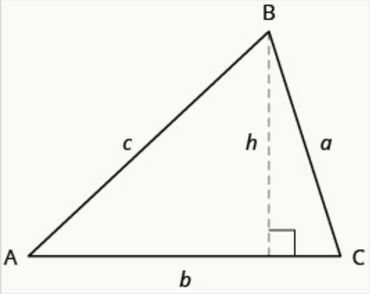
$$P = a + b + c$$

To find the area of a triangle, we need to know its base and height. The height is a line that connects the base to the opposite vertex and makes a 90° angle with the base. We will draw $\triangle ABC$ again, and now show the height, *h*. See [link].



The formula for the area of $\triangle ABC$ is $A = \frac{1}{2}bh$, where *b* is the base and *h* is the height.

Note:
Triangle Properties



For $\triangle ABC$
Angle measures:
Equation:

$$m\angle A+m\angle B+m\angle C=180$$

- The sum of the measures of the angles of a triangle is 180°.

Perimeter:
Equation:

$$P=a+b+c$$

- The perimeter is the sum of the lengths of the sides of the triangle.

Area:
Equation:

$$A=\frac{1}{2}bh, b=\text{base}, h=\text{height}$$

- The area of a triangle is one-half the base times the height.

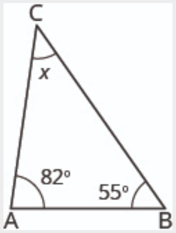
Example:
Exercise:

Problem:

The measures of two angles of a triangle are 55 and 82 degrees. Find the measure of the third angle.

Solution:

Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the measure of the third angle in a triangle
Step 3. Name. Choose a variable to represent it.	Let $x =$ the measure of the angle.
Step 4. Translate.	
Write the appropriate formula and substitute.	$m\angle A + m\angle B + m\angle C = 180$
Step 5. Solve the equation.	$55 + 82 + x = 180$ $137 + x = 180$ $x = 43$
Step 6. Check. $55 + 82 + 43 \overset{?}{=} 180$ $180 = 180\checkmark$	
Step 7. Answer the question.	The measure of the third angle is 43 degrees.

Note:
Exercise:

Problem:

The measures of two angles of a triangle are 31 and 128 degrees. Find the measure of the third angle.

Solution:

21 degrees

Note:
Exercise:

Problem:

The measures of two angles of a triangle are 49 and 75 degrees. Find the measure of the third angle.

Solution:

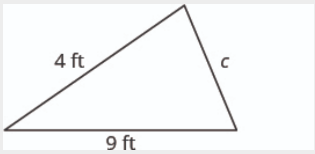
56 degrees

Example:
Exercise:

Problem:

The perimeter of a triangular garden is 24 feet. The lengths of two sides are four feet and nine feet. How long is the third side?

Solution:
Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	 $P = 24\text{ ft}$
Step 2. Identify what you are looking for.	length of the third side of a triangle
Step 3. Name. Choose a variable to represent it.	Let $c =$ the third side.
Step 4. Translate.	
Write the appropriate formula and substitute.	$\underbrace{P} = \underbrace{a} + \underbrace{b} + \underbrace{c}$
Substitute in the given information.	$24\text{ ft} = 4\text{ ft} + 9\text{ ft} + c$
Step 5. Solve the equation.	$24 = 13 + c$

	$11 = c$
Step 6. Check. $\begin{array}{rcl} \text{P} & = & a+b+c \\ 24 & \overset{?}{=} & 4+9+11 \\ 24 & = & 24\checkmark \end{array}$	
Step 7. Answer the question.	The third side is 11 feet long.

Note:
Exercise:

Problem:

The perimeter of a triangular garden is 48 feet. The lengths of two sides are 18 feet and 22 feet. How long is the third side?

Solution:

8 feet

Note:
Exercise:

Problem:

The lengths of two sides of a triangular window are seven feet and five feet. The perimeter is 18 feet. How long is the third side?

Solution:

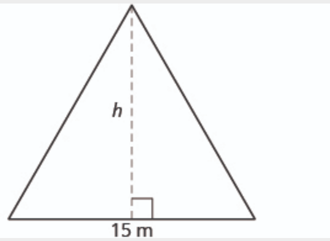
6 feet

Example:
Exercise:

Problem:

The area of a triangular church window is 90 square meters. The base of the window is 15 meters. What is the window’s height?

Solution:
Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	 Area $= 90m^2$
Step 2. Identify what you are looking for.	height of a triangle
Step 3. Name. Choose a variable to represent it.	Let $h =$ the height.
Step 4. Translate.	
Write the appropriate formula.	$\underbrace{A}\ \underbrace{=}\ \underbrace{\frac{1}{2}}\ \underbrace{\cdot}\ \underbrace{b}\ \underbrace{\cdot}\ \underbrace{h}$
Substitute in the given information.	$90\text{ m}^2 = \frac{1}{2} \cdot 15\text{ m} \cdot h$
Step 5. Solve the equation.	$90 = \frac{15}{2}h$ $12 = h$
Step 6. Check. $A = \frac{1}{2}bh$ $90 \overset{?}{=} \frac{1}{2} \cdot 15 \cdot 12$ $90 = 90 ✓$	
Step 7. Answer the question.	The height of the triangle is 12 meters.

Note:
Exercise:

Problem:

The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

Solution:

14 inches

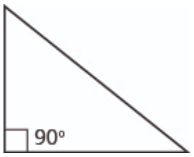
Note:
Exercise:

Problem: A triangular tent door has area 15 square feet. The height is five feet. What is the base?

Solution:

6 feet

The triangle properties we used so far apply to all triangles. Now we will look at one specific type of triangle—a right triangle. A right triangle has one 90° angle, which we usually mark with a small square in the corner.



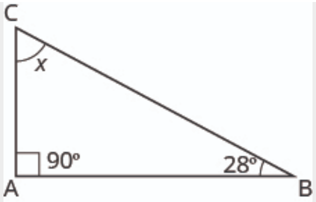
Note:
Right Triangle
A **right triangle** has one 90° angle, which is often marked with a square at the vertex.

Example:
Exercise:

Problem: One angle of a right triangle measures 28°. What is the measure of the third angle?

Solution:
Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	

	
Step 2. Identify what you are looking for.	the measure of an angle
Step 3. Name. Choose a variable to represent it.	Let $x =$ the measure of an angle.
Step 4. Translate.	$m\angle A + m\angle B + m\angle C = 180$
Write the appropriate formula and substitute.	$x + 90 + 28 = 180$
Step 5. Solve the equation.	$x + 118 = 180$ $x = 62$
Step 6. Check. $180 \overset{?}{=} 90 + 28 + 62$ $180 = 180✓$	
Step 7. Answer the question.	The measure of the third angle is 62°.

Note:
Exercise:

Problem: One angle of a right triangle measures 56°. What is the measure of the other small angle?

Solution:

34°

Note:
Exercise:

Problem: One angle of a right triangle measures 45°. What is the measure of the other small angle?

Solution:

45°

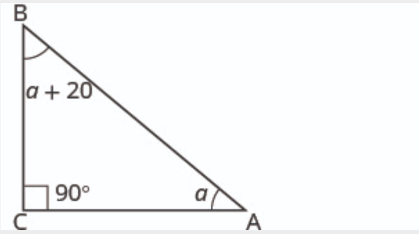
In the examples we have seen so far, we could draw a figure and label it directly after reading the problem. In the next example, we will have to define one angle in terms of another. We will wait to draw the figure until we write expressions for all the angles we are looking for.

Example:
Exercise:

Problem:

The measure of one angle of a right triangle is 20 degrees more than the measure of the smallest angle. Find the measures of all three angles.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the measures of all three angles
Step 3. Name. Choose a variable to represent it.	Let $a = 1^{\text{st}}$ angle. $a + 20 = 2^{\text{nd}}$ angle $90 = 3^{\text{rd}}$ angle (the right angle)
Draw the figure and label it with the given information	
Step 4. Translate	$m\angle A + m\angle B + m\angle C = 180$
Write the appropriate formula. Substitute into the formula.	$a + (a + 20) + 90 = 180$
Step 5. Solve the equation.	$2a + 110 = 180$ $2a = 70$ $a = 35$ first angle

	$a+20$ second angle $\color{red}{35}+20$ 55 90 third angle
Step 6. Check. $35+55+90 \overset{?}{=} 180$ $180 = 180\checkmark$	
Step 7. Answer the question.	The three angles measure 35°, 55°, and 90°.

Note:
Exercise:

Problem:

The measure of one angle of a right triangle is 50° more than the measure of the smallest angle. Find the measures of all three angles.

Solution:

$20^\circ, 70^\circ, 90^\circ$

Note:
Exercise:

Problem:

The measure of one angle of a right triangle is 30° more than the measure of the smallest angle. Find the measures of all three angles.

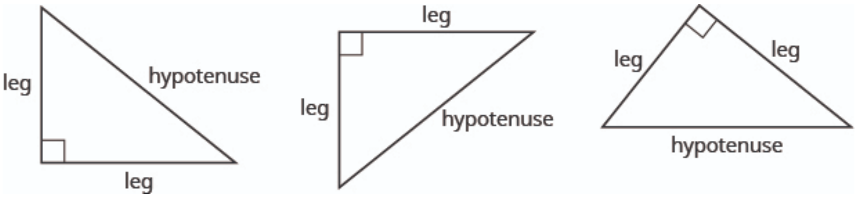
Solution:

$30^\circ, 60^\circ, 90^\circ$

Use the Pythagorean Theorem

We have learned how the measures of the angles of a triangle relate to each other. Now, we will learn how the lengths of the sides relate to each other. An important property that describes the relationship among the lengths of the three sides of a right triangle is called the Pythagorean Theorem. This theorem has been used around the world since ancient times. It is named after the Greek philosopher and mathematician, Pythagoras, who lived around 500 BC.

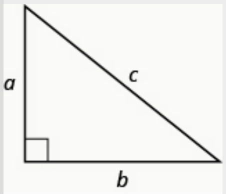
Before we state the Pythagorean Theorem, we need to introduce some terms for the sides of a triangle. Remember that a right triangle has a 90° angle, marked with a small square in the corner. The side of the triangle opposite the 90° angle is called the *hypotenuse* and each of the other sides are called *legs*.



The Pythagorean Theorem tells how the lengths of the three sides of a right triangle relate to each other. It states that in any right triangle, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse. In symbols we say: in any right triangle, $a^2 + b^2 = c^2$, where a and b are the lengths of the legs and c is the length of the hypotenuse.

Writing the formula in every exercise and saying it aloud as you write it, may help you remember the Pythagorean Theorem.

Note:
The Pythagorean Theorem
In any right triangle, $a^2 + b^2 = c^2$.



where *a* and *b* are the lengths of the legs, *c* is the length of the hypotenuse.

To solve exercises that use the Pythagorean Theorem, we will need to find square roots. We have used the notation $\sqrt{m}$ and the definition:

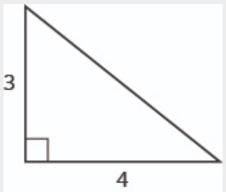
If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.

For example, we found that $\sqrt{25}$ is 5 because $25 = 5^2$.

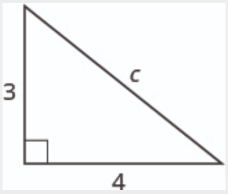
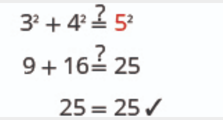
Because the Pythagorean Theorem contains variables that are squared, to solve for the length of a side in a right triangle, we will have to use square roots.

Example:
Exercise:

Problem: Use the Pythagorean Theorem to find the length of the hypotenuse shown below.



Solution:
Solution

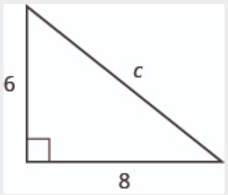
Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length of the hypotenuse of the triangle
Step 3. Name. Choose a variable to represent it. Label side *c* on the figure.	Let *c* = the length of the hypotenuse. 
Step 4. Translate.	
Write the appropriate formula.	$a^2 + b^2 = c^2$
Substitute.	$3^2 + 4^2 = c^2$
Step 5. Solve the equation.	$9 + 16 = c^2$
Simplify.	$25 = c^2$
Use the definition of square root.	$\sqrt{25} = c$
Simplify.	$5 = c$
Step 6. Check.  $3^2 + 4^2 \overset{?}{=} 5^2$ $9 + 16 \overset{?}{=} 25$ $25 = 25$ ✓	

Step 7. Answer the question.	The length of the hypotenuse is 5.

Note:
Exercise:

Problem:

Use the Pythagorean Theorem to find the length of the hypotenuse in the triangle shown below.



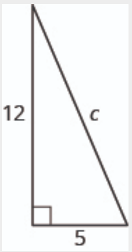
Solution:

$c = 10$

Note:
Exercise:

Problem:

Use the Pythagorean Theorem to find the length of the hypotenuse in the triangle shown below.

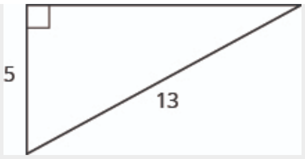


Solution:

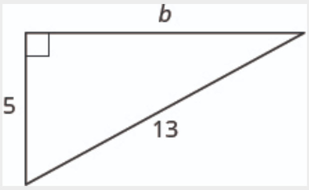
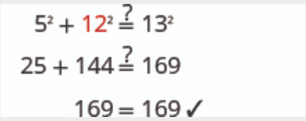
$c = 13$

Example:
Exercise:

Problem: Use the Pythagorean Theorem to find the length of the leg shown below.

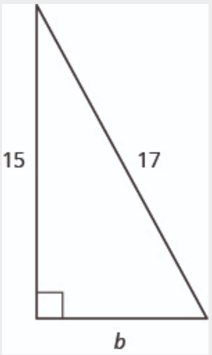


Solution:
Solution

Step 1. Read the problem.		
Step 2. Identify what you are looking for.		the length of the leg of the triangle
Step 3. Name. Choose a variable to represent it.		Let *b* = the leg of the triangle.
Lable side *b*.		
Step 4. Translate		
Write the appropriate formula.		$a^2 + b^2 = c^2$
Substitute.		$5^2 + b^2 = 13^2$
Step 5. Solve the equation.		$25 + b^2 = 169$
Isolate the variable term.		$b^2 = 144$
Use the definition of square root.		$b^2 = \sqrt{144}$
Simplify.		$b = 12$
Step 6. Check.  $5^2 + 12^2 \overset{?}{=} 13^2$ $25 + 144 \overset{?}{=} 169$ $169 = 169$ ✓		
Step 7. Answer the question.		The length of the leg is 12.

Note:
Exercise:

Problem: Use the Pythagorean Theorem to find the length of the leg in the triangle shown below.

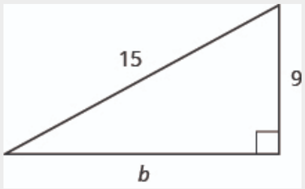


Solution:

8

Note:
Exercise:

Problem: Use the Pythagorean Theorem to find the length of the leg in the triangle shown below.

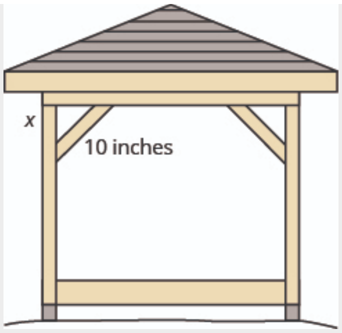


Solution:

12

Example:
Exercise:

Problem:



Kelvin is building a gazebo and wants to brace each corner by placing a $10''$ piece of wood diagonally as shown above.

If he fastens the wood so that the ends of the brace are the same distance from the corner, what is the length of the legs of the right triangle formed? Approximate to the nearest tenth of an inch.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the distance from the corner that the bracket should be attached
Step 3. Name. Choose a variable to represent it.	Let $x =$ the distance from the corner.
Step 4. Translate. Write the appropriate formula and substitute.	$a^2 + b^2 = c^2$ $x^2 + x^2 = 10^2$
Step 5. Solve the equation.	$2x^2 = 100$
Isolate the variable.	$x^2 = 50$
Use the definition of square root.	$x = \sqrt{50}$
Simplify. Approximate to the nearest tenth.	$x \approx 7.1$
Step 6. Check.	
Step 7. Answer the question.	Kelvin should fasten each piece of wood approximately $7.1''$ from the corner.

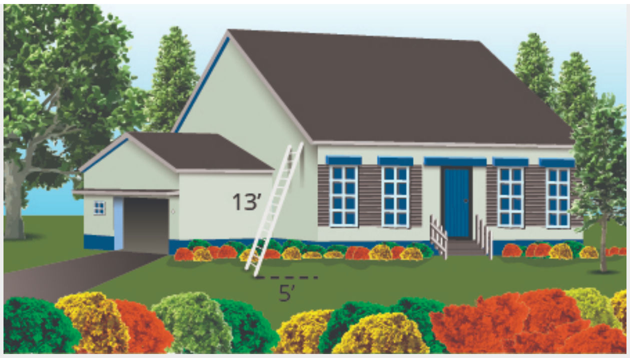
Step 6 check:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (7.1)^2 + (7.1)^2 &\approx 10^2 \text{ Yes.} \end{aligned}$$

Note:
Exercise:

Problem:

John puts the base of a 13-foot ladder five feet from the wall of his house as shown below. How far up the wall does the ladder reach?



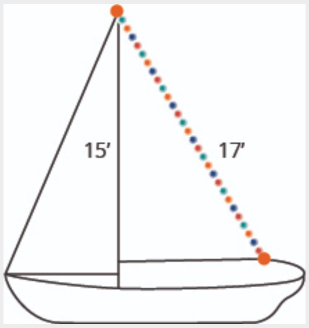
Solution:

12 feet

Note:
Exercise:

Problem:

Randy wants to attach a 17 foot string of lights to the top of the 15 foot mast of his sailboat, as shown below. How far from the base of the mast should he attach the end of the light string?

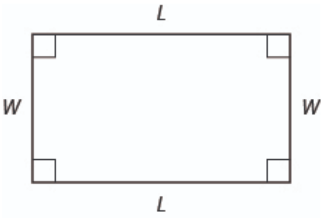


Solution:

8 feet

Solve Applications Using Rectangle Properties

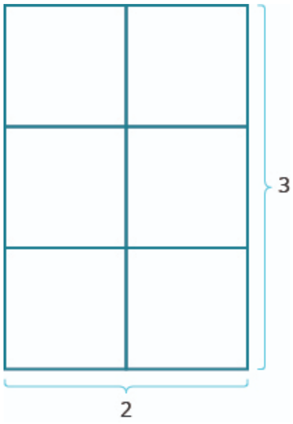
You may already be familiar with the properties of rectangles. Rectangles have four sides and four right (90°) angles. The opposite sides of a rectangle are the same length. We refer to one side of the rectangle as the length, *L*, and its adjacent side as the width, *W*.



The distance around this rectangle is $L+W+L+W$, or $2L+2W$. This is the perimeter, P, of the rectangle.
Equation:

$$P=2L+2W$$

What about the area of a rectangle? Imagine a rectangular rug that is 2-feet long by 3-feet wide. Its area is 6 square feet. There are six squares in the figure.



Equation:

$$\begin{aligned}A&=6\\A&=2\cdot 3\\A&=L\cdot W\end{aligned}$$

The area is the length times the width.

The formula for the area of a rectangle is $A=LW$.

Note:
Properties of Rectangles
Rectangles have four sides and four right (90°) angles.
The lengths of opposite sides are equal.
The perimeter of a rectangle is the sum of twice the length and twice the width.
Equation:

$$P=2L+2W$$

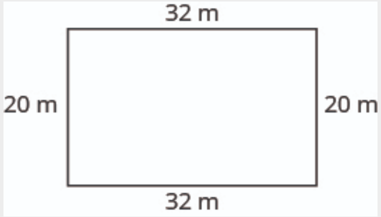
The area of a rectangle is the product of the length and the width.
Equation:

$$A = L \cdot W$$

Example:
Exercise:

Problem: The length of a rectangle is 32 meters and the width is 20 meters. What is the perimeter?

Solution:
Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	
Step 2. Identify what you are looking for.	the perimeter of a rectangle
Step 3. Name. Choose a variable to represent it.	Let P = the perimeter.
Step 4. Translate.	
Write the appropriate formula.	$\underbrace{P}\quad\underbrace{=}\quad\underbrace{2L}\quad\underbrace{+}\quad\underbrace{2W}$
Substitute.	$P \quad = \quad 2(32\text{ m}) \quad + \quad 2(20\text{ m})$
Step 5. Solve the equation.	$P = 64 + 40$ $P = 104$
Step 6. Check.	

$$\begin{aligned} P &\stackrel{?}{=} 104 \\ 20+32+20+32 &\stackrel{?}{=} 104 \\ 104 &= 104\checkmark \end{aligned}$$	
Step 7. Answer the question.	The perimeter of the rectangle is 104 meters.

Note:
Exercise:

Problem: The length of a rectangle is 120 yards and the width is 50 yards. What is the perimeter?

Solution:

340 yards

Note:
Exercise:

Problem: The length of a rectangle is 62 feet and the width is 48 feet. What is the perimeter?

Solution:

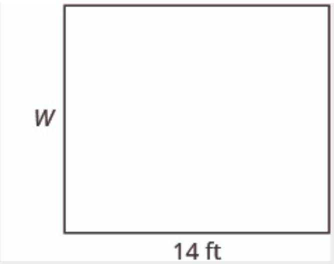
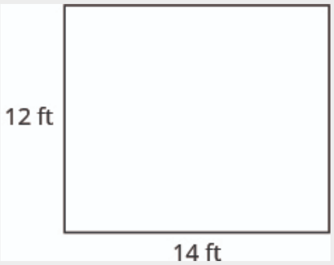
220 feet

Example:
Exercise:

Problem: The area of a rectangular room is 168 square feet. The length is 14 feet. What is the width?

Solution:
Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	

	
Step 2. Identify what you are looking for.	the width of a rectangular room
Step 3. Name. Choose a variable to represent it.	Let W = the width.
Step 4. Translate.	
Write the appropriate formula.	$A = LW$
Substitute.	$168 = 14W$
Step 5. Solve the equation.	$\frac{168}{14} = \frac{14W}{14}$ $12 = W$
Step 6. Check.  $\begin{aligned} A &= LW \\ 168 &\overset{?}{=} 14 \cdot 12 \\ 168 &= 168 ✓ \end{aligned}$	
Step 7. Answer the question.	The width of the room is 12 feet.

Note:
Exercise:

Problem: The area of a rectangle is 598 square feet. The length is 23 feet. What is the width?

Solution:

26 feet

Note:
Exercise:

Problem: The width of a rectangle is 21 meters. The area is 609 square meters. What is the length?

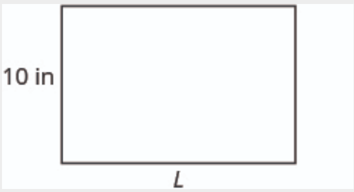
Solution:

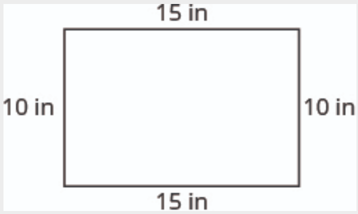
29 meters

Example:
Exercise:

Problem: Find the length of a rectangle with perimeter 50 inches and width 10 inches.

Solution:
Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	 $P = 50 \text{ in}$
Step 2. Identify what you are looking for.	the length of the rectangle
Step 3. Name. Choose a variable to represent it.	Let *L* = the length.
Step 4. Translate.	
Write the appropriate formula.	$P = 2L + 2W$
Substitute.	$50 = 2L + 2(10)$

Step 5. Solve the equation.	$50 - 20 = 2L + 20 - 20$ $30 = 2L$ $\frac{30}{2} = \frac{2L}{2}$ $15 = L$
Step 6. Check.  $P = 50$ $15 + 10 + 15 + 10 \overset{?}{=} 50$ $50 = 50\checkmark$	
Step 7. Answer the question.	The length is 15 inches.

Note:
Exercise:

Problem: Find the length of a rectangle with: perimeter 80 and width 25.

Solution:

15

Note:
Exercise:

Problem: Find the length of a rectangle with: perimeter 30 and width 6.

Solution:

9

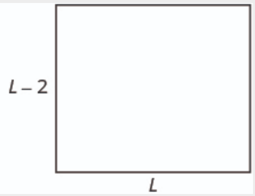
We have solved problems where either the length or width was given, along with the perimeter or area; now we will learn how to solve problems in which the width is defined in terms of the length. We will wait to draw the figure until we write an expression for the width so that we can label one side with that expression.

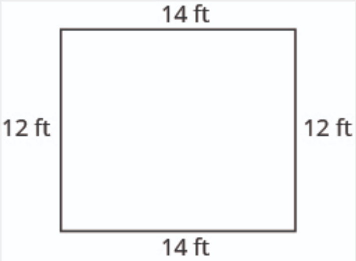
Example:
Exercise:

Problem:

The width of a rectangle is two feet less than the length. The perimeter is 52 feet. Find the length and width.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length and width of a rectangle
Step 3. Name. Choose a variable to represent it. Since the width is defined in terms of the length, we let L = length. The width is two feet less than the length, so we let $L - 2$ = width.	 $P = 52$ ft
Step 4. Translate.	
Write the appropriate formula. The formula for the perimeter of a rectangle relates all the information.	$P = 2L + 2W$
Substitute in the given information.	$52 = 2L + 2(L - 2)$
Step 5. Solve the equation.	$52 = 2L + 2L - 4$
Combine like terms.	$52 = 4L - 4$

Add 4 to each side.	$56 = 4L$
Divide by 4.	$\frac{56}{4} = \frac{4L}{4}$ $14 = L$ The length is 14 feet.
Now we need to find the width.	The width is $L - 2$. $L - 2$ $14 - 2$ 12 The width is 12 feet.
Step 6. Check. Since $14 + 12 + 14 + 12 = 52$, this works! 	
Step 7. Answer the question.	The length is 14 feet and the width is 12 feet.

Note:
Exercise:

Problem:

The width of a rectangle is seven meters less than the length. The perimeter is 58 meters. Find the length and width.

Solution:

18 meters, 11 meters

Note:
Exercise:

Problem:

The length of a rectangle is eight feet more than the width. The perimeter is 60 feet. Find the length and width.

Solution:

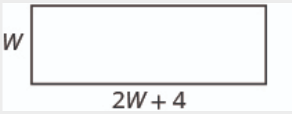
19 feet, 11 feet

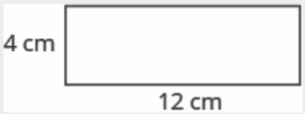
Example:
Exercise:

Problem:

The length of a rectangle is four centimeters more than twice the width. The perimeter is 32 centimeters. Find the length and width.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length and the width
Step 3. Name. Choose a variable to represent the width.	Let $W =$ width
The length is four more than twice the width.	$2W + 4 =$ length  $P = 32$ cm
Step 4. Translate	
Write the appropriate formula.	$P = 2L + 2W$

Substitute in the given information.	$32 = 2(2W + 4) + 2W$
Step 5. Solve the equation.	$32 = 4W + 8 + 2W$ $32 = 6W + 8$ $24 = 6W$ $4 = W$ (width) $2W + 4$ (length) $2(4) + 4$ 12 The length is 12 cm.
Step 6. Check.  $P = 2L + 2W$ $32 \stackrel{?}{=} 2 \cdot 12 + 2 \cdot 4$ $32 = 32\checkmark$	
Step 7. Answer the question.	The length is 12 cm and the width is 4 cm.

Note:

Exercise:

Problem:

The length of a rectangle is eight more than twice the width. The perimeter is 64. Find the length and width.

Solution:

24, 8

Note:
Exercise:

Problem:

The width of a rectangle is six less than twice the length. The perimeter is 18. Find the length and width.

Solution:

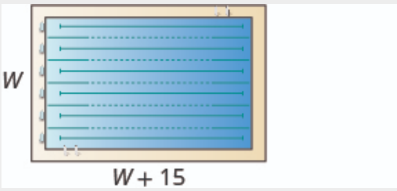
5, 4

Example:
Exercise:

Problem:

The perimeter of a rectangular swimming pool is 150 feet. The length is 15 feet more than the width. Find the length and width.

Solution:
Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	 $P = 150$ ft
Step 2. Identify what you are looking for.	the length and the width of the pool
Step 3. Name. Choose a variable to represent the width. The length is 15 feet more than the width.	Let $W =$ width $W + 15 =$ length

Step 4. Translate	
Write the appropriate formula.	$P = 2L + 2W$
Substitute.	$150 = 2(W + 15) + 2W$
Step 5. Solve the equation.	$150 = 2W + 30 + 2W$ $150 = 4W + 30$ $120 = 4W$ $30 = W$ (the width of the pool) $W + 15$ (the length of the pool) $30 + 15$ 45
Step 6. Check. $P = 2L + 2W$ $150 \overset{?}{=} 2(45) + 2(30)$ $150 = 150✓$	
Step 7. Answer the question.	The length of the pool is 45 feet and the width is 30 feet.

Note:

Exercise:

Problem:

The perimeter of a rectangular swimming pool is 200 feet. The length is 40 feet more than the width. Find the length and width.

Solution:

70 feet, 30 feet

Note:
Exercise:

Problem:

The length of a rectangular garden is 30 yards more than the width. The perimeter is 300 yards. Find the length and width.

Solution:

90 yards, 60 yards

Key Concepts

- **Problem-Solving Strategy for Geometry Applications**

  Read the problem and make sure all the words and ideas are understood. Draw the figure and label it with the given information.
  Identify what we are looking for.
  Name what we are looking for by choosing a variable to represent it.
  Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  Solve the equation using good algebra techniques.
  Check the answer in the problem and make sure it makes sense.
  Answer the question with a complete sentence.

- **Triangle Properties For $\triangle ABC$**
  Angle measures:
  - $m\angle A + m\angle B + m\angle C = 180$

  Perimeter:
  - $P = a + b + c$

  Area:
  - $A = \frac{1}{2}bh,\ \text{b} = \text{base}, \text{h} = \text{height}$

  A right triangle has one 90° angle.
- **The Pythagorean Theorem** In any right triangle, $a^2 + b^2 = c^2$ where c is the length of the hypotenuse and a and b are the lengths of the legs.
- **Properties of Rectangles**
  - Rectangles have four sides and four right (90°) angles.
  - The lengths of opposite sides are equal.

  - The perimeter of a rectangle is the sum of twice the length and twice the width: $P = 2L + 2W$. The area of a rectangle is the length times the width: $A = LW$.

Practice Makes Perfect

Solving Applications Using Triangle Properties

In the following exercises, solve using triangle properties.

Exercise:

Problem:

The measures of two angles of a triangle are 26 and 98 degrees. Find the measure of the third angle.

Solution:

56 degrees

Exercise:

Problem:

The measures of two angles of a triangle are 61 and 84 degrees. Find the measure of the third angle.

Exercise:

Problem:

The measures of two angles of a triangle are 105 and 31 degrees. Find the measure of the third angle.

Solution:

44 degrees

Exercise:

Problem:

The measures of two angles of a triangle are 47 and 72 degrees. Find the measure of the third angle.

Exercise:

Problem:

The perimeter of a triangular pool is 36 yards. The lengths of two sides are 10 yards and 15 yards. How long is the third side?

Solution:

11 feet

Exercise:

Problem:

A triangular courtyard has perimeter 120 meters. The lengths of two sides are 30 meters and 50 meters. How long is the third side?

Exercise:

Problem: If a triangle has sides 6 feet and 9 feet and the perimeter is 23 feet, how long is the third side?

Solution:

8 feet

Exercise:

Problem:

If a triangle has sides 14 centimeters and 18 centimeters and the perimeter is 49 centimeters, how long is the third side?

Exercise:

Problem: A triangular flag has base one foot and height 1.5 foot. What is its area?

Solution:

0.75 sq. ft.

Exercise:

Problem: A triangular window has base eight feet and height six feet. What is its area?

Exercise:

Problem: What is the base of a triangle with area 207 square inches and height 18 inches?

Solution:

23 inches

Exercise:

Problem: What is the height of a triangle with area 893 square inches and base 38 inches?

Exercise:

Problem:

One angle of a right triangle measures 33 degrees. What is the measure of the other small angle?

Solution:

57

Exercise:

Problem:

One angle of a right triangle measures 51 degrees. What is the measure of the other small angle?

Exercise:

Problem:

One angle of a right triangle measures 22.5 degrees. What is the measure of the other small angle?

Solution:

67.5

Exercise:

Problem:

One angle of a right triangle measures 36.5 degrees. What is the measure of the other small angle?

Exercise:

Problem:

The perimeter of a triangle is 39 feet. One side of the triangle is one foot longer than the second side. The third side is two feet longer than the second side. Find the length of each side.

Solution:

13 ft., 12 ft., 14 ft.

Exercise:

Problem:

The perimeter of a triangle is 35 feet. One side of the triangle is five feet longer than the second side. The third side is three feet longer than the second side. Find the length of each side.

Exercise:

Problem:

One side of a triangle is twice the shortest side. The third side is five feet more than the shortest side. The perimeter is 17 feet. Find the lengths of all three sides.

Solution:

3 ft., 6 ft., 8 ft.

Exercise:

Problem:

One side of a triangle is three times the shortest side. The third side is three feet more than the shortest side. The perimeter is 13 feet. Find the lengths of all three sides.

Exercise:

Problem:

The two smaller angles of a right triangle have equal measures. Find the measures of all three angles.

Solution:

$45^\circ, 45^\circ, 90^\circ$

Exercise:

Problem:

The measure of the smallest angle of a right triangle is 20° less than the measure of the next larger angle. Find the measures of all three angles.

Exercise:

Problem:

The angles in a triangle are such that one angle is twice the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

Solution:

$30^\circ, 60^\circ, 90^\circ$

Exercise:

Problem:

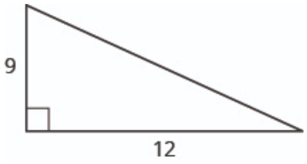
The angles in a triangle are such that one angle is 20° more than the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

Use the Pythagorean Theorem

In the following exercises, use the Pythagorean Theorem to find the length of the hypotenuse.

Exercise:

Problem:

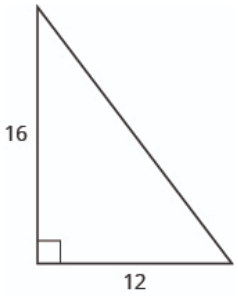


Solution:

15

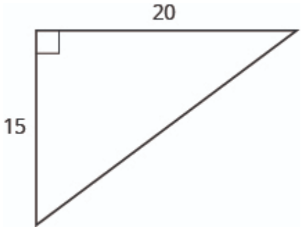
Exercise:

Problem:



Exercise:

Problem:

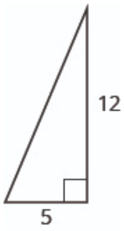


Solution:

25

Exercise:

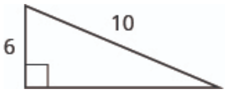
Problem:



In the following exercises, use the Pythagorean Theorem to find the length of the leg. Round to the nearest tenth, if necessary.

Exercise:

Problem:

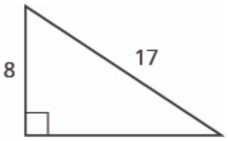


Solution:

8

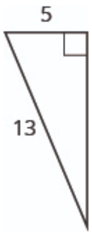
Exercise:

Problem:



Exercise:

Problem:

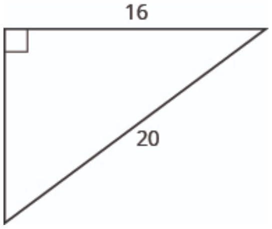


Solution:

12

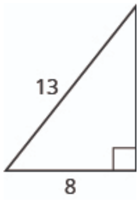
Exercise:

Problem:



Exercise:

Problem:

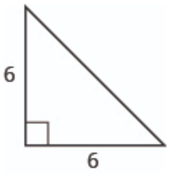


Solution:

10.2

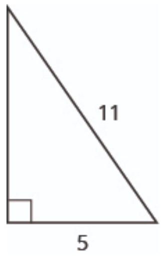
Exercise:

Problem:



Exercise:

Problem:

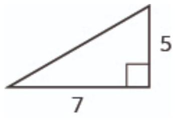


Solution:

9.8

Exercise:

Problem:

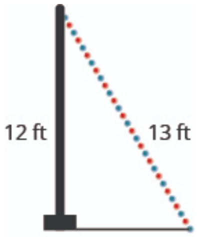


In the following exercises, solve using the Pythagorean Theorem. Approximate to the nearest tenth, if necessary.

Exercise:

Problem:

A 13-foot string of lights will be attached to the top of a 12-foot pole for a holiday display, as shown below. How far from the base of the pole should the end of the string of lights be anchored?



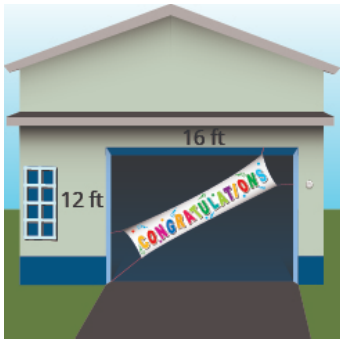
Solution:

5 feet

Exercise:

Problem:

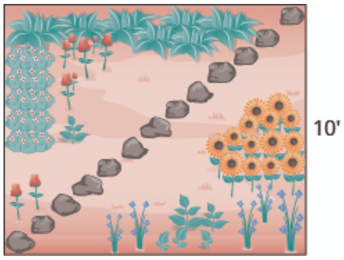
Pam wants to put a banner across her garage door, as shown below, to congratulate her son for his college graduation. The garage door is 12 feet high and 16 feet wide. How long should the banner be to fit the garage door?



Exercise:

Problem:

Chi is planning to put a path of paving stones through her flower garden, as shown below. The flower garden is a square with side 10 feet. What will the length of the path be?



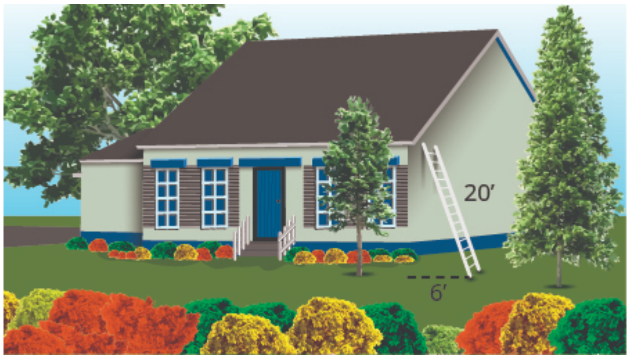
Solution:

14.1 feet

Exercise:

Problem:

Brian borrowed a 20 foot extension ladder to use when he paints his house. If he sets the base of the ladder 6 feet from the house, as shown below, how far up will the top of the ladder reach?



Solve Applications Using Rectangle Properties

In the following exercises, solve using rectangle properties.

Exercise:

Problem: The length of a rectangle is 85 feet and the width is 45 feet. What is the perimeter?

Solution:

260 feet

Exercise:

Problem: The length of a rectangle is 26 inches and the width is 58 inches. What is the perimeter?

Exercise:

Problem: A rectangular room is 15 feet wide by 14 feet long. What is its perimeter?

Solution:

58 feet

Exercise:

Problem: A driveway is in the shape of a rectangle 20 feet wide by 35 feet long. What is its perimeter?

Exercise:

Problem: The area of a rectangle is 414 square meters. The length is 18 meters. What is the width?

Solution:

23 meters

Exercise:

Problem:

The area of a rectangle is 782 square centimeters. The width is 17 centimeters. What is the length?

Exercise:

Problem:

The width of a rectangular window is 24 inches. The area is 624 square inches. What is the length?

Solution:

26 inches

Exercise:

Problem:

The length of a rectangular poster is 28 inches. The area is 1316 square inches. What is the width?

Exercise:

Problem: Find the length of a rectangle with perimeter 124 and width 38.

Solution:

24

Exercise:

Problem: Find the width of a rectangle with perimeter 92 and length 19.

Exercise:

Problem: Find the width of a rectangle with perimeter 16.2 and length 3.2.

Solution:

4.9

Exercise:

Problem: Find the length of a rectangle with perimeter 20.2 and width 7.8.

Exercise:

Problem:

The length of a rectangle is nine inches more than the width. The perimeter is 46 inches. Find the length and the width.

Solution:

16 in., 7 in.

Exercise:

Problem:

The width of a rectangle is eight inches more than the length. The perimeter is 52 inches. Find the length and the width.

Exercise:

Problem:

The perimeter of a rectangle is 58 meters. The width of the rectangle is five meters less than the length. Find the length and the width of the rectangle.

Solution:

17 m, 12 m

Exercise:

Problem:

The perimeter of a rectangle is 62 feet. The width is seven feet less than the length. Find the length and the width.

Exercise:

Problem:

The width of a rectangle is 0.7 meters less than the length. The perimeter of the rectangle is 52.6 meters. Find the dimensions of the rectangle.

Solution:

13.5 m length, 12.8 m width

Exercise:

Problem:

The length of a rectangle is 1.1 meters less than the width. The perimeter of the rectangle is 49.4 meters. Find the dimensions of the rectangle.

Exercise:

Problem:

The perimeter of a rectangle is 150 feet. The length of the rectangle is twice the width. Find the length and width of the rectangle.

Solution:

50 ft., 25 ft.

Exercise:

Problem:

The length of a rectangle is three times the width. The perimeter of the rectangle is 72 feet. Find the length and width of the rectangle.

Exercise:

Problem:

The length of a rectangle is three meters less than twice the width. The perimeter of the rectangle is 36 meters. Find the dimensions of the rectangle.

Solution:

7 m width, 11 m length

Exercise:

Problem:

The length of a rectangle is five inches more than twice the width. The perimeter is 34 inches. Find the length and width.

Exercise:

Problem:

The perimeter of a rectangular field is 560 yards. The length is 40 yards more than the width. Find the length and width of the field.

Solution:

160 yd., 120 yd.

Exercise:

Problem:

The perimeter of a rectangular atrium is 160 feet. The length is 16 feet more than the width. Find the length and width of the atrium.

Exercise:

Problem:

A rectangular parking lot has perimeter 250 feet. The length is five feet more than twice the width. Find the length and width of the parking lot.

Solution:

85 ft., 40 ft.

Exercise:

Problem:

A rectangular rug has perimeter 240 inches. The length is 12 inches more than twice the width. Find the length and width of the rug.

Everyday Math

Exercise:

Problem:

Christa wants to put a fence around her triangular flowerbed. The sides of the flowerbed are six feet, eight feet and 10 feet. How many feet of fencing will she need to enclose her flowerbed?

Solution:

24 feet

Exercise:

Problem:

Jose just removed the children’s playset from his back yard to make room for a rectangular garden. He wants to put a fence around the garden to keep out the dog. He has a 50 foot roll of fence in his garage that he plans to use. To fit in the backyard, the width of the garden must be 10 feet. How long can he make the other length?

Writing Exercises

Exercise:

Problem:

If you need to put tile on your kitchen floor, do you need to know the perimeter or the area of the kitchen? Explain your reasoning.

Solution:

area; answers will vary

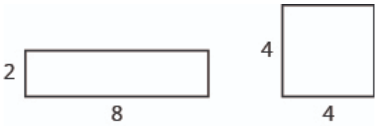
Exercise:

Problem:

If you need to put a fence around your backyard, do you need to know the perimeter or the area of the backyard? Explain your reasoning.

Exercise:

Problem: Look at the two figures below.



ⓐ Which figure looks like it has the larger area?
ⓑ Which looks like it has the larger perimeter?
ⓒ Now calculate the area and perimeter of each figure.
ⓓ Which has the larger area?
ⓔ Which has the larger perimeter?

Solution:

ⓐ Answers will vary.
ⓑ Answers will vary.
ⓒ Answers will vary.
ⓓ The areas are the same.
ⓔ The 2x8 rectangle has a larger perimeter than the 4x4 square.

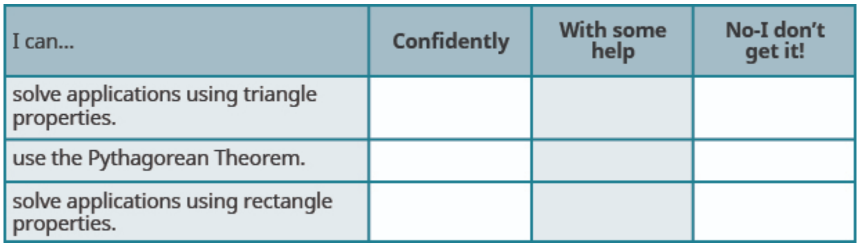
Exercise:

Problem:

Write a geometry word problem that relates to your life experience, then solve it and explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

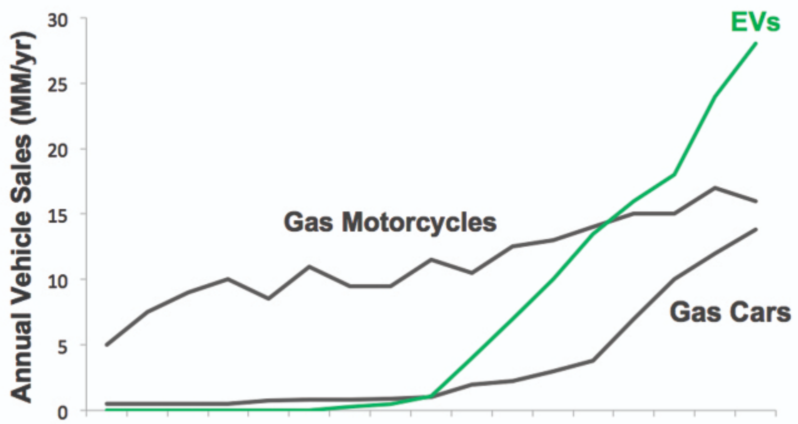


I can...	Confidently	With some help	No-I don't get it!
solve applications using triangle properties.			
use the Pythagorean Theorem.			
solve applications using rectangle properties.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Introduction class="introduction"

This graph illustrates the annual vehicle sales of gas motorcycles, gas cars, and electric vehicles from 1994 to 2010. It is a line graph with *x*- and *y*-axes, one of the most common types of graphs. (credit: Steve Jurvetson, Flickr)



Graphs are found in all areas of our lives—from commercials showing you which cell phone carrier provides the best coverage, to bank statements and news articles, to the boardroom of major corporations. In this chapter, we will study the rectangular coordinate system, which is the basis for most consumer graphs. We will look at linear graphs, slopes of lines, equations of lines, and linear inequalities.

Use the Rectangular Coordinate System
By the end of this section, you will be able to:

- Plot points in a rectangular coordinate system
- Verify solutions to an equation in two variables
- Complete a table of solutions to a linear equation
- Find solutions to a linear equation in two variables

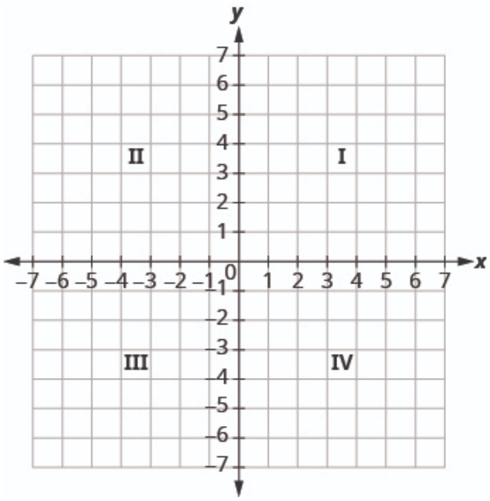
Note:
Before you get started, take this readiness quiz.

1. Evaluate $x+3$ when $x=-1$.
   If you missed this problem, review [link].
2. Evaluate $2x-5y$ when $x=3$ and $y=-2$.
   If you missed this problem, review [link].
3. Solve for y: $40-4y=20$.
   If you missed this problem, review [link].

Plot Points on a Rectangular Coordinate System

Just like maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a **rectangular coordinate system**. The rectangular coordinate system is also called the *xy*-plane or the ‘coordinate plane’.

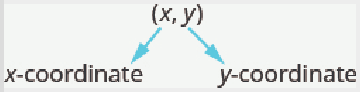
The horizontal number line is called the *x-axis*. The vertical number line is called the *y-axis*. The *x*-axis and the *y*-axis together form the rectangular coordinate system. These axes divide a plane into four regions, called **quadrants**. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise. See [link].



‘Quadrant’ has the root ‘quad,’ which means ‘four.’

In the rectangular coordinate system, every point is represented by an *ordered pair*. The first number in the ordered pair is the ***x*-coordinate** of the point, and the second number is the ***y*-coordinate** of the point.

Note:
Ordered Pair
An **ordered pair**, (x, y), gives the coordinates of a point in a rectangular coordinate system.

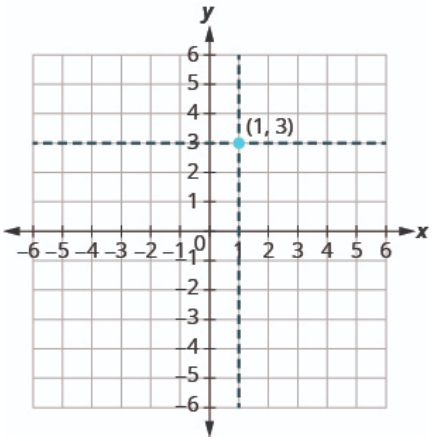


The first number is the *x*-coordinate.
The second number is the *y*-coordinate.

The phrase ‘ordered pair’ means the order is important. What is the ordered pair of the point where the axes cross? At that point both coordinates are zero, so its ordered pair is $(0, 0)$. The point $(0, 0)$ has a special name. It is called the origin.

Note:
The Origin
The point $(0, 0)$ is called the **origin**. It is the point where the *x*-axis and *y*-axis intersect.

We use the coordinates to locate a point on the *xy*-plane. Let’s plot the point $(1, 3)$ as an example. First, locate 1 on the *x*-axis and lightly sketch a vertical line through $x = 1$. Then, locate 3 on the *y*-axis and sketch a horizontal line through $y = 3$. Now, find the point where these two lines meet—that is the point with coordinates $(1, 3)$.



Notice that the vertical line through $x = 1$ and the horizontal line through $y = 3$ are not part of the graph. We just used them to help us locate the point $(1, 3)$.

Example:
Exercise:

Problem:

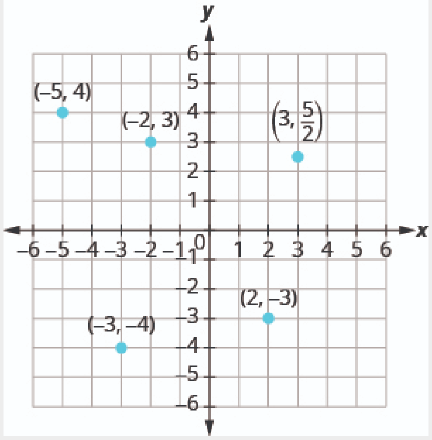
Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-5,4)$ ⓑ $(-3,-4)$ ⓒ $(2,-3)$ ⓓ $(-2,3)$ ⓔ $\left(3,\frac{5}{2}\right)$.

Solution:
Solution

The first number of the coordinate pair is the *x*-coordinate, and the second number is the *y*-coordinate.

ⓐ Since $x=-5$, the point is to the left of the *y*-axis. Also, since $y=4$, the point is above the *x*-axis. The point $(-5,4)$ is in Quadrant II.
ⓑ Since $x=-3$, the point is to the left of the *y*-axis. Also, since $y=-4$, the point is below the *x*-axis. The point $(-3,-4)$ is in Quadrant III.
ⓒ Since $x=2$, the point is to the right of the *y*-axis. Since $y=-3$, the point is below the *x*-axis. The point $(2,-3)$ is in Quadrant lV.
ⓓ Since $x=-2$, the point is to the left of the *y*-axis. Since $y=3$, the point is above the *x*-axis. The point $(-2,3)$ is in Quadrant II.
ⓔ Since $x=3$, the point is to the right of the *y*-axis. Since $y=\frac{5}{2}$, the point is above the *x*-axis. (It may be helpful to write $\frac{5}{2}$ as a mixed number or decimal.) The point $\left(3,\frac{5}{2}\right)$ is in Quadrant I.



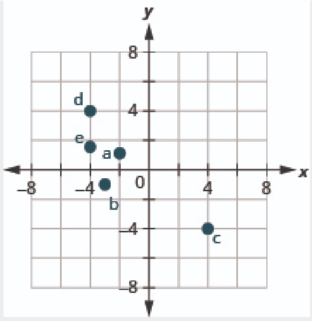
Note:
Exercise:

Problem:

Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-2,1)$ ⓑ $(-3,-1)$ ⓒ $(4,-4)$ ⓓ $(-4,4)$ ⓔ $\left(-4,\frac{3}{2}\right)$.

Solution:



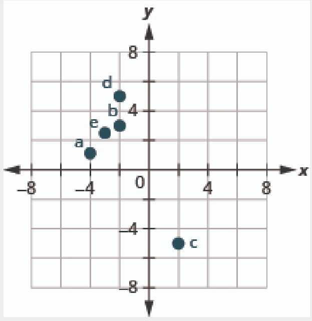
Note:
Exercise:

Problem:

Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

ⓐ $(-4, 1)$ ⓑ $(-2, 3)$ ⓒ $(2, -5)$ ⓓ $(-2, 5)$ ⓔ $\left(-3, \frac{5}{2}\right)$.

Solution:



How do the signs affect the location of the points? You may have noticed some patterns as you graphed the points in the previous example.

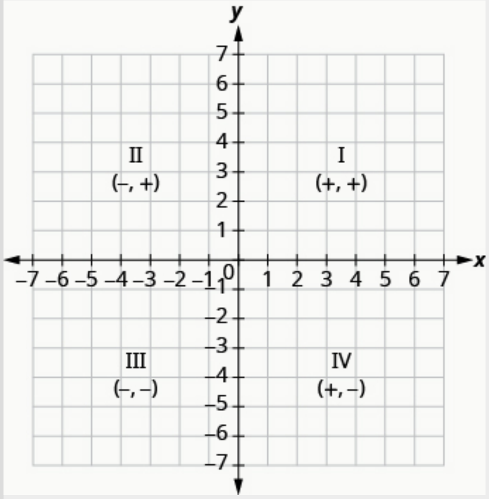
For the point in [link] in Quadrant IV, what do you notice about the signs of the coordinates? What about the signs of the coordinates of points in the third quadrant? The second quadrant? The first quadrant?

Can you tell just by looking at the coordinates in which quadrant the point $(-2, 5)$ is located? In which quadrant is $(2, -5)$ located?

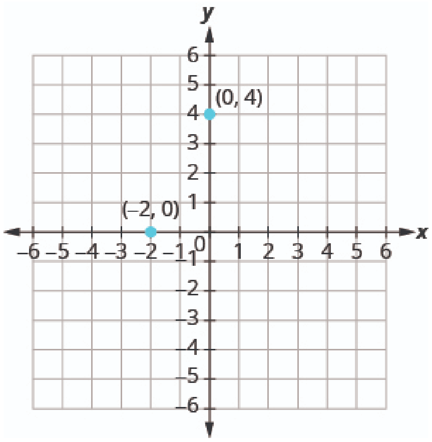
Note:
Quadrants
We can summarize sign patterns of the quadrants in this way.

Equation:

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x, y)	(x, y)	(x, y)	(x, y)
$(+, +)$	$(-, +)$	$(-, -)$	$(+, -)$



What if one coordinate is zero as shown in [link]? Where is the point $(0, 4)$ located? Where is the point $(-2, 0)$ located?



The point $(0, 4)$ is on the *y*-axis and the point $(-2, 0)$ is on the *x*-axis.

Note:
Points on the Axes
Points with a *y*-coordinate equal to 0 are on the *x*-axis, and have coordinates $(a, 0)$.
Points with an *x*-coordinate equal to 0 are on the *y*-axis, and have coordinates $(0, b)$.

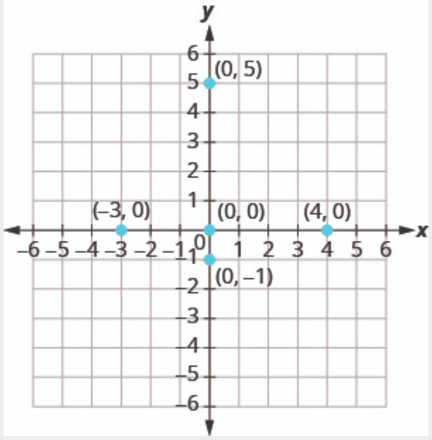
Example:
Exercise:

Problem: Plot each point:

ⓐ $(0,5)$ ⓑ $(4,0)$ ⓒ $(-3,0)$ ⓓ $(0,0)$ ⓔ $(0,-1)$.

Solution:
Solution

ⓐ Since $x=0$, the point whose coordinates are $(0,5)$ is on the *y*-axis.
ⓑ Since $y=0$, the point whose coordinates are $(4,0)$ is on the *x*-axis.
ⓒ Since $y=0$, the point whose coordinates are $(-3,0)$ is on the *x*-axis.
ⓓ Since $x=0$ and $y=0$, the point whose coordinates are $(0,0)$ is the origin.
ⓔ Since $x=0$, the point whose coordinates are $(0,-1)$ is on the *y*-axis.

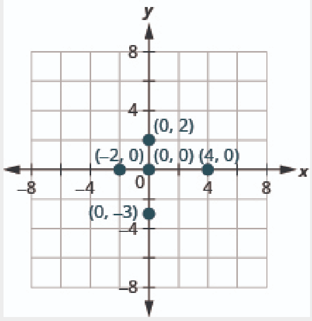


Note:
Exercise:

Problem: Plot each point:

ⓐ $(4,0)$ ⓑ $(-2,0)$ ⓒ $(0,0)$ ⓓ $(0,2)$ ⓔ $(0,-3)$.

Solution:

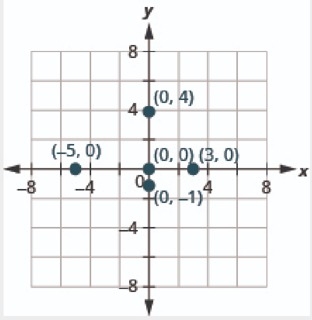


Note:
Exercise:

Problem: Plot each point:

ⓐ $(-5,0)$ ⓑ $(3,0)$ ⓒ $(0,0)$ ⓓ $(0,-1)$ ⓔ $(0,4)$.

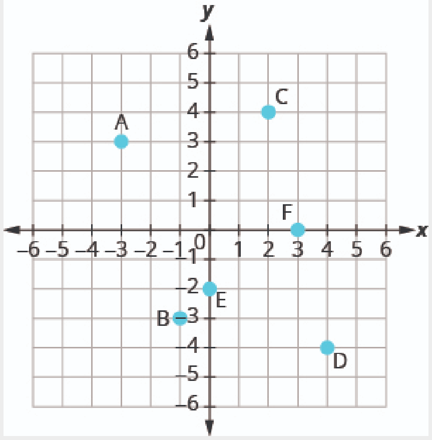
Solution:



In algebra, being able to identify the coordinates of a point shown on a graph is just as important as being able to plot points. To identify the *x*-coordinate of a point on a graph, read the number on the *x*-axis directly above or below the point. To identify the *y*-coordinate of a point, read the number on the *y*-axis directly to the left or right of the point. Remember, when you write the ordered pair use the correct order, (x,y).

Example:
Exercise:

Problem: Name the ordered pair of each point shown in the rectangular coordinate system.



Solution:
Solution

Point A is above -3 on the *x*-axis, so the *x*-coordinate of the point is -3.

- The point is to the left of 3 on the *y*-axis, so the *y*-coordinate of the point is 3.
- The coordinates of the point are $(-3, 3)$.

Point B is below -1 on the *x*-axis, so the *x*-coordinate of the point is -1.

- The point is to the left of -3 on the *y*-axis, so the *y*-coordinate of the point is -3.
- The coordinates of the point are $(-1, -3)$.

Point C is above 2 on the *x*-axis, so the *x*-coordinate of the point is 2.

- The point is to the right of 4 on the *y*-axis, so the *y*-coordinate of the point is 4.
- The coordinates of the point are $(2, 4)$.

Point D is below 4 on the *x*-axis, so the *x*-coordinate of the point is 4.

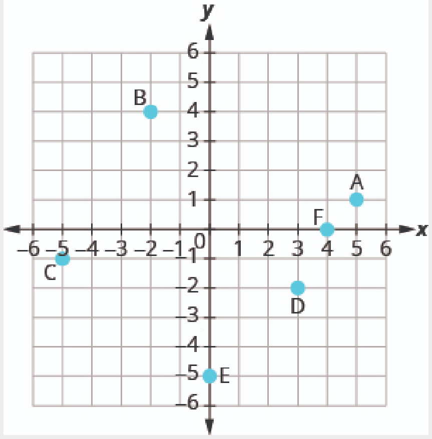
- The point is to the right of -4 on the *y*-axis, so the *y*-coordinate of the point is -4.
- The coordinates of the point are $(4, -4)$.

Point E is on the *y*-axis at $y = -2$. The coordinates of point E are $(0, -2)$.

Point F is on the *x*-axis at $x = 3$. The coordinates of point F are $(3, 0)$.

Note:
Exercise:

Problem: Name the ordered pair of each point shown in the rectangular coordinate system.

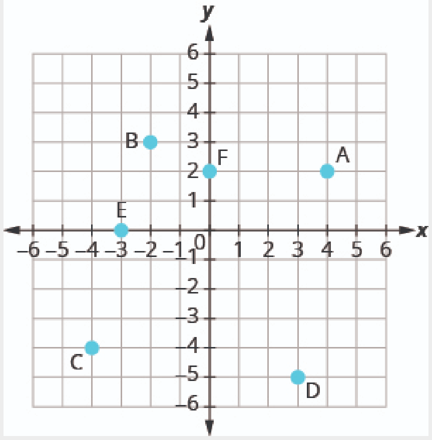


Solution:

A: $(5, 1)$ B: $(-2, 4)$ C: $(-5, -1)$ D: $(3, -2)$ E: $(0, -5)$ F: $(4, 0)$

Note:
Exercise:

Problem: Name the ordered pair of each point shown in the rectangular coordinate system.



Solution:

A: $(4, 2)$ B: $(-2, 3)$ C: $(-4, -4)$ D: $(3, -5)$ E: $(-3, 0)$ F: $(0, 2)$

Verify Solutions to an Equation in Two Variables

Up to now, all the equations you have solved were equations with just one variable. In almost every case, when you solved the equation you got exactly one solution. The process of solving an equation ended with a statement like $x = 4$. (Then, you checked the solution by substituting back into the equation.)

Here’s an example of an equation in one variable, and its one solution.
Equation:

$$\begin{aligned} 3x + 5 &= 17 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

But equations can have more than one variable. Equations with two variables may be of the form $Ax + By = C$. Equations of this form are called **linear equations in two variables**.

Note:
Linear Equation
An equation of the form $Ax + By = C$, where A and B are not both zero, is called a **linear equation in two variables**.

Notice the word *line* in **linear**. Here is an example of a linear equation in two variables, x and y.

$$Ax + By = C$$
$$x + 4y = 8$$
$$A = 1,\ B = 4,\ C = 8$$

The equation $y = -3x + 5$ is also a linear equation. But it does not appear to be in the form $Ax + By = C$. We can use the Addition Property of Equality and rewrite it in $Ax + By = C$ form.

	$y = -3x + 5$
Add to both sides.	$y + 3x = -3x + 5 + 3x$
Simplify.	$y + 3x = 5$
Use the Commutative Property to put it in $Ax + By = C$ form.	$3x + y = 5$

By rewriting $y = -3x + 5$ as $3x + y = 5$, we can easily see that it is a linear equation in two variables because it is of the form $Ax + By = C$. When an equation is in the form $Ax + By = C$, we say it is in *standard form*.

Note:
Standard Form of Linear Equation
A linear equation is in standard form when it is written $Ax + By = C$.

Most people prefer to have A, B, and C be integers and $A \geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

Linear equations have infinitely many solutions. For every number that is substituted for x there is a corresponding y value. This pair of values is a *solution* to the linear equation and is represented by the ordered pair (x, y). When we substitute these values of x and y into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

Note:
Solution of a Linear Equation in Two Variables
An **ordered pair** (x, y) is a **solution** of the linear equation $Ax + By = C$, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

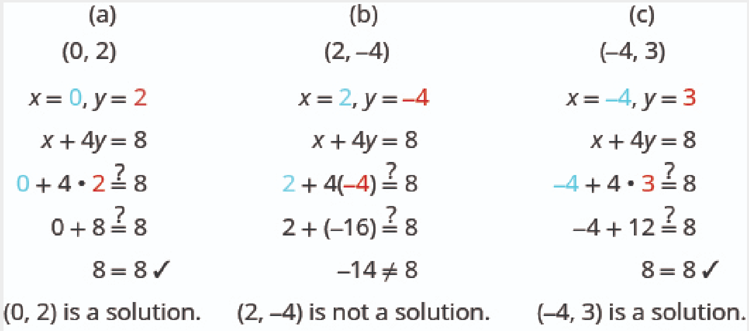
Example:
Exercise:

Problem: Determine which ordered pairs are solutions to the equation $x + 4y = 8$.

ⓐ $(0, 2)$ ⓑ $(2, -4)$ ⓒ $(-4, 3)$

Solution:
Solution

Substitute the *x*- and *y*-values from each ordered pair into the equation and determine if the result is a true statement.



(a)	(b)	(c)
(0, 2)	(2, –4)	(–4, 3)
$x=0,\ y=2$	$x=2,\ y=-4$	$x=-4,\ y=3$
$x+4y=8$	$x+4y=8$	$x+4y=8$
$0+4\cdot 2\overset{?}{=}8$	$2+4(-4)\overset{?}{=}8$	$-4+4\cdot 3\overset{?}{=}8$
$0+8\overset{?}{=}8$	$2+(-16)\overset{?}{=}8$	$-4+12\overset{?}{=}8$
$8=8$ ✓	$-14\neq 8$	$8=8$ ✓
(0, 2) is a solution.	(2, –4) is not a solution.	(–4, 3) is a solution.

Note:
Exercise:

Problem: Which of the following ordered pairs are solutions to $2x+3y=6$?

ⓐ $(3,0)$ ⓑ $(2,0)$ ⓒ $(6,-2)$

Solution:

a, c

Note:
Exercise:

Problem: Which of the following ordered pairs are solutions to the equation $4x-y=8$?

ⓐ $(0,8)$ ⓑ $(2,0)$ ⓒ $(1,-4)$

Solution:

b, c

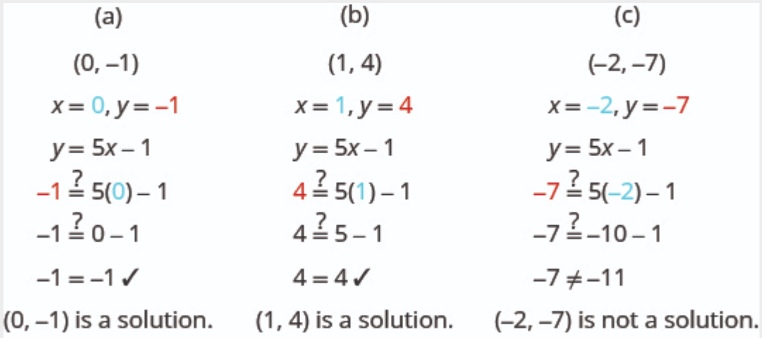
Example:
Exercise:

Problem: Which of the following ordered pairs are solutions to the equation $y=5x-1$?

ⓐ $(0,-1)$ ⓑ $(1,4)$ ⓒ $(-2,-7)$

Solution:
Solution

Substitute the *x*- and *y*-values from each ordered pair into the equation and determine if it results in a true statement.



(a)	(b)	(c)
$(0, -1)$	$(1, 4)$	$(-2, -7)$
$x = 0, y = -1$	$x = 1, y = 4$	$x = -2, y = -7$
$y = 5x - 1$	$y = 5x - 1$	$y = 5x - 1$
$-1 \overset{?}{=} 5(0) - 1$	$4 \overset{?}{=} 5(1) - 1$	$-7 \overset{?}{=} 5(-2) - 1$
$-1 \overset{?}{=} 0 - 1$	$4 \overset{?}{=} 5 - 1$	$-7 \overset{?}{=} -10 - 1$
$-1 = -1$ ✓	$4 = 4$ ✓	$-7 \neq -11$
$(0, -1)$ is a solution.	$(1, 4)$ is a solution.	$(-2, -7)$ is not a solution.

Note:
Exercise:

Problem: Which of the following ordered pairs are solutions to the equation $y = 4x - 3$?

ⓐ $(0, 3)$ ⓑ $(1, 1)$ ⓒ $(-1, -1)$

Solution:

b

Note:
Exercise:

Problem: Which of the following ordered pairs are solutions to the equation $y = -2x + 6$?

ⓐ $(0, 6)$ ⓑ $(1, 4)$ ⓒ $(-2, -2)$

Solution:

a, b

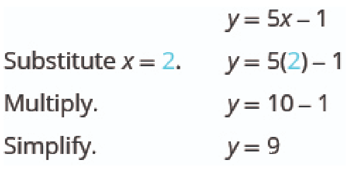
Complete a Table of Solutions to a Linear Equation in Two Variables

In the examples above, we substituted the *x*- and *y*-values of a given ordered pair to determine whether or not it was a solution to a linear equation. But how do you find the ordered pairs if they are not given? It’s easier than you might think—you can just pick a value for x and then solve the equation for y. Or, pick a value for y and then solve for x.

We’ll start by looking at the solutions to the equation $y = 5x - 1$ that we found in [link]. We can summarize this information in a table of solutions, as shown in [link].

$y=5x-1$		
x	y	(x,y)
0	-1	$(0,-1)$
1	4	$(1,4)$

To find a third solution, we'll let $x=2$ and solve for y.



	$y=5x-1$
Substitute $x=2$.	$y=5(2)-1$
Multiply.	$y=10-1$
Simplify.	$y=9$

The ordered pair $(2,9)$ is a solution to $y=5x-1$. We will add it to [link].

$y=5x-1$		
x	y	(x,y)
0	-1	$(0,-1)$
1	4	$(1,4)$
2	9	$(2,9)$

We can find more solutions to the equation by substituting in any value of x or any value of y and solving the resulting equation to get another ordered pair that is a solution. There are infinitely many solutions of this equation.

Example:
Exercise:

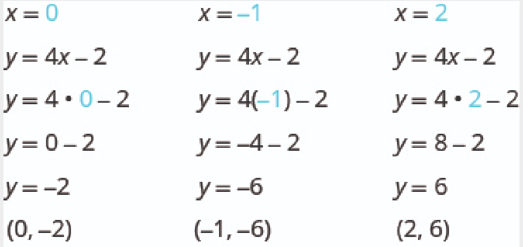
Problem: Complete [link] to find three solutions to the equation $y=4x-2$.

$y=4x-2$		
x	y	(x,y)

0		
-1		
2		

Solution:
Solution

Substitute $x = 0$, $x = -1$, and $x = 2$ into $y = 4x - 2$.



$x = 0$	$x = -1$	$x = 2$
$y = 4x - 2$	$y = 4x - 2$	$y = 4x - 2$
$y = 4 \cdot 0 - 2$	$y = 4(-1) - 2$	$y = 4 \cdot 2 - 2$
$y = 0 - 2$	$y = -4 - 2$	$y = 8 - 2$
$y = -2$	$y = -6$	$y = 6$
$(0, -2)$	$(-1, -6)$	$(2, 6)$

The results are summarized in [link].

$y = 4x - 2$		
x	y	(x, y)
0	-2	$(0, -2)$
-1	-6	$(-1, -6)$
2	6	$(2, 6)$

Note:
Exercise:

Problem: Complete the table to find three solutions to this equation: $y = 3x - 1$.

$y = 3x - 1$

x	y	(x,y)
0		
-1		
2		

Solution:

$y=3x-1$		
x	y	(x,y)
0	-1	$(0,-1)$
-1	-4	$(-1,-4)$
2	5	$(2,5)$

Note:
Exercise:

Problem: Complete the table to find three solutions to this equation: $y=6x+1$.

$y=6x+1$		
x	y	(x,y)
0		
1		
-2		

Solution:

$y = 6x + 1$		
x	y	(x, y)
0	1	$(0, 1)$
1	7	$(1, 7)$
-2	-11	$(-2, -11)$

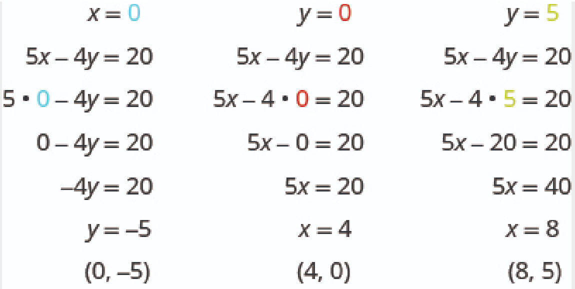
Example:
Exercise:

Problem: Complete [link] to find three solutions to the equation $5x - 4y = 20$.

$5x - 4y = 20$		
x	y	(x, y)
0		
	0	
	5	

Solution:
Solution

Substitute the given value into the equation $5x - 4y = 20$ and solve for the other variable. Then, fill in the values in the table.



$x = 0$	$y = 0$	$y = 5$
$5x - 4y = 20$	$5x - 4y = 20$	$5x - 4y = 20$
$5 \cdot 0 - 4y = 20$	$5x - 4 \cdot 0 = 20$	$5x - 4 \cdot 5 = 20$
$0 - 4y = 20$	$5x - 0 = 20$	$5x - 20 = 20$
$-4y = 20$	$5x = 20$	$5x = 40$
$y = -5$	$x = 4$	$x = 8$
$(0, -5)$	$(4, 0)$	$(8, 5)$

The results are summarized in [link].

$5x - 4y = 20$		
x	y	(x, y)
0	-5	$(0, -5)$
4	0	$(4, 0)$
8	5	$(8, 5)$

Note:
Exercise:

Problem: Complete the table to find three solutions to this equation: $2x - 5y = 20$.

$2x - 5y = 20$		
x	y	(x, y)
0		
	0	
-5		

Solution:

$2x-5y=20$		
x	y	(x,y)
0	-4	$(0,-4)$
10	0	$(10,0)$
-5	-6	$(-5,-6)$

Note:
Exercise:

Problem: Complete the table to find three solutions to this equation: $3x-4y=12$.

$3x-4y=12$		
x	y	(x,y)
0		
	0	
-4		

Solution:

$3x-4y=12$		
x	y	(x,y)
0	-3	$(0,-3)$
4	0	$(4,0)$
-4	-6	$(-4,-6)$

Find Solutions to a Linear Equation

To find a solution to a linear equation, you really can pick *any* number you want to substitute into the equation for x or y. But since you'll need to use that number to solve for the other variable it's a good idea to choose a number that's easy to work with.

When the equation is in *y*-form, with the *y* by itself on one side of the equation, it is usually easier to choose values of x and then solve for y.

Example:
Exercise:

Problem: Find three solutions to the equation $y = -3x + 2$.

Solution:
Solution

We can substitute any value we want for x or any value for y. Since the equation is in *y*-form, it will be easier to substitute in values of x. Let's pick $x = 0$, $x = 1$, and $x = -1$.

			$x = 0$	$x = 1$	$x = -1$
			$y = -3x + 2$	$y = -3x + 2$	$y = -3x + 2$
Substitute the value into the equation.			$y = -3 \cdot 0 + 2$	$y = -3 \cdot 1 + 2$	$y = -3(-1) + 2$
Simplify.			$y = 0 + 2$	$y = -3 + 2$	$y = 3 + 2$
Simplify.			$y = 2$	$y = -1$	$y = 5$
Write the ordered pair.			(0, 2)	(1, −1)	(−1, 5)
Check.					
$y = -3x + 2$	$y = -3x + 2$	$y = -3x + 2$			

$2 \overset{?}{=} -3 \cdot 0 + 2$	$-1 \overset{?}{=} -3 \cdot 1 + 2$	$5 \overset{?}{=} -3(-1) + 2$			
$2 \overset{?}{=} 0 + 2$	$-1 \overset{?}{=} -3 + 2$	$5 \overset{?}{=} 3 + 2$			
$2 = 2 \checkmark$	$-1 = -1 \checkmark$	$5 = 5 \checkmark$			

So, $(0, 2)$, $(1, -1)$ and $(-1, 5)$ are all solutions to $y = -3x + 2$. We show them in [link].

$y = -3x + 2$		
x	y	(x, y)
0	2	$(0, 2)$
1	-1	$(1, -1)$
-1	5	$(-1, 5)$

Note:
Exercise:

Problem: Find three solutions to this equation: $y = -2x + 3$.

Solution:

Answers will vary.

Note:
Exercise:

Problem: Find three solutions to this equation: $y = -4x + 1$.

Solution:

Answers will vary.

We have seen how using zero as one value of x makes finding the value of y easy. When an equation is in standard form, with both the x and y on the same side of the equation, it is usually easier to first find one solution when $x = 0$ find a second solution when $y = 0$, and then find a third solution.

Example:
Exercise:

Problem: Find three solutions to the equation $3x + 2y = 6$.

Solution:
Solution

We can substitute any value we want for x or any value for y. Since the equation is in standard form, let’s pick first $x = 0$, then $y = 0$, and then find a third point.

	$x = 0$	$y = 0$	$x = 1$
	$3x + 2y = 6$	$3x + 2y = 6$	$3x + 2y = 6$
Substitute the value into the equation.	$3(0) + 2y = 6$	$3x + 2(0) = 6$	$3(1) + 2y = 6$
Simplify.	$0 + 2y = 6$	$3x + 0 = 6$	$3 + 2y = 6$
Solve.	$2y = 6$	$3x = 6$	$2y = 3$
	$y = 3$	$x = 2$	$y = \frac{3}{2}$
Write the ordered pair.	(0, 3)	(2, 0)	$\left(1, \frac{3}{2}\right)$
Check.			

$3x + 2y = 6$	$3x + 2y = 6$	$3x + 2y = 6$
$3 \cdot 0 + 2 \cdot 3 \overset{?}{=} 6$	$3 \cdot 2 + 2 \cdot 0 \overset{?}{=} 6$	$3 \cdot 1 + 2 \cdot \frac{3}{2} \overset{?}{=} 6$
$0 + 6 \overset{?}{=} 6$	$6 + 0 \overset{?}{=} 6$	$3 + 3 \overset{?}{=} 6$
$6 = 6 \checkmark$	$6 = 6 \checkmark$	$6 = 6 \checkmark$

So $(0, 3)$, $(2, 0)$, and $\left(1, \frac{3}{2}\right)$ are all solutions to the equation $3x + 2y = 6$. We can list these three solutions in [link].

$3x+2y=6$		
x	y	(x,y)
0	3	$(0,3)$
2	0	$(2,0)$
1	$\frac{3}{2}$	$\left(1,\frac{3}{2}\right)$

Note:
Exercise:

Problem: Find three solutions to the equation $2x+3y=6$.

Solution:

Answers will vary.

Note:
Exercise:

Problem: Find three solutions to the equation $4x+2y=8$.

Solution:

Answers will vary.

Key Concepts

- **Sign Patterns of the Quadrants**

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x,y)	(x,y)	(x,y)	(x,y)
$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$

- **Points on the Axes**
    - On the *x*-axis, $y=0$. Points with a *y*-coordinate equal to 0 are on the *x*-axis, and have coordinates $(a,0)$.
    - On the *y*-axis, $x=0$. Points with an *x*-coordinate equal to 0 are on the *y*-axis, and have coordinates $(0,b)$.
- **Solution of a Linear Equation**

- An ordered pair (x, y) is a solution of the linear equation $Ax + By = C$, if the equation is a true statement when the *x*- and *y*- values of the ordered pair are substituted into the equation.

Practice Makes Perfect

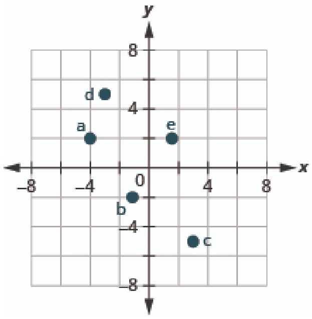
Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system and identify the quadrant in which the point is located.

Exercise:

ⓐ $(-4, 2)$
ⓑ $(-1, -2)$
ⓒ $(3, -5)$
ⓓ $(-3, 5)$
Problem: ⓔ $\left(\frac{5}{3}, 2\right)$

Solution:



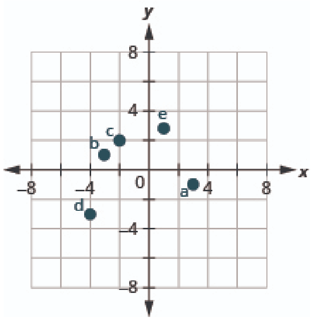
Exercise:

ⓐ $(-2, -3)$
ⓑ $(3, -3)$
ⓒ $(-4, 1)$
ⓓ $(4, -1)$
Problem: ⓔ $\left(\frac{3}{2}, 1\right)$

Exercise:

ⓐ $(3, -1)$
ⓑ $(-3, 1)$
ⓒ $(-2, 2)$
ⓓ $(-4, -3)$
Problem: ⓔ $\left(1, \frac{14}{5}\right)$

Solution:



Exercise:

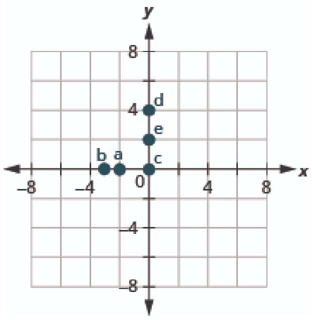
ⓐ $(-1, 1)$
ⓑ $(-2, -1)$
ⓒ $(2, 1)$
ⓓ $(1, -4)$
Problem: ⓔ $\left(3, \frac{7}{2}\right)$

In the following exercises, plot each point in a rectangular coordinate system.

Exercise:

ⓐ $(-2, 0)$
ⓑ $(-3, 0)$
ⓒ $(0, 0)$
ⓓ $(0, 4)$
Problem: ⓔ $(0, 2)$

Solution:



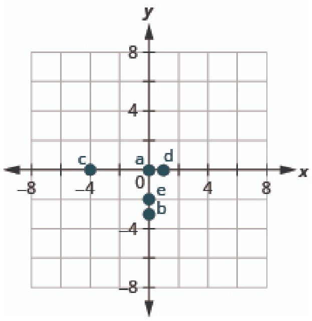
Exercise:

ⓐ $(0, 1)$
ⓑ $(0, -4)$
ⓒ $(-1, 0)$
ⓓ $(0, 0)$
Problem: ⓔ $(5, 0)$

Exercise:

ⓐ $(0,0)$
ⓑ $(0,-3)$
ⓒ $(-4,0)$
ⓓ $(1,0)$
Problem: ⓔ $(0,-2)$

Solution:



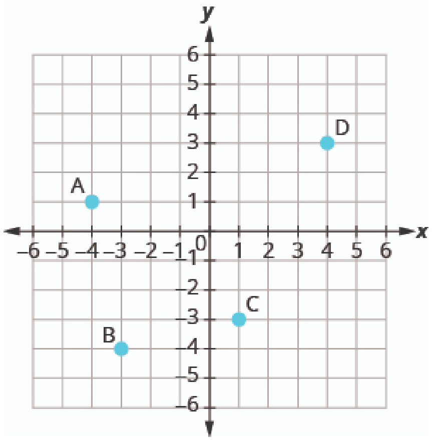
Exercise:

ⓐ $(-3,0)$
ⓑ $(0,5)$
ⓒ $(0,-2)$
ⓓ $(2,0)$
Problem: ⓔ $(0,0)$

In the following exercises, name the ordered pair of each point shown in the rectangular coordinate system.

Exercise:

Problem:

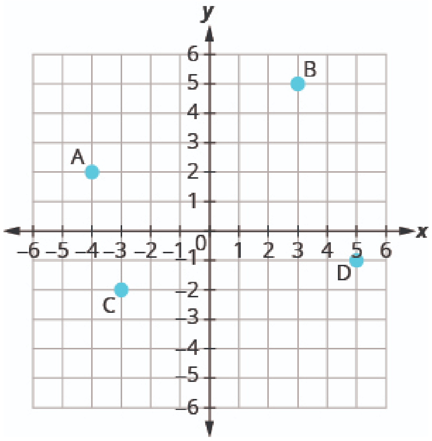


Solution:

A: $(-4,1)$ B: $(-3,-4)$ C: $(1,-3)$ D: $(4,3)$

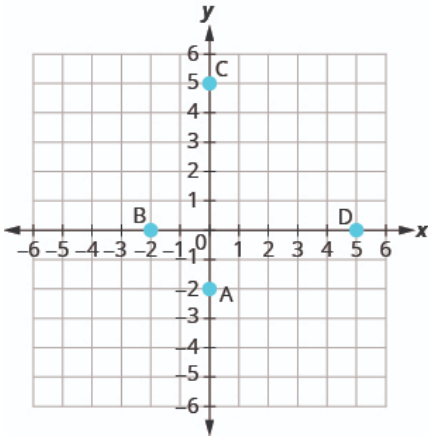
Exercise:

Problem:



Exercise:

Problem:

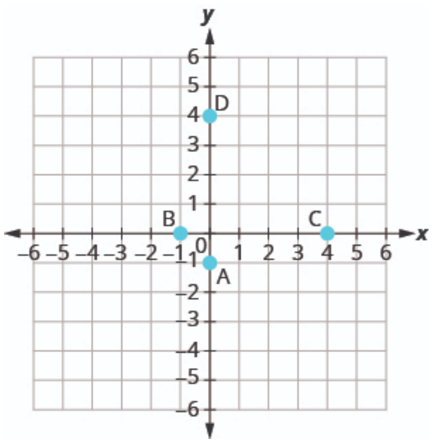


Solution:

A: $(0,-2)$ B: $(-2,0)$ C: $(0,5)$ D: $(5,0)$

Exercise:

Problem:



Verify Solutions to an Equation in Two Variables

In the following exercises, which ordered pairs are solutions to the given equations?

Exercise:

Problem: $2x+y=6$

ⓐ $(1,4)$
ⓑ $(3,0)$
ⓒ $(2,3)$

Solution:

a, b

Exercise:

Problem: $x+3y=9$

ⓐ $(0,3)$
ⓑ $(6,1)$
ⓒ $(-3,-3)$

Exercise:

Problem: $4x-2y=8$

ⓐ $(3,2)$
ⓑ $(1,4)$
ⓒ $(0,-4)$

Solution:

a, c

Exercise:

Problem: $3x-2y=12$

ⓐ $(4,0)$
ⓑ $(2,-3)$
ⓒ $(1,6)$

Exercise:

Problem: $y=4x+3$

ⓐ $(4,3)$
ⓑ $(-1,-1)$
ⓒ $\left(\frac{1}{2},5\right)$

Solution:

b, c

Exercise:

Problem: $y=2x-5$

ⓐ $(0,-5)$
ⓑ $(2,1)$
ⓒ $\left(\frac{1}{2},-4\right)$

Exercise:

Problem: $y=\frac{1}{2}x-1$

ⓐ $(2,0)$
ⓑ $(-6,-4)$
ⓒ $(-4,-1)$

Solution:

a, b

Exercise:

Problem: $y=\frac{1}{3}x+1$

ⓐ $(-3,0)$
ⓑ $(9,4)$
ⓒ $(-6,-1)$

Complete a Table of Solutions to a Linear Equation

In the following exercises, complete the table to find solutions to each linear equation.

Exercise:

Problem: $y=2x-4$

x	y	(x, y)
0		
2		
-1		

Solution:

x	y	(x, y)
0	-4	$(0, -4)$
2	0	$(2, 0)$
-1	-6	$(-1, -6)$

Exercise:

Problem: $y = 3x - 1$

x	y	(x, y)
0		
2		
-1		

Exercise:

Problem: $y = -x + 5$

x	y	(x, y)
0		
3		
-2		

Solution:

x	y	(x, y)
0	5	$(0, 5)$
3	2	$(3, 2)$
-2	7	$(-2, 7)$

Exercise:

Problem: $y = -x + 2$

x	y	(x, y)
0		
3		
-2		

Exercise:

Problem: $y = \frac{1}{3}x + 1$

x	y	(x, y)
0		
3		
6		

Solution:

x	y	(x, y)
0	1	$(0, 1)$
3	2	$(3, 2)$
6	3	$(6, 3)$

Exercise:

Problem: $y = \frac{1}{2}x + 4$

x	y	(x, y)
0		
2		
4		

Exercise:

Problem: $y = -\frac{3}{2}x - 2$

x	y	(x, y)
0		
2		
-2		

Solution:

x	y	(x, y)
0	-2	$(0, -2)$
2	-5	$(2, -5)$
-2	1	$(-2, 1)$

Exercise:

Problem: $y = -\frac{2}{3}x - 1$

x	y	(x, y)
0		
3		
-3		

Exercise:

Problem: $x + 3y = 6$

x	y	(x,y)
0		
3		
	0	

Solution:

x	y	(x,y)
0	2	$(0,2)$
3	4	$(3,1)$
6	0	$(6,0)$

Exercise:

Problem: $x+2y=8$

x	y	(x,y)
0		
4		
	0	

Exercise:

Problem: $2x-5y=10$

x	y	(x, y)
0		
10		
	0	

Solution:

x	y	(x, y)
0	-2	$(0, -2)$
10	2	$(10, 2)$
5	0	$(5, 0)$

Exercise:

Problem: $3x - 4y = 12$

x	y	(x, y)
0		
8		
	0	

Find Solutions to a Linear Equation

In the following exercises, find three solutions to each linear equation.

Exercise:

Problem: $y = 5x - 8$

Solution:

Answers will vary.

Exercise:

Problem: $y = 3x - 9$

Exercise:

Problem: $y = -4x + 5$

Solution:

Answers will vary.

Exercise:

Problem: $y = -2x + 7$

Exercise:

Problem: $x + y = 8$

Solution:

Answers will vary.

Exercise:

Problem: $x + y = 6$

Exercise:

Problem: $x + y = -2$

Solution:

Answers will vary.

Exercise:

Problem: $x + y = -1$

Exercise:

Problem: $3x + y = 5$

Solution:

Answers will vary.

Exercise:

Problem: $2x + y = 3$

Exercise:

Problem: $4x - y = 8$

Solution:

Answers will vary.

Exercise:

Problem: $5x - y = 10$

Exercise:

Problem: $2x + 4y = 8$

Solution:

Answers will vary.

Exercise:

Problem: $3x + 2y = 6$

Exercise:

Problem: $5x - 2y = 10$

Solution:

Answers will vary.

Exercise:

Problem: $4x - 3y = 12$

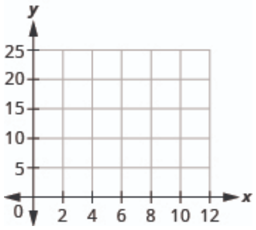
Everyday Math

Exercise:

Problem:

Weight of a baby. Mackenzie recorded her baby’s weight every two months. The baby’s age, in months, and weight, in pounds, are listed in the table below, and shown as an ordered pair in the third column.

ⓐ Plot the points on a coordinate plane.

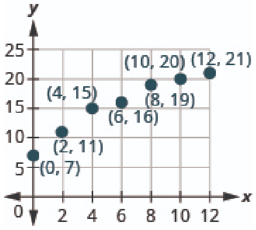


ⓑ Why is only Quadrant I needed?

Age x	**Weight** y	(x, y)
0	7	(0, 7)
2	11	(2, 11)
4	15	(4, 15)
6	16	(6, 16)
8	19	(8, 19)
10	20	(10, 20)
12	21	(12, 21)

Solution:

ⓐ



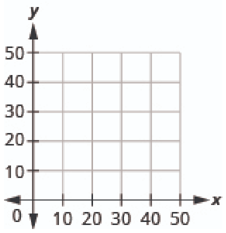
ⓑ Age and weight are only positive.

Exercise:

Problem:

Weight of a child. Latresha recorded her son’s height and weight every year. His height, in inches, and weight, in pounds, are listed in the table below, and shown as an ordered pair in the third column.

ⓐ Plot the points on a coordinate plane.



ⓑ Why is only Quadrant I needed?

Height x	**Weight** y	(x, y)
28	22	(28, 22)
31	27	(31, 27)
33	33	(33, 33)
37	35	(37, 35)
40	41	(40, 41)
42	45	(42, 45)

Writing Exercises

Exercise:

Problem: Explain in words how you plot the point $(4, -2)$ in a rectangular coordinate system.

Solution:

Answers will vary.

Exercise:

Problem: How do you determine if an ordered pair is a solution to a given equation?

Exercise:

Problem: Is the point $(-3, 0)$ on the *x*-axis or *y*-axis? How do you know?

Solution:

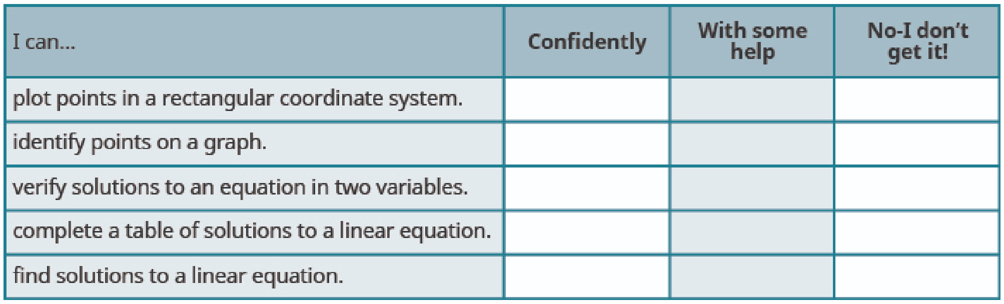
Answers will vary.

Exercise:

Problem: Is the point $(0, 8)$ on the *x*-axis or *y*-axis? How do you know?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
plot points in a rectangular coordinate system.			
identify points on a graph.			
verify solutions to an equation in two variables.			
complete a table of solutions to a linear equation.			
find solutions to a linear equation.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no, I don't get it. This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

linear equation

A linear equation is of the form $Ax + By = C$, where A and B are not both zero, is called a linear equation in two variables.

ordered pair

An ordered pair (x, y) gives the coordinates of a point in a rectangular coordinate system.

origin

The point $(0, 0)$ is called the origin. It is the point where the *x*-axis and *y*-axis intersect.

quadrant

The *x*-axis and the *y*-axis divide a plane into four regions, called quadrants.

rectangular coordinate system

A grid system is used in algebra to show a relationship between two variables; also called the *xy*-plane or the ‘coordinate plane’.

x-coordinate
The first number in an ordered pair (x, y).

y-coordinate
The second number in an ordered pair (x, y).

Graph Linear Equations in Two Variables
By the end of this section, you will be able to:

- Recognize the relationship between the solutions of an equation and its graph.
- Graph a linear equation by plotting points.
- Graph vertical and horizontal lines.

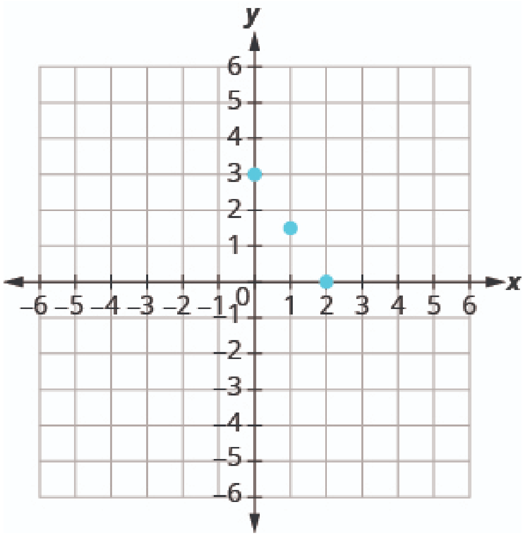
Note:
Before you get started, take this readiness quiz.

1. Evaluate $3x+2$ when $x=-1$.
   If you missed this problem, review [link].
2. Solve $3x+2y=12$ for y in general.
   If you missed this problem, review [link].

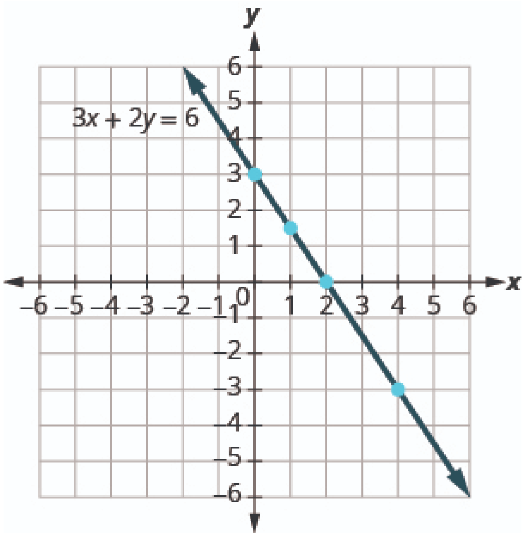
Recognize the Relationship Between the Solutions of an Equation and its Graph

In the previous section, we found several solutions to the equation $3x+2y=6$. They are listed in [link]. So, the ordered pairs $(0,3)$, $(2,0)$, and $\left(1,\frac{3}{2}\right)$ are some solutions to the equation $3x+2y=6$. We can plot these solutions in the rectangular coordinate system as shown in [link].

$3x+2y=6$		
x	y	(x,y)
0	3	$(0,3)$
2	0	$(2,0)$
1	$\frac{3}{2}$	$\left(1,\frac{3}{2}\right)$

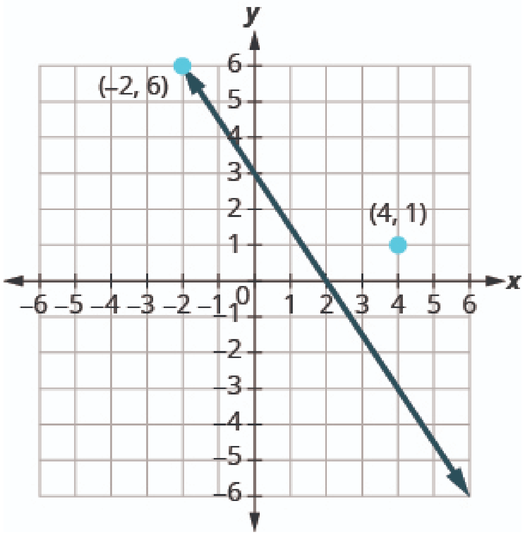


Notice how the points line up perfectly? We connect the points with a line to get the graph of the equation $3x + 2y = 6$. See [link]. Notice the arrows on the ends of each side of the line. These arrows indicate the line continues.



Every point on the line is a solution of the equation. Also, every solution of this equation is a point on this line. Points *not* on the line are not solutions.

Notice that the point whose coordinates are $(-2, 6)$ is on the line shown in [link]. If you substitute $x = -2$ and $y = 6$ into the equation, you find that it is a solution to the equation.



Test (–2, 6)

$$\begin{aligned} 3x+2y&=6\\ 3(-2)+2(6)&=6\\ -6+12&=6\\ 6&=6\ \checkmark \end{aligned}$$

So the point $(-2,6)$ is a solution to the equation $3x+2y=6$. (The phrase "the point whose coordinates are $(-2,6)$" is often shortened to "the point $(-2,6)$.")

What about (4, 1) ?

$$\begin{aligned} 3x+2y&=6\\ 3\cdot 4+2\cdot 1&=6\\ 12+2&\overset{?}{=}6\\ 14&\neq 6 \end{aligned}$$

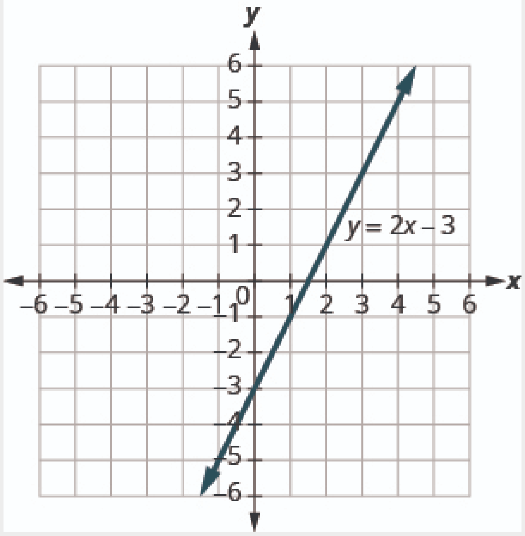
So $(4,1)$ is not a solution to the equation $3x+2y=6$. Therefore, the point $(4,1)$ is not on the line. See [link]. This is an example of the saying, "A picture is worth a thousand words." The line shows you *all* the solutions to the equation. Every point on the line is a solution of the equation. And, every solution of this equation is on this line. This line is called the *graph* of the equation $3x+2y=6$.

Note:
Graph of a Linear Equation
The **graph of a linear equation** $Ax+By=C$ is a line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

Example:
Exercise:

Problem: The graph of $y = 2x - 3$ is shown.



For each ordered pair, decide:

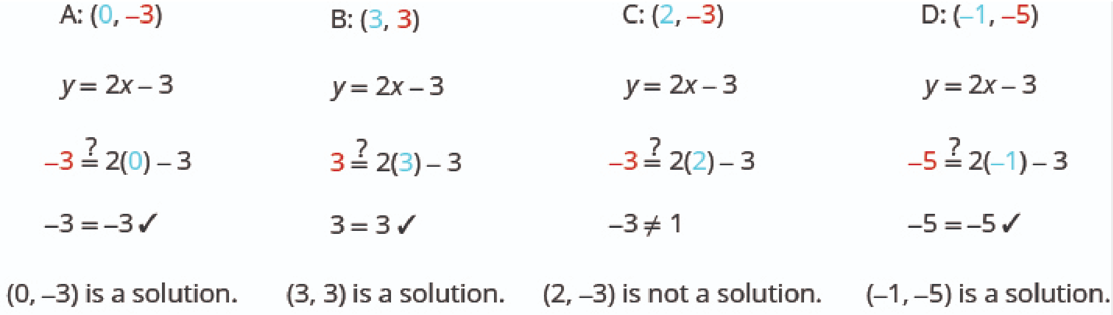
ⓐ Is the ordered pair a solution to the equation?
ⓑ Is the point on the line?

A $(0, -3)$ B $(3, 3)$ C $(2, -3)$ D $(-1, -5)$

Solution:
Solution

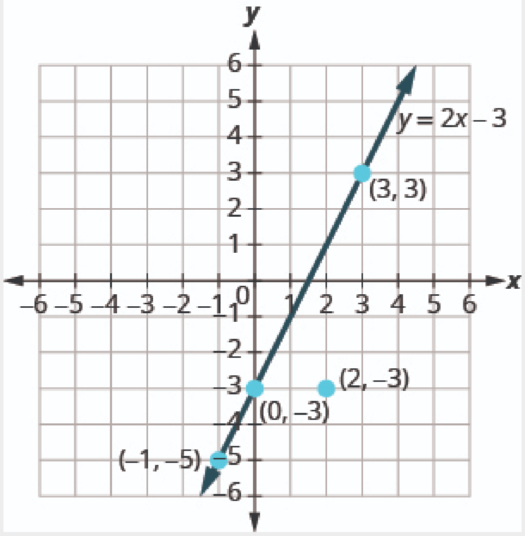
Substitute the *x*- and *y*- values into the equation to check if the ordered pair is a solution to the equation.

ⓐ



A: $(0, -3)$	B: $(3, 3)$	C: $(2, -3)$	D: $(-1, -5)$
$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$
$-3 \overset{?}{=} 2(0) - 3$	$3 \overset{?}{=} 2(3) - 3$	$-3 \overset{?}{=} 2(2) - 3$	$-5 \overset{?}{=} 2(-1) - 3$
$-3 = -3$ ✓	$3 = 3$ ✓	$-3 \neq 1$	$-5 = -5$ ✓
$(0, -3)$ is a solution.	$(3, 3)$ is a solution.	$(2, -3)$ is not a solution.	$(-1, -5)$ is a solution.

ⓑ Plot the points A $(0, 3)$, B $(3, 3)$, C $(2, -3)$, and D $(-1, -5)$.



The points $(0, 3)$, $(3, 3)$, and $(-1, -5)$ are on the line $y = 2x - 3$, and the point $(2, -3)$ is not on the line.

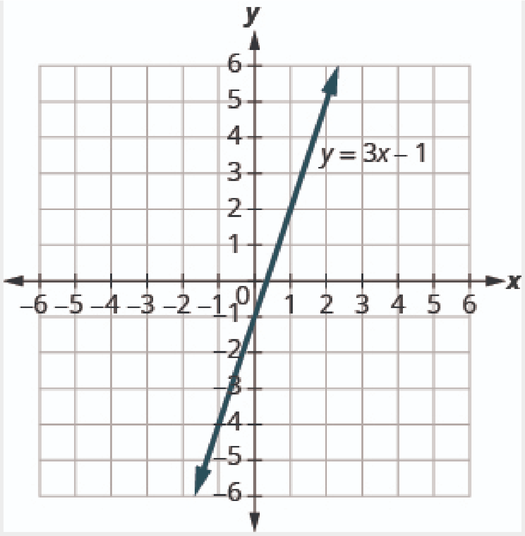
The points that are solutions to $y = 2x - 3$ are on the line, but the point that is not a solution is not on the line.

Note:
Exercise:

Problem: Use the graph of $y = 3x - 1$ to decide whether each ordered pair is:

- a solution to the equation.
- on the line.

ⓐ $(0, -1)$ ⓑ $(2, 5)$



Solution:

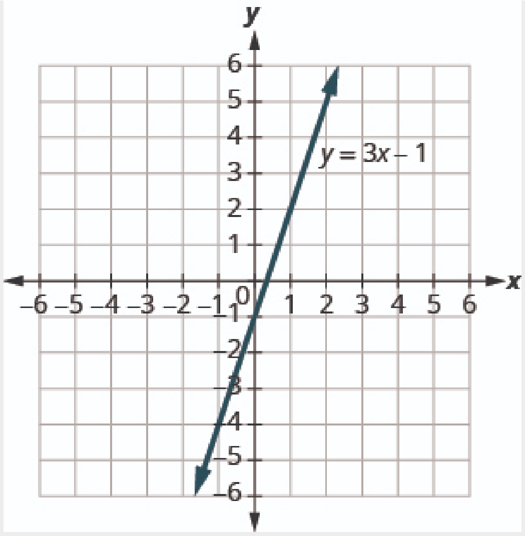
ⓐ yes, yes ⓑ yes, yes

Note:
Exercise:

Problem: Use graph of $y=3x-1$ to decide whether each ordered pair is:

- a solution to the equation
- on the line

ⓐ $(3,-1)$ ⓑ $(-1,-4)$



Solution:

ⓐ no, no ⓑ yes, yes

Graph a Linear Equation by Plotting Points

There are several methods that can be used to graph a linear equation. The method we used to graph $3x + 2y = 6$ is called plotting points, or the Point–Plotting Method.

Example:
How To Graph an Equation By Plotting Points
Exercise:

Problem: Graph the equation $y = 2x + 1$ by plotting points.

Solution:
Solution

Step 1. Find three points whose coordinates are solutions to the equation.

You can choose any values for x or y.

In this case, since y is isolated on the left side of the equation, it is easier to choose values for x.

$$y = 2x + 1$$

$$x = 0$$
$$y = 2x + 1$$
$$y = 2 \cdot 0 + 1$$
$$y = 0 + 1$$
$$y = 1$$

$$x = 1$$
$$y = 2x + 1$$
$$y = 2 \cdot 1 + 1$$
$$y = 2 + 1$$
$$y = 3$$

$$x = -2$$
$$y = 2x + 1$$
$$y = 2(-2) + 1$$
$$y = -4 + 1$$
$$y = -3$$

Organize the solutions in a table.

Put the three solutions in a table.

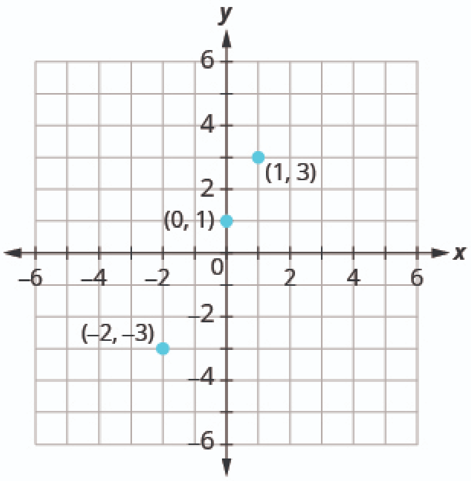
$y = 2x + 1$		
x	y	(x, y)
0	1	(0, 1)
1	3	(1, 3)
–2	–3	(–2, –3)

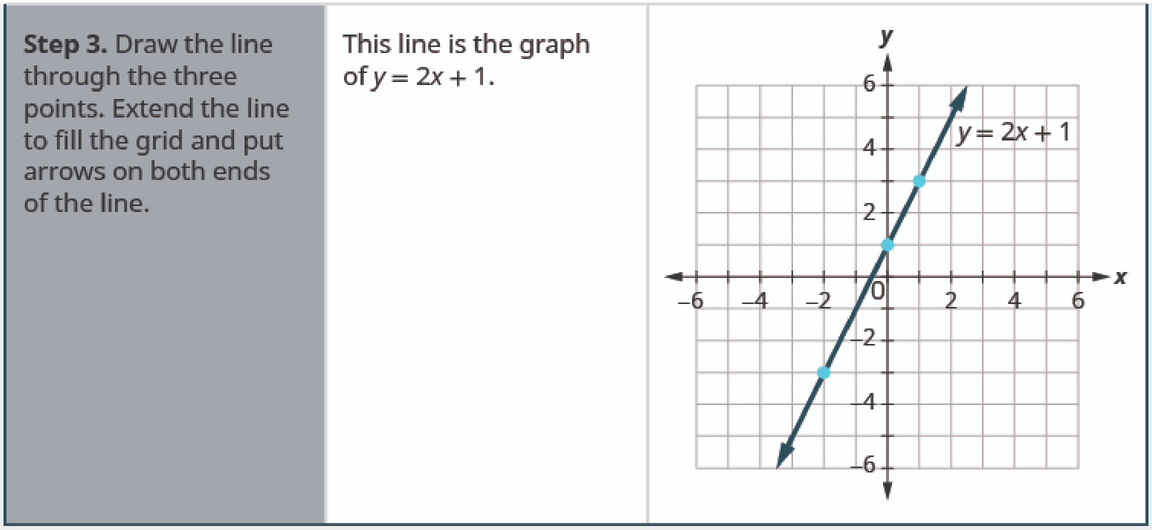
Step 2. Plot the points in a rectangular coordinate system.

Check that the points line up. If they do not, carefully check your work!

Plot:
(0, 1), (1, 3), (–2, –3).

Do the points line up?
Yes, the points line up.





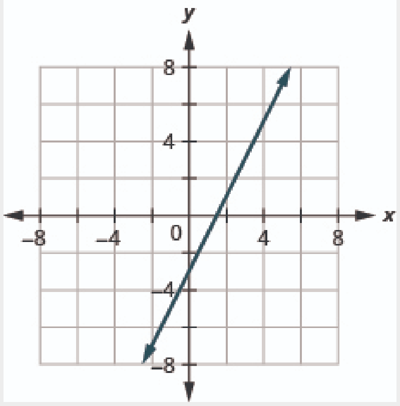
Step 3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.	This line is the graph of $y = 2x + 1$.	

y
x
y = 2x + 1
6
4
2
0
–2
–4
–6
–6
–4
–2
2
4
6

Note:
Exercise:

Problem: Graph the equation by plotting points: $y = 2x - 3$.

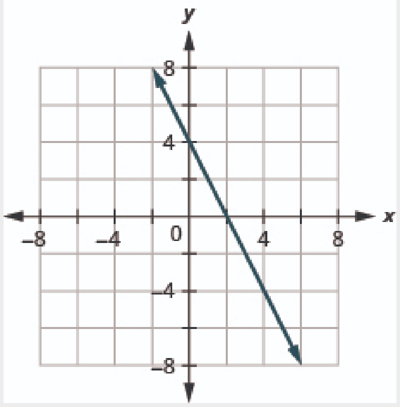
Solution:



Note:
Exercise:

Problem: Graph the equation by plotting points: $y = -2x + 4$.

Solution:



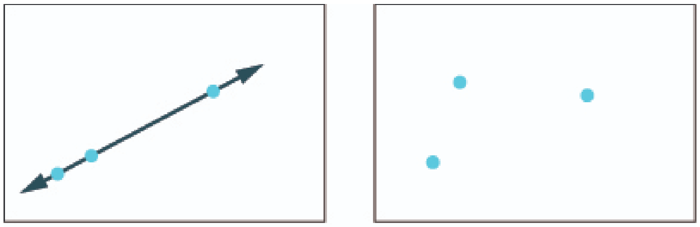
The steps to take when graphing a linear equation by plotting points are summarized below.

Note:
Graph a linear equation by plotting points.

Find three points whose coordinates are solutions to the equation. Organize them in a table.
Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If you only plot two points and one of them is incorrect, you can still draw a line but it will not represent the solutions to the equation. It will be the wrong line.

If you use three points, and one is incorrect, the points will not line up. This tells you something is wrong and you need to check your work. Look at the difference between part (a) and part (b) in [link].



(a) (b)

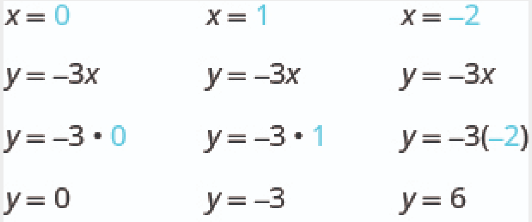
Let’s do another example. This time, we’ll show the last two steps all on one grid.

Example:
Exercise:

Problem: Graph the equation $y = -3x$.

Solution:
Solution

Find three points that are solutions to the equation. Here, again, it’s easier to choose values for x. Do you see why?



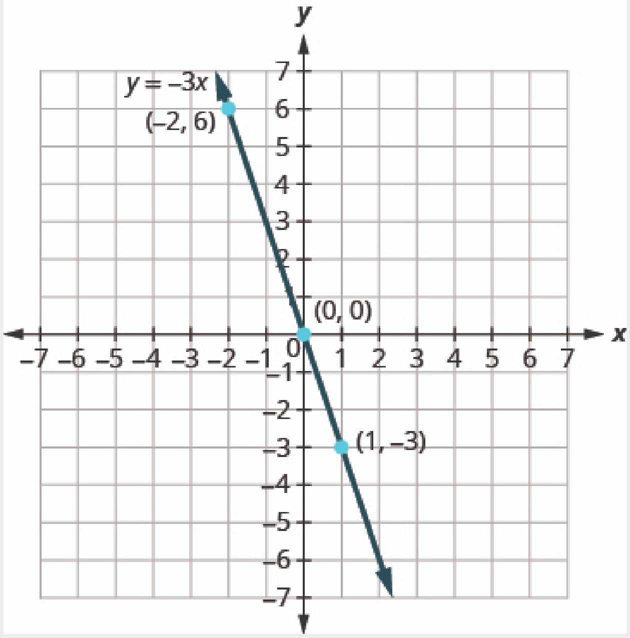
$x = 0$	$x = 1$	$x = -2$
$y = -3x$	$y = -3x$	$y = -3x$
$y = -3 \cdot 0$	$y = -3 \cdot 1$	$y = -3(-2)$
$y = 0$	$y = -3$	$y = 6$

We list the points in [link].

$y = -3x$		
x	y	(x, y)
0	0	$(0, 0)$
1	-3	$(1, -3)$

-2	6	$(-2, 6)$

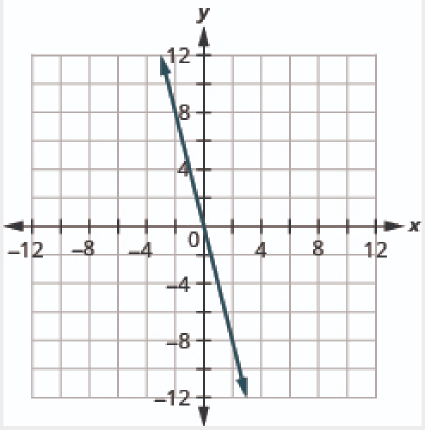
Plot the points, check that they line up, and draw the line.



Note:
Exercise:

Problem: Graph the equation by plotting points: $y = -4x$.

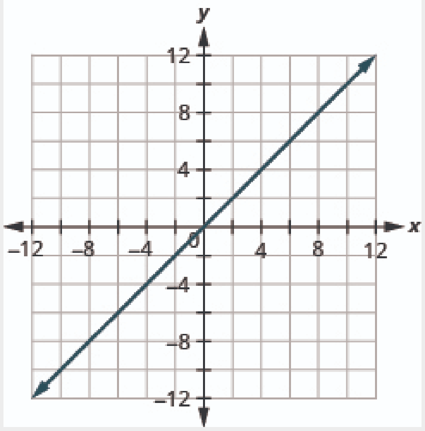
Solution:



Note:
Exercise:

Problem: Graph the equation by plotting points: $y = x$.

Solution:



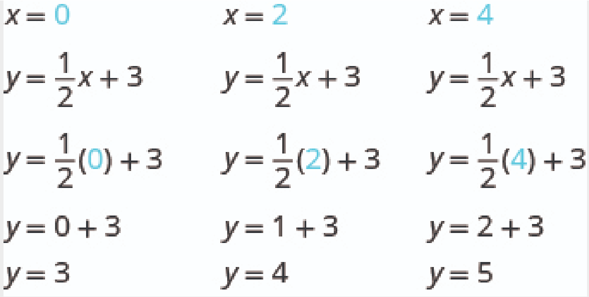
When an equation includes a fraction as the coefficient of x, we can still substitute any numbers for x. But the math is easier if we make ‘good’ choices for the values of x. This way we will avoid fraction answers, which are hard to graph precisely.

Example:
Exercise:

Problem: Graph the equation $y = \frac{1}{2}x + 3$.

Solution:
Solution

Find three points that are solutions to the equation. Since this equation has the fraction $\frac{1}{2}$ as a coefficient of x, we will choose values of x carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of 2 a good choice for values of x?

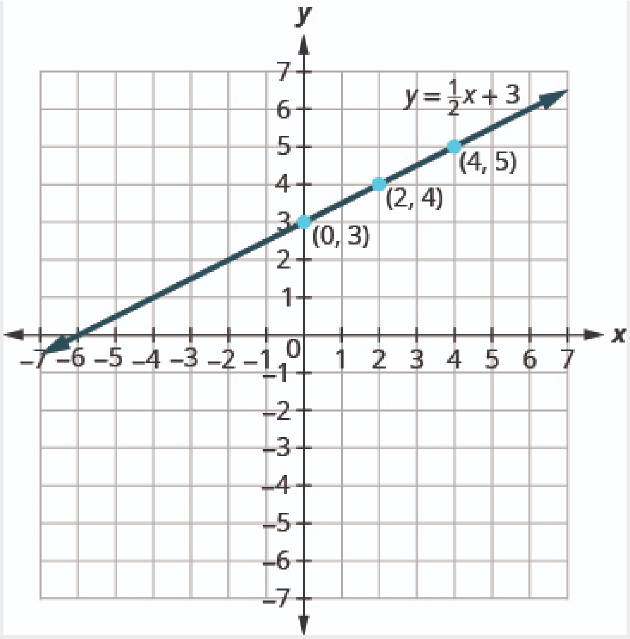


$x = 0$	$x = 2$	$x = 4$
$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$
$y = \frac{1}{2}(0) + 3$	$y = \frac{1}{2}(2) + 3$	$y = \frac{1}{2}(4) + 3$
$y = 0 + 3$	$y = 1 + 3$	$y = 2 + 3$
$y = 3$	$y = 4$	$y = 5$

The points are shown in [link].

$y = \frac{1}{2}x + 3$		
x	y	(x, y)
0	3	$(0, 3)$
2	4	$(2, 4)$
4	5	$(4, 5)$

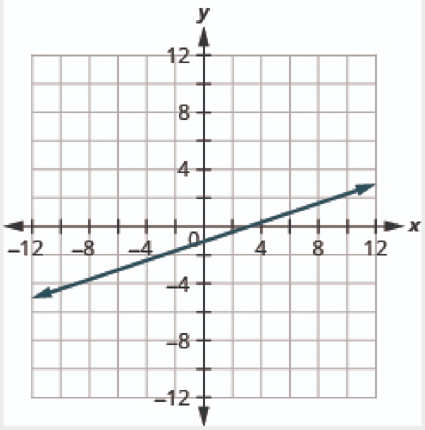
Plot the points, check that they line up, and draw the line.



Note:
Exercise:

Problem: Graph the equation $y = \frac{1}{3}x - 1$.

Solution:

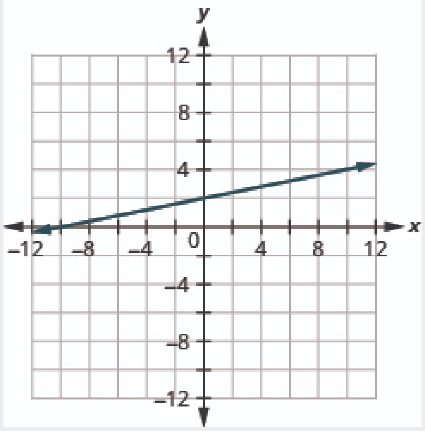


Note:

Exercise:

Problem: Graph the equation $y = \frac{1}{4}x + 2$.

Solution:



So far, all the equations we graphed had y given in terms of x. Now we'll graph an equation with x and y on the same side. Let's see what happens in the equation $2x + y = 3$. If $y = 0$ what is the value of x?

$$\begin{aligned} y &= {\color{red}0} \\ 2x + y &= 3 \\ 2x + {\color{red}0} &= 3 \\ 2x &= 3 \\ x &= \frac{3}{2} \\ &\left(\frac{3}{2}, 0\right) \end{aligned}$$

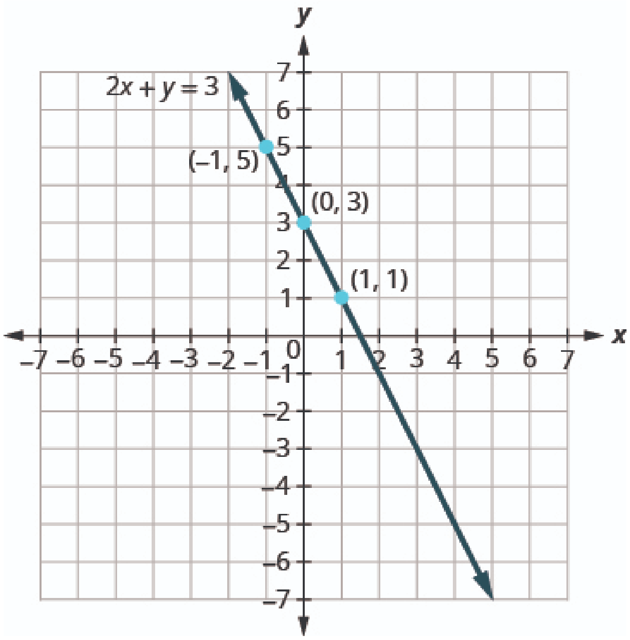
This point has a fraction for the *x*- coordinate and, while we could graph this point, it is hard to be precise graphing fractions. Remember in the example $y = \frac{1}{2}x + 3$, we carefully chose values for x so as not to graph fractions at all. If we solve the equation $2x + y = 3$ for y, it will be easier to find three solutions to the equation.

Equation:

$$\begin{aligned} 2x + y &= 3 \\ y &= -2x + 3 \end{aligned}$$

The solutions for $x = 0$, $x = 1$, and $x = -1$ are shown in the [link]. The graph is shown in [link].

$2x + y = 3$		
x	y	(x, y)
0	3	$(0, 3)$
1	1	$(1, 1)$
-1	5	$(-1, 5)$



Can you locate the point $\left(\frac{3}{2}, 0\right)$, which we found by letting $y = 0$, on the line?

Example:
Exercise:

Problem: Graph the equation $3x + y = -1$.

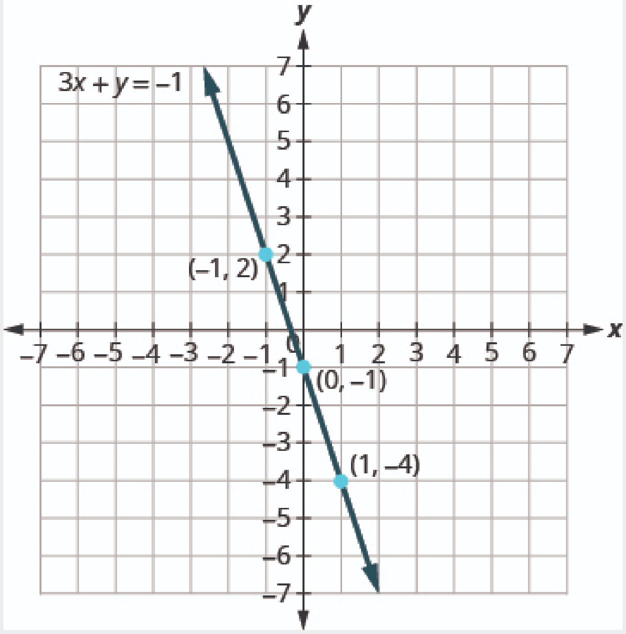
Solution:
Solution

Find three points that are solutions to the equation. $3x + y = -1$

First solve the equation for y. $y = -3x - 1$

We'll let x be 0, 1, and -1 to find 3 points. The ordered pairs are shown in [link]. Plot the points, check that they line up, and draw the line. See [link].

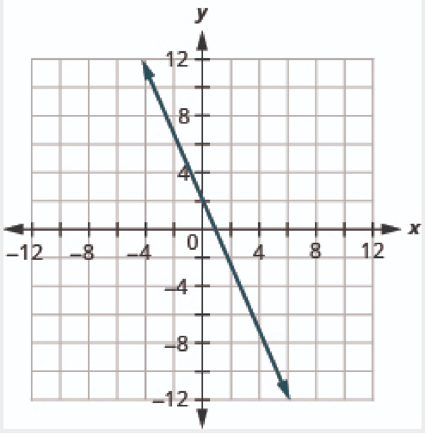
$3x + y = -1$		
x	y	(x, y)
0	-1	$(0, -1)$
1	-4	$(1, -4)$
-1	2	$(-1, 2)$



Note:
Exercise:

Problem: Graph the equation $2x + y = 2$.

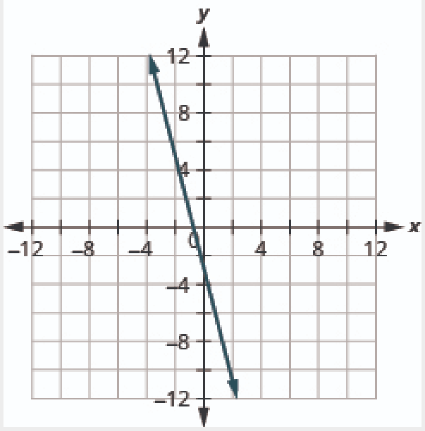
Solution:



Note:
Exercise:

Problem: Graph the equation $4x + y = -3$.

Solution:



If you can choose any three points to graph a line, how will you know if your graph matches the one shown in the answers in the book? If the points where the graphs cross the *x*- and *y*-axis are the same, the graphs match!

The equation in [link] was written in standard form, with both x and y on the same side. We solved that equation for y in just one step. But for other equations in standard form it is not that easy to solve for y, so we will leave them in standard form. We can still find a first point to plot by letting $x = 0$ and solving for y. We can plot a second point by letting $y = 0$ and then solving for x. Then we will plot a third point by using some other value for x or y.

Example:
Exercise:

Problem: Graph the equation $2x - 3y = 6$.

Solution:
Solution

Find three points that are solutions to the equation.

$$2x - 3y = 6$$

First let $x = 0$.

Solve for y.

$$\begin{aligned} 2(0) - 3y &= 6 \\ -3y &= 6 \\ y &= -2 \end{aligned}$$

Now let $y = 0$.

Solve for x.

$$\begin{aligned} 2x - 3(0) &= 6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

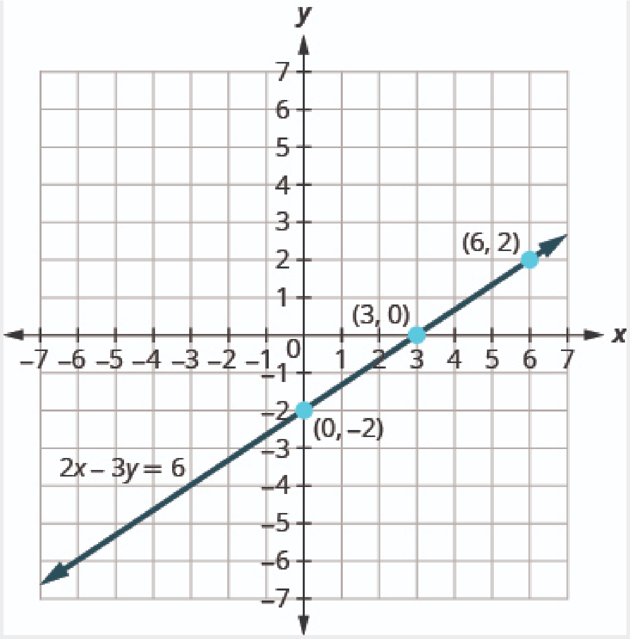
We need a third point. Remember, we can choose any value for x or y. We'll let $x = 6$.

Solve for y.

$$\begin{aligned} 2(6) - 3y &= 6 \\ 12 - 3y &= 6 \\ -3y &= -6 \\ y &= 2 \end{aligned}$$

We list the ordered pairs in [link]. Plot the points, check that they line up, and draw the line. See [link].

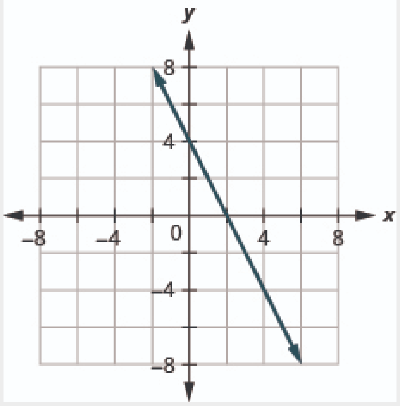
$2x - 3y = 6$		
x	y	(x, y)
0	-2	$(0, -2)$
3	0	$(3, 0)$
6	2	$(6, 2)$



Note:
Exercise:

Problem: Graph the equation $4x + 2y = 8$.

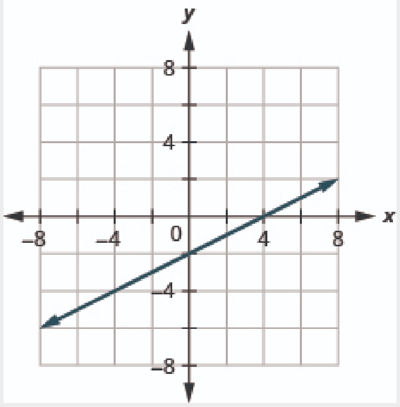
Solution:



Note:
Exercise:

Problem: Graph the equation $2x - 4y = 8$.

Solution:



Graph Vertical and Horizontal Lines

Can we graph an equation with only one variable? Just x and no y, or just y without an x ? How will we make a table of values to get the points to plot?

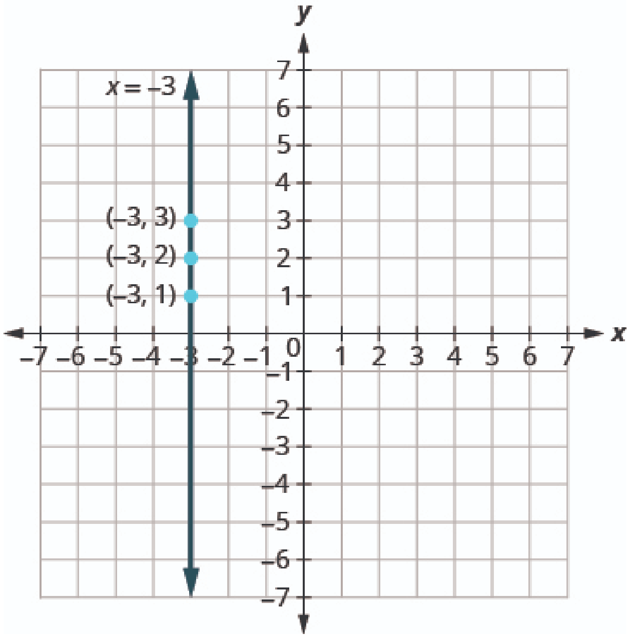
Let’s consider the equation $x = -3$. This equation has only one variable, x. The equation says that x is *always* equal to -3, so its value does not depend on y. No matter what y is, the value of x is always -3.

So to make a table of values, write -3 in for all the x values. Then choose any values for y. Since x does not depend on y, you can choose any numbers you like. But to fit the points on our coordinate graph, we’ll use 1, 2, and 3 for the *y*-coordinates. See [link].

$x = -3$		
x	y	(x, y)
-3	1	$(-3, 1)$

-3	2	$(-3, 2)$
-3	3	$(-3, 3)$

Plot the points from [link] and connect them with a straight line. Notice in [link] that we have graphed a *vertical line*.



Note:
Vertical Line
A **vertical line** is the graph of an equation of the form $x = a$.
The line passes through the *x*-axis at $(a, 0)$.

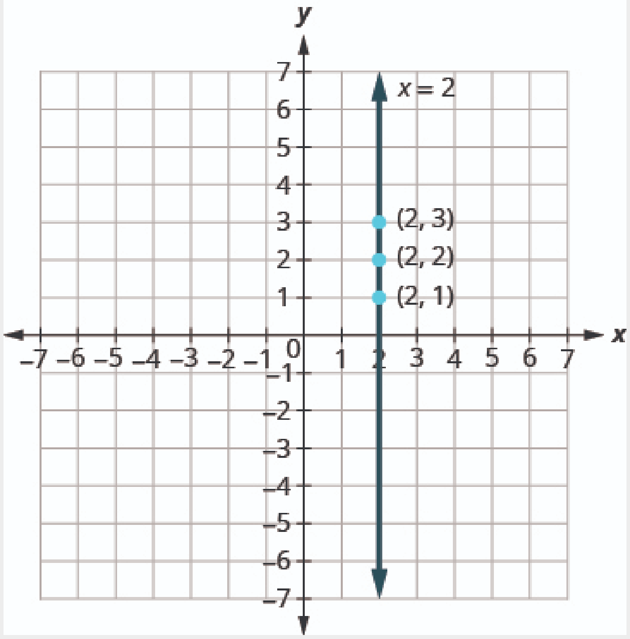
Example:
Exercise:

Problem: Graph the equation $x = 2$.

Solution:
Solution

The equation has only one variable, x, and x is always equal to 2. We create [link] where x is always 2 and then put in any values for y. The graph is a vertical line passing through the x-axis at 2. See [link].

$x = 2$		
x	y	(x, y)
2	1	$(2, 1)$
2	2	$(2, 2)$
2	3	$(2, 3)$

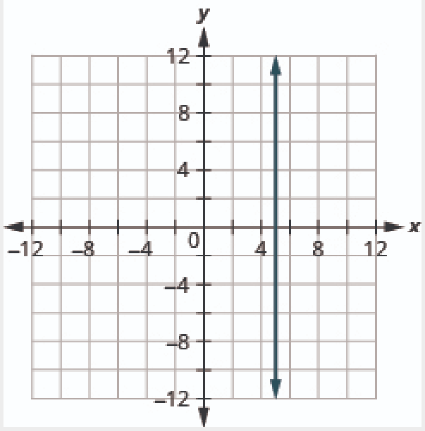


Note:

Exercise:

Problem: Graph the equation $x = 5$.

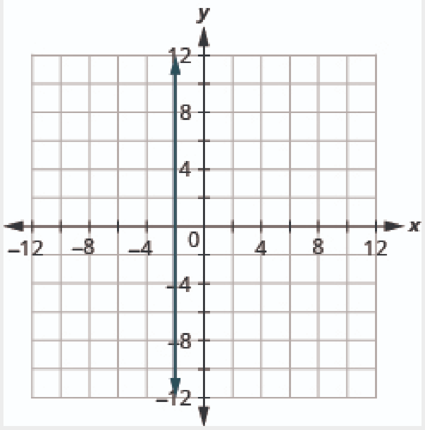
Solution:



Note:
Exercise:

Problem: Graph the equation $x = -2$.

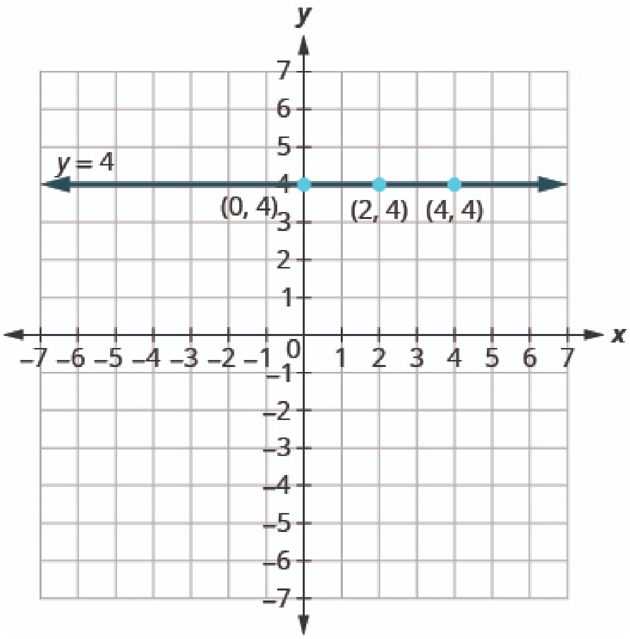
Solution:



What if the equation has y but no x? Let's graph the equation $y=4$. This time the *y*-value is a constant, so in this equation, y does not depend on x. Fill in 4 for all the y's in [link] and then choose any values for x. We'll use 0, 2, and 4 for the *x*-coordinates.

$y=4$		
x	y	(x,y)
0	4	$(0,4)$
2	4	$(2,4)$
4	4	$(4,4)$

The graph is a horizontal line passing through the *y*-axis at 4. See [link].



Note:

Horizontal Line

A **horizontal line** is the graph of an equation of the form $y = b$.

The line passes through the *y*-axis at $(0, b)$.

Example:

Exercise:

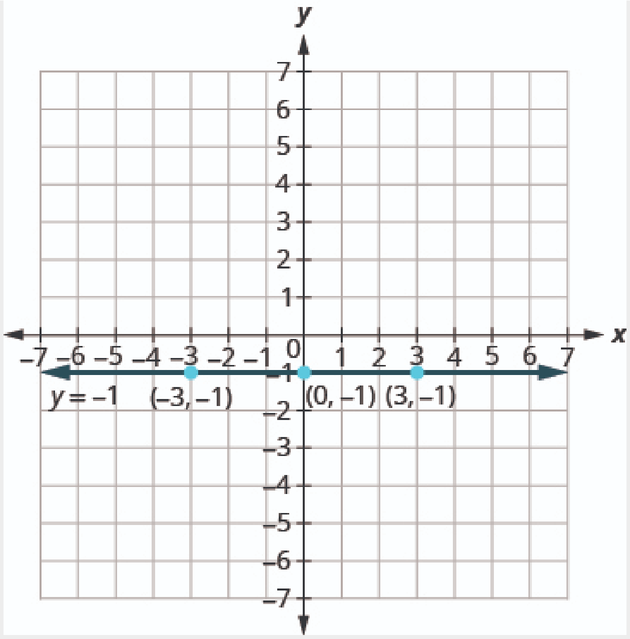
Problem: Graph the equation $y = -1$.

Solution:

Solution

The equation $y = -1$ has only one variable, y. The value of y is constant. All the ordered pairs in [link] have the same *y*-coordinate. The graph is a horizontal line passing through the *y*-axis at -1, as shown in [link].

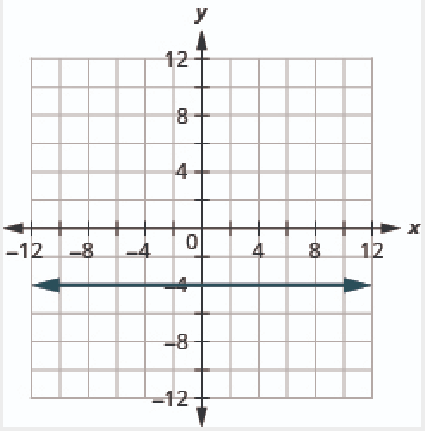
$y = -1$		
x	y	(x, y)
0	-1	$(0, -1)$
3	-1	$(3, -1)$
-3	-1	$(-3, -1)$



Note:
Exercise:

Problem: Graph the equation $y = -4$.

Solution:

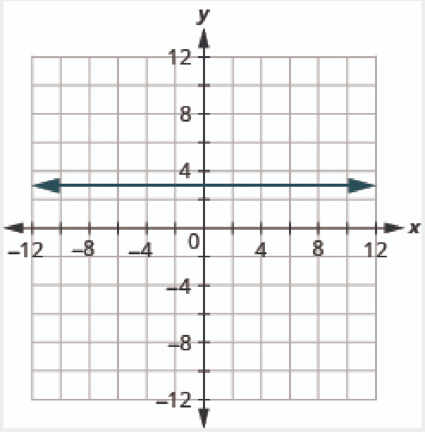


Note:

Exercise:

Problem: Graph the equation $y = 3$.

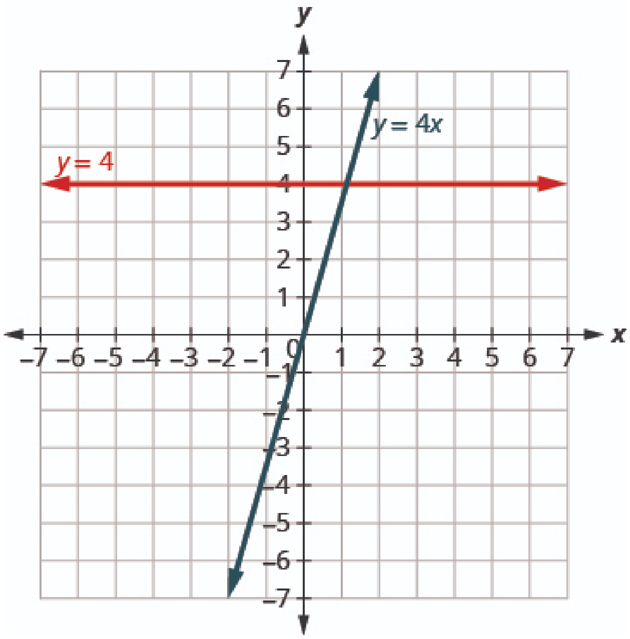
Solution:



The equations for vertical and horizontal lines look very similar to equations like $y = 4x$. What is the difference between the equations $y = 4x$ and $y = 4$?

The equation $y = 4x$ has both x and y. The value of y depends on the value of x. The y-coordinate changes according to the value of x. The equation $y = 4$ has only one variable. The value of y is constant. The y-coordinate is always 4. It does not depend on the value of x. See [link].

$y = 4x$				$y = 4$		
x	y	(x, y)		x	y	(x, y)
0	0	$(0, 0)$		0	4	$(0, 4)$
1	4	$(1, 4)$		1	4	$(1, 4)$
2	8	$(2, 8)$		2	4	$(2, 4)$



Notice, in [link], the equation $y = 4x$ gives a slanted line, while $y = 4$ gives a horizontal line.

Example:
Exercise:

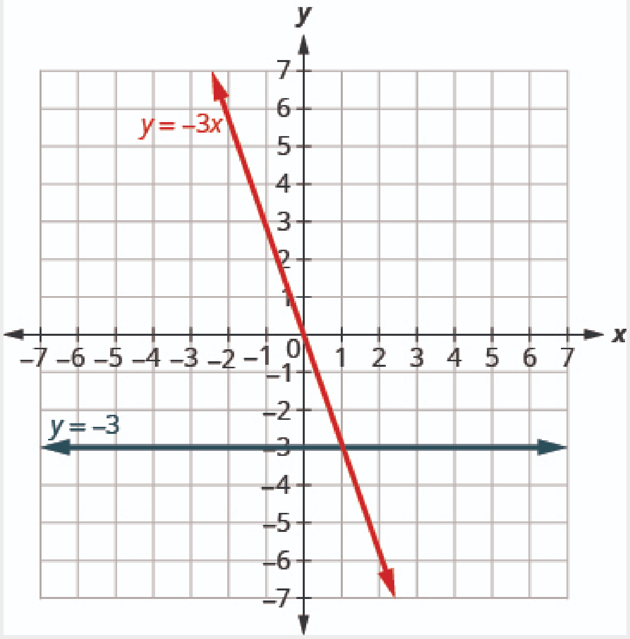
Problem: Graph $y = -3x$ and $y = -3$ in the same rectangular coordinate system.

Solution:
Solution

Notice that the first equation has the variable x, while the second does not. See [link]. The two graphs are shown in [link].

$y = -3x$				$y = -3$		
x	y	(x, y)		x	y	(x, y)
0	0	$(0, 0)$		0	-3	$(0, -3)$

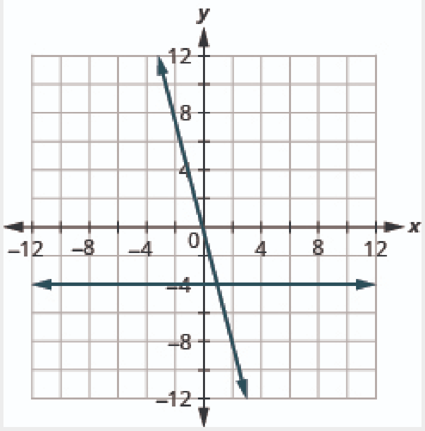
1	-3	$(1, -3)$		1	-3	$(1, -3)$
2	-6	$(2, -6)$		2	-3	$(2, -3)$



Note:
Exercise:

Problem: Graph $y = -4x$ and $y = -4$ in the same rectangular coordinate system.

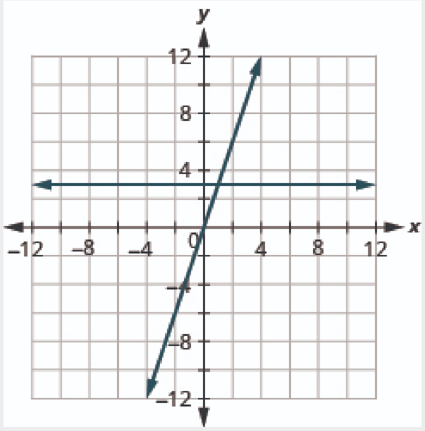
Solution:



Note:
Exercise:

Problem: Graph $y = 3$ and $y = 3x$ in the same rectangular coordinate system.

Solution:



Key Concepts

- **Graph a Linear Equation by Plotting Points**

  Find three points whose coordinates are solutions to the equation. Organize them in a table.
  Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work!
  Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

Practice Makes Perfect

Recognize the Relationship Between the Solutions of an Equation and its Graph

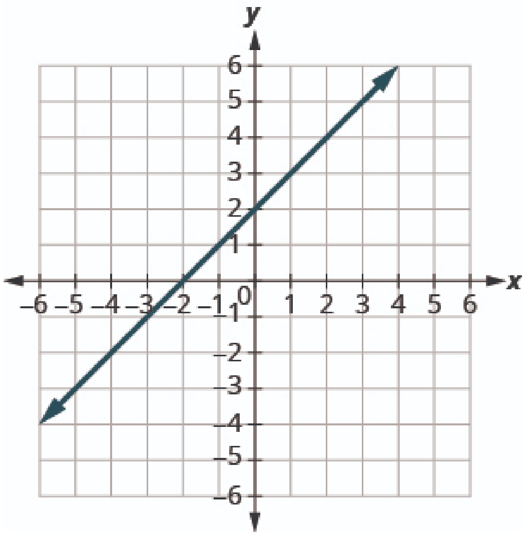
In the following exercises, for each ordered pair, decide:

ⓐ Is the ordered pair a solution to the equation? ⓑ Is the point on the line?

Exercise:

Problem: $y = x + 2$

ⓐ $(0, 2)$
ⓑ $(1, 2)$
ⓒ $(-1, 1)$
ⓓ $(-3, -1)$



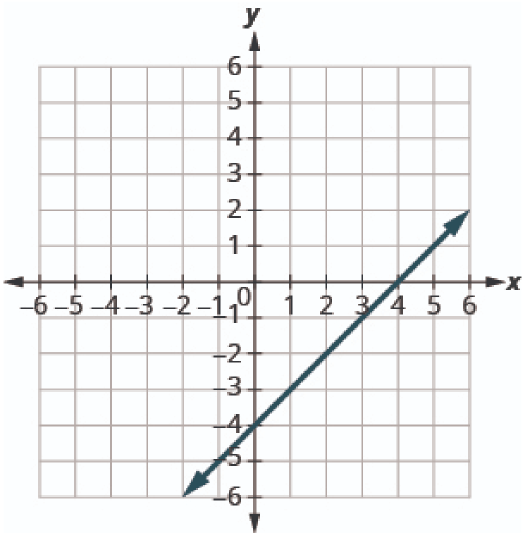
Solution:

ⓐ yes; no ⓑ no; no ⓒ yes; yes ⓓ yes; yes

Exercise:

Problem: $y = x - 4$

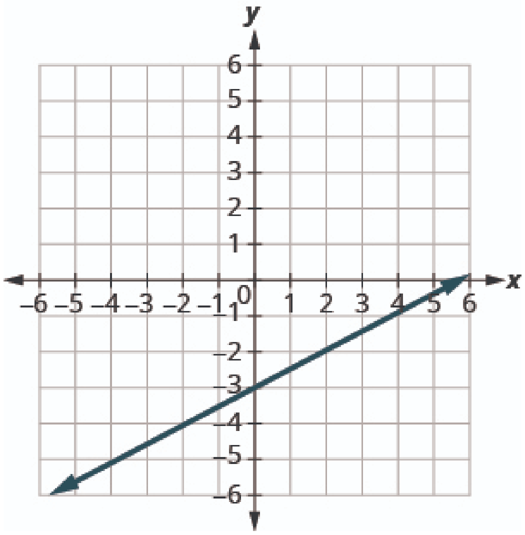
ⓐ $(0, -4)$
ⓑ $(3, -1)$
ⓒ $(2, 2)$
ⓓ $(1, -5)$



Exercise:

Problem: $y=\frac{1}{2}x-3$

ⓐ $(0,-3)$
ⓑ $(2,-2)$
ⓒ $(-2,-4)$
ⓓ $(4,1)$



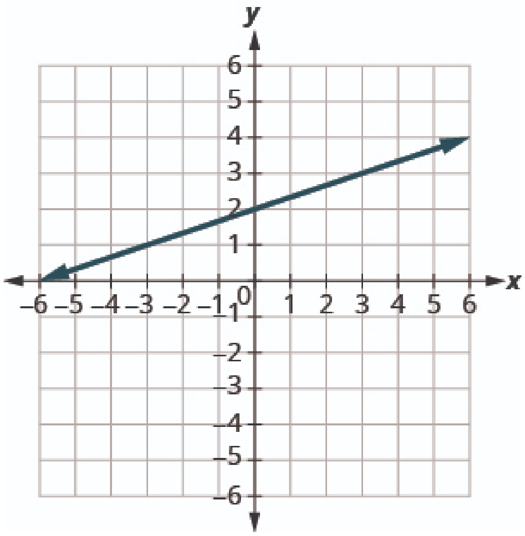
Solution:

ⓐ yes; yes ⓑ yes; yes ⓒ yes; yes ⓓ no; no

Exercise:

Problem: $y = \frac{1}{3}x + 2$

ⓐ $(0, 2)$
ⓑ $(3, 3)$
ⓒ $(-3, 2)$
ⓓ $(-6, 0)$



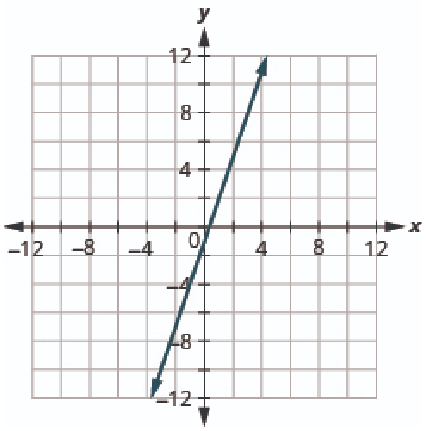
Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

Exercise:

Problem: $y = 3x - 1$

Solution:



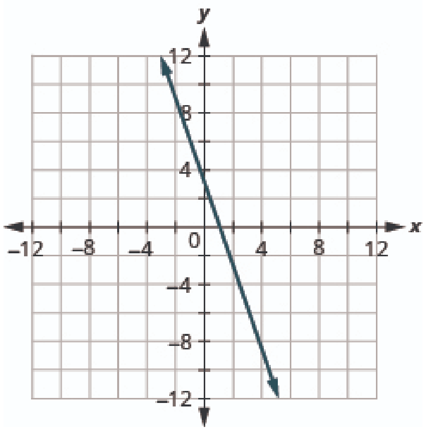
Exercise:

Problem: $y = 2x + 3$

Exercise:

Problem: $y = -2x + 2$

Solution:



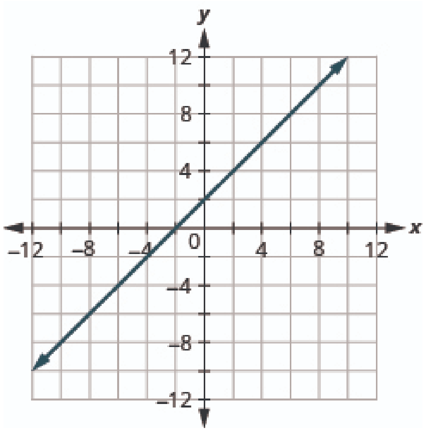
Exercise:

Problem: $y = -3x + 1$

Exercise:

Problem: $y = x + 2$

Solution:



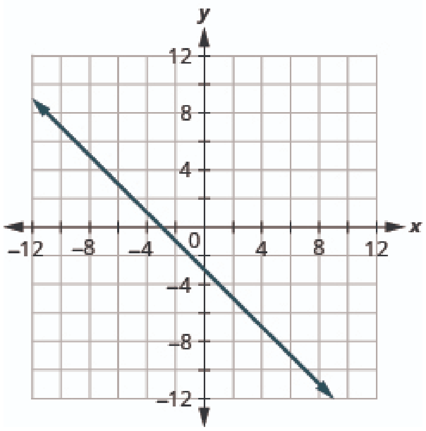
Exercise:

Problem: $y = x - 3$

Exercise:

Problem: $y = -x - 3$

Solution:



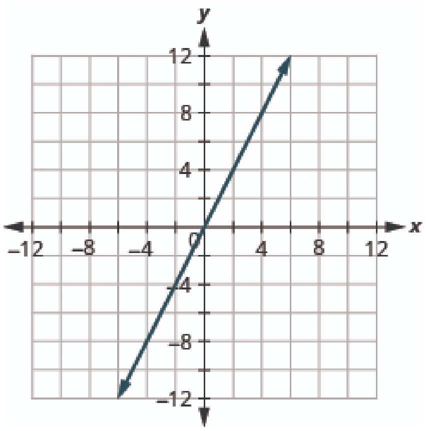
Exercise:

Problem: $y = -x - 2$

Exercise:

Problem: $y = 2x$

Solution:



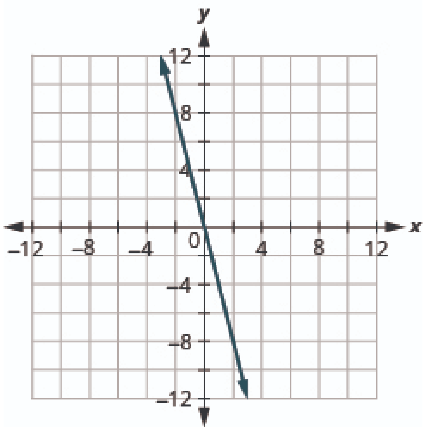
Exercise:

Problem: $y = 3x$

Exercise:

Problem: $y = -4x$

Solution:



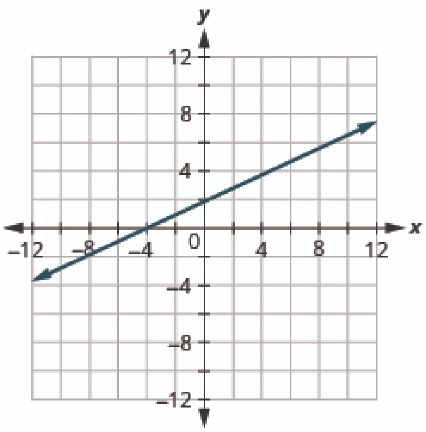
Exercise:

Problem: $y = -2x$

Exercise:

Problem: $y = \frac{1}{2}x + 2$

Solution:



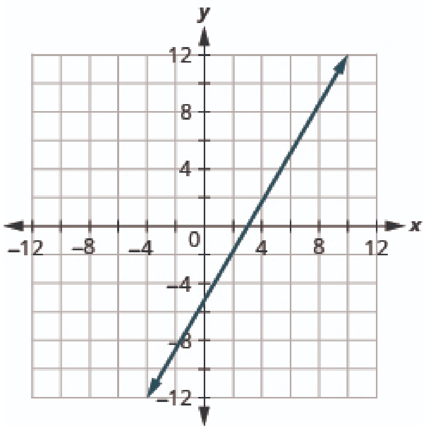
Exercise:

Problem: $y = \frac{1}{3}x - 1$

Exercise:

Problem: $y = \frac{4}{3}x - 5$

Solution:



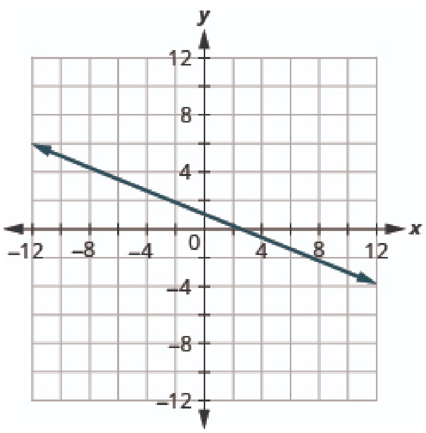
Exercise:

Problem: $y = \frac{3}{2}x - 3$

Exercise:

Problem: $y = -\frac{2}{5}x + 1$

Solution:



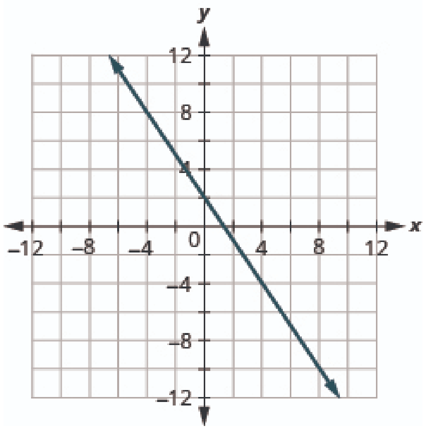
Exercise:

Problem: $y=-\frac{4}{5}x-1$

Exercise:

Problem: $y=-\frac{3}{2}x+2$

Solution:



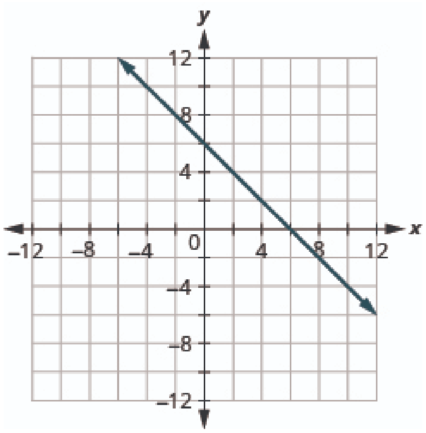
Exercise:

Problem: $y=-\frac{5}{3}x+4$

Exercise:

Problem: $x + y = 6$

Solution:



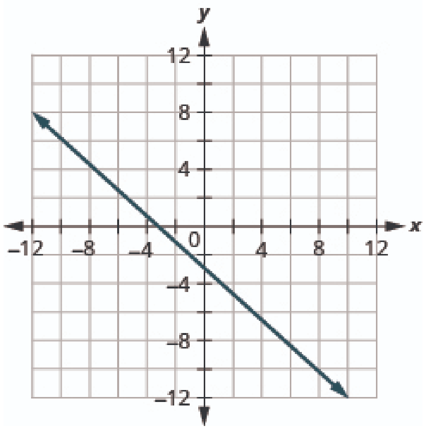
Exercise:

Problem: $x + y = 4$

Exercise:

Problem: $x + y = -3$

Solution:



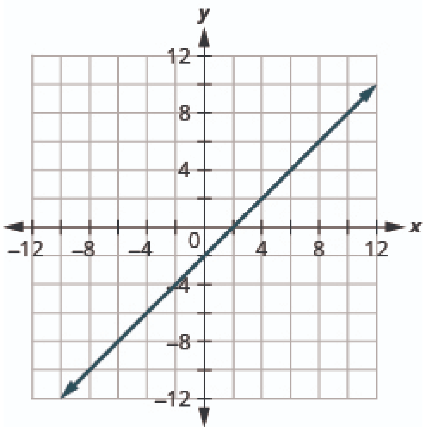
Exercise:

Problem: $x + y = -2$

Exercise:

Problem: $x - y = 2$

Solution:



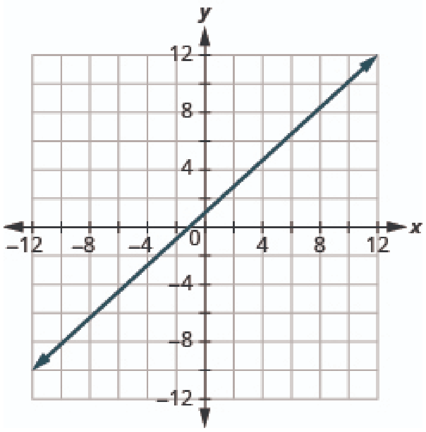
Exercise:

Problem: $x - y = 1$

Exercise:

Problem: $x - y = -1$

Solution:



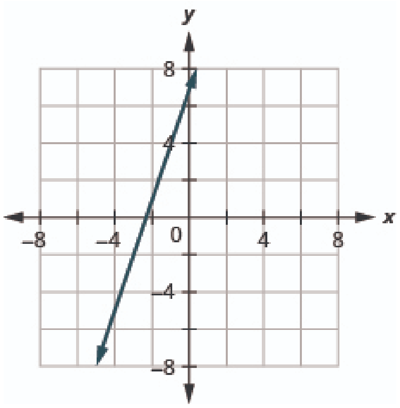
Exercise:

Problem: $x - y = -3$

Exercise:

Problem: $3x + y = 7$

Solution:



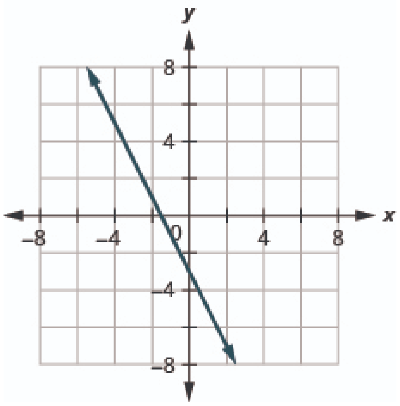
Exercise:

Problem: $5x + y = 6$

Exercise:

Problem: $2x + y = -3$

Solution:



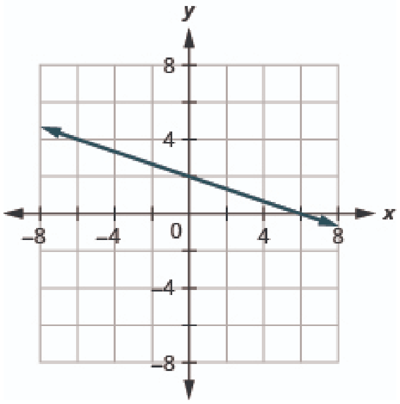
Exercise:

Problem: $4x + y = -5$

Exercise:

Problem: $\frac{1}{3}x + y = 2$

Solution:



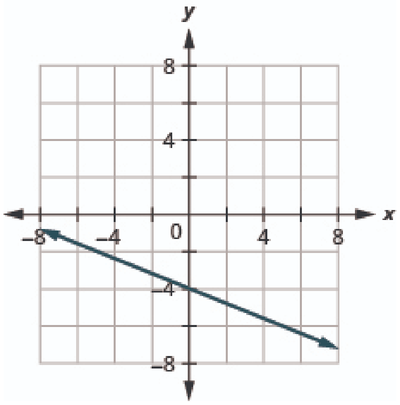
Exercise:

Problem: $\frac{1}{2}x + y = 3$

Exercise:

Problem: $\frac{2}{5}x - y = 4$

Solution:



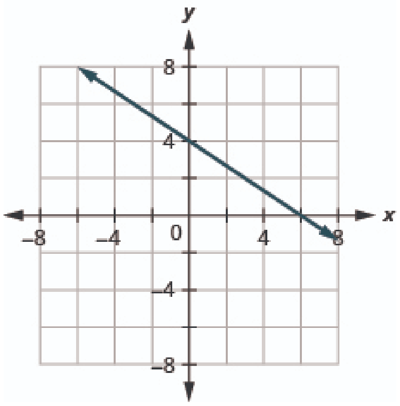
Exercise:

Problem: $\frac{3}{4}x - y = 6$

Exercise:

Problem: $2x + 3y = 12$

Solution:



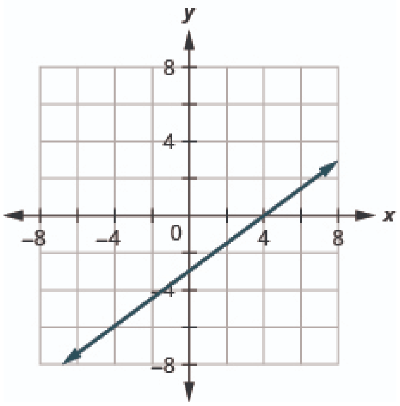
Exercise:

Problem: $4x + 2y = 12$

Exercise:

Problem: $3x - 4y = 12$

Solution:



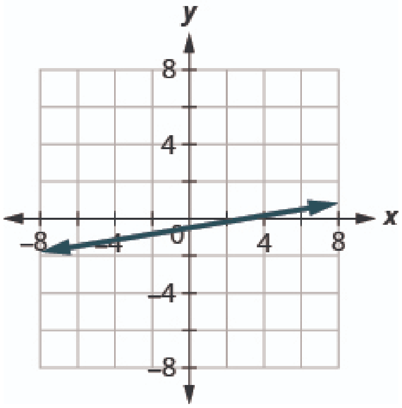
Exercise:

Problem: $2x - 5y = 10$

Exercise:

Problem: $x - 6y = 3$

Solution:



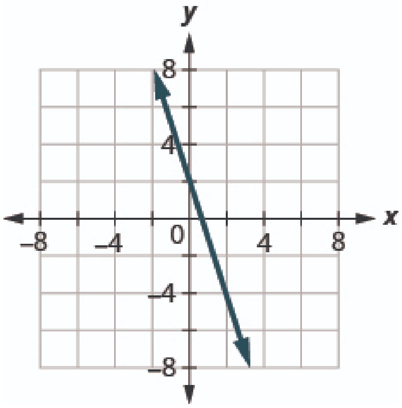
Exercise:

Problem: $x - 4y = 2$

Exercise:

Problem: $5x + 2y = 4$

Solution:



Exercise:

Problem: $3x + 5y = 5$

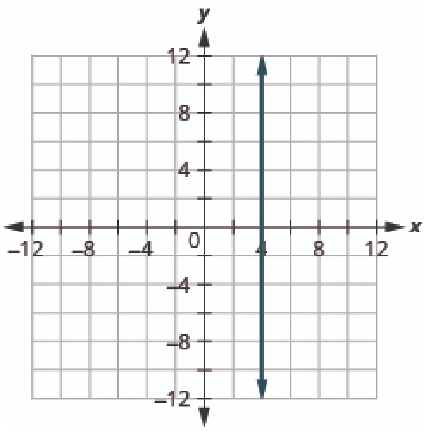
Graph Vertical and Horizontal Lines

In the following exercises, graph each equation.

Exercise:

Problem: $x = 4$

Solution:



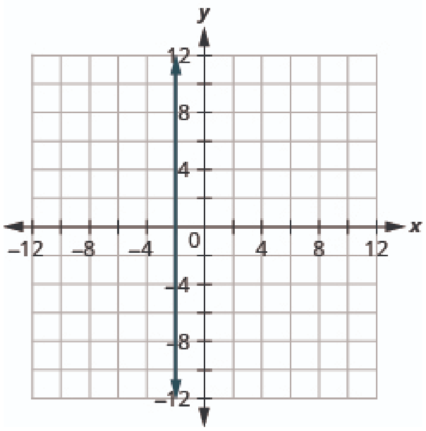
Exercise:

Problem: $x = 3$

Exercise:

Problem: $x = -2$

Solution:



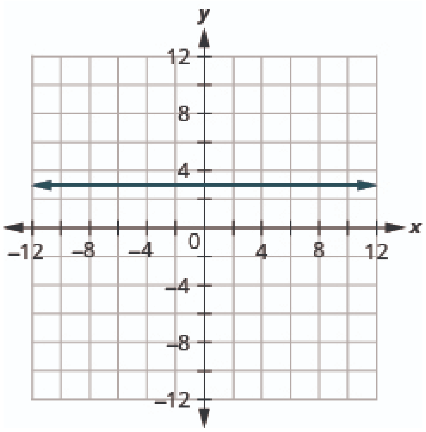
Exercise:

Problem: $x = -5$

Exercise:

Problem: $y = 3$

Solution:



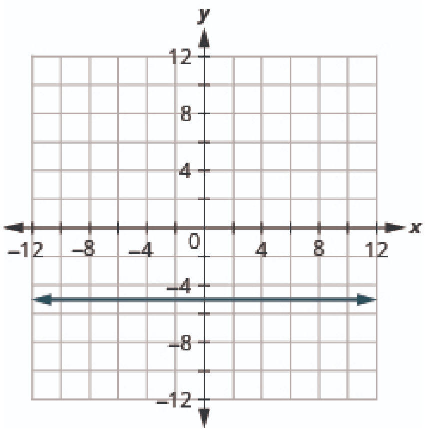
Exercise:

Problem: $y = 1$

Exercise:

Problem: $y = -5$

Solution:



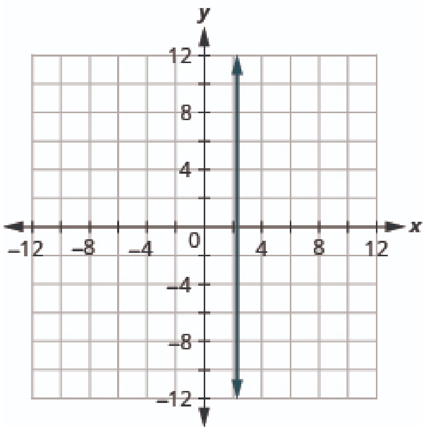
Exercise:

Problem: $y=-2$

Exercise:

Problem: $x=\frac{7}{3}$

Solution:



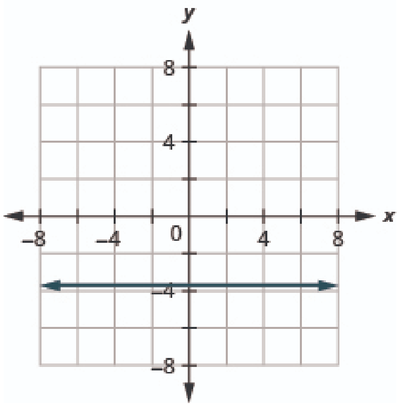
Exercise:

Problem: $x=\frac{5}{4}$

Exercise:

Problem: $y=-\frac{15}{4}$

Solution:



Exercise:

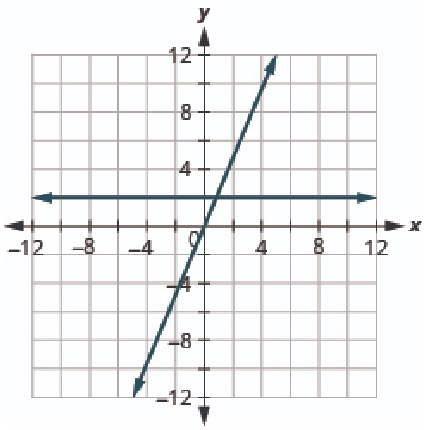
Problem: $y=-\frac{5}{3}$

In the following exercises, graph each pair of equations in the same rectangular coordinate system.

Exercise:

Problem: $y=2x$ and $y=2$

Solution:



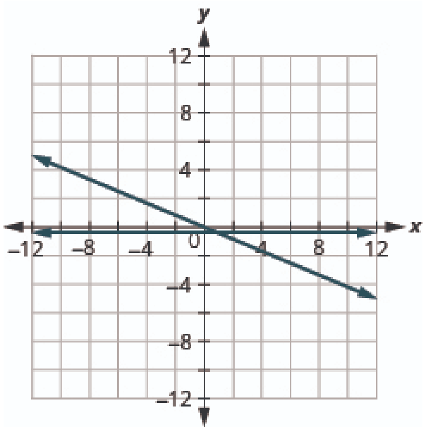
Exercise:

Problem: $y=5x$ and $y=5$

Exercise:

Problem: $y=-\frac{1}{2}x$ and $y=-\frac{1}{2}$

Solution:



Exercise:

Problem: $y=-\frac{1}{3}x$ and $y=-\frac{1}{3}$

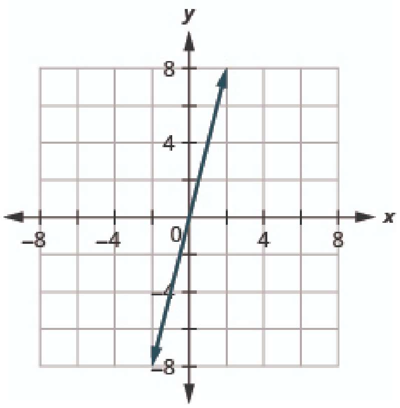
Mixed Practice

In the following exercises, graph each equation.

Exercise:

Problem: $y=4x$

Solution:



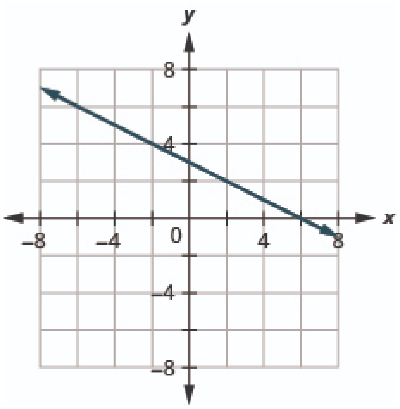
Exercise:

Problem: $y=2x$

Exercise:

Problem: $y = -\frac{1}{2}x + 3$

Solution:



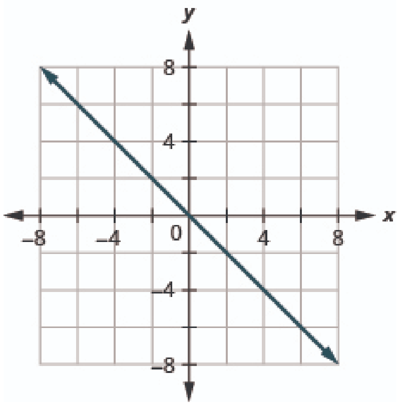
Exercise:

Problem: $y = \frac{1}{4}x - 2$

Exercise:

Problem: $y = -x$

Solution:



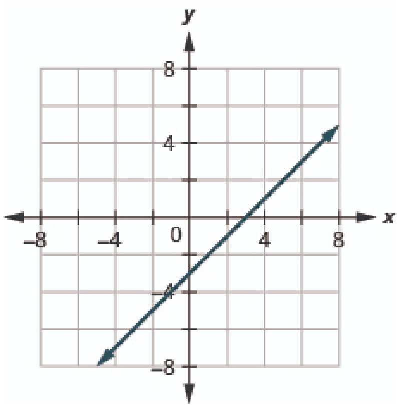
Exercise:

Problem: $y = x$

Exercise:

Problem: $x - y = 3$

Solution:



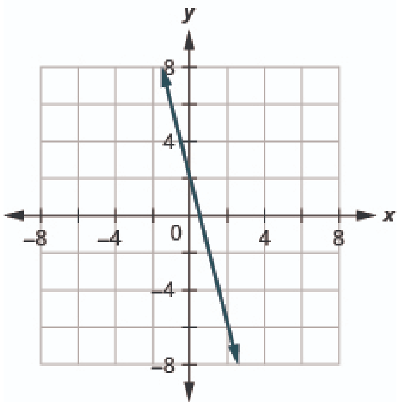
Exercise:

Problem: $x + y = -5$

Exercise:

Problem: $4x + y = 2$

Solution:



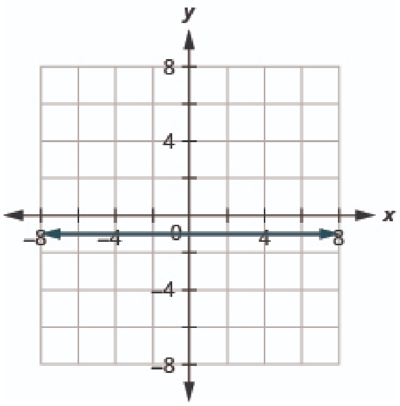
Exercise:

Problem: $2x + y = 6$

Exercise:

Problem: $y = -1$

Solution:



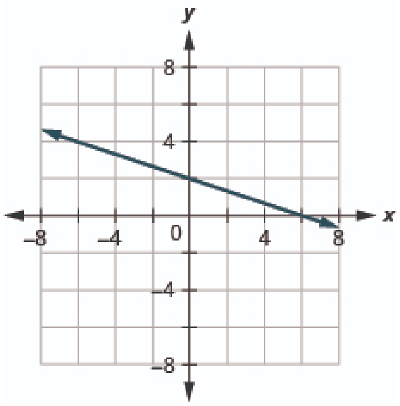
Exercise:

Problem: $y = 5$

Exercise:

Problem: $2x + 6y = 12$

Solution:



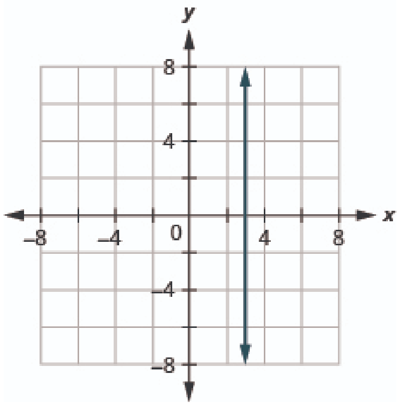
Exercise:

Problem: $5x + 2y = 10$

Exercise:

Problem: $x = 3$

Solution:



Exercise:

Problem: $x = -4$

Everyday Math

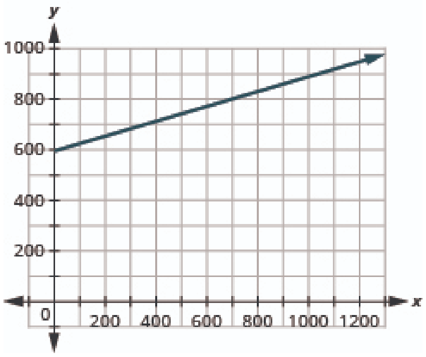
Exercise:

Problem:

Motor home cost. The Robinsons rented a motor home for one week to go on vacation. It cost them $594 plus $0.32 per mile to rent the motor home, so the linear equation $y = 594 + 0.32x$ gives the cost, y, for driving x miles. Calculate the rental cost for driving 400, 800, and 1200 miles, and then graph the line.

Solution:

$722, $850, $978



Exercise:

Problem:

Weekly earnings. At the art gallery where he works, Salvador gets paid $200 per week plus 15% of the sales he makes, so the equation $y = 200 + 0.15x$ gives the amount, y, he earns for selling x dollars of artwork. Calculate the amount Salvador earns for selling $900, $1600, and $2000, and then graph the line.

Writing Exercises

Exercise:

Problem:

Explain how you would choose three *x*- values to make a table to graph the line $y = \frac{1}{5}x - 2$.

Solution:

Answers will vary.

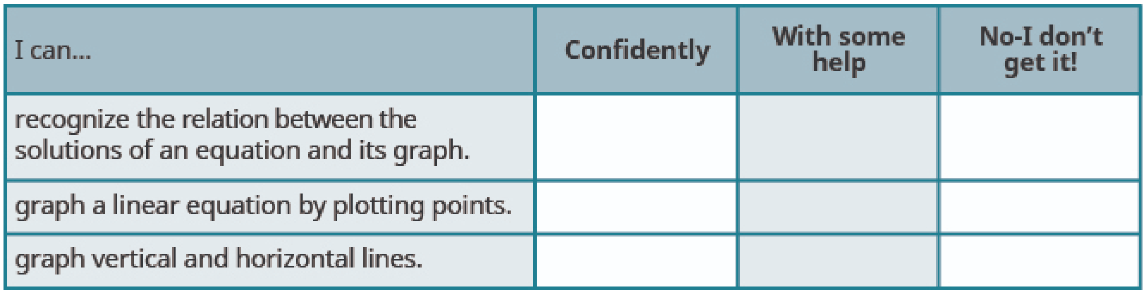
Exercise:

Problem:

What is the difference between the equations of a vertical and a horizontal line?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
recognize the relation between the solutions of an equation and its graph.			
graph a linear equation by plotting points.			
graph vertical and horizontal lines.			

ⓑ After reviewing this checklist, what will you do to become confident for all goals?

Glossary

graph of a linear equation

The graph of a linear equation $Ax + By = C$ is a straight line. Every point on the line is a solution of the equation. Every solution of this equation is a point on this line.

horizontal line

A horizontal line is the graph of an equation of the form $y = b$. The line passes through the *y*-axis at $(0, b)$.

vertical line

A vertical line is the graph of an equation of the form $x = a$. The line passes through the *x*-axis at $(a, 0)$.

Graph with Intercepts
By the end of this section, you will be able to:

- Identify the x- and y- intercepts on a graph
- Find the x- and y- intercepts from an equation of a line
- Graph a line using the intercepts

Note:
Before you get started, take this readiness quiz.

1. Solve: $3 \cdot 0 + 4y = -2$.
   If you missed this problem, review [link].

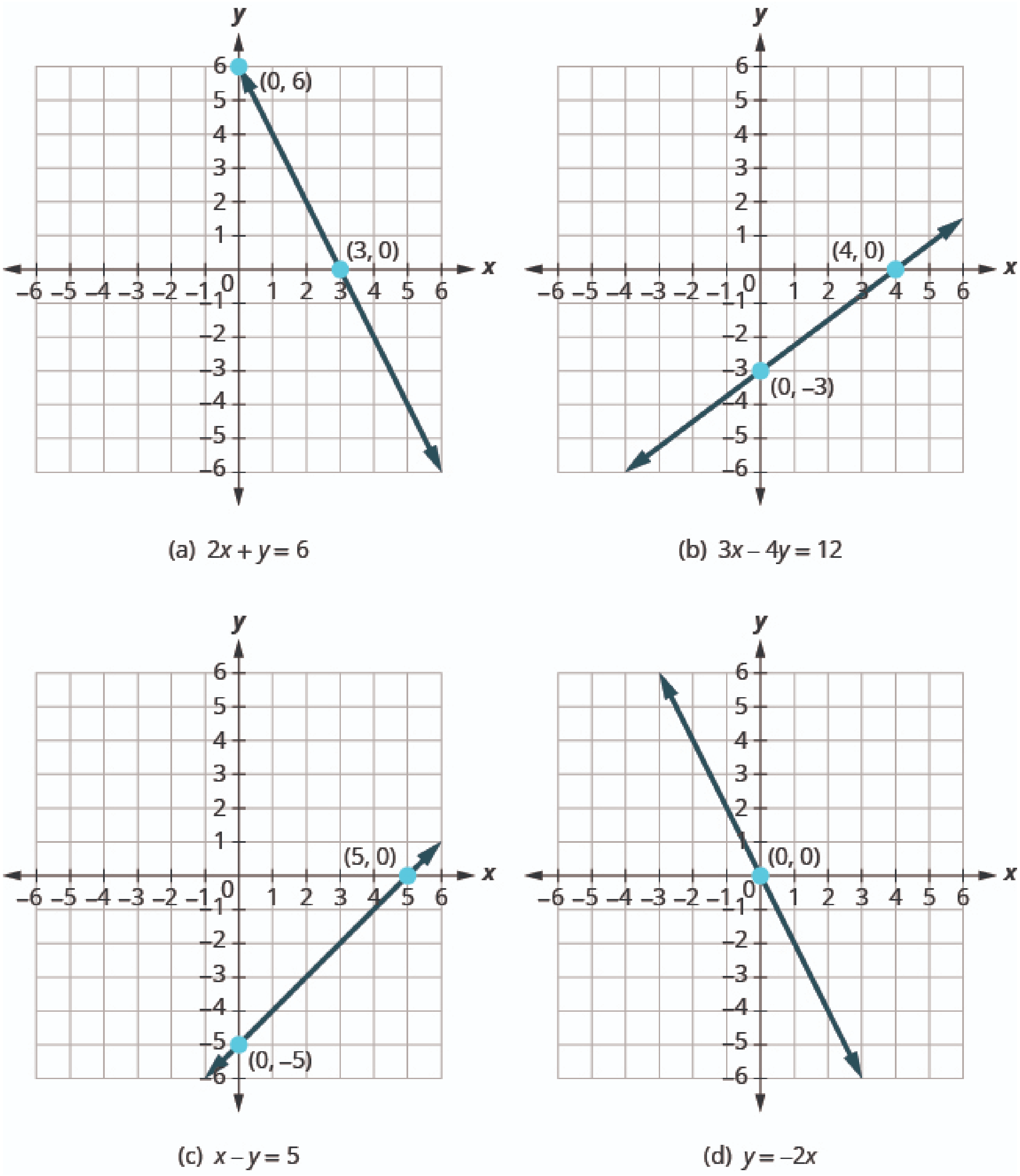
Identify the *x*- and *y*- Intercepts on a Graph

Every linear equation can be represented by a unique line that shows all the solutions of the equation. We have seen that when graphing a line by plotting points, you can use any three solutions to graph. This means that two people graphing the line might use different sets of three points.

At first glance, their two lines might not appear to be the same, since they would have different points labeled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at where the line crosses the *x*- axis and the *y*- axis. These points are called the *intercepts* of the line.

Note:
Intercepts of a Line
The points where a line crosses the *x*- axis and the *y*- axis are called the **intercepts of a line**.

Let’s look at the graphs of the lines in [link].



(a) $2x + y = 6$

(b) $3x - 4y = 12$

(c) $x - y = 5$

(d) $y = -2x$

Examples of graphs crossing the x-negative axis.

First, notice where each of these lines crosses the x negative axis. See [link].

Figure	**The line crosses the *x*- axis at:**	**Ordered pair of this point**
Figure (a)	3	$(3,0)$
Figure (b)	4	$(4,0)$
Figure (c)	5	$(5,0)$
Figure (d)	0	$(0,0)$

Do you see a pattern?

For each row, the *y*- coordinate of the point where the line crosses the *x*- axis is zero. The point where the line crosses the *x*- axis has the form $(a,0)$ and is called the ***x*- intercept of a line**. The *x*- intercept occurs when y is zero.

Now, let's look at the points where these lines cross the *y*- axis. See [link].

Figure	**The line crosses the *y*- axis at:**	**Ordered pair for this point**
Figure (a)	6	$(0, 6)$
Figure (b)	-3	$(0, -3)$
Figure (c)	-5	$(0, 5)$
Figure (d)	0	$(0, 0)$

What is the pattern here?

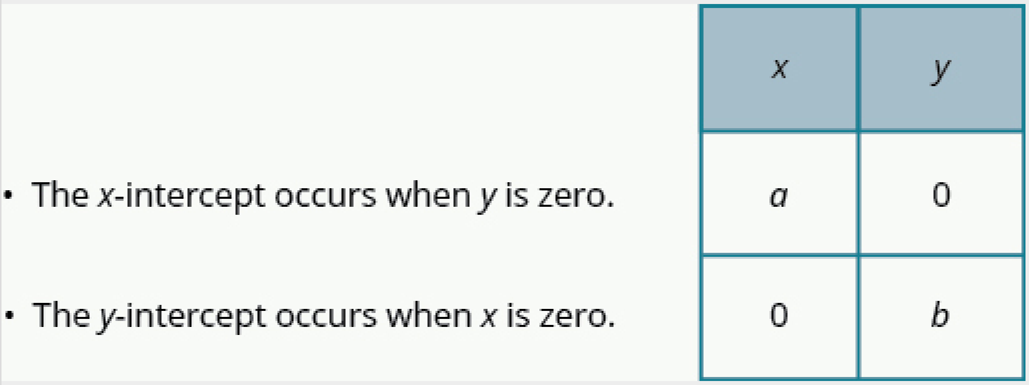
In each row, the *x*- coordinate of the point where the line crosses the *y*- axis is zero. The point where the line crosses the *y*- axis has the form $(0, b)$ and is called the *y*- *intercept* of the line. The *y*- intercept occurs when x is zero.

Note:

x- intercept and ***y*- intercept of a line**

The *x*- intercept is the point $(a, 0)$ where the line crosses the *x*- axis.

The *y*- intercept is the point $(0, b)$ where the line crosses the *y*- axis.

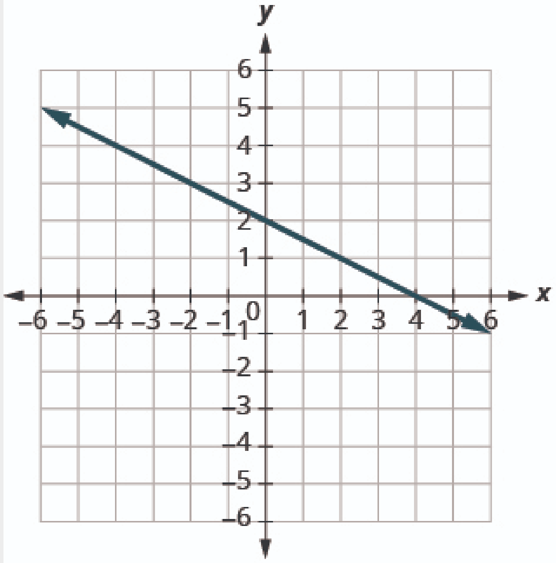


- The *x*-intercept occurs when *y* is zero.
- The *y*-intercept occurs when *x* is zero.

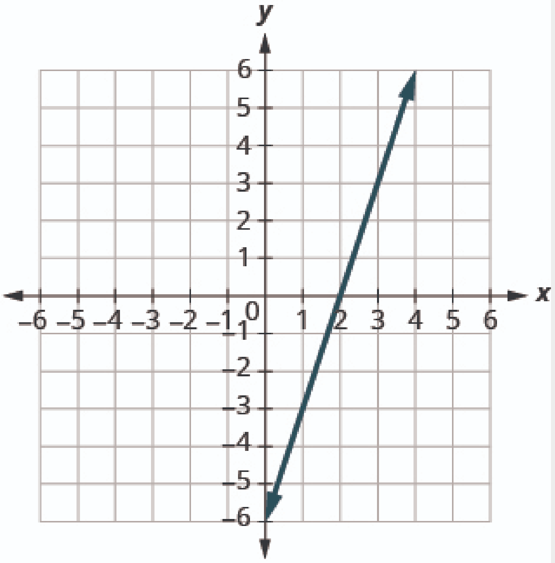
x	*y*
a	0
0	*b*

Example:
Exercise:

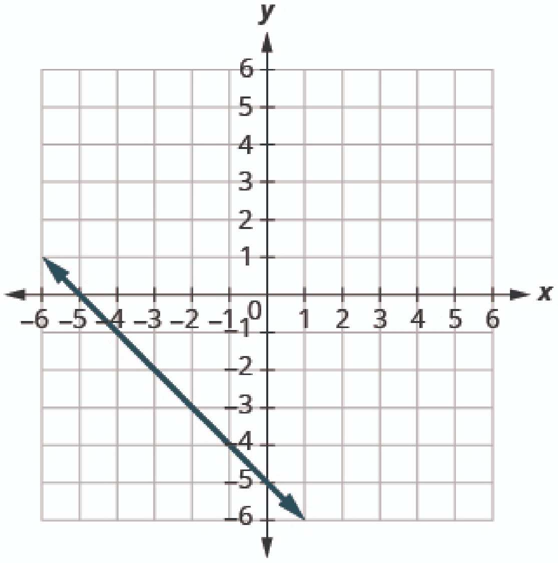
Problem: Find the *x*- and *y*- intercepts on each graph.



(a)



(b)



(c)

Solution:
Solution

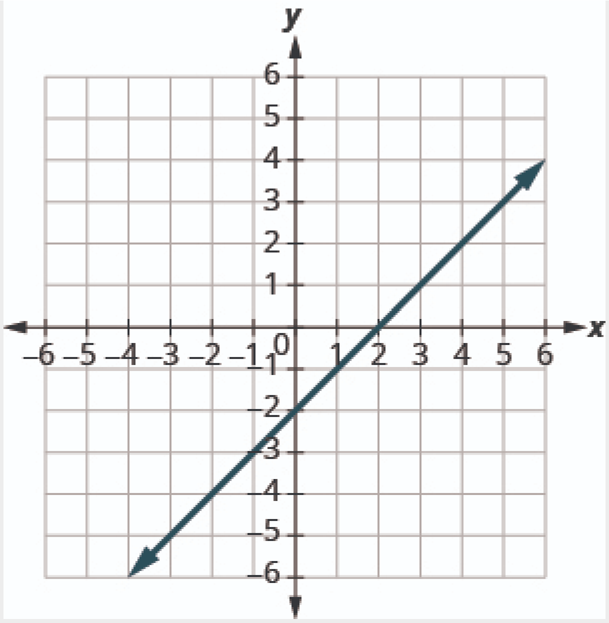
ⓐ The graph crosses the *x*- axis at the point $(4,0)$. The *x*- intercept is $(4,0)$.
The graph crosses the *y*- axis at the point $(0,2)$. The *y*- intercept is $(0,2)$.

ⓑ The graph crosses the *x*- axis at the point $(2,0)$. The *x*- intercept is $(2,0)$
The graph crosses the *y*- axis at the point $(0,-6)$. The *y*- intercept is $(0,-6)$.

ⓒ The graph crosses the *x*- axis at the point $(-5,0)$. The *x*- intercept is $(-5,0)$.
The graph crosses the *y*- axis at the point $(0,-5)$. The *y*- intercept is $(0,-5)$.

Note:
Exercise:

Problem: Find the *x*- and *y*- intercepts on the graph.

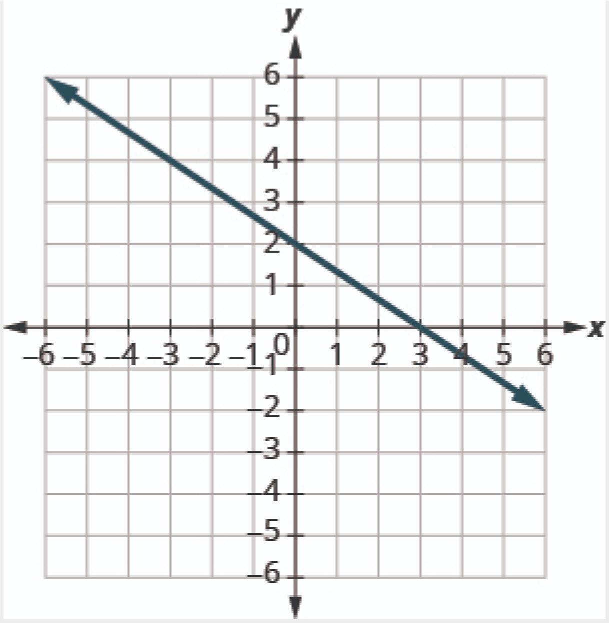


Solution:

x- intercept: $(2, 0)$; *y*- intercept: $(0, -2)$

Note:
Exercise:

Problem: Find the *x*- and *y*- intercepts on the graph.



Solution:

x- intercept: $(3, 0)$, *y*- intercept: $(0, 2)$

Find the *x*- and *y*- Intercepts from an Equation of a Line

Recognizing that the *x*- intercept occurs when *y* is zero and that the *y*- intercept occurs when *x* is zero, gives us a method to find the intercepts of a line from its equation. To find the *x*- intercept, let $y = 0$ and solve for *x*. To find the *y*- intercept, let $x = 0$ and solve for *y*.

Note:
Find the *x*- and *y*- Intercepts from the Equation of a Line
Use the equation of the line. To find:

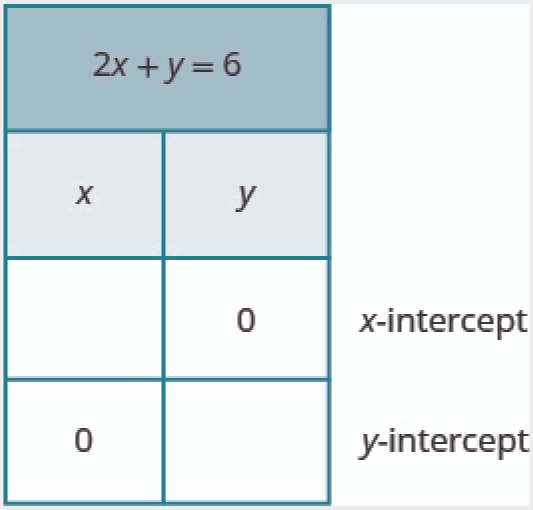
- the *x*- intercept of the line, let $y = 0$ and solve for x.
- the *y*- intercept of the line, let $x = 0$ and solve for y.

Example:
Exercise:

Problem: Find the intercepts of $2x + y = 6$.

Solution:
Solution

We will let $y = 0$ to find the *x*- intercept, and let $x = 0$ to find the *y*-intercept. We will fill in the table, which reminds us of what we need to find.



$2x+y=6$		
x	y	
	0	*x*-intercept
0		*y*-intercept

To find the *x*- intercept, let $y = 0$.

	$2x+y=6$

Let $y = 0$.	$2x + 0 = 6$
Simplify.	$2x = 6$
	$x = 3$
The *x*-intercept is	(3, 0)
To find the *y*-intercept, let $x = 0$.	
	$2x + y = 6$
Let $x = 0$.	$2 \cdot 0 + y = 6$
Simplify.	$0 + y = 6$
	$y = 6$
The *y*-intercept is	(0, 6)

The intercepts are the points $(3, 0)$ and $(0, 6)$ as shown in [link].

$2x + y = 6$	
x	y
3	0
0	6

Note:
Exercise:

Problem: Find the intercepts of $3x + y = 12$.

Solution:

x- intercept: $(4, 0)$, *y*- intercept: $(0, 12)$

Note:
Exercise:

Problem: Find the intercepts of $x + 4y = 8$.

Solution:

x- intercept: $(8, 0)$, *y*- intercept: $(0, 2)$

Example:
Exercise:

Problem: Find the intercepts of $4x - 3y = 12$.

Solution:
Solution

To find the *x*-intercept, let *y* = 0.	
	$4x - 3y = 12$
Let *y* = 0.	$4x - 3 \cdot 0 = 12$
Simplify.	$4x - 0 = 12$
	$4x = 12$
	$x = 3$
The *x*-intercept is	(3, 0)
To find the *y*-intercept, let *x* = 0.	
	$4x - 3y = 12$

Let *x* = 0.	$4 \cdot 0 - 3y = 12$
Simplify.	$0 - 3y = 12$
	$-3y = 12$
	$y = -4$
The *y*-intercept is	(0, −4)

The intercepts are the points (3, 0) and (0, −4) as shown in [link].

$4x - 3y = 12$	
x	y
3	0
0	-4

Note:
Exercise:

Problem: Find the intercepts of $3x – 4y = 12$.

Solution:

x- intercept: $(4, 0)$, *y*- intercept: $(0, -3)$

Note:
Exercise:

Problem: Find the intercepts of $2x – 4y = 8$.

Solution:

x- intercept: $(4, 0)$, *y*- intercept: $(0, -2)$

Graph a Line Using the Intercepts

To graph a linear equation by plotting points, you need to find three points whose coordinates are solutions to the equation. You can use the *x*- and *y*-intercepts as two of your three points. Find the intercepts, and then find a third point to ensure accuracy. Make sure the points line up—then draw the line. This method is often the quickest way to graph a line.

Example:
How to Graph a Line Using Intercepts
Exercise:

Problem: Graph $-x + 2y = 6$ using the intercepts.

Solution:
Solution

Step 1. Find the x- and y-intercepts of the line. Let $y = 0$ and solve for x. Let $x = 0$ and solve for y.	Find the x-intercept.	Let $y = 0$ $-x + 2y = 6$ $-x + 2(0) = 6$ $-x = 6$ $x = -6$ The x-intercept is $(-6, 0)$.
	Find the y-intercept.	Let $x = 0$ $-x + 2y = 6$ $-0 + 2y = 6$ $2y = 6$ $y = 3$ The y-intercept is $(0, 3)$.

Step 2. Find another solution to the equation.	We'll use $x = 2$.	Let $x = 2$ $-x + 2y = 6$ $-2 + 2y = 6$ $2y = 8$ $y = 4$ A third point is $(2, 4)$.

<table>
<tr>
<td>Step 3. Plot the three points. Check that the points line up.</td>
<td>
<table>
<tr><th>x</th><th>y</th><th>(x, y)</th></tr>
<tr><td>–6</td><td>0</td><td>(–6, 0)</td></tr>
<tr><td>0</td><td>3</td><td>(0, 3)</td></tr>
<tr><td>2</td><td>4</td><td>(2, 4)</td></tr>
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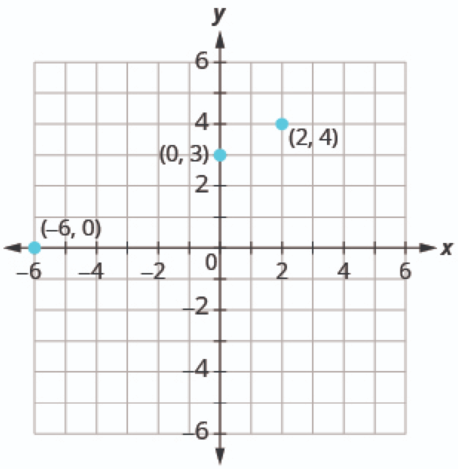
</td>
<td>


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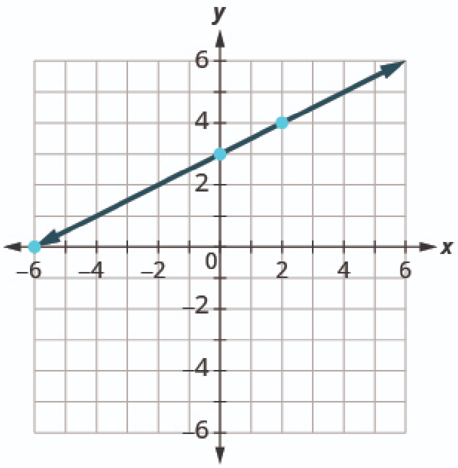
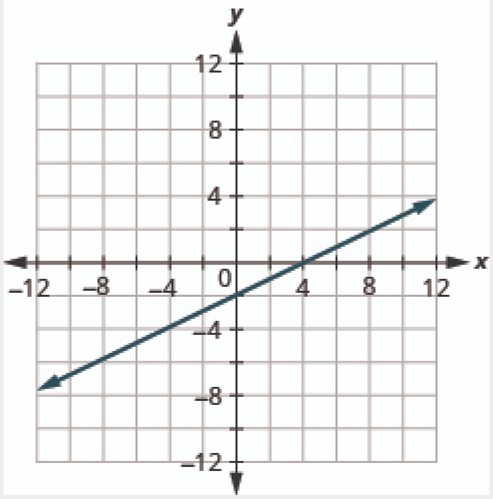
</tr>
</table>

Step 4. Draw the line.	See the graph.	

Note:
Exercise:

Problem: Graph $x - 2y = 4$ using the intercepts.

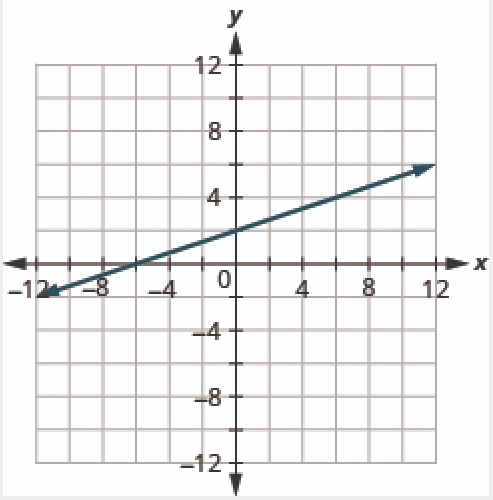
Solution:



Note:
Exercise:

Problem: Graph $-x + 3y = 6$ using the intercepts.

Solution:



The steps to graph a linear equation using the intercepts are summarized below.

Note:
Graph a linear equation using the intercepts.

Find the *x*- and *y*- intercepts of the line.

- Let $y = 0$ and solve for x
- Let $x = 0$ and solve for y.

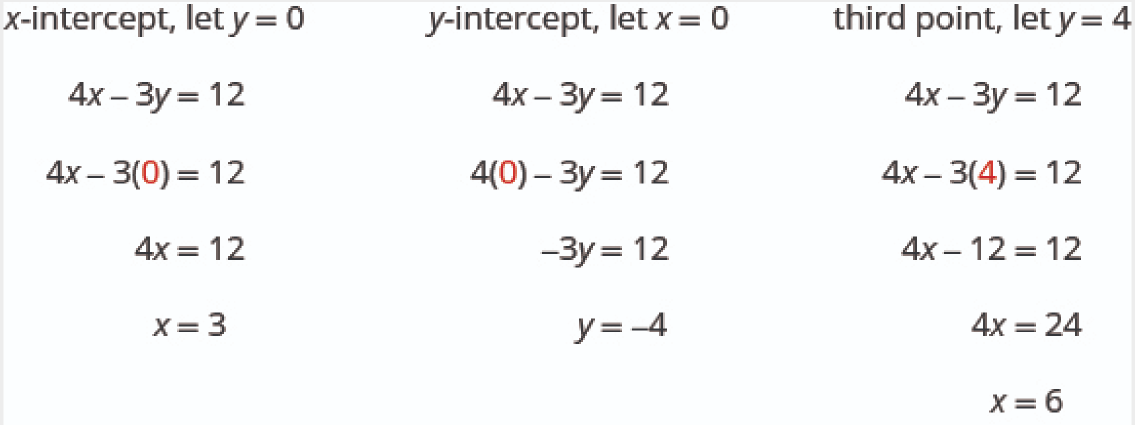
Find a third solution to the equation.
Plot the three points and check that they line up.
Draw the line.

Example:
Exercise:

Problem: Graph $4x - 3y = 12$ using the intercepts.

Solution:
Solution

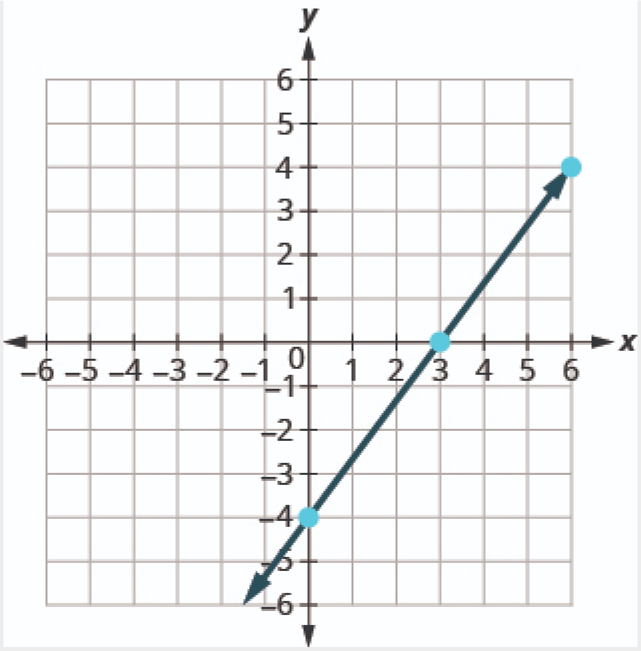
Find the intercepts and a third point.



x-intercept, let $y = 0$	y-intercept, let $x = 0$	third point, let $y = 4$
$4x - 3y = 12$	$4x - 3y = 12$	$4x - 3y = 12$
$4x - 3(0) = 12$	$4(0) - 3y = 12$	$4x - 3(4) = 12$
$4x = 12$	$-3y = 12$	$4x - 12 = 12$
$x = 3$	$y = -4$	$4x = 24$
		$x = 6$

We list the points in [link] and show the graph below.

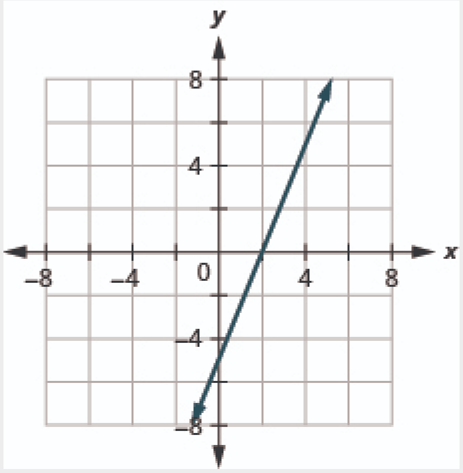
$4x - 3y = 12$		
x	y	(x, y)
3	0	$(3, 0)$
0	-4	$(0, -4)$
6	4	$(6, 4)$



Note:
Exercise:

Problem: Graph $5x - 2y = 10$ using the intercepts.

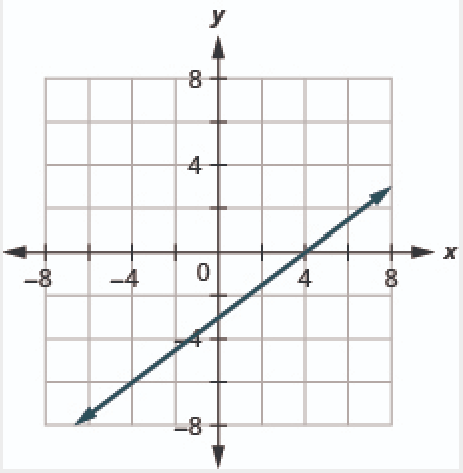
Solution:



Note:
Exercise:

Problem: Graph $3x-4y=12$ using the intercepts.

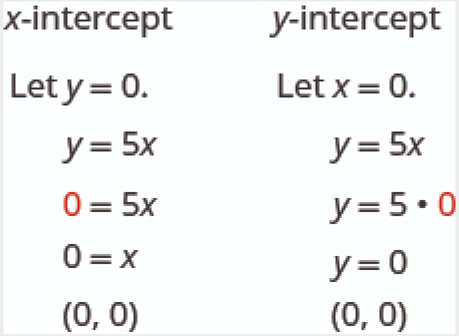
Solution:



Example:
Exercise:

Problem: Graph $y=5x$ using the intercepts.

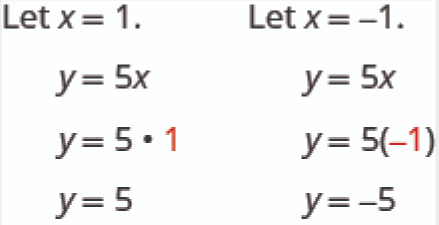
Solution:
Solution



x-intercept	y-intercept
Let $y=0$.	Let $x=0$.
$y=5x$	$y=5x$
$0=5x$	$y=5\cdot 0$
$0=x$	$y=0$
$(0,0)$	$(0,0)$

This line has only one intercept. It is the point $(0, 0)$.

To ensure accuracy we need to plot three points. Since the *x*- and *y*-intercepts are the same point, we need *two* more points to graph the line.

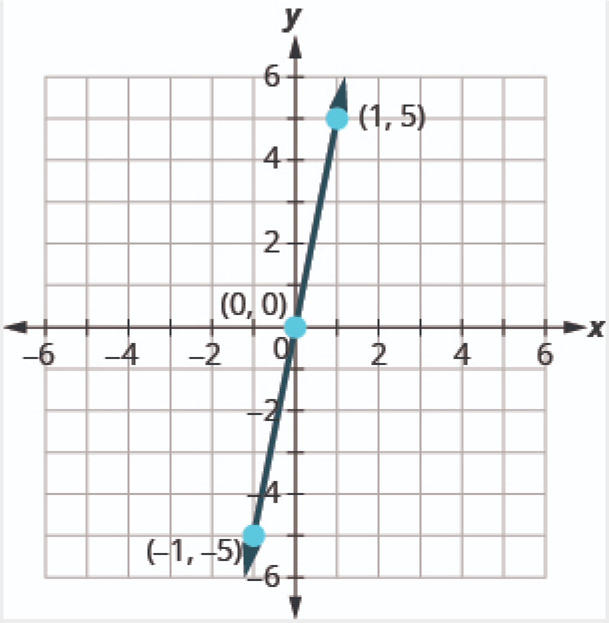


Let $x = 1$.	Let $x = -1$.
$y = 5x$	$y = 5x$
$y = 5 \cdot 1$	$y = 5(-1)$
$y = 5$	$y = -5$

See [link].

$y = 5x$		
x	y	(x, y)
0	0	$(0, 0)$
1	5	$(1, 5)$
-1	-5	$(-1, -5)$

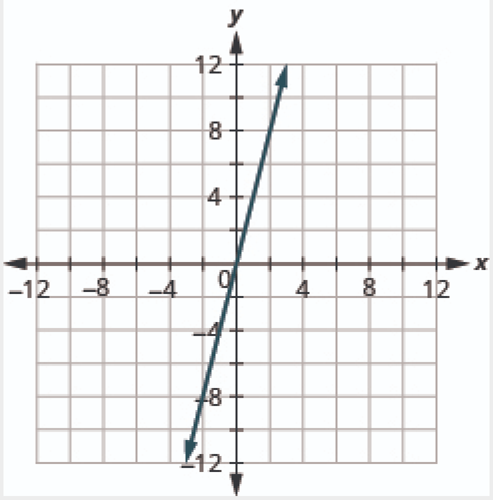
Plot the three points, check that they line up, and draw the line.



Note:
Exercise:

Problem: Graph $y = 4x$ using the intercepts.

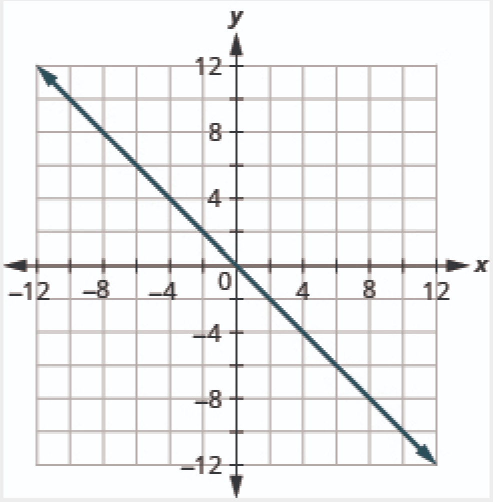
Solution:



Note:
Exercise:

Problem: Graph $y = -x$ the intercepts.

Solution:



Key Concepts

- **Find the *x*- and *y*- Intercepts from the Equation of a Line**
    - Use the equation of the line to find the *x*- intercept of the line, let $y = 0$ and solve for *x*.
    - Use the equation of the line to find the *y*- intercept of the line, let $x = 0$ and solve for *y*.
- **Graph a Linear Equation using the Intercepts**

Find the	*x*- and	*y*- intercepts of the line.	Let $y = 0$	and solve for	*x*.	Let $x = 0$ and solve for *y*.

Find a third solution to the equation.
Plot the three points and then check that they line up.

Draw the line.

- **Strategy for Choosing the Most Convenient Method to Graph a Line:**
    - Consider the form of the equation.
    - If it only has one variable, it is a vertical or horizontal line.
      $x = a$ is a vertical line passing through the *x*- axis at a
      $y = b$ is a horizontal line passing through the *y*- axis at b.
    - If *y* is isolated on one side of the equation, graph by plotting points.
    - Choose any three values for *x* and then solve for the corresponding *y*- values.
    - If the equation is of the form $ax + by = c$, find the intercepts. Find the *x*- and *y*- intercepts and then a third point.

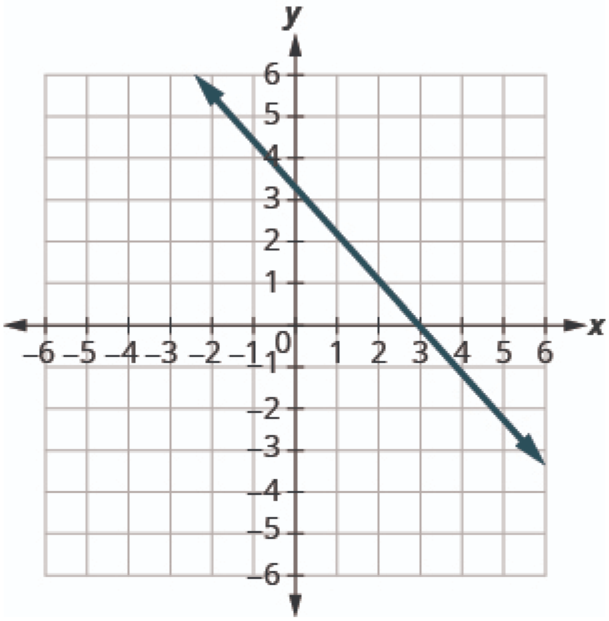
Practice Makes Perfect

Identify the *x*- and *y*- Intercepts on a Graph

In the following exercises, find the *x*- and *y*- intercepts on each graph.

Exercise:

Problem:

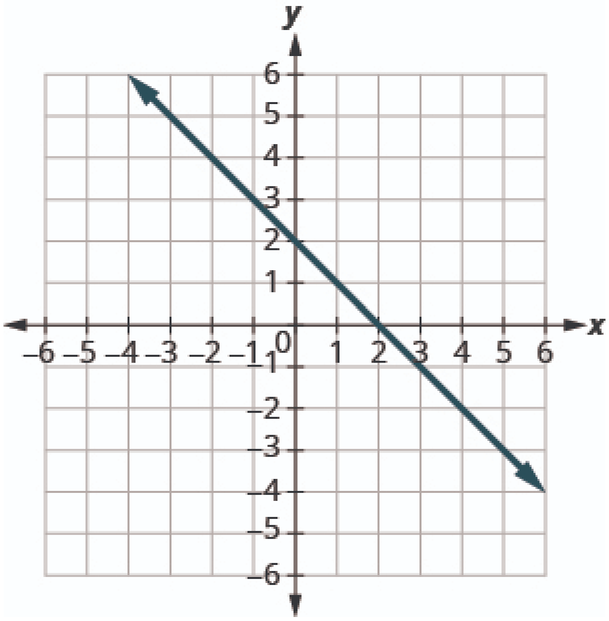


Solution:

$(3, 0), (0, 3)$

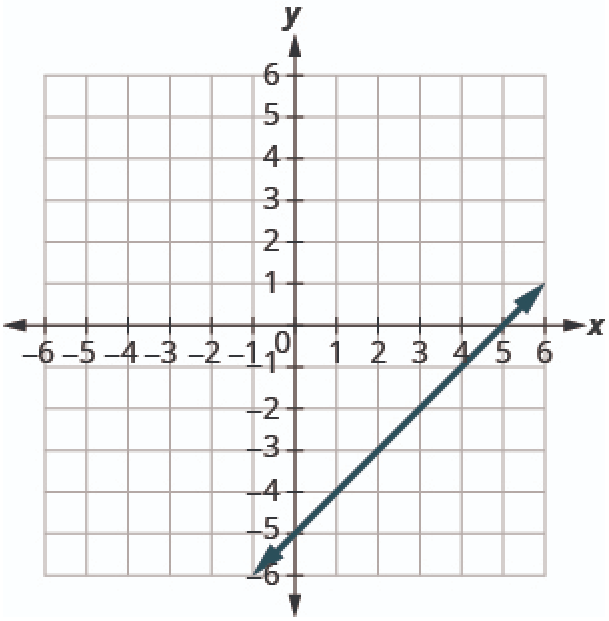
Exercise:

Problem:



Exercise:

Problem:

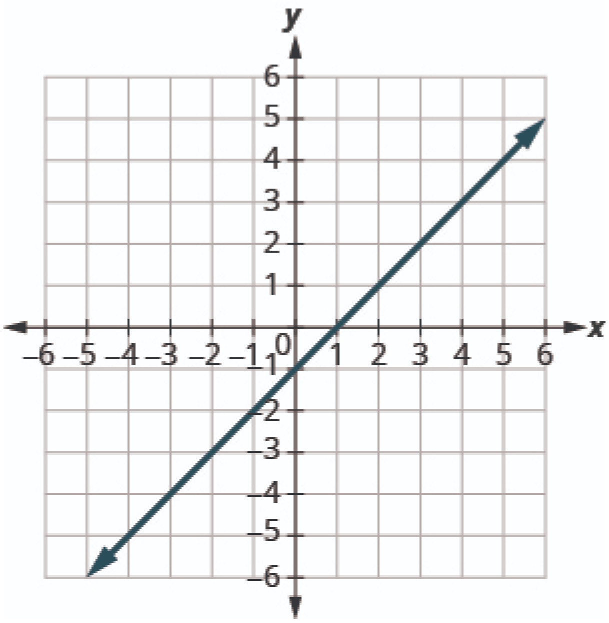


Solution:

$(5,0),(0,-5)$

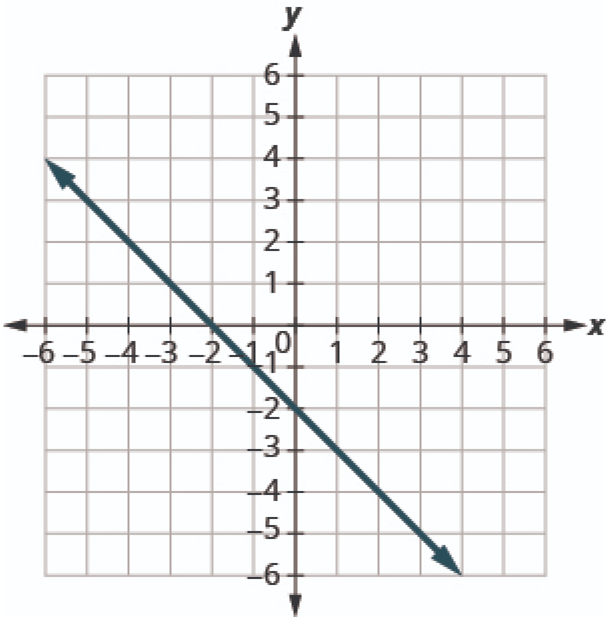
Exercise:

Problem:



Exercise:

Problem:

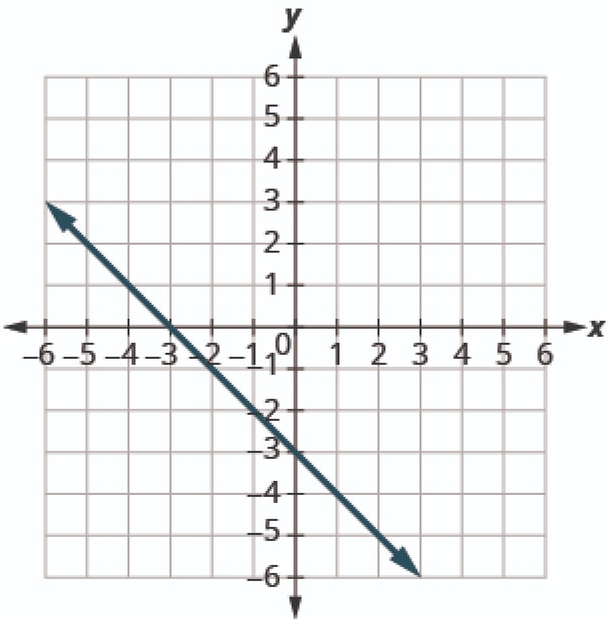


Solution:

$(-2, 0), (0, -2)$

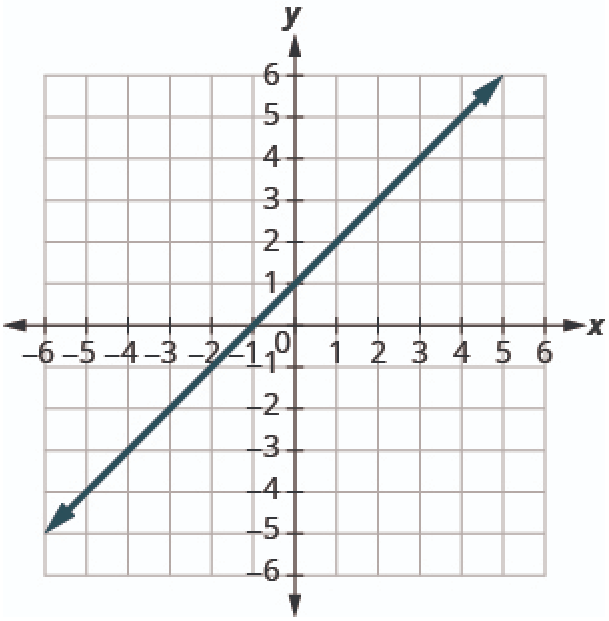
Exercise:

Problem:



Exercise:

Problem:

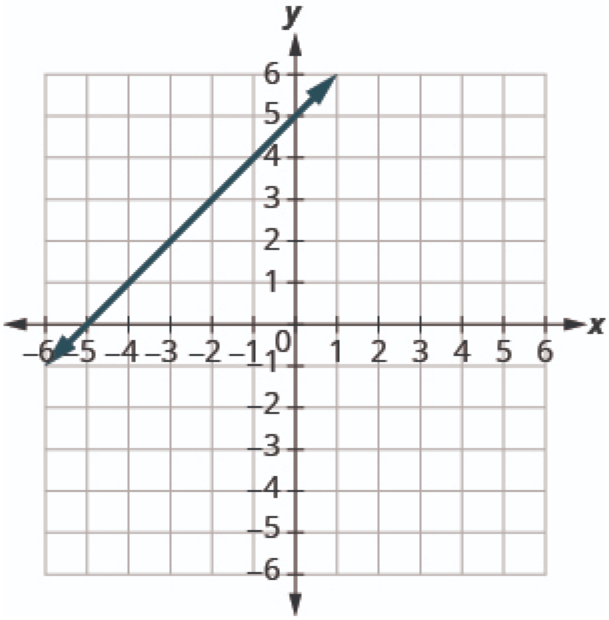


Solution:

$(-1, 0), (0, 1)$

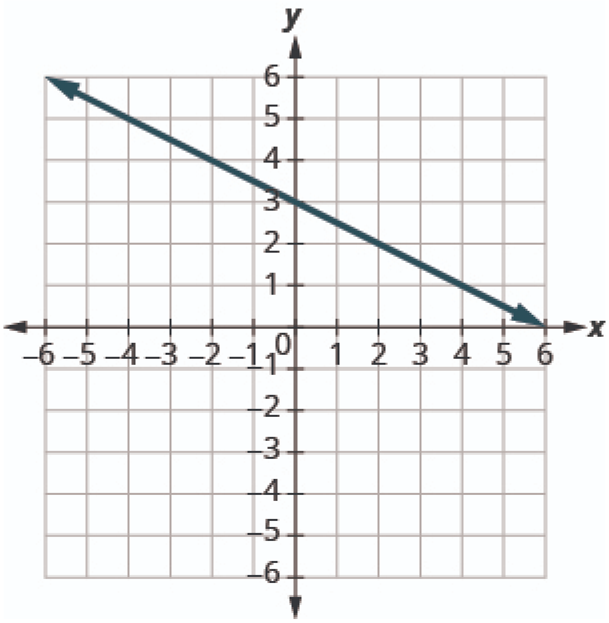
Exercise:

Problem:



Exercise:

Problem:

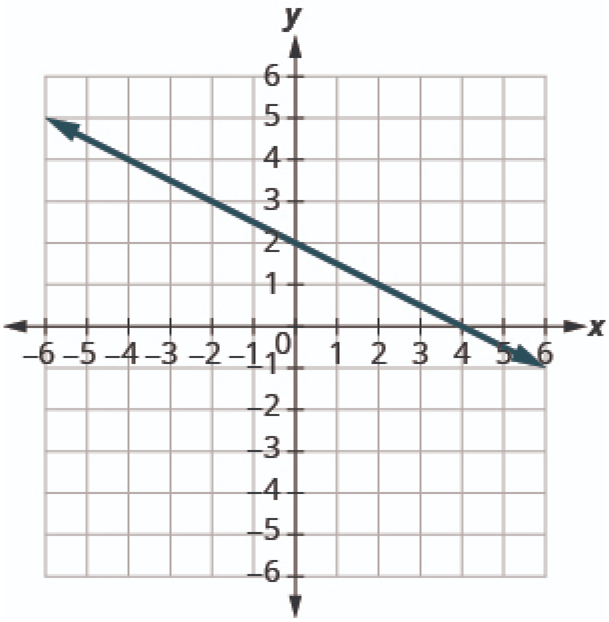


Solution:

$(6, 0), (0, 3)$

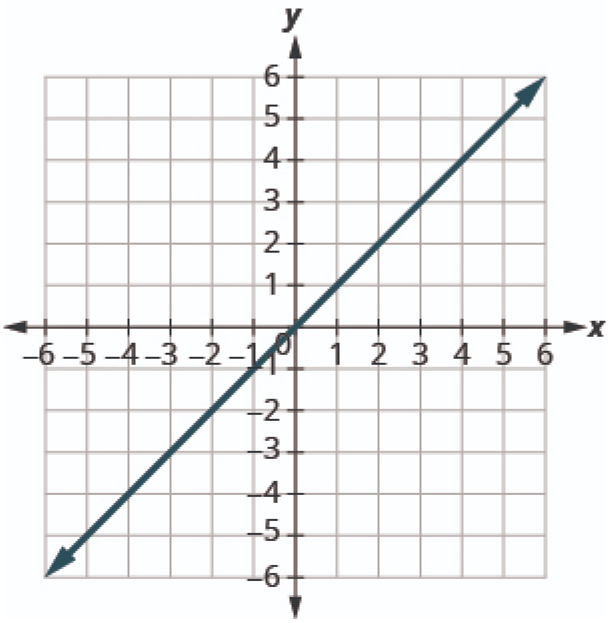
Exercise:

Problem:



Exercise:

Problem:

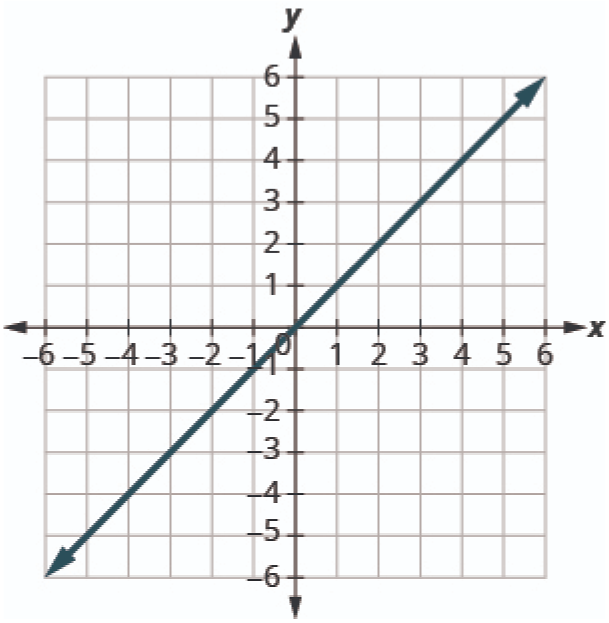


Solution:

$(0, 0)$

Exercise:

Problem:



Find the *x*- and *y*- Intercepts from an Equation of a Line

In the following exercises, find the intercepts for each equation.

Exercise:

Problem: $x + y = 4$

Solution:

$(4, 0), (0, 4)$

Exercise:

Problem: $x + y = 3$

Exercise:

Problem: $x + y = -2$

Solution:

$(-2, 0), (0, -2)$

Exercise:

Problem: $x + y = -5$

Exercise:

Problem: $x - y = 5$

Solution:

$(5,0),(0,-5)$

Exercise:

Problem: $x - y = 1$

Exercise:

Problem: $x - y = -3$

Solution:

$(-3,0),\quad(0,3)$

Exercise:

Problem: $x - y = -4$

Exercise:

Problem: $x + 2y = 8$

Solution:

$(8,0),(0,4)$

Exercise:

Problem: $x + 2y = 10$

Exercise:

Problem: $3x + y = 6$

Solution:

$(2, 0), (0, 6)$

Exercise:

Problem: $3x + y = 9$

Exercise:

Problem: $x - 3y = 12$

Solution:

$(12, 0), (0, -4)$

Exercise:

Problem: $x - 2y = 8$

Exercise:

Problem: $4x - y = 8$

Solution:

$(2, 0), (0, -8)$

Exercise:

Problem: $5x - y = 5$

Exercise:

Problem: $2x + 5y = 10$

Solution:

$(5, 0), (0, 2)$

Exercise:

Problem: $2x + 3y = 6$

Exercise:

Problem: $3x – 2y = 12$

Solution:

$(4, 0), (0, -6)$

Exercise:

Problem: $3x – 5y = 30$

Exercise:

Problem: $y = \frac{1}{3}x + 1$

Solution:

$(3, 0), (0, -1)$

Exercise:

Problem: $y = \frac{1}{4}x - 1$

Exercise:

Problem: $y = \frac{1}{5}x + 2$

Solution:

$(-10, 0), (0, 2)$

Exercise:

Problem: $y = \frac{1}{3}x + 4$

Exercise:

Problem: $y = 3x$

Solution:

$(0, 0)$

Exercise:

Problem: $y = -2x$

Exercise:

Problem: $y = -4x$

Solution:

$(0, 0)$

Exercise:

Problem: $y = 5x$

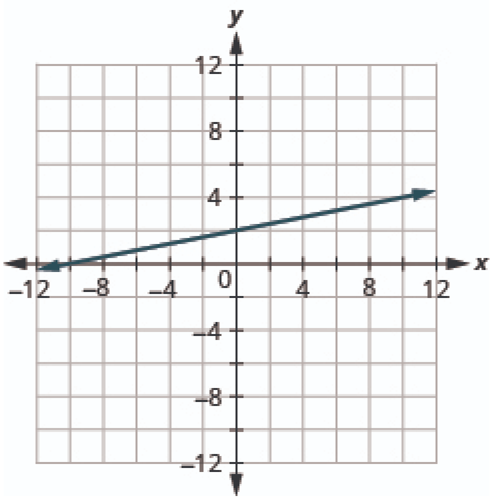
Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

Exercise:

Problem: $-x+5y=10$

Solution:



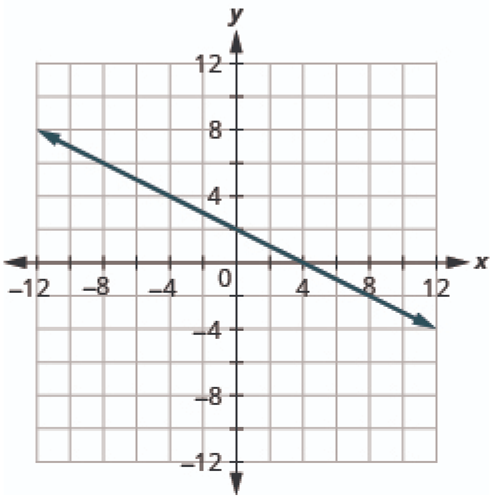
Exercise:

Problem: $-x+4y=8$

Exercise:

Problem: $x+2y=4$

Solution:



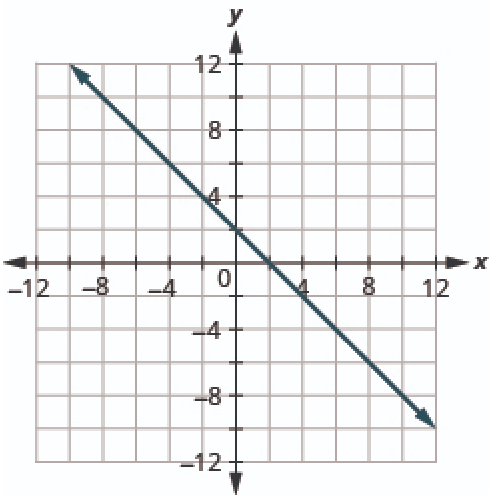
Exercise:

Problem: $x + 2y = 6$

Exercise:

Problem: $x + y = 2$

Solution:



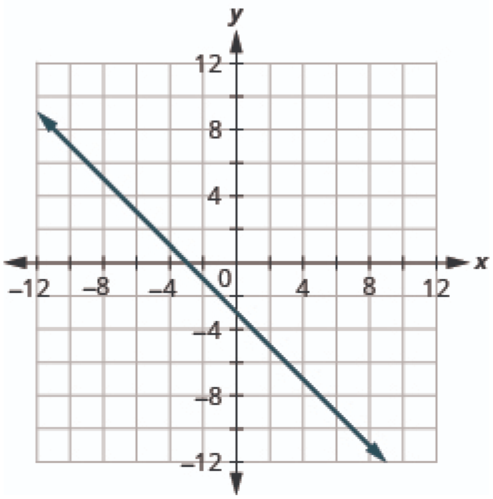
Exercise:

Problem: $x + y = 5$

Exercise:

Problem: $x + y = -3$

Solution:



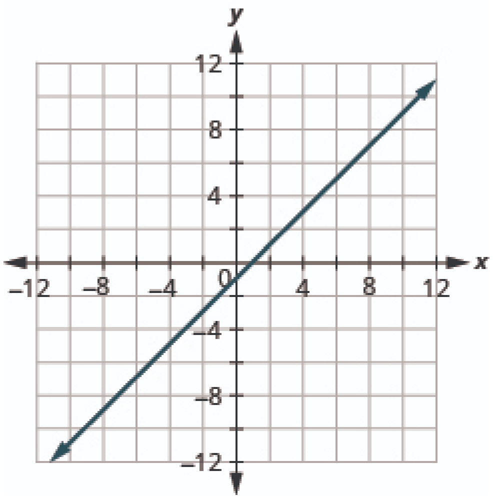
Exercise:

Problem: $x + y = -1$

Exercise:

Problem: $x - y = 1$

Solution:



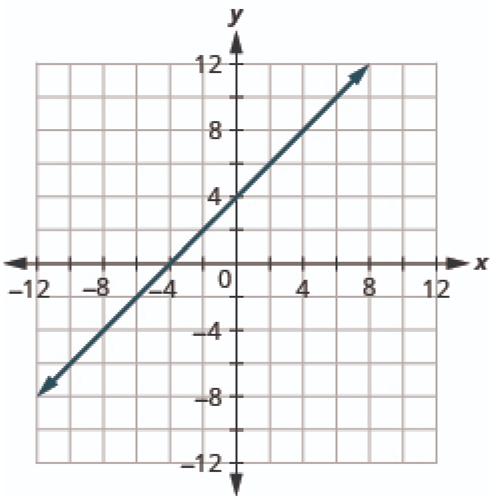
Exercise:

Problem: $x - y = 2$

Exercise:

Problem: $x - y = -4$

Solution:



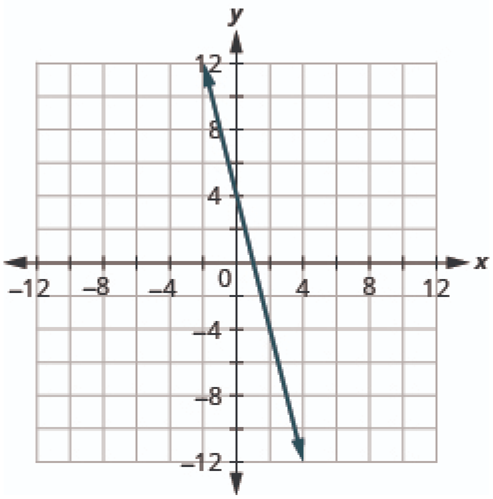
Exercise:

Problem: $x - y = -3$

Exercise:

Problem: $4x + y = 4$

Solution:



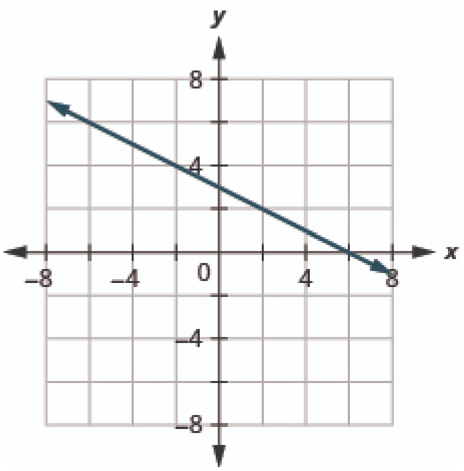
Exercise:

Problem: $3x + y = 3$

Exercise:

Problem: $2x + 4y = 12$

Solution:



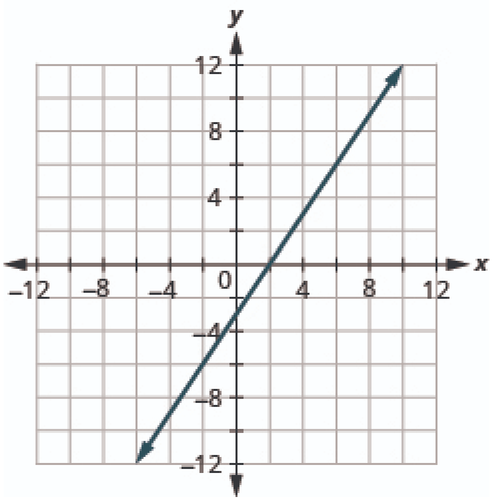
Exercise:

Problem: $3x + 2y = 12$

Exercise:

Problem: $3x – 2y = 6$

Solution:



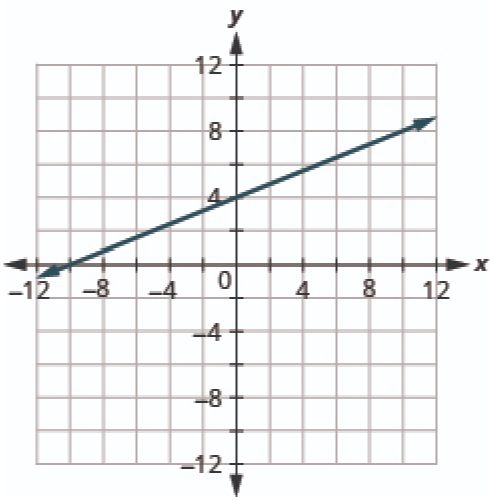
Exercise:

Problem: $5x – 2y = 10$

Exercise:

Problem: $2x - 5y = -20$

Solution:



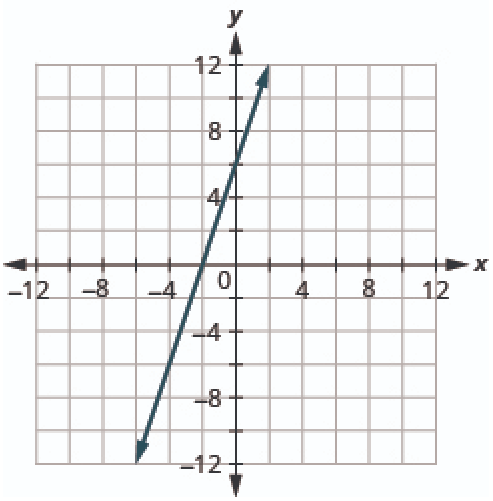
Exercise:

Problem: $3x - 4y = -12$

Exercise:

Problem: $3x - y = -6$

Solution:



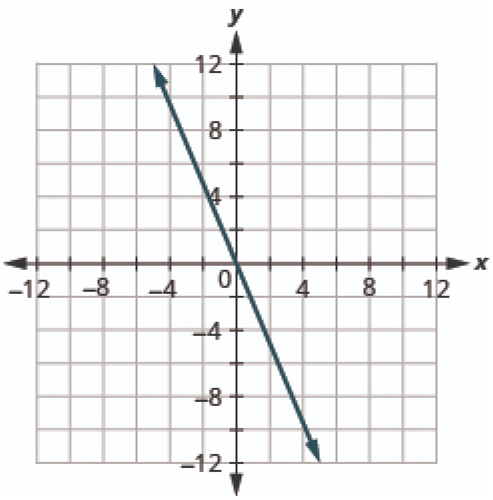
Exercise:

Problem: $2x – y = -8$

Exercise:

Problem: $y = -2x$

Solution:



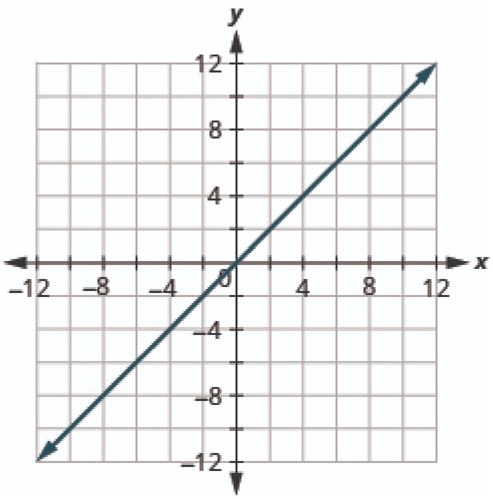
Exercise:

Problem: $y = -4x$

Exercise:

Problem: $y = x$

Solution:



Exercise:

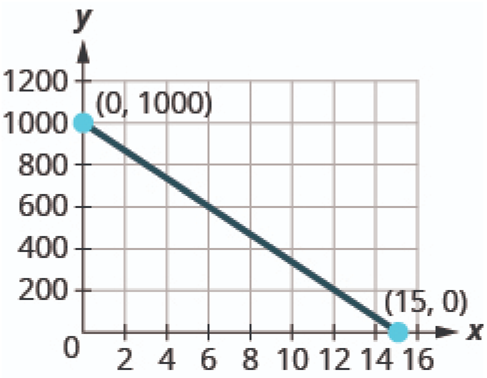
Problem: $y = 3x$

Everyday Math

Exercise:

Problem:

Road trip. Damien is driving from Chicago to Denver, a distance of 1000 miles. The *x*- axis on the graph below shows the time in hours since Damien left Chicago. The *y*- axis represents the distance he has left to drive.



ⓐ Find the *x*- and *y*- intercepts.
ⓑ Explain what the *x*- and *y*- intercepts mean for Damien.

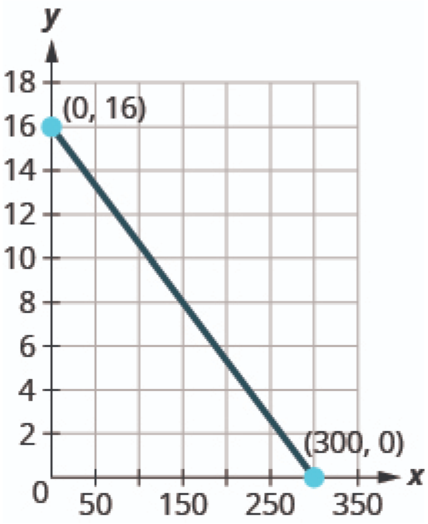
Solution:

ⓐ $(0, 1000), (15, 0)$
ⓑ At $(0, 1000)$, he has been gone 0 hours and has 1000 miles left. At $(15, 0)$, he has been gone 15 hours and has 0 miles left to go.

Exercise:

Problem:

Road trip. Ozzie filled up the gas tank of his truck and headed out on a road trip. The *x*- axis on the graph below shows the number of miles Ozzie drove since filling up. The *y*- axis represents the number of gallons of gas in the truck’s gas tank.



ⓐ Find the *x*- and *y*- intercepts.
ⓑ Explain what the *x*- and *y*- intercepts mean for Ozzie.

Writing Exercises

Exercise:

Problem:

How do you find the *x*- intercept of the graph of $3x-2y=6$?

Solution:

Answers will vary.

Exercise:

Problem:

Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $4x+y=-4$? Why?

Exercise:

Problem:

Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $y = \frac{2}{3}x - 2$? Why?

Solution:

Answers will vary.

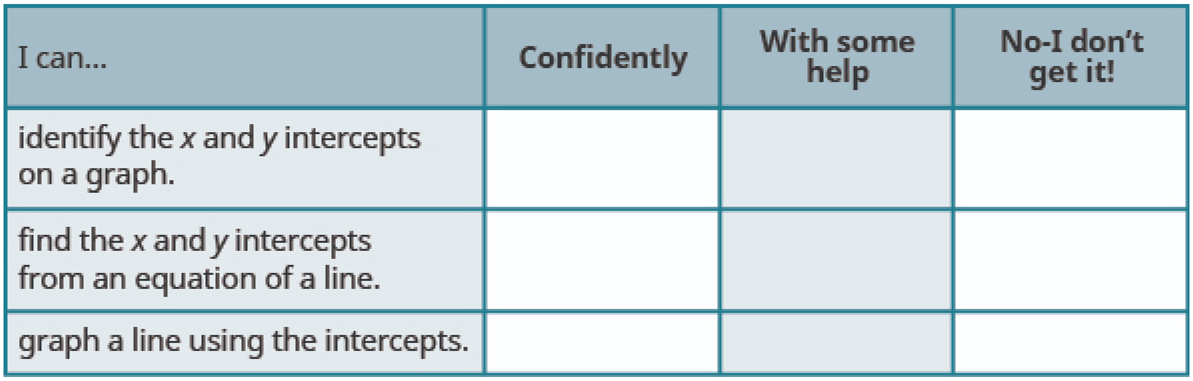
Exercise:

Problem:

Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $y = 6$? Why?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
identify the x and y intercepts on a graph.			
find the x and y intercepts from an equation of a line.			
graph a line using the intercepts.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

intercepts of a line

The points where a line crosses the *x*- axis and the *y*- axis are called the intercepts of the line.

x- intercept

The point $(a, 0)$ where the line crosses the *x*- axis; the *x*- intercept occurs when y is zero.

y-intercept

The point $(0, b)$ where the line crosses the *y*- axis; the *y*- intercept occurs when x is zero.

Understand Slope of a Line

By the end of this section, you will be able to:

- Use geoboards to model slope
- Use $m=\frac{rise}{run}$ to find the slope of a line from its graph
- Find the slope of horizontal and vertical lines
- Use the slope formula to find the slope of a line between two points
- Graph a line given a point and the slope
- Solve slope applications

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\frac{1-4}{8-2}$.
   If you missed this problem, review [link].
2. Divide: $\frac{0}{4}$, $\frac{4}{0}$.
   If you missed this problem, review [link].
3. Simplify: $\frac{15}{-3}$, $\frac{-15}{3}$, $\frac{-15}{-3}$.
   If you missed this problem, review [link].

When you graph linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter. What determines whether a line tilts up or down or if it is steep or flat?

In mathematics, the ‘tilt’ of a line is called the *slope* of the line. The concept of slope has many applications in the real world. The pitch of a roof, grade of a highway, and a ramp for a wheelchair are some examples where you literally see slopes. And when you ride a bicycle, you feel the slope as you pump uphill or coast downhill.

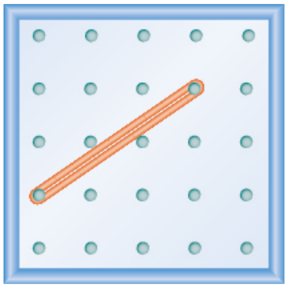
In this section, we will explore the concept of slope.

Use Geoboards to Model Slope

A **geoboard** is a board with a grid of pegs on it. Using rubber bands on a geoboard gives us a concrete way to model lines on a coordinate grid. By stretching a rubber band between two pegs on a geoboard, we can discover how to find the slope of a line.

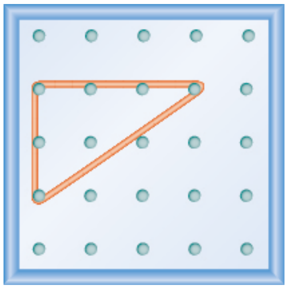
Note: Doing the Manipulative Mathematics activity “Exploring Slope” will help you develop a better understanding of the slope of a line. (Graph paper can be used instead of a geoboard, if needed.)

We’ll start by stretching a rubber band between two pegs as shown in [link].



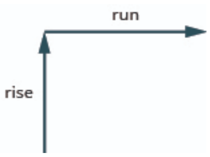
Doesn’t it look like a line?

Now we stretch one part of the rubber band straight up from the left peg and around a third peg to make the sides of a right triangle, as shown in [link]

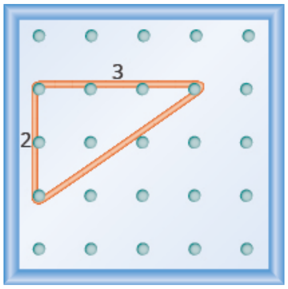


We carefully make a 90º angle around the third peg, so one of the newly formed lines is vertical and the other is horizontal.

To find the slope of the line, we measure the distance along the vertical and horizontal sides of the triangle. The vertical distance is called the rise and the horizontal distance is called the run, as shown in [link].



If our geoboard and rubber band look just like the one shown in [link], the rise is 2. The rubber band goes up 2 units. (Each space is one unit.)



The rise on this

geoboard is 2, as the
rubber band goes up
two units.

What is the run?

The rubber band goes across 3 units. The run is 3 (see [link]).

The slope of a line is the ratio of the rise to the run. In mathematics, it is always referred to with the letter m.

Note:
Slope of a Line
The **slope of a line** of a line is $m=\frac{\text{rise}}{\text{run}}$.
The **rise** measures the vertical change and the **run** measures the horizontal change between two points on the line.

What is the slope of the line on the geoboard in [link]?
Equation:

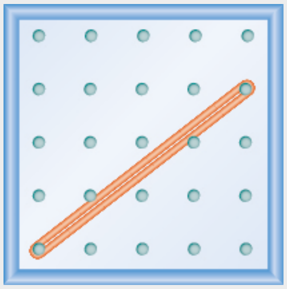
$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ m &= \frac{2}{3} \end{aligned}$$

The line has slope $\frac{2}{3}$. This means that the line rises 2 units for every 3 units of run.

When we work with geoboards, it is a good idea to get in the habit of starting at a peg on the left and connecting to a peg to the right. If the rise goes up it is positive and if it goes down it is negative. The run will go from left to right and be positive.

Example:
Exercise:

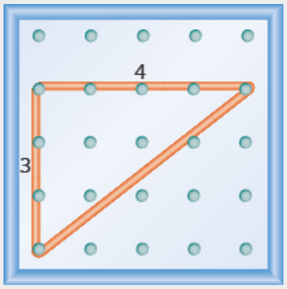
Problem: What is the slope of the line on the geoboard shown?



Solution:
Solution

Use the definition of slope: $m=\frac{\text{rise}}{\text{run}}$.

Start at the left peg and count the spaces up and to the right to reach the second peg.



The rise is 3. $m = \frac{3}{\text{run}}$

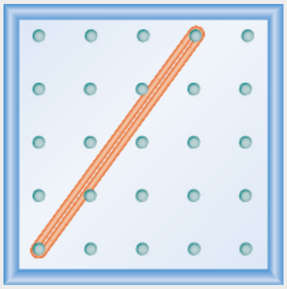
The run is 4. $m = \frac{3}{4}$

The slope is $\frac{3}{4}$.

This means that the line rises 3 units for every 4 units of run.

Note:
Exercise:

Problem: What is the slope of the line on the geoboard shown?

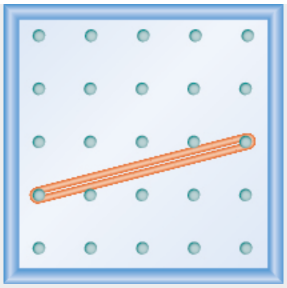


Solution:

$\frac{4}{3}$

Note:
Exercise:

Problem: What is the slope of the line on the geoboard shown?

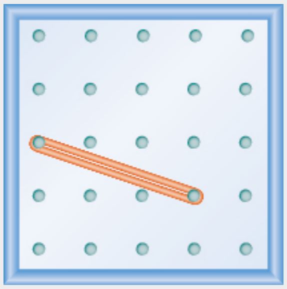


Solution:

$\frac{1}{4}$

Example:
Exercise:

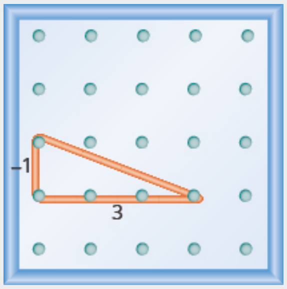
Problem: What is the slope of the line on the geoboard shown?



Solution:
Solution

Use the definition of slope: $m = \frac{\text{rise}}{\text{run}}$.

Start at the left peg and count the units down and to the right to reach the second peg.



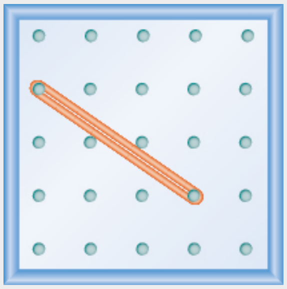
The rise is -1.	$m = \frac{-1}{\text{run}}$
The run is 3.	$m = \frac{-1}{3}$
	$m = -\frac{1}{3}$

The slope is $-\frac{1}{3}$.

This means that the line drops 1 unit for every 3 units of run.

Note:
Exercise:

Problem: What is the slope of the line on the geoboard?

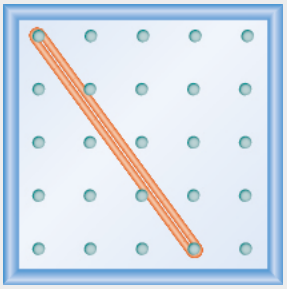


Solution:

$-\frac{2}{3}$

Note:
Exercise:

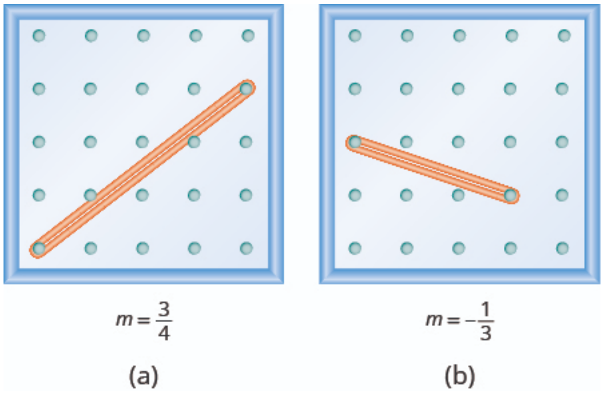
Problem: What is the slope of the line on the geoboard?



Solution:

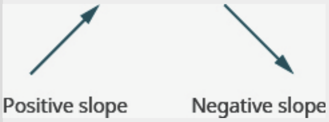
$-\frac{4}{3}$

Notice that in [link] the slope is positive and in [link] the slope is negative. Do you notice any difference in the two lines shown in [link](a) and [link](b)?



We ‘read’ a line from left to right just like we read words in English. As you read from left to right, the line in [link](a) is going up; it has **positive slope**. The line in [link](b) is going down; it has **negative slope**.

Note:
Positive and Negative Slopes



Example:
Exercise:

Problem: Use a geoboard to model a line with slope $\frac{1}{2}$.

Solution:
Solution

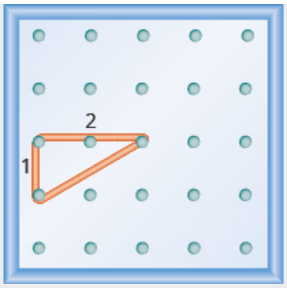
To model a line on a geoboard, we need the rise and the run.

Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Replace m with $\frac{1}{2}$.	$\frac{1}{2} = \frac{\text{rise}}{\text{run}}$

So, the rise is 1 and the run is 2.

Start at a peg in the lower left of the geoboard.

Stretch the rubber band up 1 unit, and then right 2 units.

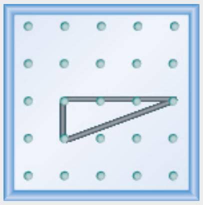


The hypotenuse of the right triangle formed by the rubber band represents a line whose slope is $\frac{1}{2}$.

Note:
Exercise:

Problem: Model the slope $m = \frac{1}{3}$. Draw a picture to show your results.

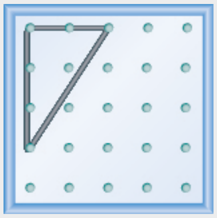
Solution:



Note:
Exercise:

Problem: Model the slope $m = \frac{3}{2}$. Draw a picture to show your results.

Solution:



Example:
Exercise:

Problem: Use a geoboard to model a line with slope $\frac{-1}{4}$.

Solution:
Solution

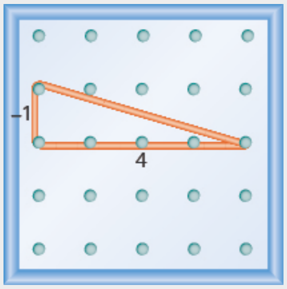
Use the slope formula. $m = \frac{\text{rise}}{\text{run}}$

Replace m with $\frac{-1}{4}$. $\frac{-1}{4} = \frac{\text{rise}}{\text{run}}$

So, the rise is -1 and the run is 4.

Since the rise is negative, we choose a starting peg on the upper left that will give us room to count down.

We stretch the rubber band down 1 unit, then go to the right 4 units, as shown.

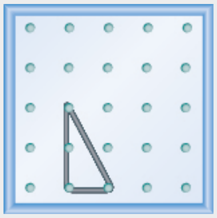


The hypotenuse of the right triangle formed by the rubber band represents a line whose slope is $\frac{-1}{4}$.

Note:
Exercise:

Problem: Model the slope $m = \frac{-2}{3}$. Draw a picture to show your results.

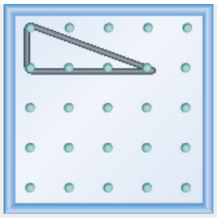
Solution:



Note:
Exercise:

Problem: Model the slope $m = \frac{-1}{3}$. Draw a picture to show your results.

Solution:



Use $m = \frac{\text{rise}}{\text{run}}$ to Find the Slope of a Line from its Graph

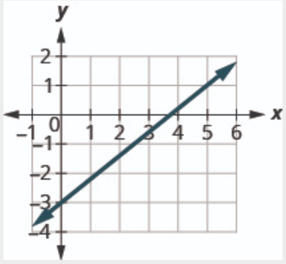
Now, we’ll look at some graphs on the xy-coordinate plane and see how to find their slopes. The method will be very similar to what we just modeled on our geoboards.

To find the slope, we must count out the rise and the run. But where do we start?

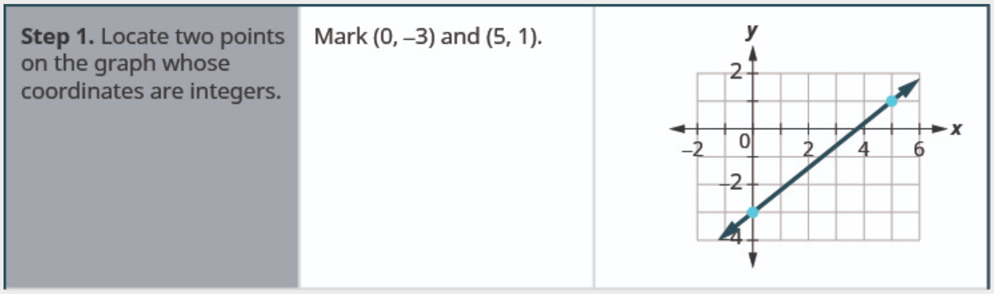
We locate two points on the line whose coordinates are integers. We then start with the point on the left and sketch a right triangle, so we can count the rise and run.

Example:
How to Use $m = \frac{\text{rise}}{\text{run}}$ to Find the Slope of a Line from its Graph
Exercise:

Problem: Find the slope of the line shown.

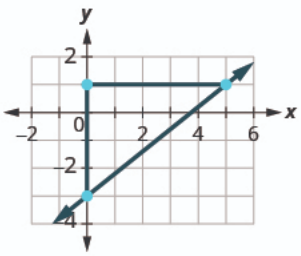
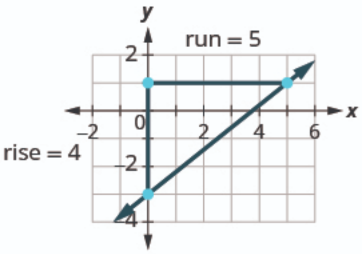


Solution:
Solution



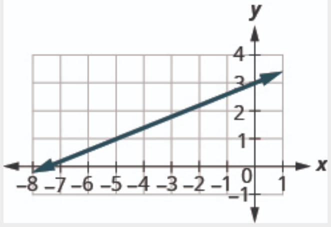
Step 1. Locate two points on the graph whose coordinates are integers.	Mark (0, –3) and (5, 1).	

y
x
2
–2
0
2
4
6
–2
–4

Step 2. Starting with the point on the left, sketch a right triangle, going from the first point to the second point.	Starting at (0, –3), sketch a right triangle to (5, 1).	
Step 3. Count the rise and the run on the legs of the triangle.	Count the rise. Count the run.	 The rise is 4. The run is 5.
Step 4. Take the ratio of rise to run to find the slope. $m = \frac{\text{rise}}{\text{run}}$	Use the slope formula. Substitute the values of the rise and run.	$m = \frac{\text{rise}}{\text{run}}$ $m = \frac{4}{5}$ The slope of the line is $\frac{4}{5}$. This means that y increases 4 units as x increases 5 units.

Note:
Exercise:

Problem: Find the slope of the line shown.

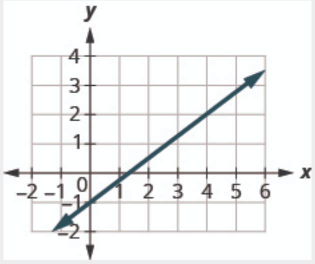


Solution:

$\frac{2}{5}$

Note:
Exercise:

Problem: Find the slope of the line shown.



Solution:

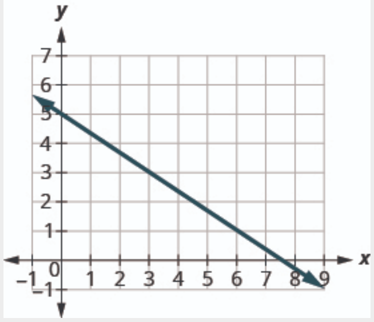
$\frac{3}{4}$

Note:
Find the slope of a line from its graph using $m=\frac{\text{rise}}{\text{run}}$.

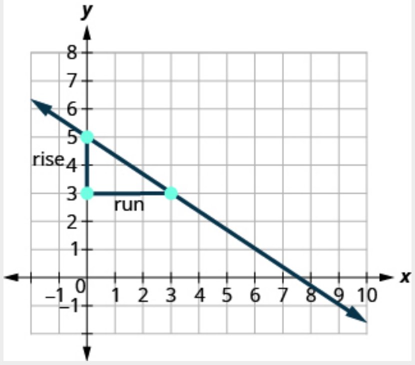
Locate two points on the line whose coordinates are integers.
Starting with the point on the left, sketch a right triangle, going from the first point to the second point.
Count the rise and the run on the legs of the triangle.
Take the ratio of rise to run to find the slope,$m=\frac{\text{rise}}{\text{run}}$.

Example:
Exercise:

Problem: Find the slope of the line shown.

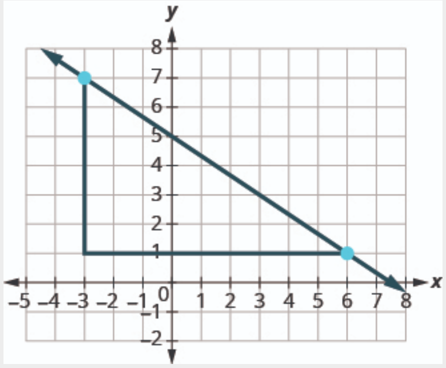


Solution:
Solution

Locate two points on the graph whose coordinates are integers.	$(0,5)$ and $(3,3)$
Which point is on the left?	$(0,5)$
Starting at $(0,5)$, sketch a right triangle to $(3,3)$.	
Count the rise—it is negative.	The rise is -2.
Count the run.	The run is 3.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{-2}{3}$
Simplify.	$m = -\frac{2}{3}$
	The slope of the line is $-\frac{2}{3}$.

So y increases by 3 units as x decreases by 2 units.

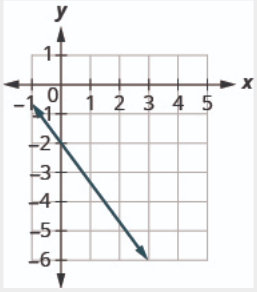
What if we used the points $(-3,7)$ and $(6,1)$ to find the slope of the line?



The rise would be -6 and the run would be 9. Then $m = \frac{-6}{9}$, and that simplifies to $m = -\frac{2}{3}$. Remember, it does not matter which points you use—the slope of the line is always the same.

Note:
Exercise:

Problem: Find the slope of the line shown.

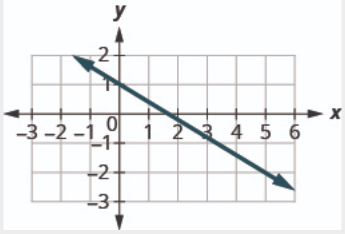


Solution:

$-\frac{4}{3}$

Note:
Exercise:

Problem: Find the slope of the line shown.



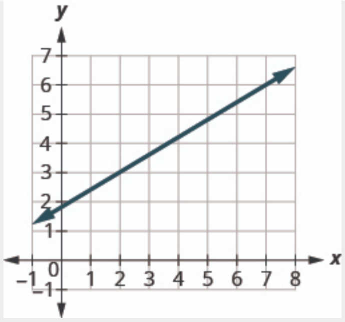
Solution:

$-\frac{3}{5}$

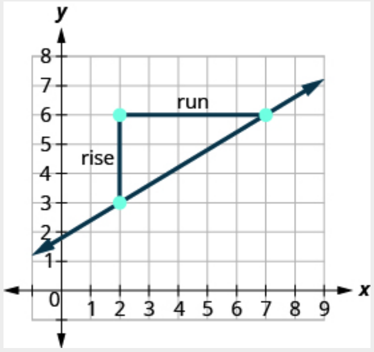
In the last two examples, the lines had *y*-intercepts with integer values, so it was convenient to use the *y*-intercept as one of the points to find the slope. In the next example, the *y*-intercept is a fraction. Instead of using that point, we’ll look for two other points whose coordinates are integers. This will make the slope calculations easier.

Example:
Exercise:

Problem: Find the slope of the line shown.



Solution:
Solution

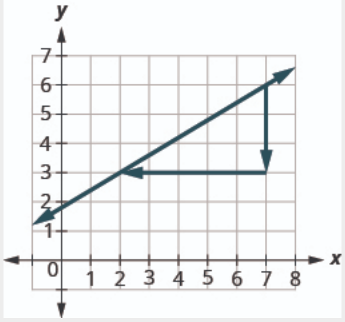
Locate two points on the graph whose coordinates are integers.	$(2,3)$ and $(7,6)$
Which point is on the left?	$(2,3)$
Starting at $(2,3)$, sketch a right triangle to $(7,6)$.	
Count the rise.	The rise is 3.
Count the run.	The run is 5.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{3}{5}$
	The slope of the line is $\frac{3}{5}$.

This means that y increases 5 units as x increases 3 units.

When we used geoboards to introduce the concept of slope, we said that we would always start with the point on the left and count the rise and the run to get to the point on the right. That way the run was always positive and the rise determined whether the slope was positive or negative.

What would happen if we started with the point on the right?

Let’s use the points $(2,3)$ and $(7,6)$ again, but now we’ll start at $(7,6)$.

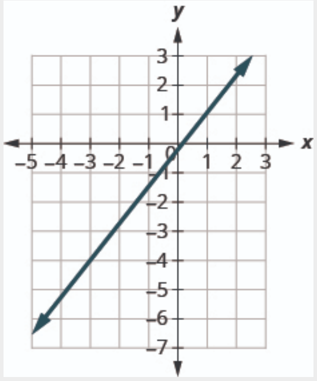


Count the rise.	The rise is -3.
Count the run. It goes from right to left, so it is negative.	The run is -5.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{-3}{-5}$
	The slope of the line is $\frac{3}{5}$.

It does not matter where you start—the slope of the line is always the same.

Note:
Exercise:

Problem: Find the slope of the line shown.

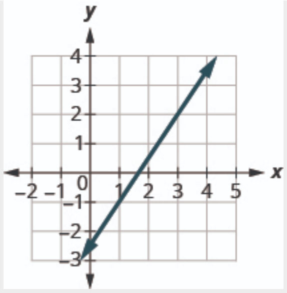


Solution:

$\frac{5}{4}$

Note:
Exercise:

Problem: Find the slope of the line shown.



Solution:

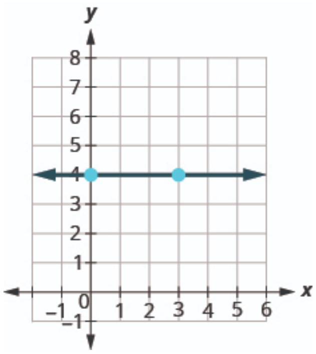
$\frac{3}{2}$

Find the Slope of Horizontal and Vertical Lines

Do you remember what was special about horizontal and vertical lines? Their equations had just one variable.
Equation:

Horizontal line $y=b$	**Vertical line** $x=a$
y-coordinates are the same.	x-coordinates are the same.

So how do we find the slope of the horizontal line $y=4$? One approach would be to graph the horizontal line, find two points on it, and count the rise and the run. Let’s see what happens when we do this.



What is the rise?	The rise is 0.
What is the run?	The run is 3.
	$m = \frac{\text{rise}}{\text{run}}$
What is the slope?	$m = \frac{0}{3}$
	$m = 0$
	The slope of the horizontal line $y=4$ is 0.

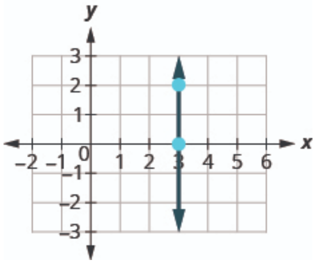
All horizontal lines have slope 0. When the *y*-coordinates are the same, the rise is 0.

Note:
Slope of a Horizontal Line

The slope of a horizontal line, $y = b$, is 0.

The floor of your room is horizontal. Its slope is 0. If you carefully placed a ball on the floor, it would not roll away.

Now, we'll consider a vertical line, the line.



What is the rise?	The rise is 2.
What is the run?	The run is 0.
What is the slope?	$m = \frac{\text{rise}}{\text{run}}$ $m = \frac{2}{0}$

But we can't divide by 0. Division by 0 is not defined. So we say that the slope of the vertical line $x = 3$ is undefined.

The slope of any vertical line is undefined. When the *x*-coordinates of a line are all the same, the run is 0.

Note:
Slope of a Vertical Line
The slope of a vertical line, $x = a$, is undefined.

Example:
Exercise:

Problem: Find the slope of each line:

ⓐ $x = 8$ ⓑ $y = -5.$

Solution:
Solution

ⓐ $x = 8$
This is a vertical line.
Its slope is undefined.

ⓑ $y = -5$
This is a horizontal line.
It has slope 0.

Note:
Exercise:

Problem: Find the slope of the line: $x = -4$.

Solution:

undefined

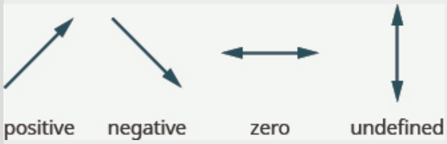
Note:
Exercise:

Problem: Find the slope of the line: $y = 7$.

Solution:

0

Note:
Quick Guide to the Slopes of Lines



Remember, we ‘read’ a line from left to right, just like we read written words in English.

Use the Slope Formula to find the Slope of a Line Between Two Points

Note: Doing the Manipulative Mathematics activity “Slope of Lines Between Two Points” will help you develop a better understanding of how to find the slope of a line between two points.

Sometimes we’ll need to find the slope of a line between two points when we don’t have a graph to count out the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but as we’ll see, there is a way to find the slope without graphing. Before we get to it, we need to introduce some algebraic notation.

We have seen that an ordered pair (x, y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x, y) be used to represent two different points? Mathematicians use subscripts to distinguish the points.
Equation:

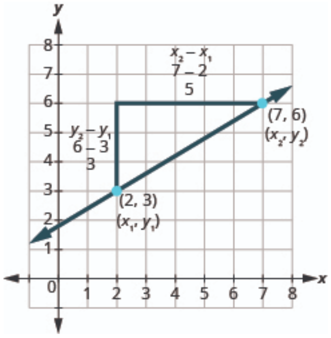
$$\begin{array}{ll}(x_1, y_1) & \text{read ‘ } x \text{ sub } 1, y \text{ sub } 1\text{’}\\ (x_2, y_2) & \text{read ‘ } x \text{ sub } 2, y \text{ sub } 2\text{’}\end{array}$$

The use of subscripts in math is very much like the use of last name initials in elementary school. Maybe you remember Laura C. and Laura M. in your third grade class?

We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point.

If we had more than two points, we could use (x_3, y_3), (x_4, y_4), and so on.

Let’s see how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points $(2, 3)$ and $(7, 6)$.



Since we have two points, we will use subscript notation, $\left(\overset{x_1, y_1}{2, 3}\right)\left(\overset{x_2, y_2}{7, 6}\right)$.

On the graph, we counted the rise of 3 and the run of 5.

Notice that the rise of 3 can be found by subtracting the *y*-coordinates 6 and 3.
Equation:

$$3 = 6 - 3$$

And the run of 5 can be found by subtracting the *x*-coordinates 7 and 2.
Equation:

$$5 = 7 - 2$$

We know $m = \frac{\text{rise}}{\text{run}}$. So $m = \frac{3}{5}$.

We rewrite the rise and run by putting in the coordinates $m = \frac{6-3}{7-2}$.

But 6 is y_2, the *y*-coordinate of the second point and 3 is y_1, the *y*-coordinate of the first point.

So we can rewrite the slope using subscript notation. $m = \frac{y_2 - y_1}{7-2}$

Also, 7 is x_2, the *x*-coordinate of the second point and 2 is x_1, the *x*-coordinate of the first point.

So, again, we rewrite the slope using subscript notation. $m = \frac{y_2 - y_1}{x_2 - x_1}$

We’ve shown that $m = \frac{y_2 - y_1}{x_2 - x_1}$ is really another version of $m = \frac{\text{rise}}{\text{run}}$. We can use this formula to find the slope of a line when we have two points on the line.

Note:
Slope Formula
The slope of the line between two points (x_1, y_1) and (x_2, y_2) is
Equation:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

This is the **slope formula**.
The slope is:
Equation:

$$\begin{gathered} y \text{ of the second point minus } y \text{ of the first point} \\ \text{over} \\ x \text{ of the second point minus } x \text{ of the first point.} \end{gathered}$$

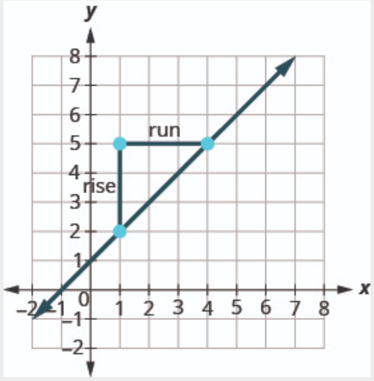
Example:
Exercise:

Problem: Use the slope formula to find the slope of the line between the points $(1, 2)$ and $(4, 5)$.

Solution:
Solution

We'll call $(1, 2)$ point #1 and $(4, 5)$ point #2.	$\left(\overset{x_1, y_1}{1, 2}\right) \left(\overset{x_2, y_2}{4, 5}\right)$
Use the slope formula.	$m = \frac{y_2 - y_1}{x_2 - x_1}$
Substitute the values.	
y of the second point minus y of the first point	$m = \frac{5 - 2}{x_2 - x_1}$
x of the second point minus x of the first point	$m = \frac{5 - 2}{4 - 1}$
Simplify the numerator and the denominator.	$m = \frac{3}{3}$
Simplify.	$m = 1$

Let's confirm this by counting out the slope on a graph using $m = \frac{\text{rise}}{\text{run}}$.



It doesn’t matter which point you call point #1 and which one you call point #2. The slope will be the same. Try the calculation yourself.

Note:
Exercise:

Problem: Use the slope formula to find the slope of the line through the points: $(8, 5)$ and $(6, 3)$.

Solution:

1

Note:
Exercise:

Problem: Use the slope formula to find the slope of the line through the points: $(1, 5)$ and $(5, 9)$.

Solution:

1

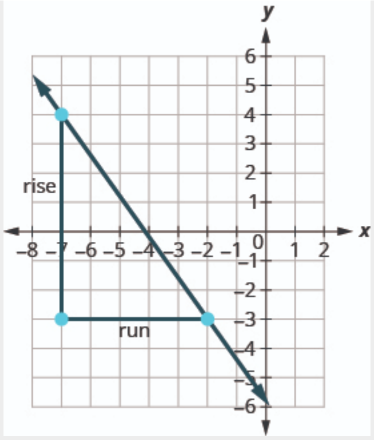
Example:
Exercise:

Problem: Use the slope formula to find the slope of the line through the points $(-2, -3)$ and $(-7, 4)$.

Solution:
Solution

We’ll call $(-2, -3)$ point $\#1$ and $(-7, 4)$ point $\#2$.	$\left(\overset{x_1,}{-2,}\ \overset{y_1}{-3}\right)\ \Big(\overset{x_2,}{-}$
Use the slope formula.	$m = \frac{y_2 - y_1}{x_2 - x_1}$
Substitute the values.	
y of the second point minus y of the first point	$m = \frac{4-(-3)}{x_2 - x_1}$
x of the second point minus x of the first point	$m = \frac{4-(-3)}{-7-(-2)}$
Simplify.	$m = \frac{7}{-5}$
	$m = -\frac{7}{5}$

Let’s verify this slope on the graph shown.



Equation:

$$\begin{aligned} m &= \tfrac{\text{rise}}{\text{run}} \\ m &= \tfrac{-7}{5} \\ m &= -\tfrac{7}{5} \end{aligned}$$

Note:
Exercise:

Problem: Use the slope formula to find the slope of the line through the points: $(-3,4)$ and $(2,-1)$.

Solution:

-1

Note:
Exercise:

Problem:

Use the slope formula to find the slope of the line through the pair of points: $(-2,6)$ and $(-3,-4)$.

Solution:

10

Graph a Line Given a Point and the Slope

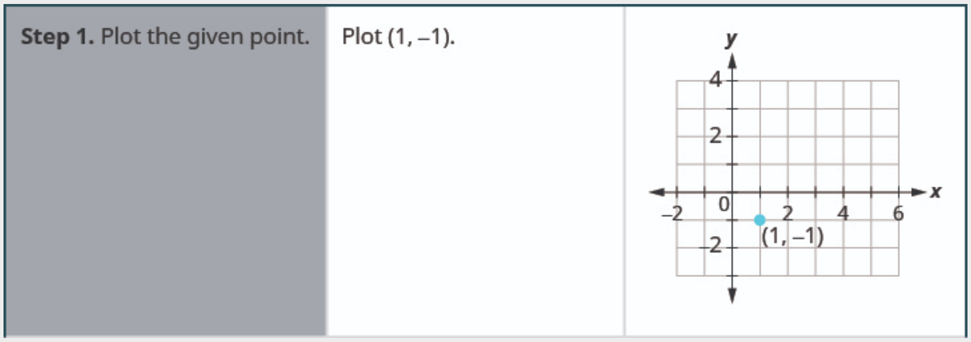
Up to now, in this chapter, we have graphed lines by plotting points, by using intercepts, and by recognizing horizontal and vertical lines.

One other method we can use to graph lines is called the **point–slope method**. We will use this method when we know one point and the slope of the line. We will start by plotting the point and then use the definition of slope to draw the graph of the line.

Example:
How To Graph a Line Given a Point and The Slope
Exercise:

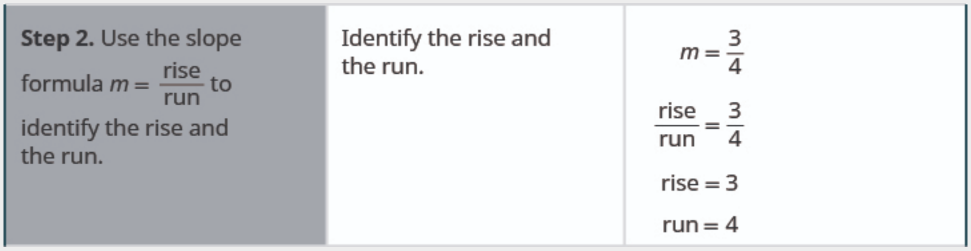
Problem: Graph the line passing through the point $(1, -1)$ whose slope is $m = \frac{3}{4}$.

Solution:
Solution

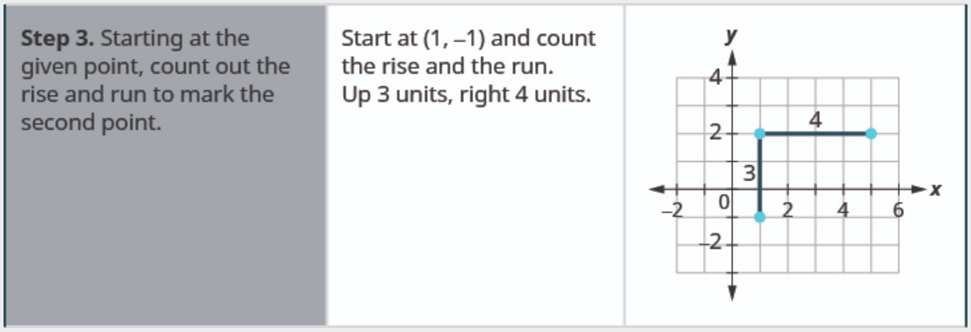


Step 1. Plot the given point.	Plot (1, –1).	

y
4
2
x
–2
0
2
4
6
–2
(1, –1)

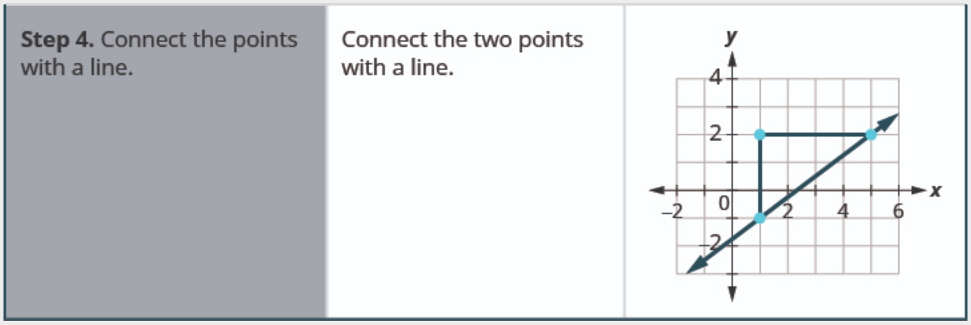


Step 2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m = \frac{3}{4}$ $\frac{\text{rise}}{\text{run}} = \frac{3}{4}$ $\text{rise} = 3$ $\text{run} = 4$



Step 3. Starting at the given point, count out the rise and run to mark the second point.	Start at (1, –1) and count the rise and the run. Up 3 units, right 4 units.	

y
4
2
4
3
x
–2
0
2
4
6
–2



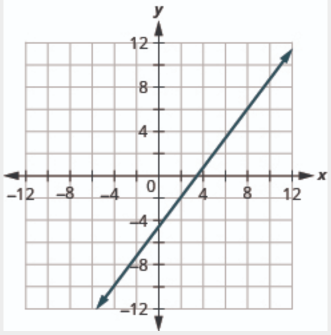
Step 4. Connect the points with a line.	Connect the two points with a line.	

y
4
2
x
–2
0
2
4
6
–2

Note:
Exercise:

Problem: Graph the line passing through the point $(2, -2)$ with the slope $m = \frac{4}{3}$.

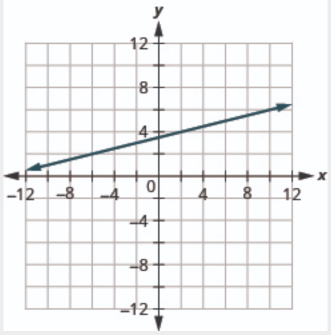
Solution:



Note:
Exercise:

Problem: Graph the line passing through the point $(-2, 3)$ with the slope $m = \frac{1}{4}$.

Solution:



Note:
Graph a line given a point and the slope.

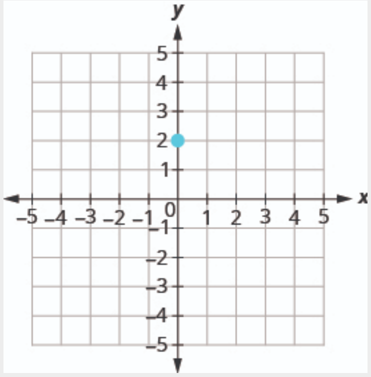
Plot the given point.
Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
Starting at the given point, count out the rise and run to mark the second point.
Connect the points with a line.

Example:
Exercise:

Problem: Graph the line with y-intercept 2 whose slope is $m = -\frac{2}{3}$.

Solution:
Solution

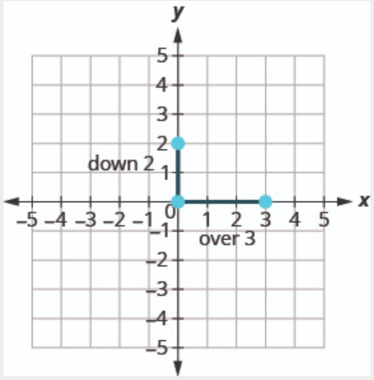
Plot the given point, the y-intercept, $(0, 2)$.



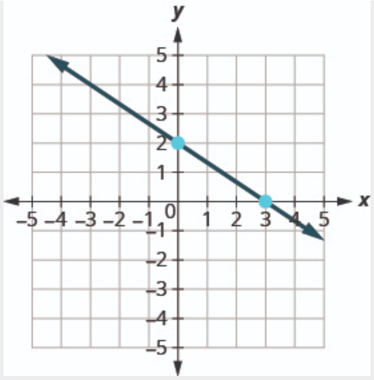
Identify the rise and the run.

$$\begin{aligned} m &= -\frac{2}{3} \\ \frac{\text{rise}}{\text{run}} &= \frac{-2}{3} \\ \text{rise} &= -2 \\ \text{run} &= 3 \end{aligned}$$

Count the rise and the run. Mark the second point.



Connect the two points with a line.

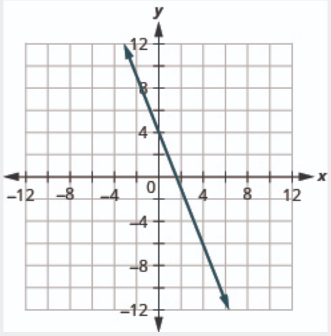


You can check your work by finding a third point. Since the slope is $m = -\frac{2}{3}$, it can be written as $m = \frac{2}{-3}$. Go back to $(0, 2)$ and count out the rise, 2, and the run, -3.

Note:
Exercise:

Problem: Graph the line with the *y*-intercept 4 and slope $m = -\frac{5}{2}$.

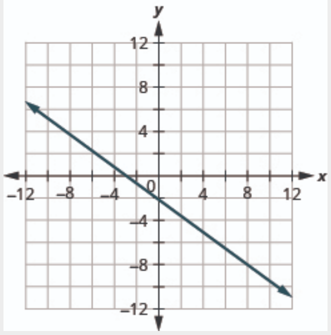
Solution:



Note:
Exercise:

Problem: Graph the line with the *x*-intercept -3 and slope $m = -\frac{3}{4}$.

Solution:

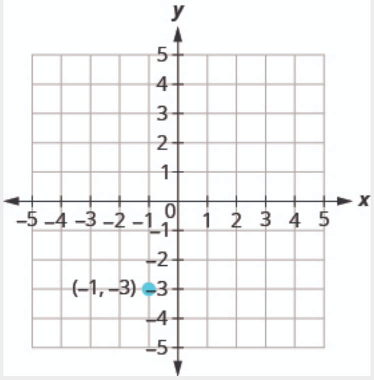


Example:
Exercise:

Problem: Graph the line passing through the point $(-1,-3)$ whose slope is $m=4$.

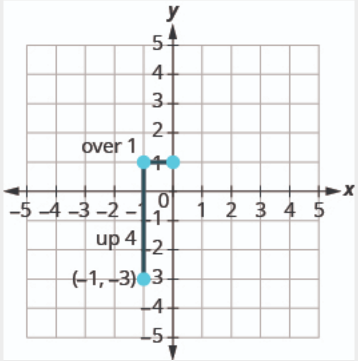
Solution:
Solution

Plot the given point.

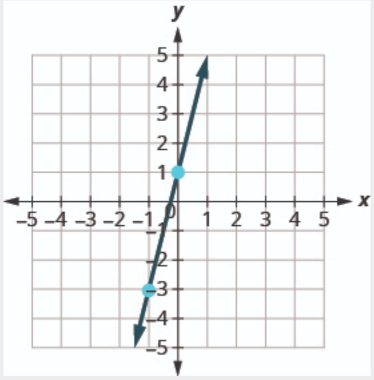


$$\begin{array}{lrcl} \text{Identify the rise and the run.} & m & = & 4 \\ \text{Write 4 as a fraction.} & \frac{\text{rise}}{\text{run}} & = & \frac{4}{1} \\ & \text{rise} & = & 4 \;\; \text{run} = 1 \end{array}$$

Count the rise and run and mark the second point.



Connect the two points with a line.

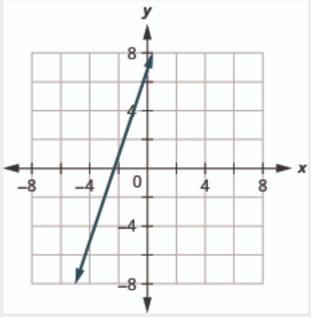


You can check your work by finding a third point. Since the slope is $m = 4$, it can be written as $m = \frac{-4}{-1}$. Go back to $(-1, -3)$ and count out the rise, -4, and the run, -1.

Note:
Exercise:

Problem: Graph the line with the point $(-2, 1)$ and slope $m = 3$.

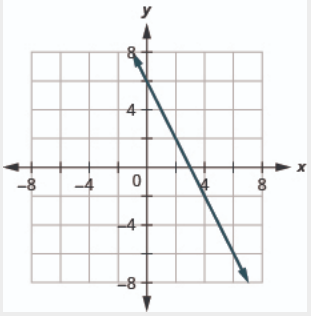
Solution:



Note:
Exercise:

Problem: Graph the line with the point $(4, -2)$ and slope $m = -2$.

Solution:



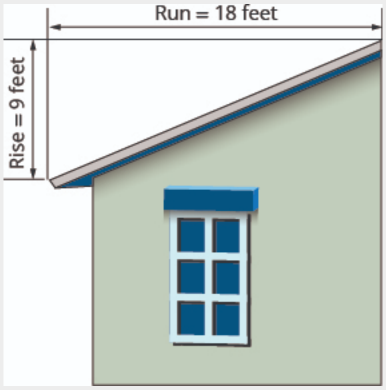
Solve Slope Applications

At the beginning of this section, we said there are many applications of slope in the real world. Let’s look at a few now.

Example:
Exercise:

Problem:

The ‘pitch’ of a building’s roof is the slope of the roof. Knowing the pitch is important in climates where there is heavy snowfall. If the roof is too flat, the weight of the snow may cause it to collapse. What is the slope of the roof shown?



Solution:
Solution

Use the slope formula.	$m=\frac{\text{rise}}{\text{run}}$
Substitute the values for rise and run.	$m=\frac{9}{18}$
Simplify.	$m=\frac{1}{2}$
The slope of the roof is $\frac{1}{2}$.	
	The roof rises 1 foot for every 2 feet of horizontal run.

Note:
Exercise:

Problem: Use [link], substituting the rise = 14 and run = 24.

Solution:

$\frac{7}{12}$

Note:
Exercise:

Problem: Use [link], substituting rise = 15 and run = 36.

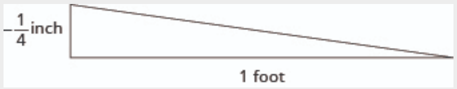
Solution:

$\frac{5}{12}$

Example:
Exercise:

Problem:

Have you ever thought about the sewage pipes going from your house to the street? They must slope down $\frac{1}{4}$ inch per foot in order to drain properly. What is the required slope?



Solution:
Solution

Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{-\frac{1}{4}\text{inch}}{1\text{ foot}}$
	$m = \frac{-\frac{1}{4}\text{inch}}{12\text{ inches}}$
Simplify.	$m = -\frac{1}{48}$

The slope of the pipe is $-\frac{1}{48}$.

The pipe drops 1 inch for every 48 inches of horizontal run.

Note:
Exercise:

Problem: Find the slope of a pipe that slopes down $\frac{1}{3}$ inch per foot.

Solution:

$-\frac{1}{36}$

Note:
Exercise:

Problem: Find the slope of a pipe that slopes down $\frac{3}{4}$ inch per yard.

Solution:

$-\frac{1}{48}$

Note:
Access these online resources for additional instruction and practice with understanding slope of a line.

- Practice Slope with a Virtual Geoboard
- Small, Medium, and Large Virtual Geoboards
- Explore Area and Perimeter with a Geoboard

Key Concepts

- **Find the Slope of a Line from its Graph using $m = \frac{\text{rise}}{\text{run}}$**

  Locate two points on the line whose coordinates are integers.
  Starting with the point on the left, sketch a right triangle, going from the first point to the second point.
  Count the rise and the run on the legs of the triangle.
  Take the ratio of rise to run to find the slope.

- **Graph a Line Given a Point and the Slope**

  Plot the given point.
  Use the slope formula$m = \frac{\text{rise}}{\text{run}}$to identify the rise and the run.
  Starting at the given point, count out the rise and run to mark the second point.
  Connect the points with a line.

- **Slope of a Horizontal Line**
  - The slope of a horizontal line, $y = b$, is 0.
- **Slope of a vertical line**
  - The slope of a vertical line, $x = a$, is undefined

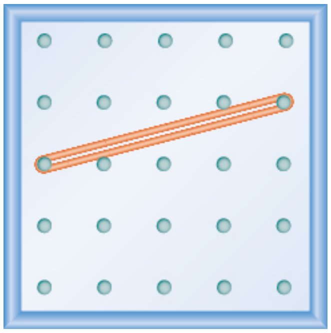
Practice Makes Perfect

Use Geoboards to Model Slope

In the following exercises, find the slope modeled on each geoboard.

Exercise:

Problem:

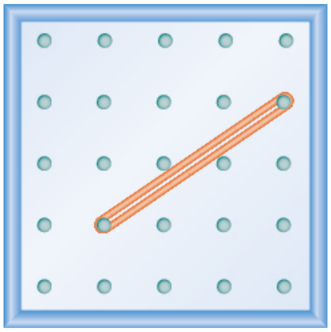


Solution:

$\frac{1}{4}$

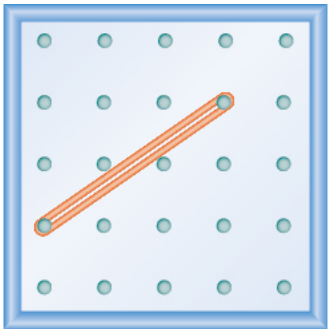
Exercise:

Problem:



Exercise:

Problem:

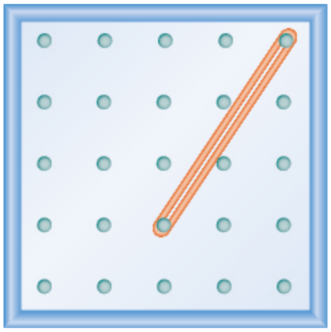


Solution:

$\frac{2}{3}$

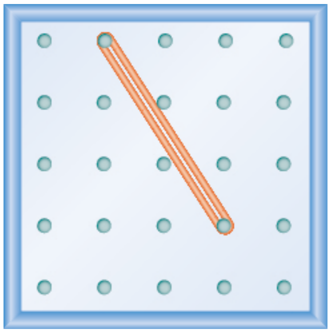
Exercise:

Problem:



Exercise:

Problem:

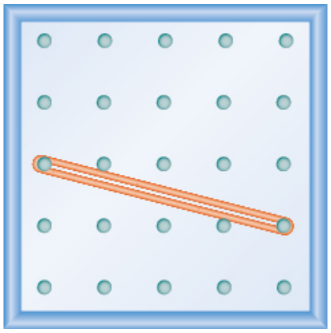


Solution:

$\frac{-3}{2} = -\frac{3}{2}$

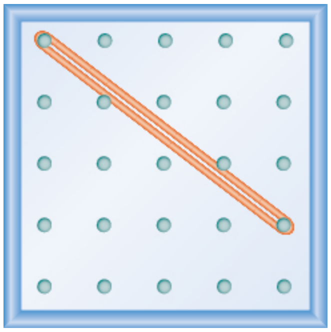
Exercise:

Problem:



Exercise:

Problem:

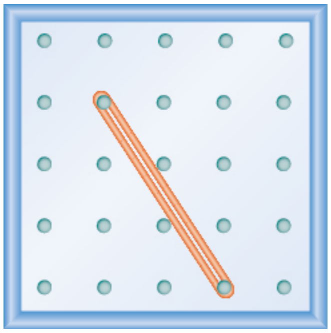


Solution:

$-\frac{2}{3}$

Exercise:

Problem:

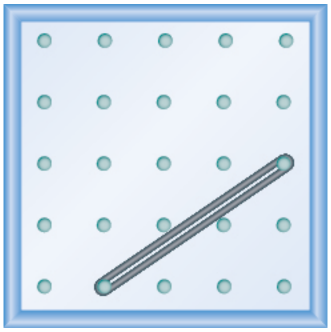


In the following exercises, model each slope. Draw a picture to show your results.

Exercise:

Problem: $\frac{2}{3}$

Solution:



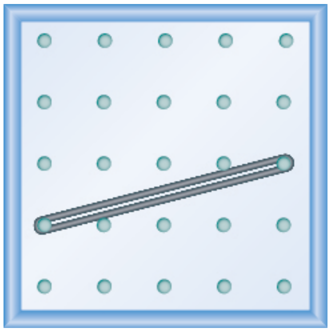
Exercise:

Problem: $\frac{3}{4}$

Exercise:

Problem: $\frac{1}{4}$

Solution:



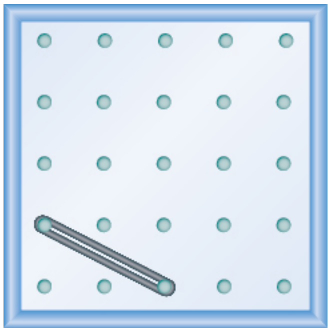
Exercise:

Problem: $\frac{4}{3}$

Exercise:

Problem: $-\frac{1}{2}$

Solution:



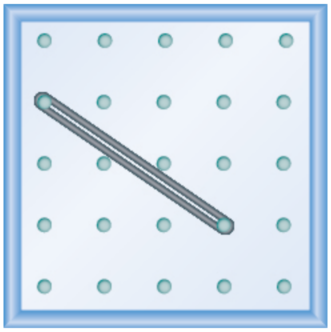
Exercise:

Problem: $-\frac{3}{4}$

Exercise:

Problem: $-\frac{2}{3}$

Solution:



Exercise:

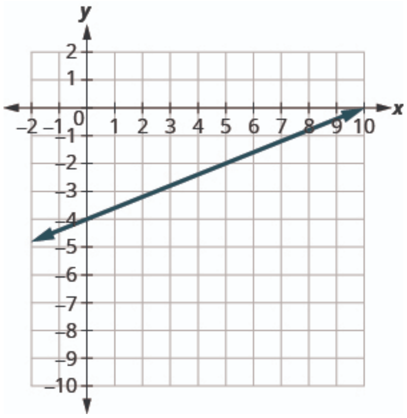
Problem: $-\frac{3}{2}$

Use $m = \frac{\text{rise}}{\text{run}}$ to find the Slope of a Line from its Graph

In the following exercises, find the slope of each line shown.

Exercise:

Problem:

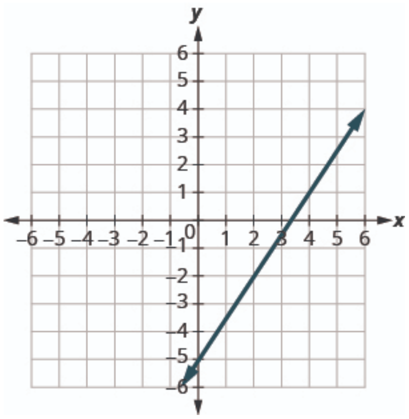


Solution:

$\frac{2}{5}$

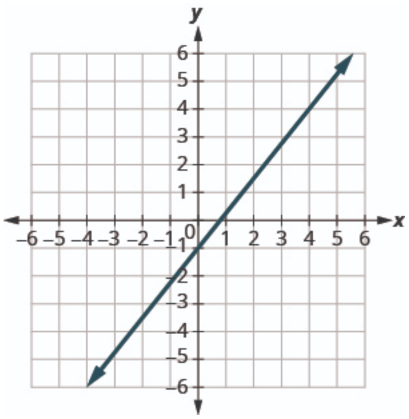
Exercise:

Problem:



Exercise:

Problem:

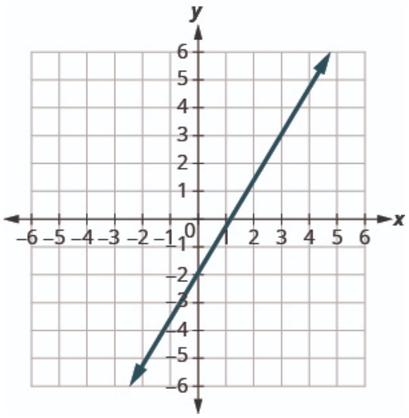


Solution:

$\frac{5}{4}$

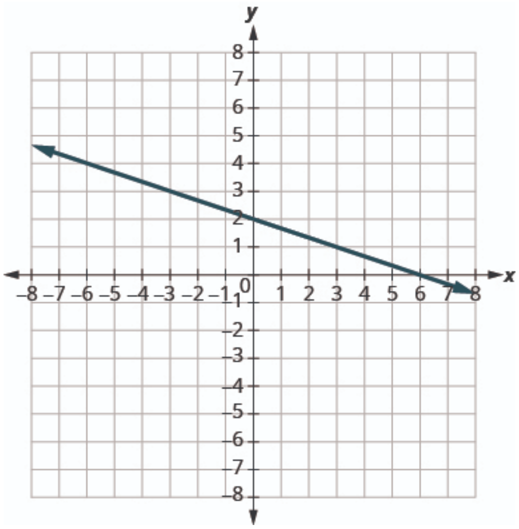
Exercise:

Problem:



Exercise:

Problem:

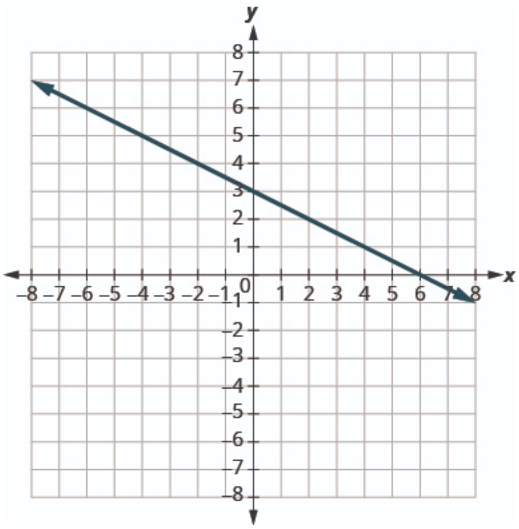


Solution:

$-\frac{1}{3}$

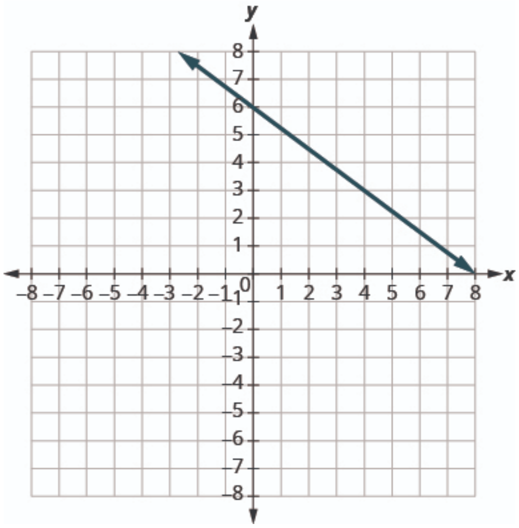
Exercise:

Problem:



Exercise:

Problem:

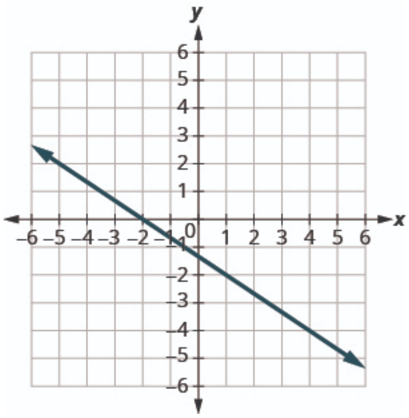


Solution:

$-\frac{3}{4}$

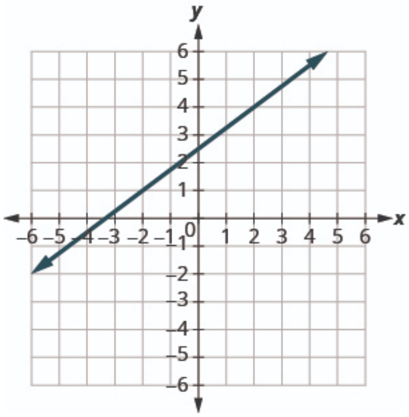
Exercise:

Problem:



Exercise:

Problem:

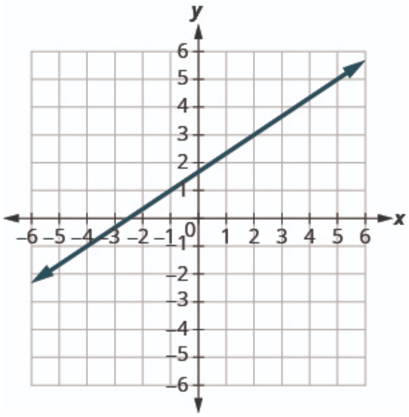


Solution:

$\frac{3}{4}$

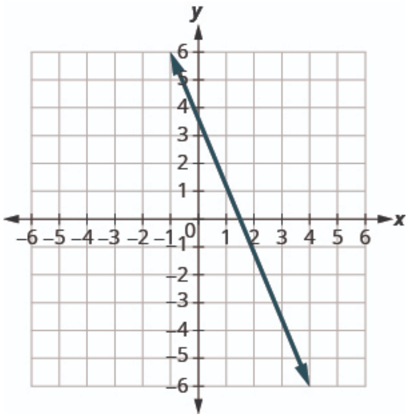
Exercise:

Problem:



Exercise:

Problem:

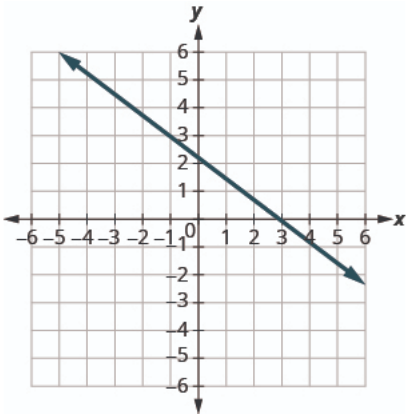


Solution:

$-\frac{5}{2}$

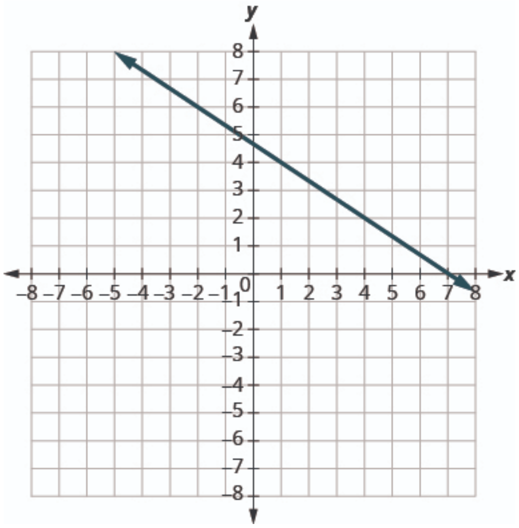
Exercise:

Problem:



Exercise:

Problem:

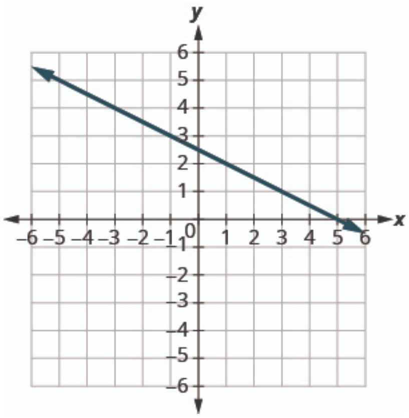


Solution:

$-\frac{2}{3}$

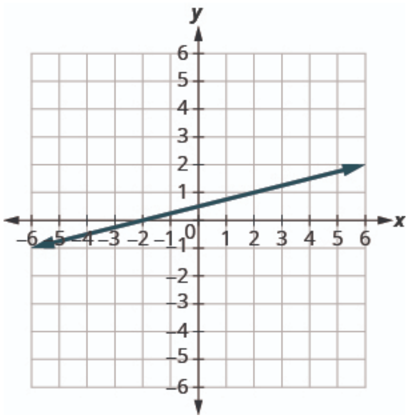
Exercise:

Problem:



Exercise:

Problem:

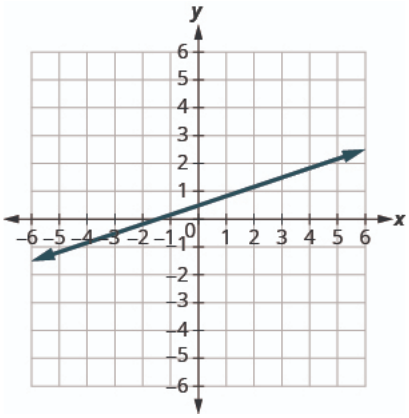


Solution:

$\frac{1}{4}$

Exercise:

Problem:



Find the Slope of Horizontal and Vertical Lines

In the following exercises, find the slope of each line.

Exercise:

Problem: $y = 3$

Solution:

0

Exercise:

Problem: $y = 1$

Exercise:

Problem: $x = 4$

Solution:

undefined

Exercise:

Problem: $x = 2$

Exercise:

Problem: $y = -2$

Solution:

0

Exercise:

Problem: $y = -3$

Exercise:

Problem: $x = -5$

Solution:

undefined

Exercise:

Problem: $x = -4$

Use the Slope Formula to find the Slope of a Line between Two Points

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

Exercise:

Problem: $(1, 4), (3, 9)$

Solution:

$\frac{5}{2}$

Exercise:

Problem: $(2, 3), (5, 7)$

Exercise:

Problem: $(0, 3), (4, 6)$

Solution:

$\frac{3}{4}$

Exercise:

Problem: $(0, 1), (5, 4)$

Exercise:

Problem: $(2, 5), (4, 0)$

Solution:

$-\frac{5}{2}$

Exercise:

Problem: $(3, 6), (8, 0)$

Exercise:

Problem: $(-3, 3), (4, -5)$

Solution:

$-\frac{8}{7}$

Exercise:

Problem: $(-2, 4), (3, -1)$

Exercise:

Problem: $(-1, -2), (2, 5)$

Solution:

$\frac{7}{3}$

Exercise:

Problem: $(-2, -1), (6, 5)$

Exercise:

Problem: $(4,-5),(1,-2)$

Solution:

-1

Exercise:

Problem: $(3,-6),(2,-2)$

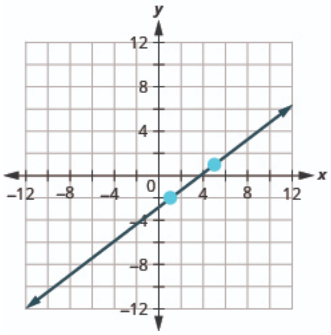
Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

Exercise:

Problem: $(1,-2);\ m=\frac{3}{4}$

Solution:



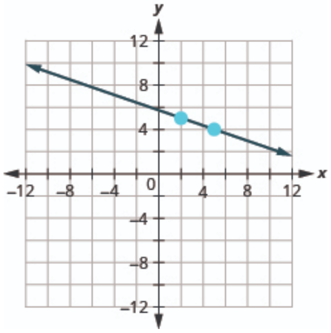
Exercise:

Problem: $(1,-1);\ m=\frac{2}{3}$

Exercise:

Problem: $(2,5);\ m=-\frac{1}{3}$

Solution:



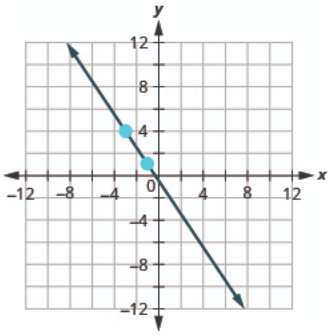
Exercise:

Problem: $(1,4); m=-\frac{1}{2}$

Exercise:

Problem: $(-3,4); m=-\frac{3}{2}$

Solution:



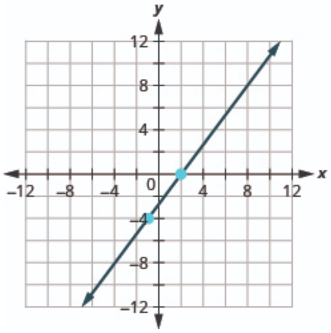
Exercise:

Problem: $(-2,5); m=-\frac{5}{4}$

Exercise:

Problem: $(-1,-4); m=\frac{4}{3}$

Solution:



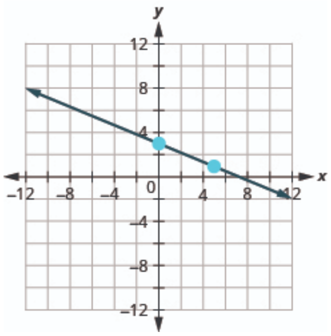
Exercise:

Problem: $(-3,-5); m=\frac{3}{2}$

Exercise:

Problem: *y*-intercept 3; $m=-\frac{2}{5}$

Solution:



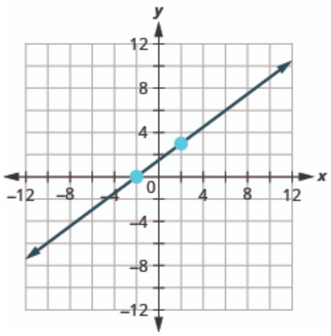
Exercise:

Problem: y-intercept 5; $m=-\frac{4}{3}$

Exercise:

Problem: x-intercept -2; $m=\frac{3}{4}$

Solution:



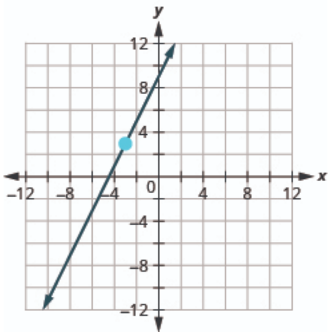
Exercise:

Problem: x-intercept -1; $m=\frac{1}{5}$

Exercise:

Problem: $(-3,3)$; $m=2$

Solution:



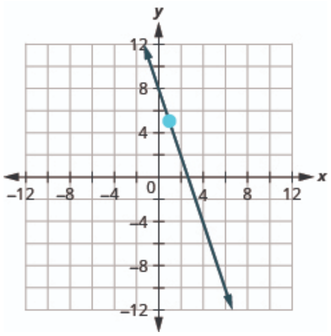
Exercise:

Problem: $(-4, 2); m = 4$

Exercise:

Problem: $(1, 5); m = -3$

Solution:



Exercise:

Problem: $(2, 3); m = -1$

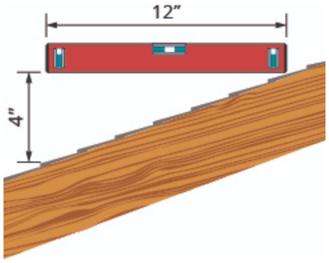
Everyday Math

Exercise:

Problem:

Slope of a roof. An easy way to determine the slope of a roof is to set one end of a 12 inch level on the roof surface and hold it level. Then take a tape measure or ruler and measure from the other end of the level down to the roof surface. This will give you the slope of the roof. Builders, sometimes, refer to this as pitch and state it as an “x 12 pitch” meaning $\frac{x}{12}$, where x is the measurement from the roof to the level—the rise. It is also sometimes stated as an “x-in-12 pitch”.

ⓐ What is the slope of the roof in this picture?
ⓑ What is the pitch in construction terms?



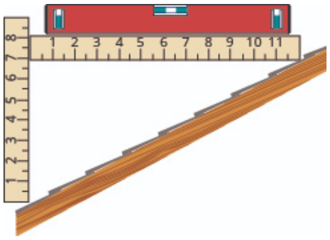
Solution:

ⓐ $\frac{1}{3}$ ⓑ 4 12 pitch or 4-in-12 pitch

Exercise:

Problem:

The slope of the roof shown here is measured with a 12” level and a ruler. What is the slope of this roof?



Exercise:

Problem:

Road grade. A local road has a grade of 6%. The grade of a road is its slope expressed as a percent. Find the slope of the road as a fraction and then simplify. What rise and run would reflect this slope or grade?

Solution:

$\frac{3}{50}$; rise = 3, run = 50

Exercise:

Problem: **Highway grade.** A local road rises 2 feet for every 50 feet of highway.

ⓐ What is the slope of the highway?
ⓑ The grade of a highway is its slope expressed as a percent. What is the grade of this highway?

Exercise:

Problem:

Wheelchair ramp. The rules for wheelchair ramps require a maximum 1-inch rise for a 12-inch run.

ⓐ How long must the ramp be to accommodate a 24-inch rise to the door?
ⓑ Create a model of this ramp.

Solution:

ⓐ 288 inches (24 feet) ⓑ Models will vary.

Exercise:

Problem:

Wheelchair ramp. A 1-inch rise for a 16-inch run makes it easier for the wheelchair rider to ascend a ramp.

ⓐ How long must a ramp be to easily accommodate a 24-inch rise to the door?
ⓑ Create a model of this ramp.

Writing Exercises

Exercise:

Problem: What does the sign of the slope tell you about a line?

Solution:

When the slope is a positive number the line goes up from left to right. When the slope is a negative number the line goes down from left to right.

Exercise:

Problem: How does the graph of a line with slope $m=\frac{1}{2}$ differ from the graph of a line with slope $m=2$?

Exercise:

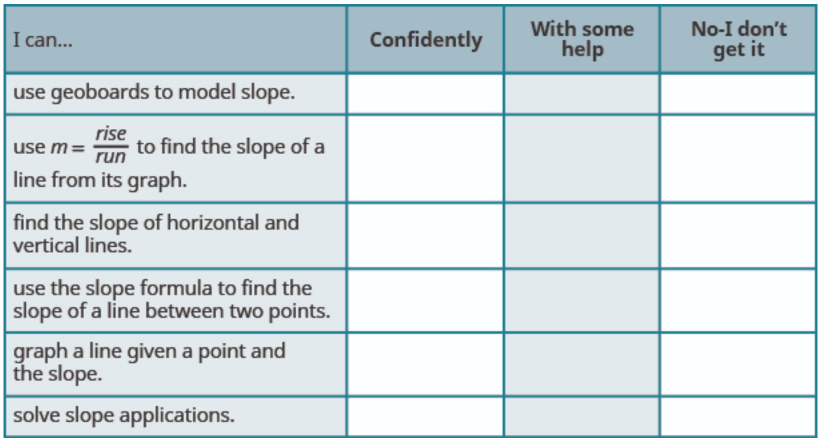
Problem: Why is the slope of a vertical line “undefined”?

Solution:

A vertical line has 0 run and since division by 0 is undefined the slope is undefined.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don’t get it
use geoboards to model slope.			
use $m=\frac{rise}{run}$ to find the slope of a line from its graph.			
find the slope of horizontal and vertical lines.			
use the slope formula to find the slope of a line between two points.			
graph a line given a point and the slope.			
solve slope applications.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

geoboard

A geoboard is a board with a grid of pegs on it.

negative slope

A negative slope of a line goes down as you read from left to right.

positive slope

A positive slope of a line goes up as you read from left to right.

rise

The rise of a line is its vertical change.

run
The run of a line is its horizontal change.

slope formula
The slope of the line between two points (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$.

slope of a line
The slope of a line is $m = \frac{\text{rise}}{\text{run}}$. The rise measures the vertical change and the run measures the horizontal change.

Use the Slope–Intercept Form of an Equation of a Line
By the end of this section, you will be able to:

- Recognize the relation between the graph and the slope–intercept form of an equation of a line
- Identify the slope and y-intercept form of an equation of a line
- Graph a line using its slope and intercept
- Choose the most convenient method to graph a line
- Graph and interpret applications of slope–intercept
- Use slopes to identify parallel lines
- Use slopes to identify perpendicular lines

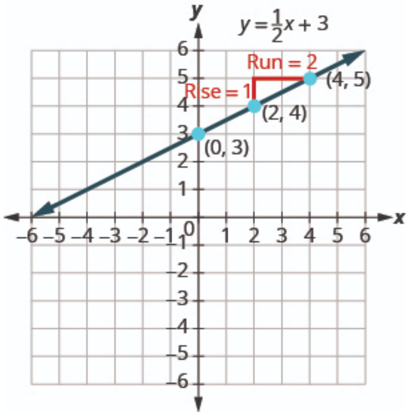
Note:
Before you get started, take this readiness quiz.

1. Add: $\frac{x}{4}+\frac{1}{4}$.
   If you missed this problem, review [link].
2. Find the reciprocal of $\frac{3}{7}$.
   If you missed this problem, review [link].
3. Solve $2x-3y=12$ for y.
   If you missed this problem, review [link].

Recognize the Relation Between the Graph and the Slope–Intercept Form of an Equation of a Line

We have graphed linear equations by plotting points, using intercepts, recognizing horizontal and vertical lines, and using the point–slope method. Once we see how an equation in slope–intercept form and its graph are related, we'll have one more method we can use to graph lines.

In Graph Linear Equations in Two Variables, we graphed the line of the equation $y=\frac{1}{2}x+3$ by plotting points. See [link]. Let's find the slope of this line.



The red lines show us the rise is 1 and the run is 2. Substituting into the slope formula:
Equation:

$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ m &= \frac{1}{2} \end{aligned}$$

What is the y-intercept of the line? The y-intercept is where the line crosses the y-axis, so y-intercept is $(0, 3)$. The equation of this line is:

$y = \frac{1}{2}x + 3$

Notice, the line has:

slope $m = \frac{1}{2}$ and y-intercept $(0, 3)$

When a linear equation is solved for y, the coefficient of the x term is the slope and the constant term is the y-coordinate of the y-intercept. We say that the equation $y = \frac{1}{2}x + 3$ is in slope–intercept form.

$m = \frac{1}{2}$; y-intercept is $(0, 3)$

$y = \frac{1}{2}x + 3$

$y = mx + b$

Note:
Slope-Intercept Form of an Equation of a Line
The **slope–intercept form** of an equation of a line with slope m and y-intercept, $(0, b)$ is,
Equation:

$$y = mx + b$$

Sometimes the slope–intercept form is called the “y-form.”

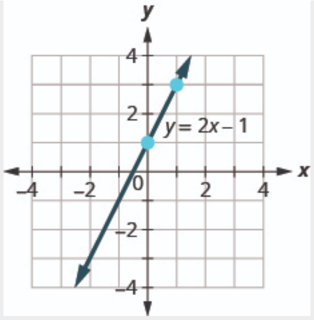
Example:
Exercise:

Problem: Use the graph to find the slope and y-intercept of the line, $y = 2x + 1$.

Compare these values to the equation$y = mx + b$.

Solution:
Solution

To find the slope of the line, we need to choose two points on the line. We’ll use the points $(0, 1)$ and $(1, 3)$.

	
Find the rise and run.	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{1}$
	$m = 2$
Find the *y*-intercept of the line.	The *y*-intercept is the point (0, 1).
We found slope $m = 2$ and *y*-intercept (0, 1).	$y = 2x + 1$ $y = mx + b$

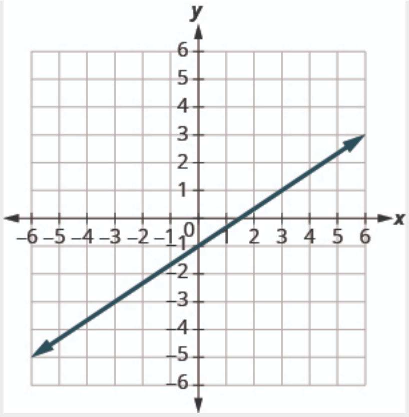
The slope is the same as the coefficient of x and the *y*-coordinate of the *y*-intercept is the same as the constant term.

Note:

Exercise:

Problem:

Use the graph to find the slope and *y*-intercept of the line $y = \frac{2}{3}x - 1$. Compare these values to the equation $y = mx + b$.



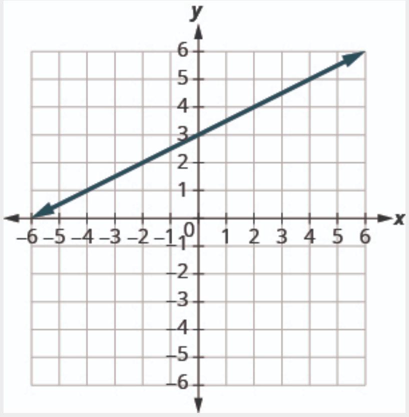
Solution:

slope $m = \frac{2}{3}$ and *y*-intercept $(0, -1)$

Note:
Exercise:

Problem:

Use the graph to find the slope and *y*-intercept of the line $y = \frac{1}{2}x + 3$. Compare these values to the equation $y = mx + b$.



Solution:

slope $m = \frac{1}{2}$ and *y*-intercept $(0, 3)$

Identify the Slope and *y*-Intercept From an Equation of a Line

In Understand Slope of a Line, we graphed a line using the slope and a point. When we are given an equation in slope–intercept form, we can use the *y*-intercept as the point, and then count out the slope from there. Let’s practice finding the values of the slope and *y*-intercept from the equation of a line.

Example:
Exercise:

Problem: Identify the slope and y-intercept of the line with equation $y = -3x + 5$.

Solution:
Solution

We compare our equation to the slope–intercept form of the equation.

	$y = mx + b$
Write the equation of the line.	$y = -3x + 5$
Identify the slope.	$m = -3$
Identify the *y*-intercept.	*y*-intercept is (0, 5)

Note:
Exercise:

Problem: Identify the slope and y-intercept of the line $y = \frac{2}{5}x - 1$.

Solution:

$\frac{2}{5}; (0, -1)$

Note:
Exercise:

Problem: Identify the slope and y-intercept of the line $y = -\frac{4}{3}x + 1$.

Solution:

$-\frac{4}{3}; (0, 1)$

When an equation of a line is not given in slope–intercept form, our first step will be to solve the equation for y.

Example:
Exercise:

Problem: Identify the slope and y-intercept of the line with equation $x + 2y = 6$.

Solution:
Solution

This equation is not in slope–intercept form. In order to compare it to the slope–intercept form we must first solve the equation for y.

Solve for y.	$x + 2y = 6$
Subtract x from each side.	$2y = -x + 6$
Divide both sides by 2.	$\frac{2y}{2} = \frac{-x + 6}{2}$
Simplify.	$\frac{2y}{2} = \frac{-x}{2} + \frac{6}{2}$
$\left(\text{Remember: } \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}\right)$	
Simplify.	$y = -\frac{1}{2}x + 3$
Write the slope–intercept form of the equation of the line.	$y = mx + b$
Write the equation of the line.	$y = -\frac{1}{2}x + 3$
Identify the slope.	$m = -\frac{1}{2}$
Identify the y-intercept.	y-intercept is $(0, 3)$

Note:
Exercise:

Problem: Identify the slope and *y*-intercept of the line $x + 4y = 8$.

Solution:

$-\frac{1}{4}$; $(0, 2)$

Note:
Exercise:

Problem: Identify the slope and *y*-intercept of the line $3x + 2y = 12$.

Solution:

$-\frac{3}{2}$; $(0, 6)$

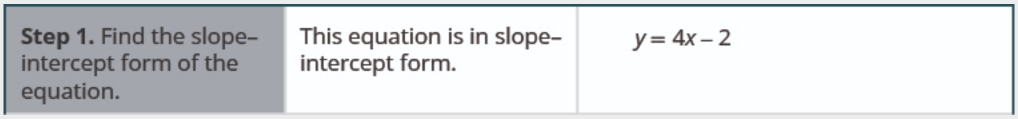
Graph a Line Using its Slope and Intercept

Now that we know how to find the slope and *y*-intercept of a line from its equation, we can graph the line by plotting the *y*-intercept and then using the slope to find another point.

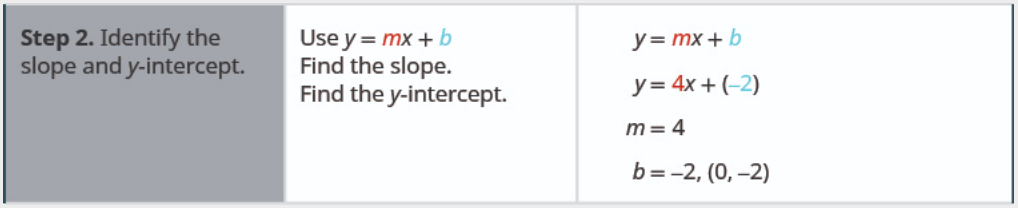
Example:
How to Graph a Line Using its Slope and Intercept
Exercise:

Problem: Graph the line of the equation $y = 4x - 2$ using its slope and *y*-intercept.

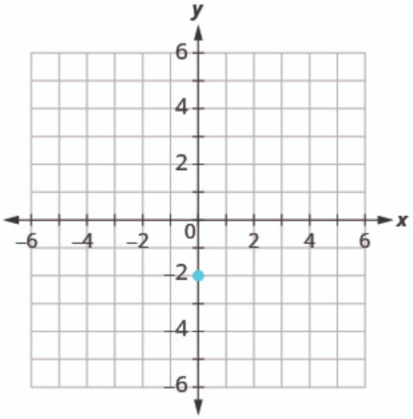
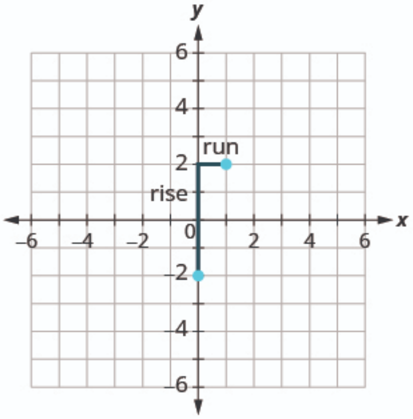
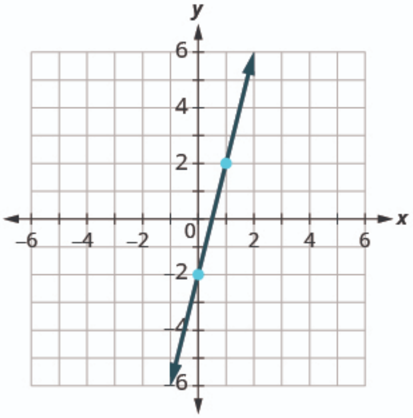
Solution:
Solution



Step 1. Find the slope–intercept form of the equation.	This equation is in slope–intercept form.	$y = 4x - 2$



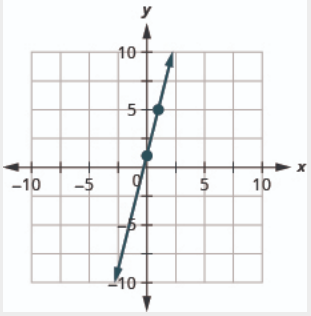
Step 2. Identify the slope and *y*-intercept.	Use $y = mx + b$ Find the slope. Find the *y*-intercept.	$y = mx + b$ $y = 4x + (-2)$ $m = 4$ $b = -2, (0, -2)$

Step 3. Plot the *y*-intercept.	Plot (0, –2).	
Step 4. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m = 4$ $\frac{\text{rise}}{\text{run}} = \frac{4}{1}$ rise = 4 run = 1
Step 5. Starting at the *y*-intercept, count out the rise and run to mark the second point.	Start at (0, –2) and count the rise and the run. Up 4, right 1.	
Step 6. Connect the points with a line.	Connect the two points with a line.	

Note:
Exercise:

Problem: Graph the line of the equation $y = 4x + 1$ using its slope and *y*-intercept.

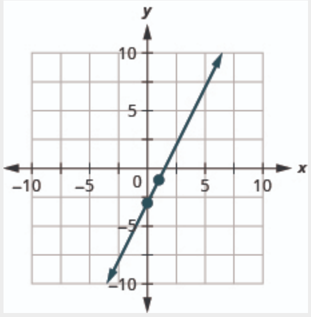
Solution:



Note:
Exercise:

Problem: Graph the line of the equation $y = 2x - 3$ using its slope and *y*-intercept.

Solution:



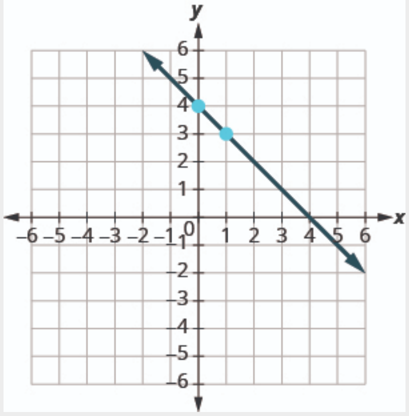
Note:
Graph a line using its slope and *y*-intercept.

Find the slope-intercept form of the equation of the line.
Identify the slope and *y*-intercept.
Plot the *y*-intercept.
Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
Starting at the *y*-intercept, count out the rise and run to mark the second point.
Connect the points with a line.

Example:
Exercise:

Problem: Graph the line of the equation $y = -x + 4$ using its slope and *y*-intercept.

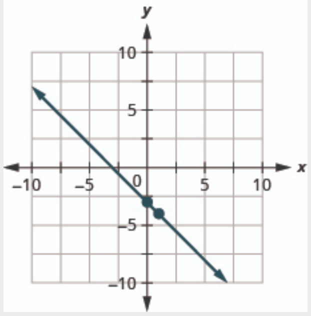
Solution:
Solution

	$y = mx + b$
The equation is in slope–intercept form.	$y = -x + 4$
Identify the slope and *y*-intercept.	$m = -1$
	y-intercept is (0, 4)
Plot the *y*-intercept.	See graph below.
Identify the rise and the run.	$m = \frac{-1}{1}$
Count out the rise and run to mark the second point.	rise −1, run 1
Draw the line.	
To check your work, you can find another point on the line and make sure it is a solution of the equation. In the graph we see the line goes through (4, 0).	
Check. $y = -x + 4$ $0 \overset{?}{=} -4 + 4$ $0 = 0 \checkmark$	

Note:
Exercise:

Problem: Graph the line of the equation $y = -x - 3$ using its slope and *y*-intercept.

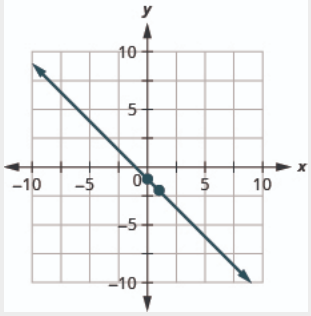
Solution:



Note:
Exercise:

Problem: Graph the line of the equation $y = -x - 1$ using its slope and *y*-intercept.

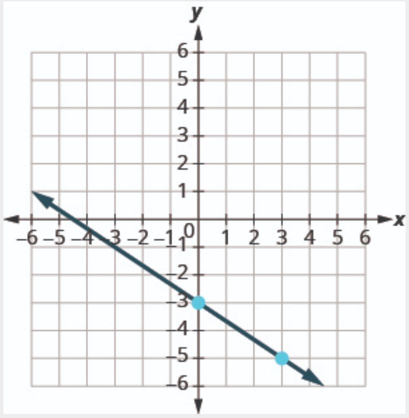
Solution:



Example:
Exercise:

Problem: Graph the line of the equation $y = -\frac{2}{3}x - 3$ using its slope and *y*-intercept.

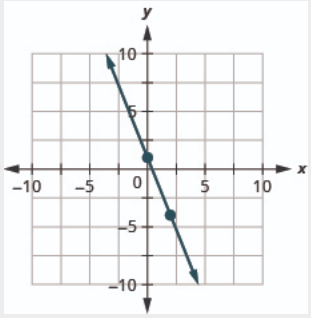
Solution:
Solution

	$y = mx + b$
The equation is in slope–intercept form.	$y = -\frac{2}{3}x - 3$
Identify the slope and *y*-intercept.	$m = -\frac{2}{3}$; *y*-intercept is (0, −3)
Plot the *y*-intercept.	See graph below.
Identify the rise and the run.	
Count out the rise and run to mark the second point.	
Draw the line.	

Note:
Exercise:

Problem: Graph the line of the equation $y = -\frac{5}{2}x + 1$ using its slope and *y*-intercept.

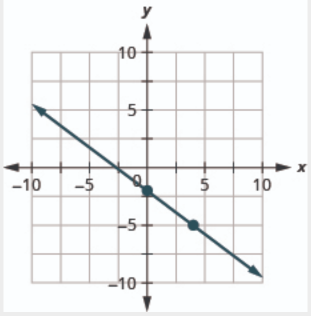
Solution:



Note:
Exercise:

Problem: Graph the line of the equation $y = -\frac{3}{4}x - 2$ using its slope and *y*-intercept.

Solution:

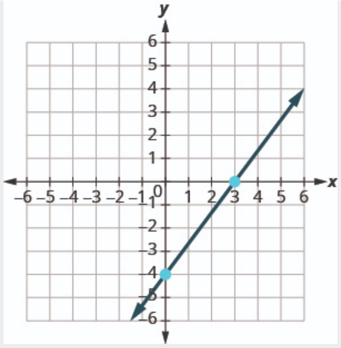


Example:
Exercise:

Problem: Graph the line of the equation $4x - 3y = 12$ using its slope and *y*-intercept.

Solution:
Solution

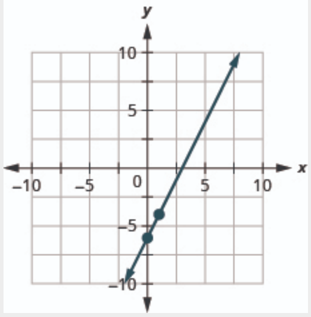
	$4x - 3y = 12$
Find the slope–intercept form of the equation.	$-3y = -4x + 12$
	$-\frac{3y}{3} = \frac{-4x+12}{-3}$
The equation is now in slope–intercept form.	$y = \frac{4}{3}x - 4$
Identify the slope and *y*-intercept.	$m = \frac{4}{3}$
	y-intercept is (0, −4)
Plot the *y*-intercept.	See graph below.
Identify the rise and the run; count out the rise and run to mark the second point.	
Draw the line.	



Note:
Exercise:

Problem: Graph the line of the equation $2x - y = 6$ using its slope and y-intercept.

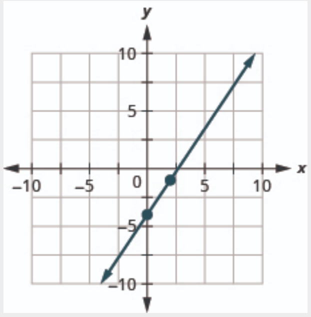
Solution:



Note:
Exercise:

Problem: Graph the line of the equation $3x - 2y = 8$ using its slope and y-intercept.

Solution:



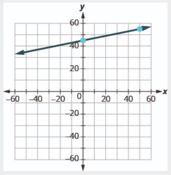
We have used a grid with x and y both going from about -10 to 10 for all the equations we've graphed so far. Not all linear equations can be graphed on this small grid. Often, especially in applications with real-world data, we'll need to extend the axes to bigger positive or smaller negative numbers.

Example:
Exercise:

Problem: Graph the line of the equation $y = 0.2x + 45$ using its slope and *y*-intercept.

Solution:
Solution

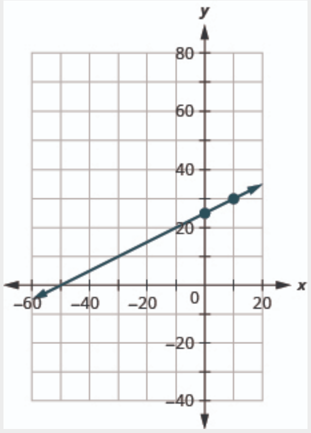
We'll use a grid with the axes going from about -80 to 80.

	$y = mx + b$
The equation is in slope–intercept form.	$y = 0.2x + 45$
Identify the slope and *y*-intercept.	$m = 0.2$
	The *y*-intercept is (0, 45)
Plot the *y*-intercept.	See graph below.
Count out the rise and run to mark the second point. The slope is $m = 0.2$; in fraction form this means $m = \frac{2}{10}$. Given the scale of our graph, it would be easier to use the equivalent fraction $m = \frac{10}{50}$.	
Draw the line.	

Note:
Exercise:

Problem: Graph the line of the equation $y = 0.5x + 25$ using its slope and *y*-intercept.

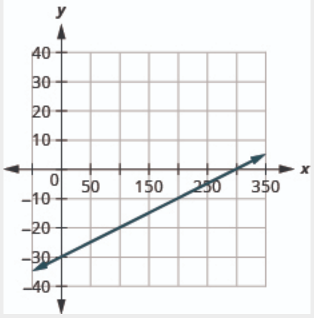
Solution:



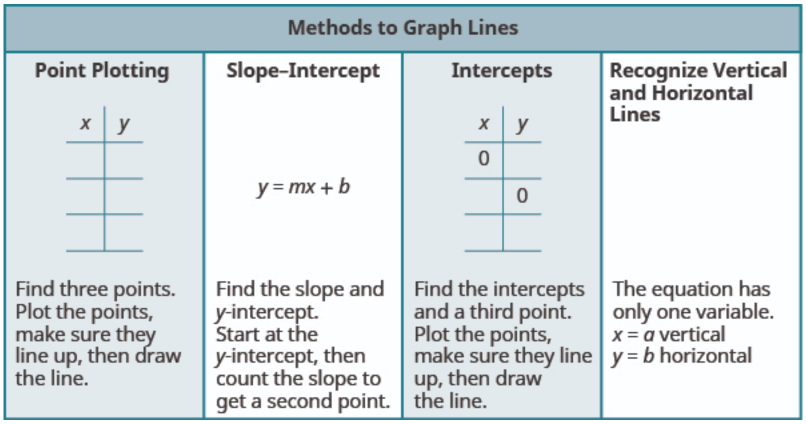
Note:
Exercise:

Problem: Graph the line of the equation $y = 0.1x - 30$ using its slope and *y*-intercept.

Solution:



Now that we have graphed lines by using the slope and *y*-intercept, let’s summarize all the methods we have used to graph lines. See [link].



Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
x | y	$y = mx + b$	x | y 0 | | 0	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and y-intercept. Start at the y-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Choose the Most Convenient Method to Graph a Line

Now that we have seen several methods we can use to graph lines, how do we know which method to use for a given equation?

While we could plot points, use the slope–intercept form, or find the intercepts for *any* equation, if we recognize the most convenient way to graph a certain type of equation, our work will be easier. Generally, plotting points is not the most efficient way to graph a line. We saw better methods in sections 4.3, 4.4, and earlier in this section. Let’s look for some patterns to help determine the most convenient method to graph a line.

Here are six equations we graphed in this chapter, and the method we used to graph each of them.
Equation:

	Equation	**Method**
#1	$x = 2$	Vertical line
#2	$y = 4$	Horizontal line
#3	$-x + 2y = 6$	Intercepts
#4	$4x - 3y = 12$	Intercepts
#5	$y = 4x - 2$	Slope–intercept
#6	$y = -x + 4$	Slope–intercept

Equations #1 and #2 each have just one variable. Remember, in equations of this form the value of that one variable is constant; it does not depend on the value of the other variable. Equations of this form have graphs that are vertical or horizontal lines.

In equations #3 and #4, both x and y are on the same side of the equation. These two equations are of the form $Ax + By = C$. We substituted $y = 0$ to find the x-intercept and $x = 0$ to find the y-intercept, and then found a third point by choosing another value for x or y.

Equations #5 and #6 are written in slope–intercept form. After identifying the slope and y-intercept from the equation we used them to graph the line.

This leads to the following strategy.

Note:
Strategy for Choosing the Most Convenient Method to Graph a Line
Consider the form of the equation.

- If it only has one variable, it is a vertical or horizontal line.
    - $x = a$ is a vertical line passing through the *x*-axis at a.
    - $y = b$ is a horizontal line passing through the *y*-axis at b.
- If y is isolated on one side of the equation, in the form $y = mx + b$, graph by using the slope and y-intercept.
    - Identify the slope and *y*-intercept and then graph.
- If the equation is of the form $Ax + By = C$, find the intercepts.
    - Find the *x*- and *y*-intercepts, a third point, and then graph.

Example:
Exercise:

Problem: Determine the most convenient method to graph each line.

ⓐ $y = -6$ ⓑ $5x - 3y = 15$ ⓒ $x = 7$ ⓓ $y = \frac{2}{5}x - 1$.

Solution:
Solution

ⓐ $y = -6$
This equation has only one variable,y. Its graph is a horizontal line crossing the *y*-axis at -6.
ⓑ $5x - 3y = 15$
This equation is of the form $Ax + By = C$. The easiest way to graph it will be to find the intercepts and one more point.
ⓒ $x = 7$
There is only one variable, x. The graph is a vertical line crossing the *x*-axis at 7.
ⓓ $y = \frac{2}{5}x - 1$
Since this equation is in $y = mx + b$ form, it will be easiest to graph this line by using the slope and *y*-intercept.

Note:
Exercise:

Problem:

Determine the most convenient method to graph each line: ⓐ $3x + 2y = 12$ ⓑ $y = 4$ ⓒ $y = \frac{1}{5}x - 4$ ⓓ $x = -7$.

Solution:

ⓐ intercepts ⓑ horizontal line ⓒ slope–intercept ⓓ vertical line

Note:

Exercise:

Problem:

Determine the most convenient method to graph each line: ⓐ $x=6$ ⓑ $y=-\frac{3}{4}x+1$ ⓒ $y=-8$ ⓓ $4x-3y=-1$.

Solution:

ⓐ vertical line ⓑ slope–intercept ⓒ horizontal line ⓓ intercepts

Graph and Interpret Applications of Slope–Intercept

Many real-world applications are modeled by linear equations. We will take a look at a few applications here so you can see how equations written in slope–intercept form relate to real-world situations.

Usually when a linear equation models a real-world situation, different letters are used for the variables, instead of *x* and *y*. The variable names remind us of what quantities are being measured.

Example:
Exercise:

Problem:

The equation $F=\frac{9}{5}C+32$ is used to convert temperatures, C, on the Celsius scale to temperatures, F, on the Fahrenheit scale.

ⓐ Find the Fahrenheit temperature for a Celsius temperature of 0.
ⓑ Find the Fahrenheit temperature for a Celsius temperature of 20.
ⓒ Interpret the slope and *F*-intercept of the equation.
ⓓ Graph the equation.

Solution:
Solution

ⓐ

Find the Fahrenheit temperature for a Celsius temperature of 0.	$F=\frac{9}{5}C+32$
Find F when $C=0$.	$F=\frac{9}{5}(0)+32$
Simplify.	$F=32$

ⓑ

Find the Fahrenheit temperature for a Celsius temperature of 20.	$F=\frac{9}{5}C+32$
Find F when $C=20$.	$F=\frac{9}{5}(20)+32$
Simplify.	$F=36+32$
Simplify.	$F=68$

ⓒ Interpret the slope and *F*-intercept of the equation.

Even though this equation uses $F$and C, it is still in slope–intercept form.

$$y = mx + b$$

$$F = mC + b$$

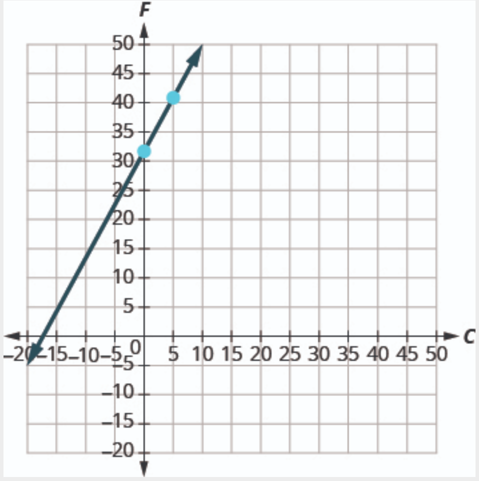
$$F = \frac{9}{5}C + 32$$

The slope, $\frac{9}{5}$, means that the temperature Fahrenheit (F) increases 9 degrees when the temperature Celsius (C) increases 5 degrees.

The F-intercept means that when the temperature is 0° on the Celsius scale, it is 32° on the Fahrenheit scale.

ⓓ Graph the equation.

We'll need to use a larger scale than our usual. Start at the F-intercept $(0, 32)$ then count out the rise of 9 and the run of 5 to get a second point. See [link].



Note:
Exercise:

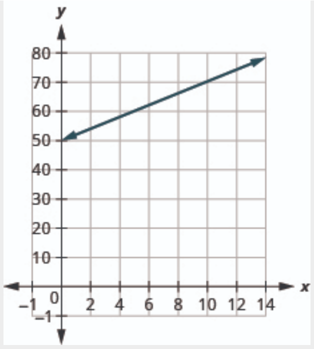
Problem:

The equation $h = 2s + 50$ is used to estimate a woman's height in inches, h, based on her shoe size, s.

ⓐ Estimate the height of a child who wears women's shoe size 0.
ⓑ Estimate the height of a woman with shoe size 8.
ⓒ Interpret the slope and h-intercept of the equation.
ⓓ Graph the equation.

Solution:

ⓐ 50 inches
ⓑ 66 inches
ⓒ The slope, 2, means that the height, h, increases by 2 inches when the shoe size, s, increases by 1. The h-intercept means that when the shoe size is 0, the height is 50 inches.
ⓓ



Note:
Exercise:

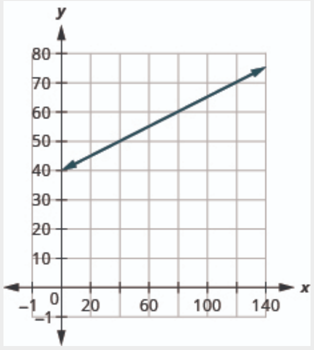
Problem:

The equation $T = \frac{1}{4}n + 40$ is used to estimate the temperature in degrees Fahrenheit, *T*, based on the number of cricket chirps, *n*, in one minute.

ⓐ Estimate the temperature when there are no chirps.
ⓑ Estimate the temperature when the number of chirps in one minute is 100.
ⓒ Interpret the slope and *T*-intercept of the equation.
ⓓ Graph the equation.

Solution:

ⓐ 40 degrees
ⓑ 65 degrees
ⓒ The slope, $\frac{1}{4}$, means that the temperature Fahrenheit (*F*) increases 1 degree when the number of chirps, *n*, increases by 4. The *T*-intercept means that when the number of chirps is 0, the temperature is 40°.
ⓓ



The cost of running some types business has two components—a *fixed cost* and a *variable cost*. The fixed cost is always the same regardless of how many units are produced. This is the cost of rent, insurance, equipment, advertising, and other items that must be paid regularly. The variable cost depends on the number of units produced. It is for the material and labor needed to produce each item.

Example:
Exercise:

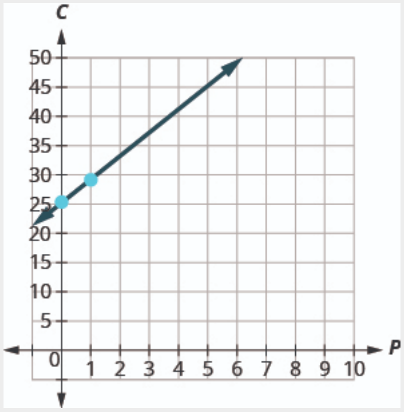
Problem:

Stella has a home business selling gourmet pizzas. The equation $C = 4p + 25$ models the relation between her weekly cost, *C*, in dollars and the number of pizzas, *p*, that she sells.

ⓐ Find Stella’s cost for a week when she sells no pizzas.
ⓑ Find the cost for a week when she sells 15 pizzas.
ⓒ Interpret the slope and *C*-intercept of the equation.
ⓓ Graph the equation.

Solution:
Solution

ⓐ Find Stella's cost for a week when she sells no pizzas.	$C = 4p + 25$
Find *C* when $p = 0$.	$C = 4(0) + 25$
Simplify.	$C = 25$
	Stella's fixed cost is $25 when she sells no pizzas.
ⓑ Find the cost for a week when she sells 15 pizzas.	$C = 4p + 25$
Find *C* when $p = 15$.	$C = 4(15) + 25$
Simplify.	$C = 60 + 25$
	$C = 85$
	Stella's costs are $85 when she sells 15 pizzas.
ⓒ Interpret the slope and *C*-intercept of the equation.	$y = mx + b$ $C = 4p + 25$
	The slope, 4, means that the cost increases by $4

	for each pizza Stella sells. The C-intercept means that even when Stella sells no pizzas, her costs for the week are $25.
ⓓ Graph the equation. We'll need to use a larger scale than our usual. Start at the C-intercept (0, 25) then count out the rise of 4 and the run of 1 to get a second point.	

Note:
Exercise:

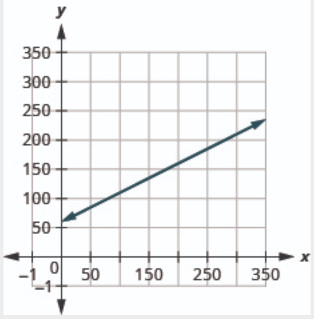
Problem:

Sam drives a delivery van. The equation $C = 0.5m + 60$ models the relation between his weekly cost, C, in dollars and the number of miles, m, that he drives.

ⓐ Find Sam’s cost for a week when he drives 0 miles.
ⓑ Find the cost for a week when he drives 250 miles.
ⓒ Interpret the slope and C-intercept of the equation.
ⓓ Graph the equation.

Solution:

ⓐ $60
ⓑ $185
ⓒ The slope, 0.5, means that the weekly cost, C, increases by $0.50 when the number of miles driven, n, increases by 1. The C-intercept means that when the number of miles driven is 0, the weekly cost is $60
ⓓ



Note:
Exercise:

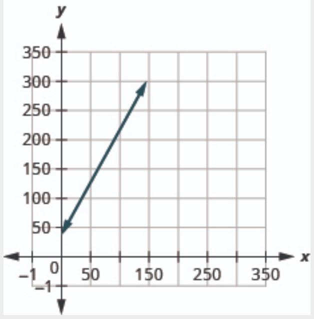
Problem:

Loreen has a calligraphy business. The equation $C = 1.8n + 35$ models the relation between her weekly cost, *C*, in dollars and the number of wedding invitations, *n*, that she writes.

ⓐ Find Loreen’s cost for a week when she writes no invitations.
ⓑ Find the cost for a week when she writes 75 invitations.
ⓒ Interpret the slope and *C*-intercept of the equation.
ⓓ Graph the equation.

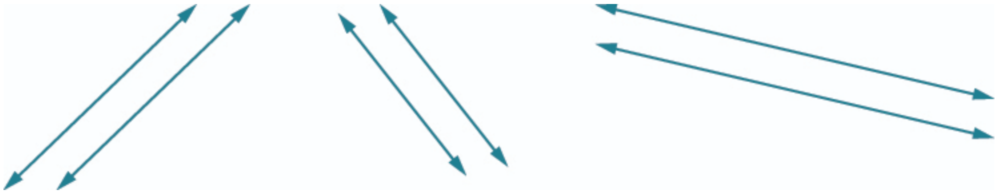
Solution:

ⓐ $35
ⓑ $170
ⓒ The slope, 1.8, means that the weekly cost, C, increases by $1.80 when the number of invitations, *n*, increases by 1.80.
The *C*-intercept means that when the number of invitations is 0, the weekly cost is $35.;
ⓓ

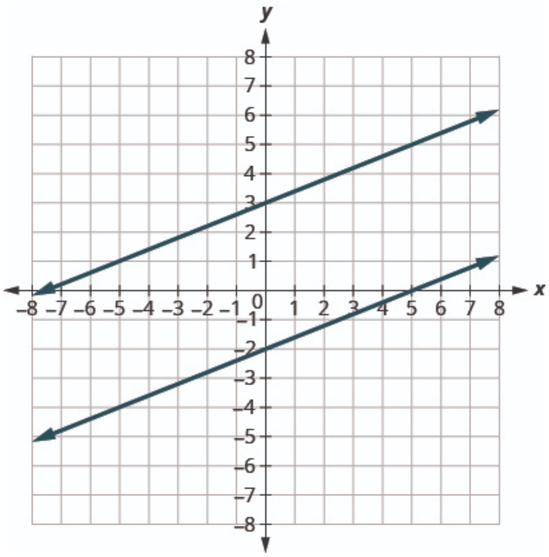


Use Slopes to Identify Parallel Lines

The slope of a line indicates how steep the line is and whether it rises or falls as we read it from left to right. Two lines that have the same slope are called parallel lines. Parallel lines never intersect.

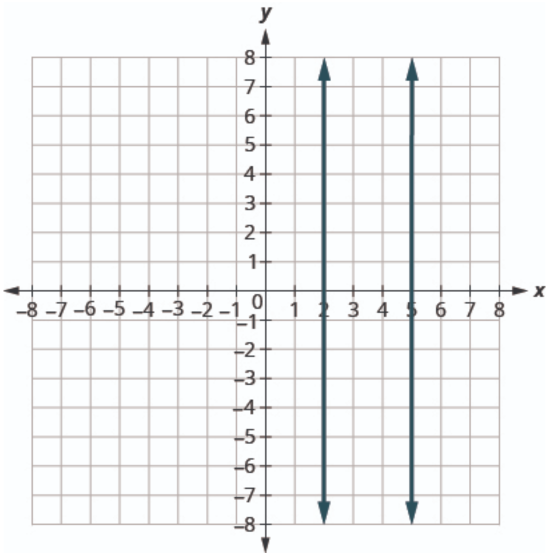


We say this more formally in terms of the rectangular coordinate system. Two lines that have the same slope and different *y*-intercepts are called **parallel lines**. See [link].



Verify that both lines have the same slope, $m = \frac{2}{5}$, and different *y*-intercepts.

What about vertical lines? The slope of a vertical line is undefined, so vertical lines don’t fit in the definition above. We say that vertical lines that have different *x*-intercepts are parallel. See [link].



Vertical lines with diferent *x*-intercepts are parallel.

Note:

Parallel Lines

Parallel lines are lines in the same plane that do not intersect.

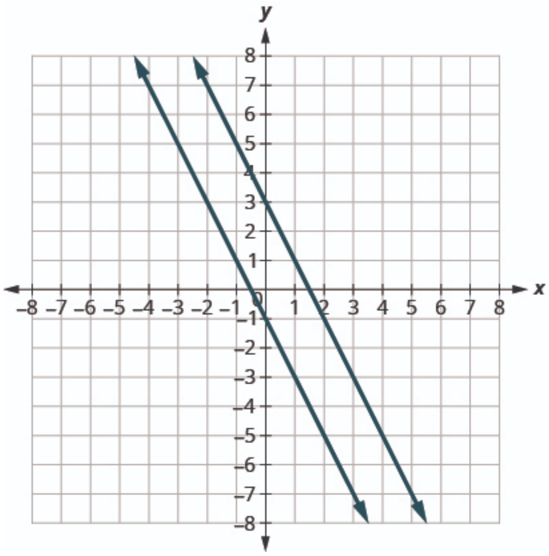
- Parallel lines have the same slope and different *y*-intercepts.
- If m_1 and m_2 are the slopes of two parallel lines then$m_1 = m_2$.
- Parallel vertical lines have different *x*-intercepts.

Let’s graph the equations $y = -2x + 3$ and $2x + y = -1$ on the same grid. The first equation is already in slope–intercept form: $y = -2x + 3$. We solve the second equation for y:

Equation:

$$\begin{aligned} 2x + y &= -1 \\ y &= -2x - 1 \end{aligned}$$

Graph the lines.



Notice the lines look parallel. What is the slope of each line? What is the *y*-intercept of each line?

Equation:

$$\begin{aligned} y &= mx + b \\ y &= -2x + 3 \\ m &= -2 \\ b &= 3, (0, 3) \end{aligned} \qquad \begin{aligned} y &= mx + b \\ y &= -2x - 1 \\ m &= -2 \\ b &= -1, (0, -1) \end{aligned}$$

The slopes of the lines are the same and the *y*-intercept of each line is different. So we know these lines are parallel.

Since parallel lines have the same slope and different *y*-intercepts, we can now just look at the slope–intercept form of the equations of lines and decide if the lines are parallel.

Example:

Exercise:

Problem: Use slopes and y-intercepts to determine if the lines $3x-2y=6$ and $y=\frac{3}{2}x+1$ are parallel.

Solution:
Solution

Solve the first equation for y.

$$\begin{aligned} 3x-2y &= 6 \\ -2y &= -3x+6 \\ \frac{-2y}{-2} &= \frac{-3x+6}{-2} \\ y &= \frac{3}{2}x-3 \end{aligned}$$

and $y=\frac{3}{2}$

The equation is now in slope–intercept form.

The equation of the second line is already in slope–intercept form.

$y=\frac{3}{2}$

Identify the slope and y-intercept of both lines.

$$\begin{aligned} y &= \frac{3}{2}x-3 \\ y &= mx+b \\ m &= \frac{3}{2} \end{aligned}$$

y-intercept is $(0,-3)$

$y=\frac{3}{2}$

$y=m$

$m=\frac{3}{2}$

y-inter

The lines have the same slope and different y-intercepts and so they are parallel. You may want to graph the lines to confirm whether they are parallel.

Note:
Exercise:

Problem: Use slopes and y-intercepts to determine if the lines $2x+5y=5$ and $y=-\frac{2}{5}x-4$ are parallel.

Solution:

parallel

Note:
Exercise:

Problem: Use slopes and y-intercepts to determine if the lines $4x-3y=6$ and $y=\frac{4}{3}x-1$ are parallel.

Solution:

parallel

Example:
Exercise:

Problem: Use slopes and y-intercepts to determine if the lines $y=-4$ and $y=3$ are parallel.

Solution:
Solution

	$y=-4$	and	$y=3$
	$y=0x-4$		$y=0x+$
Write each equation in slope–intercept form.	$y=0x-4$		$y=0x+$
Since there is no x term we write $0x$.	$y=mx+b$		$y=mx+$
Identify the slope and y-intercept of both lines.	$m=0$		$m=0$
	y-intercept is $(0,4)$		y-intercep

The lines have the same slope and different y-intercepts and so they are parallel.

There is another way you can look at this example. If you recognize right away from the equations that these are horizontal lines, you know their slopes are both 0. Since the horizontal lines cross the y-axis at $y=-4$ and at $y=3$, we know the y-intercepts are $(0,-4)$ and $(0,3)$. The lines have the same slope and different y-intercepts and so they are parallel.

Note:
Exercise:

Problem: Use slopes and y-intercepts to determine if the lines $y=8$ and $y=-6$ are parallel.

Solution:

parallel

Note:
Exercise:

Problem: Use slopes and y-intercepts to determine if the lines $y=1$ and $y=-5$ are parallel.

Solution:

parallel

Example:
Exercise:

Problem: Use slopes and y-intercepts to determine if the lines $x=-2$ and $x=-5$ are parallel.

Solution:
Solution
Equation:

$$x=-2 \text{ and } x=-5$$

Since there is no$y$, the equations cannot be put in slope–intercept form. But we recognize them as equations of vertical lines. Their x-intercepts are -2 and -5. Since their x-intercepts are different, the vertical lines are parallel.

Note:
Exercise:

Problem: Use slopes and y-intercepts to determine if the lines $x = 1$ and $x = -5$ are parallel.

Solution:

parallel

Note:
Exercise:

Problem: Use slopes and y-intercepts to determine if the lines $x = 8$ and $x = -6$ are parallel.

Solution:

parallel

Example:
Exercise:

Problem:

Use slopes and y-intercepts to determine if the lines $y = 2x - 3$ and $-6x + 3y = -9$ are parallel. You may want to graph these lines, too, to see what they look like.

Solution:
Solution

$$\begin{array}{rcl} y & = & 2x - 3 \end{array} \quad \text{and} \quad -$$

The first equation is already in slope–intercept form.
Solve the second equation for y.

$$\begin{array}{rcl} y & = & 2x - 3 \\ -6x + 3y & = & -9 \\ 3y & = & 6x - 9 \\ \frac{3y}{3} & = & \frac{6x-9}{3} \end{array}$$

$$\begin{array}{rcl} y & = & 2x - 3 \end{array}$$

The second equation is now in slope–intercept form.
Identify the slope and y-intercept of both lines.

$$\begin{array}{rcl} y & = & 2x - 3 \\ y & = & mx + b \\ m & = & 2 \end{array} \qquad \begin{array}{l} y \\ y \\ m \end{array}$$

y-intercept is $(0, -3)$ y-i

The lines have the same slope, but they also have the same y-intercepts. Their equations represent the same line. They are not parallel; they are the same line.

Note:
Exercise:

Problem: Use slopes and y-intercepts to determine if the lines $y=-\frac{1}{2}x-1$ and $x+2y=2$ are parallel.

Solution:

not parallel; same line

Note:
Exercise:

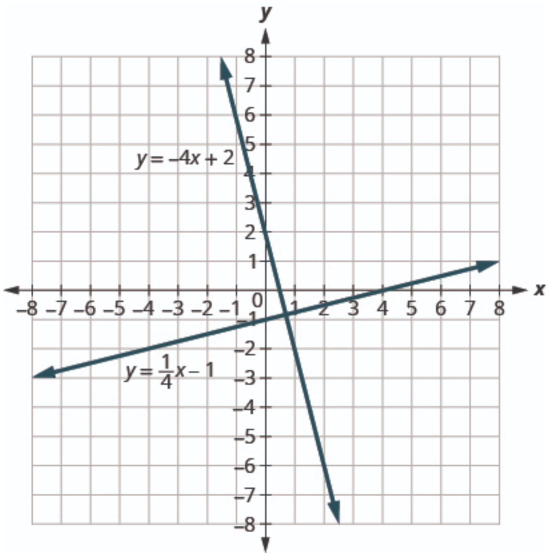
Problem: Use slopes and y-intercepts to determine if the lines $y=\frac{3}{4}x-3$ and $3x-4y=12$ are parallel.

Solution:

not parallel; same line

Use Slopes to Identify Perpendicular Lines

Let's look at the lines whose equations are $y=\frac{1}{4}x-1$ and $y=-4x+2$, shown in [link].



These lines lie in the same plane and intersect in right angles. We call these lines **perpendicular**.

What do you notice about the slopes of these two lines? As we read from left to right, the line $y=\frac{1}{4}x-1$ rises, so its slope is positive. The line$y=-4x+2$ drops from left to right, so it has a negative slope. Does it make sense to you that the slopes of two perpendicular lines will have opposite signs?

If we look at the slope of the first line, $m_1 = \frac{1}{4}$, and the slope of the second line, $m_2 = -4$, we can see that they are *negative reciprocals* of each other. If we multiply them, their product is -1.
Equation:

$$\begin{array}{l} m_1 \cdot m_2 \\ \frac{1}{4}(-4) \\ -1 \end{array}$$

This is always true for perpendicular lines and leads us to this definition.

Note:
Perpendicular Lines
Perpendicular lines are lines in the same plane that form a right angle.
If m_1 and m_2 are the slopes of two perpendicular lines, then:
Equation:

$$m_1 \cdot m_2 = -1 \text{ and } m_1 = \frac{-1}{m_2}$$

Vertical lines and horizontal lines are always perpendicular to each other.

We were able to look at the slope–intercept form of linear equations and determine whether or not the lines were parallel. We can do the same thing for perpendicular lines.

We find the slope–intercept form of the equation, and then see if the slopes are negative reciprocals. If the product of the slopes is -1, the lines are perpendicular. Perpendicular lines may have the same *y*-intercepts.

Example:
Exercise:

Problem: Use slopes to determine if the lines, $y = -5x - 4$ and $x - 5y = 5$ are perpendicular.

Solution:
Solution

The first equation is in slope–intercept form.

$$\begin{array}{rcl} y & = & -5x - 4 \end{array}$$

Solve the second equation for y.

$$\begin{array}{rcl} x - 5y & = & 5 \\ -5y & = & -x + 5 \\ \frac{-5y}{-5} & = & \frac{-x+5}{-5} \\ y & = & \frac{1}{5}x - 1 \end{array}$$

Identify the slope of each line.

$$\begin{array}{rclcrcl} y & = & -5x - 4 & \quad & y & = & \frac{1}{5}x - 1 \\ y & = & mx + b & & y & = & mx + b \\ m_1 & = & -5 & & m_2 & = & \frac{1}{5} \end{array}$$

The slopes are negative reciprocals of each other, so the lines are perpendicular. We check by multiplying the slopes,
Equation:

$$m_1 \cdot m_2$$
$$-5\left(\frac{1}{5}\right)$$
$$-1\checkmark$$

Note:
Exercise:

Problem: Use slopes to determine if the lines $y=-3x+2$ and $x-3y=4$ are perpendicular.

Solution:

perpendicular

Note:
Exercise:

Problem: Use slopes to determine if the lines $y=2x-5$ and $x+2y=-6$ are perpendicular.

Solution:

perpendicular

Example:
Exercise:

Problem: Use slopes to determine if the lines, $7x+2y=3$ and $2x+7y=5$ are perpendicular.

Solution:
Solution

Solve the equations for y.	$\begin{aligned} 7x+2y &= 3 \\ 2y &= -7x+3 \\ \frac{2y}{2} &= \frac{-7x+3}{2} \\ y &= -\frac{7}{2}x+\frac{3}{2} \end{aligned}$	$\begin{aligned} 2x+7y &= 5 \\ 7y &= -2x+5 \\ \frac{7y}{7} &= \frac{-2x+5}{7} \\ y &= -\frac{2}{7}x+\frac{5}{7} \end{aligned}$
Identify the slope of each line.	$\begin{aligned} y &= mx+b \\ m_1 &= -\frac{7}{2} \end{aligned}$	$\begin{aligned} y &= mx+b \\ m_2 &= -\frac{2}{7} \end{aligned}$

The slopes are reciprocals of each other, but they have the same sign. Since they are not negative reciprocals, the lines are not perpendicular.

Note:
Exercise:

Problem: Use slopes to determine if the lines $5x + 4y = 1$ and $4x + 5y = 3$ are perpendicular.

Solution:

not perpendicular

Note:
Exercise:

Problem: Use slopes to determine if the lines $2x - 9y = 3$ and $9x - 2y = 1$ are perpendicular.

Solution:

not perpendicular

Note:
Access this online resource for additional instruction and practice with graphs.

- Explore the Relation Between a Graph and the Slope–Intercept Form of an Equation of a Line

Key Concepts

- The slope–intercept form of an equation of a line with slope m and *y*-intercept, $(0, b)$ is, $y = mx + b$.
- **Graph a Line Using its Slope and *y*-Intercept**

  Find the slope-intercept form of the equation of the line.
  Identify the slope and *y*-intercept.
  Plot the *y*-intercept.
  Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
  Starting at the *y*-intercept, count out the rise and run to mark the second point.
  Connect the points with a line.

- **Strategy for Choosing the Most Convenient Method to Graph a Line:** Consider the form of the equation.
  - If it only has one variable, it is a vertical or horizontal line.
    $x = a$ is a vertical line passing through the *x*-axis at a.
    $y = b$ is a horizontal line passing through the *y*-axis at b.
  - If y is isolated on one side of the equation, in the form $y = mx + b$, graph by using the slope and *y*-intercept.
    Identify the slope and *y*-intercept and then graph.
  - If the equation is of the form $Ax + By = C$, find the intercepts.
    Find the *x*- and *y*-intercepts, a third point, and then graph.
- Parallel lines are lines in the same plane that do not intersect.
  - Parallel lines have the same slope and different *y*-intercepts.

  - If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.
  - Parallel vertical lines have different *x*-intercepts.

- Perpendicular lines are lines in the same plane that form a right angle.
  - If m_1 and m_2 are the slopes of two perpendicular lines, then $m_1 \cdot m_2 = -1$ and $m_1 = \frac{-1}{m_2}$.
  - Vertical lines and horizontal lines are always perpendicular to each other.

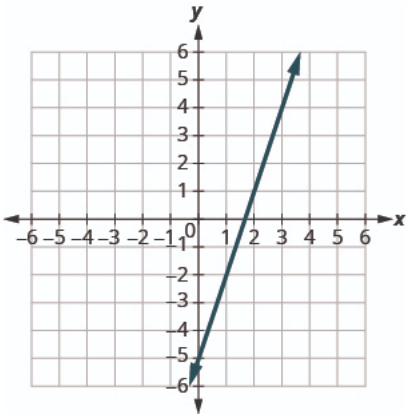
Practice Makes Perfect

Recognize the Relation Between the Graph and the Slope–Intercept Form of an Equation of a Line

In the following exercises, use the graph to find the slope and y-intercept of each line. Compare the values to the equation $y = mx + b$.

Exercise:

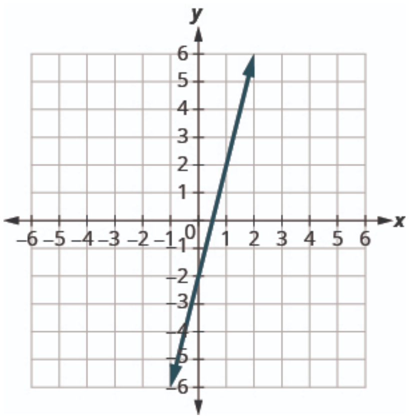
Problem:



$y = 3x - 5$

Exercise:

Problem:



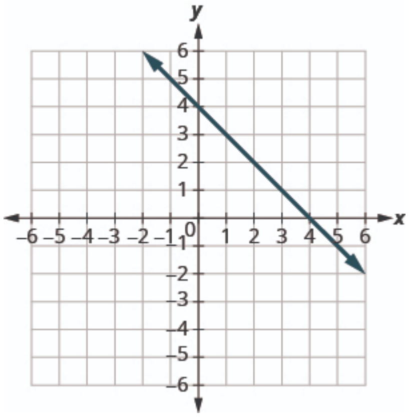
$y = 4x - 2$

Solution:

slope $m = 4$ and *y*-intercept $(0, -2)$

Exercise:

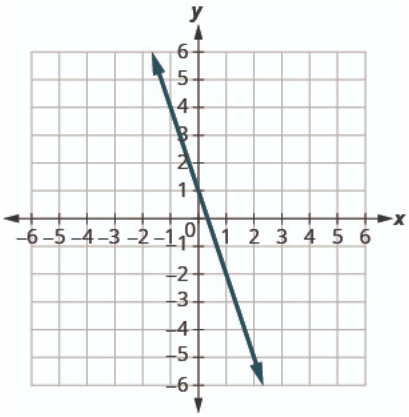
Problem:



$y = -x + 4$

Exercise:

Problem:



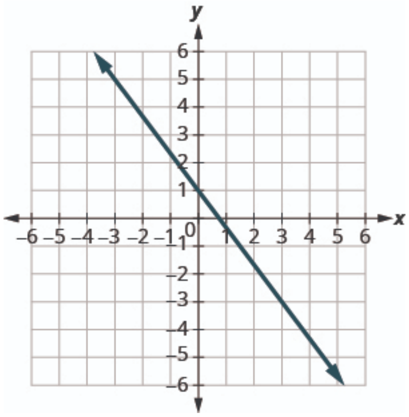
$y = -3x + 1$

Solution:

slope $m = -3$ and *y*-intercept $(0, 1)$

Exercise:

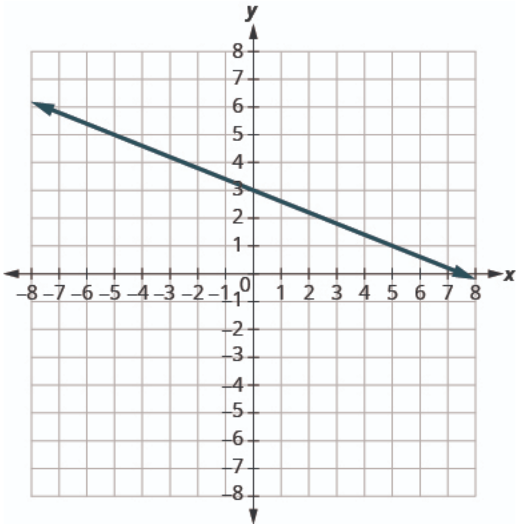
Problem:



$y=-\frac{4}{3}x+1$

Exercise:

Problem:



$y=-\frac{2}{5}x+3$

Solution:

slope $m=-\frac{2}{4}$ and y-intercept $(0,3)$

Identify the Slope and y-Intercept From an Equation of a Line

In the following exercises, identify the slope and y-intercept of each line.

Exercise:

Problem: $y=-7x+3$

Exercise:

Problem: $y=-9x+7$

Solution:

$-9; (0,7)$

Exercise:

Problem: $y = 6x - 8$

Exercise:

Problem: $y = 4x - 10$

Solution:

$4; (0,-10)$

Exercise:

Problem: $3x + y = 5$

Exercise:

Problem: $4x + y = 8$

Solution:

$-4; (0,8)$

Exercise:

Problem: $6x + 4y = 12$

Exercise:

Problem: $8x + 3y = 12$

Solution:

$-\frac{8}{3}; (0,4)$

Exercise:

Problem: $5x - 2y = 6$

Exercise:

Problem: $7x - 3y = 9$

Solution:

$\frac{7}{3}; (0,-3)$

Graph a Line Using Its Slope and Intercept

In the following exercises, graph the line of each equation using its slope and y-intercept.

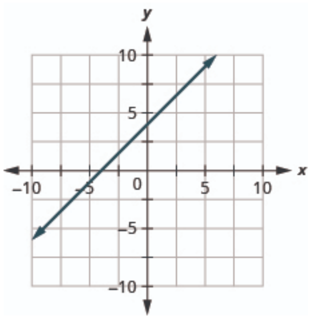
Exercise:

Problem: $y = x + 3$

Exercise:

Problem: $y = x + 4$

Solution:



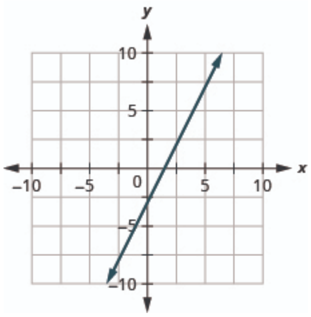
Exercise:

Problem: $y = 3x - 1$

Exercise:

Problem: $y = 2x - 3$

Solution:



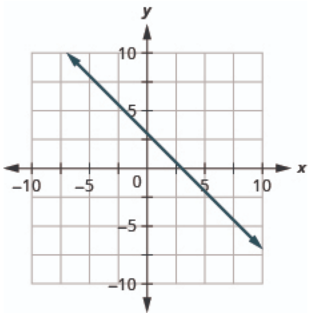
Exercise:

Problem: $y = -x + 2$

Exercise:

Problem: $y = -x + 3$

Solution:



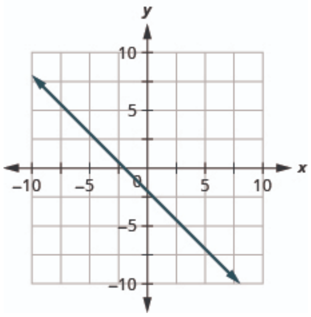
Exercise:

Problem: $y = -x - 4$

Exercise:

Problem: $y = -x - 2$

Solution:



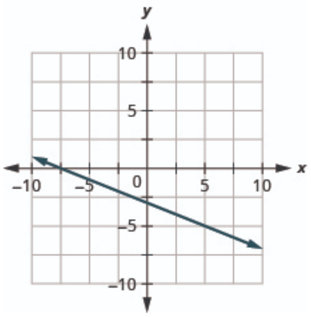
Exercise:

Problem: $y = -\frac{3}{4} - 1$

Exercise:

Problem: $y = -\frac{2}{5} - 3$

Solution:



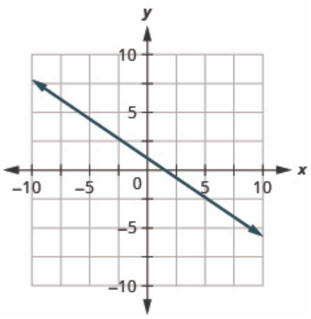
Exercise:

Problem: $y = -\frac{3}{5} + 2$

Exercise:

Problem: $y = -\frac{2}{3} + 1$

Solution:



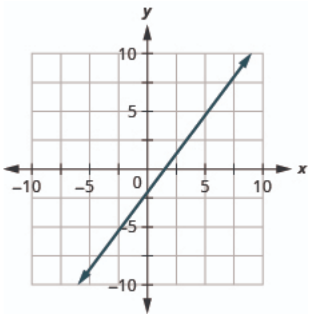
Exercise:

Problem: $3x - 4y = 8$

Exercise:

Problem: $4x - 3y = 6$

Solution:



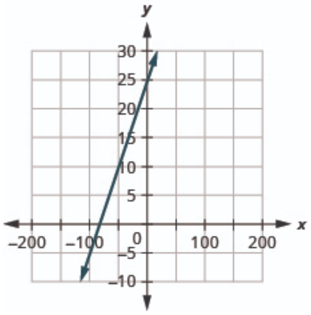
Exercise:

Problem: $y = 0.1x + 15$

Exercise:

Problem: $y = 0.3x + 25$

Solution:



Choose the Most Convenient Method to Graph a Line

In the following exercises, determine the most convenient method to graph each line.

Exercise:

Problem: $x = 2$

Exercise:

Problem: $y = 4$

Solution:

horizontal line

Exercise:

Problem: $y = 5$

Exercise:

Problem: $x = -3$

Solution:

vertical line

Exercise:

Problem: $y = -3x + 4$

Exercise:

Problem: $y = -5x + 2$

Solution:

slope–intercept

Exercise:

Problem: $x - y = 5$

Exercise:

Problem: $x - y = 1$

Solution:

intercepts

Exercise:

Problem: $y = \frac{2}{3}x - 1$

Exercise:

Problem: $y = \frac{4}{5}x - 3$

Solution:

slope–intercept

Exercise:

Problem: $y = -3$

Exercise:

Problem: $y = -1$

Solution:

horizontal line

Exercise:

Problem: $3x - 2y = -12$

Exercise:

Problem: $2x - 5y = -10$

Solution:

intercepts

Exercise:

Problem: $y = -\frac{1}{4} + 3$

Exercise:

Problem: $y = -\frac{1}{3}x + 5$

Solution:

slope–intercept

Graph and Interpret Applications of Slope–Intercept

Exercise:

Problem:

The equation $P = 31 + 1.75w$ models the relation between the amount of Tuyet’s monthly water bill payment, P, in dollars, and the number of units of water, w, used.

ⓐ Find Tuyet’s payment for a month when 0 units of water are used.
ⓑ Find Tuyet’s payment for a month when 12 units of water are used.
ⓒ Interpret the slope and P-intercept of the equation.
ⓓ Graph the equation.

Exercise:

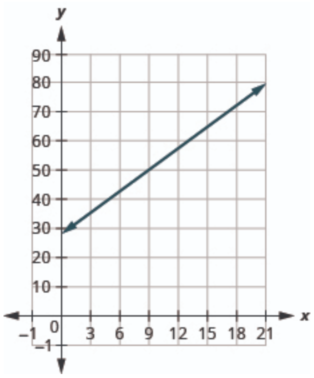
Problem:

The equation $P = 28 + 2.54w$ models the relation between the amount of Randy’s monthly water bill payment, P, in dollars, and the number of units of water, w, used.

ⓐ Find the payment for a month when Randy used 0 units of water.
ⓑ Find the payment for a month when Randy used 15 units of water.
ⓒ Interpret the slope and P-intercept of the equation.
ⓓ Graph the equation.

Solution:

ⓐ \$28
ⓑ \$66.10
ⓒ The slope, 2.54, means that Randy’s payment, P, increases by \$2.54 when the number of units of water he used, w, increases by 1. The P–intercept means that if the number units of water Randy used was 0, the payment would be \$28.
ⓓ



Exercise:

Problem:

Bruce drives his car for his job. The equation $R = 0.575m + 42$ models the relation between the amount in dollars, R, that he is reimbursed and the number of miles, m, he drives in one day.

ⓐ Find the amount Bruce is reimbursed on a day when he drives 0 miles.
ⓑ Find the amount Bruce is reimbursed on a day when he drives 220 miles.
ⓒ Interpret the slope and R-intercept of the equation.
ⓓ Graph the equation.

Exercise:

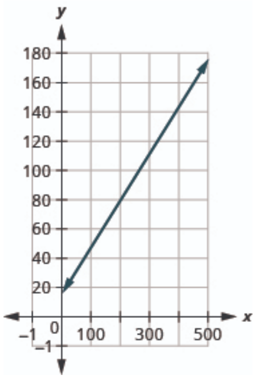
Problem:

Janelle is planning to rent a car while on vacation. The equation $C = 0.32m + 15$ models the relation between the cost in dollars, *C*, per day and the number of miles, *m*, she drives in one day.

ⓐ Find the cost if Janelle drives the car 0 miles one day.
ⓑ Find the cost on a day when Janelle drives the car 400 miles.
ⓒ Interpret the slope and *C*–intercept of the equation.
ⓓ Graph the equation.

Solution:

ⓐ $15
ⓑ $143
ⓒ The slope, 0.32, means that the cost, *C*, increases by $0.32 when the number of miles driven, *m*, increases by 1. The *C*-intercept means that if Janelle drives 0 miles one day, the cost would be $15.
ⓓ



Exercise:

Problem:

Cherie works in retail and her weekly salary includes commission for the amount she sells. The equation $S = 400 + 0.15c$ models the relation between her weekly salary, *S*, in dollars and the amount of her sales, *c*, in dollars.

ⓐ Find Cherie’s salary for a week when her sales were 0.
ⓑ Find Cherie’s salary for a week when her sales were 3600.
ⓒ Interpret the slope and *S*–intercept of the equation.
ⓓ Graph the equation.

Exercise:

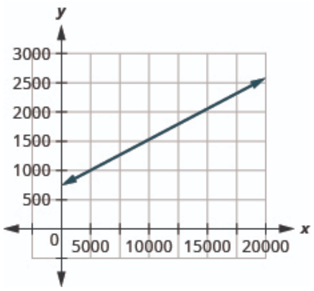
Problem:

Patel’s weekly salary includes a base pay plus commission on his sales. The equation $S = 750 + 0.09c$ models the relation between his weekly salary, *S*, in dollars and the amount of his sales, *c*, in dollars.

ⓐ Find Patel’s salary for a week when his sales were 0.
ⓑ Find Patel’s salary for a week when his sales were 18,540.
ⓒ Interpret the slope and *S*-intercept of the equation.
ⓓ Graph the equation.

Solution:

ⓐ $750
ⓑ $2418.60
ⓒ The slope, 0.09, means that Patel's salary, S, increases by $0.09 for every $1 increase in his sales. The S-intercept means that when his sales are $0, his salary is $750.
ⓓ



Exercise:

Problem:

Costa is planning a lunch banquet. The equation $C = 450 + 28g$ models the relation between the cost in dollars, C, of the banquet and the number of guests, g.

ⓐ Find the cost if the number of guests is 40.
ⓑ Find the cost if the number of guests is 80.
ⓒ Interpret the slope and C-intercept of the equation.
ⓓ Graph the equation.

Exercise:

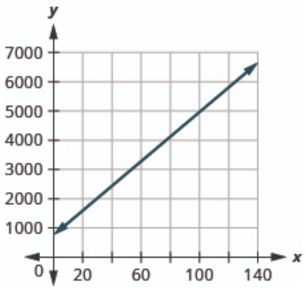
Problem:

Margie is planning a dinner banquet. The equation $C = 750 + 42g$ models the relation between the cost in dollars, C of the banquet and the number of guests, g.

ⓐ Find the cost if the number of guests is 50.
ⓑ Find the cost if the number of guests is 100.
ⓒ Interpret the slope and C–intercept of the equation.
ⓓ Graph the equation.

Solution:

ⓐ $2850
ⓑ $4950
ⓒ The slope, 42, means that the cost, C, increases by $42 for when the number of guests increases by 1. The C-intercept means that when the number of guests is 0, the cost would be $750.
ⓓ



Use Slopes to Identify Parallel Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel.

Exercise:

Problem: $y = \frac{3}{4}x - 3;\ \ 3x - 4y = -2$

Exercise:

Problem: $y = \frac{2}{3}x - 1;\ \ 2x - 3y = -2$

Solution:

parallel

Exercise:

Problem: $2x - 5y = -3;\ \ y = \frac{2}{5}x + 1$

Exercise:

Problem: $3x - 4y = -2;\ \ y = \frac{3}{4}x - 3$

Solution:

parallel

Exercise:

Problem: $2x - 4y = 6;\ \ x - 2y = 3$

Exercise:

Problem: $6x - 3y = 9;\ \ 2x - y = 3$

Solution:

not parallel

Exercise:

Problem: $4x + 2y = 6;\ \ 6x + 3y = 3$

Exercise:

Problem: $8x + 6y = 6;\ \ 12x + 9y = 12$

Solution:

parallel

Exercise:

Problem: $x = 5; \;\; x = -6$

Exercise:

Problem: $x = 7; \;\; x = -8$

Solution:

parallel

Exercise:

Problem: $x = -4; \;\; x = -1$

Exercise:

Problem: $x = -3; \;\; x = -2$

Solution:

parallel

Exercise:

Problem: $y = 2; \;\; y = 6$

Exercise:

Problem: $y = 5; \;\; y = 1$

Solution:

parallel

Exercise:

Problem: $y = -4; \;\; y = 3$

Exercise:

Problem: $y = -1; \;\; y = 2$

Solution:

parallel

Exercise:

Problem: $x - y = 2; \;\; 2x - 2y = 4$

Exercise:

Problem: $4x + 4y = 8; \;\; x + y = 2$

Solution:

not parallel

Exercise:

Problem: $x - 3y = 6; \;\; 2x - 6y = 12$

Exercise:

Problem: $5x - 2y = 11; \;\; 5x - y = 7$

Solution:

not parallel

Exercise:

Problem: $3x - 6y = 12; \;\; 6x - 3y = 3$

Exercise:

Problem: $4x - 8y = 16; \;\; x - 2y = 4$

Solution:

not parallel

Exercise:

Problem: $9x - 3y = 6; \;\; 3x - y = 2$

Exercise:

Problem: $x - 5y = 10; \;\; 5x - y = -10$

Solution:

not parallel

Exercise:

Problem: $7x - 4y = 8; \;\; 4x + 7y = 14$

Exercise:

Problem: $9x - 5y = 4; \;\; 5x + 9y = -1$

Solution:

not parallel

Use Slopes to Identify Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are perpendicular.

Exercise:

Problem: $3x - 2y = 8; 2x + 3y = 6$

Exercise:

Problem: $x - 4y = 8; 4x + y = 2$

Solution:

perpendicular

Exercise:

Problem: $2x + 5y = 3; 5x - 2y = 6$

Exercise:

Problem: $2x + 3y = 5; 3x - 2y = 7$

Solution:

perpendicular

Exercise:

Problem: $3x - 2y = 1; 2x - 3y = 2$

Exercise:

Problem: $3x - 4y = 8; 4x - 3y = 6$

Solution:

not perpendicular

Exercise:

Problem: $5x + 2y = 6; 2x + 5y = 8$

Exercise:

Problem: $2x + 4y = 3; 6x + 3y = 2$

Solution:

not perpendicular

Exercise:

Problem: $4x - 2y = 5; 3x + 6y = 8$

Exercise:

Problem: $2x - 6y = 4; 12x + 4y = 9$

Solution:

perpendicular

Exercise:

Problem: $6x - 4y = 5; 8x + 12y = 3$

Exercise:

Problem: $8x - 2y = 7; 3x + 12y = 9$

Solution:

perpendicular

Everyday Math

Exercise:

Problem:

The equation $C = \frac{5}{9}F - 17.8$ can be used to convert temperatures *F*, on the Fahrenheit scale to temperatures, *C*, on the Celsius scale.

ⓐ Explain what the slope of the equation means.
ⓑ Explain what the *C*–intercept of the equation means.

Exercise:

Problem:

The equation $n = 4T - 160$ is used to estimate the number of cricket chirps, *n*, in one minute based on the temperature in degrees Fahrenheit, *T*.

ⓐ Explain what the slope of the equation means.
ⓑ Explain what the *n*–intercept of the equation means. Is this a realistic situation?

Solution:

ⓐ For every increase of one degree Fahrenheit, the number of chirps increases by four.
ⓑ There would be -160 chirps when the Fahrenheit temperature is 0°. (Notice that this does not make sense; this model cannot be used for all possible temperatures.)

Writing Exercises

Exercise:

Problem: Explain in your own words how to decide which method to use to graph a line.

Exercise:

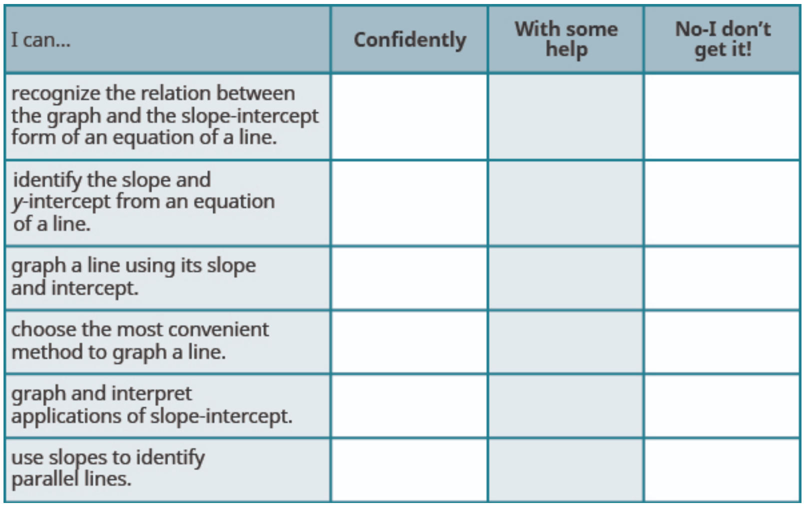
Problem: Why are all horizontal lines parallel?

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
recognize the relation between the graph and the slope-intercept form of an equation of a line.			
identify the slope and *y*-intercept from an equation of a line.			
graph a line using its slope and intercept.			
choose the most convenient method to graph a line.			
graph and interpret applications of slope-intercept.			
use slopes to identify parallel lines.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

parallel lines

Lines in the same plane that do not intersect.

perpendicular lines

Lines in the same plane that form a right angle.

slope-intercept form of an equation of a line

The slope–intercept form of an equation of a line with slope m and y-intercept, $(0, b)$ is, $y = mx + b$.

Find the Equation of a Line
By the end of this section, you will be able to:

- Find an equation of the line given the slope and y-intercept
- Find an equation of the line given the slope and a point
- Find an equation of the line given two points
- Find an equation of a line parallel to a given line
- Find an equation of a line perpendicular to a given line

Note:
Before you get started, take this readiness quiz.

1. Solve: $\frac{2}{3}=\frac{x}{5}$.
   If you missed this problem, review [link].
2. Simplify: $-\frac{2}{5}(x-15)$.
   If you missed this problem, review [link].

How do online retailers know that ‘you may also like’ a particular item based on something you just ordered? How can economists know how a rise in the minimum wage will affect the unemployment rate? How do medical researchers create drugs to target cancer cells? How can traffic engineers predict the effect on your commuting time of an increase or decrease in gas prices? It’s all mathematics.

You are at an exciting point in your mathematical journey as the mathematics you are studying has interesting applications in the real world.

The physical sciences, social sciences, and the business world are full of situations that can be modeled with linear equations relating two variables. Data is collected and graphed. If the data points appear to form a straight line, an equation of that line can be used to predict the value of one variable based on the value of the other variable.

To create a mathematical model of a linear relation between two variables, we must be able to find the equation of the line. In this section we will look at several ways to write the equation of a line. The specific method we use will be determined by what information we are given.

Find an Equation of the Line Given the Slope and *y*-Intercept

We can easily determine the slope and intercept of a line if the equation was written in slope–intercept form, $y=mx+b$. Now, we will do the reverse—we will start with the slope and *y*-intercept and use them to find the equation of the line.

Example:
Exercise:

Problem: Find an equation of a line with slope -7 and *y*-intercept $(0, -1)$.

Solution:
Solution

Since we are given the slope and *y*-intercept of the line, we can substitute the needed values into the slope–intercept form, $y = mx + b$.

Name the slope.	$m = -7$
Name the *y*-intercept.	*y*-intercept $(0, -1)$
Substitute the values into $y = mx + b$.	$y = mx + b$
	$y = -7x + (-1)$
	$y = -7x + -1$

Note:
Exercise:

Problem: Find an equation of a line with slope $\frac{2}{5}$ and *y*-intercept $(0, 4)$.

Solution:

$y = \frac{2}{5}x + 4$

Note:
Exercise:

Problem: Find an equation of a line with slope -1 and *y*-intercept $(0, -3)$.

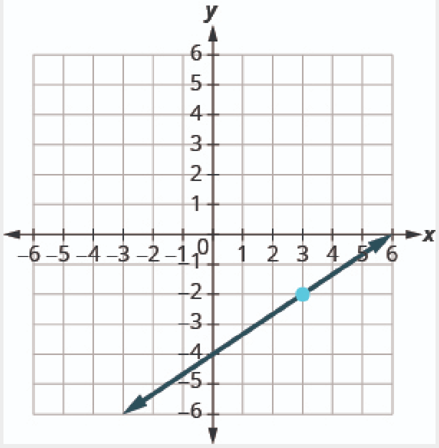
Solution:

$y = -x - 3$

Sometimes, the slope and intercept need to be determined from the graph.

Example:
Exercise:

Problem: Find the equation of the line shown.



Solution:
Solution

We need to find the slope and *y*-intercept of the line from the graph so we can substitute the needed values into the slope–intercept form, $y = mx + b$.

To find the slope, we choose two points on the graph.

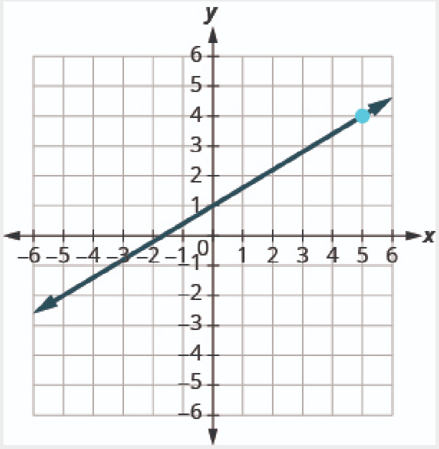
The *y*-intercept is $(0, -4)$ and the graph passes through $(3, -2)$.

Find the slope by counting the rise and run.	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{3}$

Find the y-intercept.	y-intercept $(0, -4)$
Substitute the values into $y = mx + b$.	$y = mx + b$
	$y = \frac{2}{3}x - 4$

Note:
Exercise:

Problem: Find the equation of the line shown in the graph.

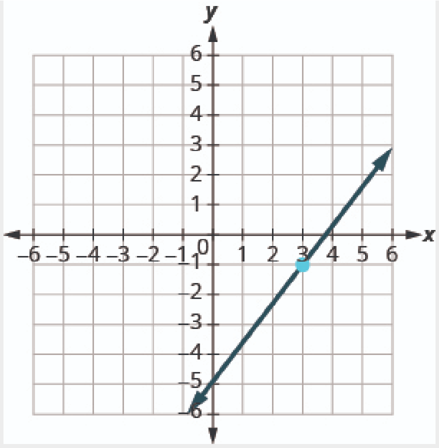


Solution:

$y = \frac{3}{5}x + 1$

Note:
Exercise:

Problem: Find the equation of the line shown in the graph.



Solution:

$y = \frac{4}{3}x - 5$

Find an Equation of the Line Given the Slope and a Point

Finding an equation of a line using the slope–intercept form of the equation works well when you are given the slope and *y*-intercept or when you read them off a graph. But what happens when you have another point instead of the *y*-intercept?

We are going to use the slope formula to derive another form of an equation of the line. Suppose we have a line that has slope m and that contains some specific point (x_1, y_1) and some other point, which we will just call (x, y). We can write the slope of this line and then change it to a different form.

	$m = \frac{y-y_1}{x-x_1}$
Multiply both sides of the equation by $x - x_1$.	$m(x - x_1) = \left(\frac{y-y_1}{x-x_1}\right)(x - x_1)$
Simplify.	$m(x - x_1) = y - y_1$
Rewrite the equation with the y terms on the left.	$y - y_1 = m(x - x_1)$

This format is called the point–slope form of an equation of a line.

Note:
Point–slope Form of an Equation of a Line
The **point–slope form** of an equation of a line with slope m and containing the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

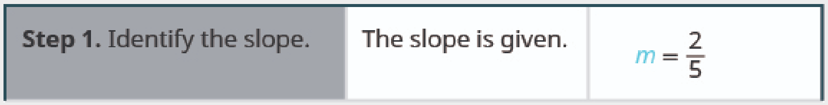
We can use the point–slope form of an equation to find an equation of a line when we are given the slope and one point. Then we will rewrite the equation in slope–intercept form. Most applications of linear equations use the the slope–intercept form.

Example:
Find an Equation of a Line Given the Slope and a Point
Exercise:

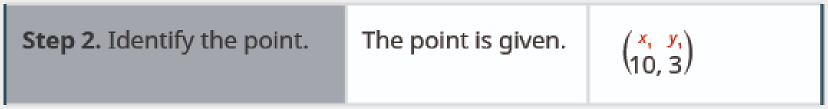
Problem:

Find an equation of a line with slope $m = \frac{2}{5}$ that contains the point $(10, 3)$. Write the equation in slope–intercept form.

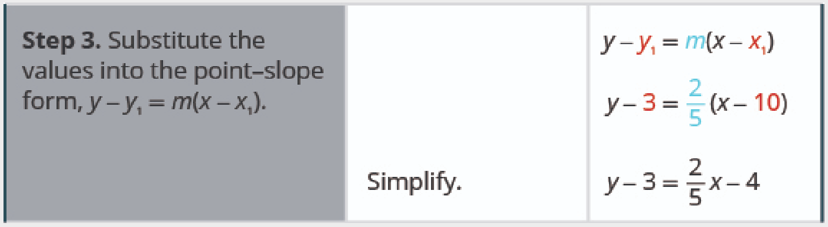
Solution:
Solution



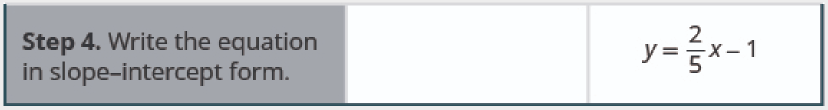
Step 1. Identify the slope.	The slope is given.	$m = \frac{2}{5}$



Step 2. Identify the point.	The point is given.	$\left(\overset{x_1}{10}, \overset{y_1}{3}\right)$



Step 3. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.		$y - y_1 = m(x - x_1)$ $y - 3 = \frac{2}{5}(x - 10)$
	Simplify.	$y - 3 = \frac{2}{5}x - 4$



Step 4. Write the equation in slope–intercept form.		$y = \frac{2}{5}x - 1$

Note:
Exercise:

Problem: Find an equation of a line with slope $m = \frac{5}{6}$ and containing the point $(6, 3)$.

Solution:

$y = \frac{5}{6}x - 2$

Note:
Exercise:

Problem: Find an equation of a line with slope $m = \frac{2}{3}$ and containing thepoint $(9, 2)$.

Solution:

$y = \frac{2}{3}x - 4$

Note:
Find an equation of a line given the slope and a point.

Identify the slope.
Identify the point.
Substitute the values into the point-slope form,$y - y_1 = m\,(x - x_1)$.
Write the equation in slope–intercept form.

Example:
Exercise:

Problem:

Find an equation of a line with slope $m = -\frac{1}{3}$ that contains the point $(6, -4)$. Write the equation in slope–intercept form.

Solution:
Solution

Since we are given a point and the slope of the line, we can substitute the needed values into the point–slope form, $y - y_1 = m\,(x - x_1)$.

Identify the slope.	$m = -\frac{1}{3}$
Identify the point.	$\left(\overset{x_1}{6}, \overset{y_1}{-4}\right)$
Substitute the values into $y - y_1 = m(x - x_1)$.	$y - y_1 = m(x - x_1)$

	$y-(-4)=-\frac{1}{3}(x-6)$
Simplify.	$y+4=-\frac{1}{3}x+2$
Write in slope–intercept form.	$y=-\frac{1}{3}x-2$

Note:
Exercise:

Problem: Find an equation of a line with slope $m=-\frac{2}{5}$ and containing the point $(10,-5)$.

Solution:

$y=-\frac{2}{5}x-1$

Note:
Exercise:

Problem: Find an equation of a line with slope $m=-\frac{3}{4}$, and containing the point $(4,-7)$.

Solution:

$y=-\frac{3}{4}x-4$

Example:
Exercise:

Problem:

Find an equation of a horizontal line that contains the point $(-1,2)$. Write the equation in slope–intercept form.

Solution:
Solution

Every horizontal line has slope 0. We can substitute the slope and points into the point–slope form, $y-y_1=m\left(x-x_1\right)$.

Identify the slope.	$m=0$
Identify the point.	$\left(\overset{x_1}{-1}, \overset{y_1}{2}\right)$
Substitute the values into $y-y_1=m(x-x_1)$.	$y-y_1=m(x-x_1)$
	$y-2=0(x-(-1))$
Simplify.	$y-2=0(x+1)$
	$y-2=0$
	$y=2$
Write in slope–intercept form.	It is in *y*-form, but could be written $y=0x+2$.

Did we end up with the form of a horizontal line, $y=a$?

Note:
Exercise:

Problem: Find an equation of a horizontal line containing the point $(-3,8)$.

Solution:

$y=8$

Note:
Exercise:

Problem: Find an equation of a horizontal line containing the point $(-1,4)$.

Solution:

$y = 4$

Find an Equation of the Line Given Two Points

When real-world data is collected, a linear model can be created from two data points. In the next example we'll see how to find an equation of a line when just two points are given.

We have two options so far for finding an equation of a line: slope–intercept or point–slope. Since we will know two points, it will make more sense to use the point–slope form.

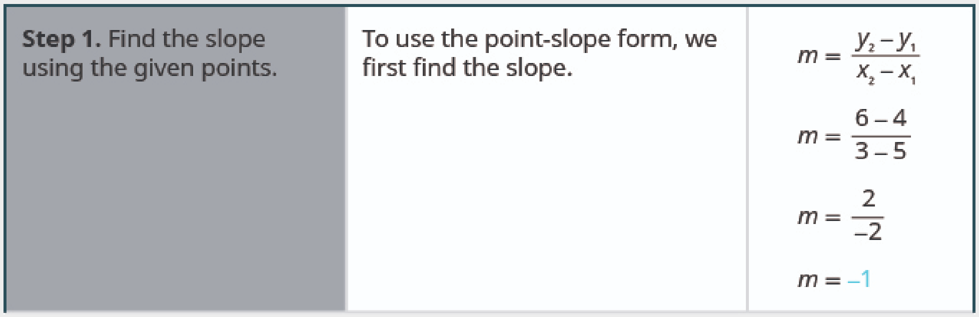
But then we need the slope. Can we find the slope with just two points? Yes. Then, once we have the slope, we can use it and one of the given points to find the equation.

Example:
Find an Equation of a Line Given Two Points
Exercise:

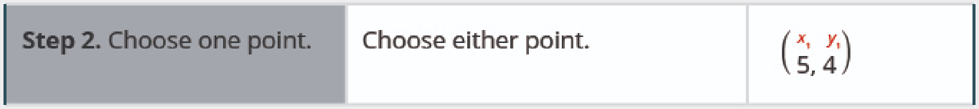
Problem:

Find an equation of a line that contains the points $(5, 4)$ and $(3, 6)$. Write the equation in slope–intercept form.

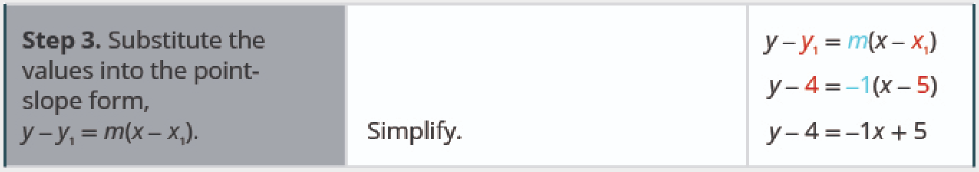
Solution:



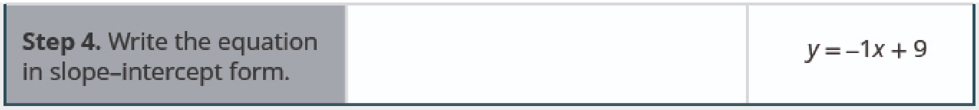
Step 1. Find the slope using the given points.	To use the point-slope form, we first find the slope.	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{6 - 4}{3 - 5}$ $m = \frac{2}{-2}$ $m = -1$



Step 2. Choose one point.	Choose either point.	$\left(\overset{x_1}{5}, \overset{y_1}{4}\right)$



Step 3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.		$y - y_1 = m(x - x_1)$ $y - 4 = -1(x - 5)$
	Simplify.	$y - 4 = -1x + 5$



Step 4. Write the equation in slope–intercept form.		$y = -1x + 9$

Use the point $(3,6)$ and see that you get the same equation.

Note:
Exercise:

Problem: Find an equation of a line containing the points $(3,1)$ and $(5,6)$.

Solution:

$y = \frac{5}{2}x - \frac{13}{2}$

Note:
Exercise:

Problem: Find an equation of a line containing the points $(1,4)$ and $(6,2)$.

Solution:

$y = -\frac{2}{5}x + \frac{22}{5}$

Note:
Find an equation of a line given two points.

Find the slope using the given points.
Choose one point.
Substitute the values into the point-slope form,$y - y_1 = m\left(x - x_1\right)$.
Write the equation in slope–intercept form.

Example:
Exercise:

Problem:

Find an equation of a line that contains the points $(-3,-1)$ and $(2,-2)$. Write the equation in slope–intercept form.

Solution:
Solution

Since we have two points, we will find an equation of the line using the point–slope form. The first step will be to find the slope.

Find the slope of the line through (−3, −1) and (2, −2).	$m = \frac{y_2 - y_1}{x_2 - x_1}$
	$m = \frac{-2 - (-1)}{2 - (-3)}$
	$m = \frac{-1}{5}$
	$m = -\frac{1}{5}$
Choose either point.	$\left(\overset{x_1}{2}, \overset{y_1}{-2}\right)$
Substitute the values into $y - y_1 = m(x - x_1)$.	$y - y_1 = m(x - x_1)$
	$y - (-2) = -\frac{1}{5}(x - 2)$
	$y + 2 = -\frac{1}{5}x + \frac{2}{5}$
Write in slope–intercept form.	$y = -\frac{1}{5}x - \frac{8}{5}$

Note:
Exercise:

Problem: Find an equation of a line containing the points $(-2,-4)$ and $(1,-3)$.

Solution:

$y=\frac{1}{3}x-\frac{10}{3}$

Note:
Exercise:

Problem: Find an equation of a line containing the points $(-4,-3)$ and $(1,-5)$.

Solution:

$y=-\frac{2}{5}x-\frac{23}{5}$

Example:
Exercise:

Problem:

Find an equation of a line that contains the points $(-2,4)$ and $(-2,-3)$. Write the equation in slope–intercept form.

Solution:

Again, the first step will be to find the slope.

Find the slope of the line through $(-2,4)$ and $(-2,-3)$.

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ m &= \frac{-3-4}{-2-(-2)} \\ m &= \frac{-7}{0} \end{aligned}$$

The slope is undefined.

This tells us it is a vertical line. Both of our points have an *x*-coordinate of -2. So our equation of the line is $x=-2$. Since there is no y, we cannot write it in slope–intercept form.

You may want to sketch a graph using the two given points. Does the graph agree with our conclusion that this is a vertical line?

Note:
Exercise:

Problem: Find an equation of a line containing the points $(5,1)$ and $(5,-4)$.

Solution:

$x=5$

Note:
Exercise:

Problem: Find an equaion of a line containing the points $(-4,4)$ and $(-4,3)$.

Solution:

$x=-4$

We have seen that we can use either the slope–intercept form or the point–slope form to find an equation of a line. Which form we use will depend on the information we are given. This is summarized in [link].

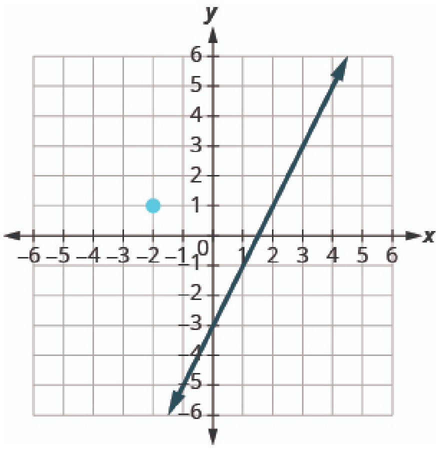
To Write an Equation of a Line		
If given:	Use:	Form:
Slope and y-intercept	slope–intercept	$y=mx+b$
Slope and a point	point–slope	$y-y_1=m\left(x-x_1\right)$
Two points	point–slope	$y-y_1=m\left(x-x_1\right)$

Find an Equation of a Line Parallel to a Given Line

Suppose we need to find an equation of a line that passes through a specific point and is parallel to a given line. We can use the fact that parallel lines have the same slope. So we will have a point and the slope—just what we need to use the point–slope equation.

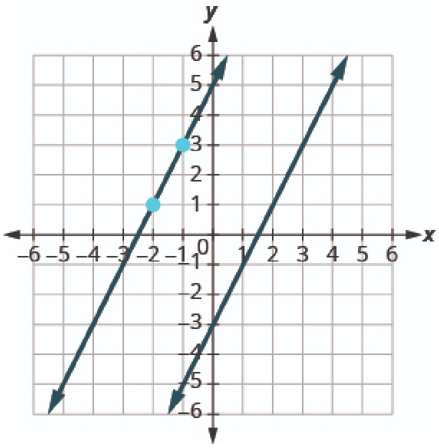
First let's look at this graphically.

The graph shows the graph of $y=2x-3$. We want to graph a line parallel to this line and passing through the point $(-2,1)$.



We know that parallel lines have the same slope. So the second line will have the same slope as $y = 2x - 3$. That slope is $m_{\parallel} = 2$. We'll use the notation $m_{\parallel}$ to represent the slope of a line parallel to a line with slope m. (Notice that the subscript $\parallel$ looks like two parallel lines.)

The second line will pass through $(-2, 1)$ and have $m = 2$. To graph the line, we start at $(-2, 1)$ and count out the rise and run. With $m = 2$ (or $m = \frac{2}{1}$), we count out the rise 2 and the run 1. We draw the line.



Do the lines appear parallel? Does the second line pass through $(-2, 1)$?

Now, let's see how to do this algebraically.

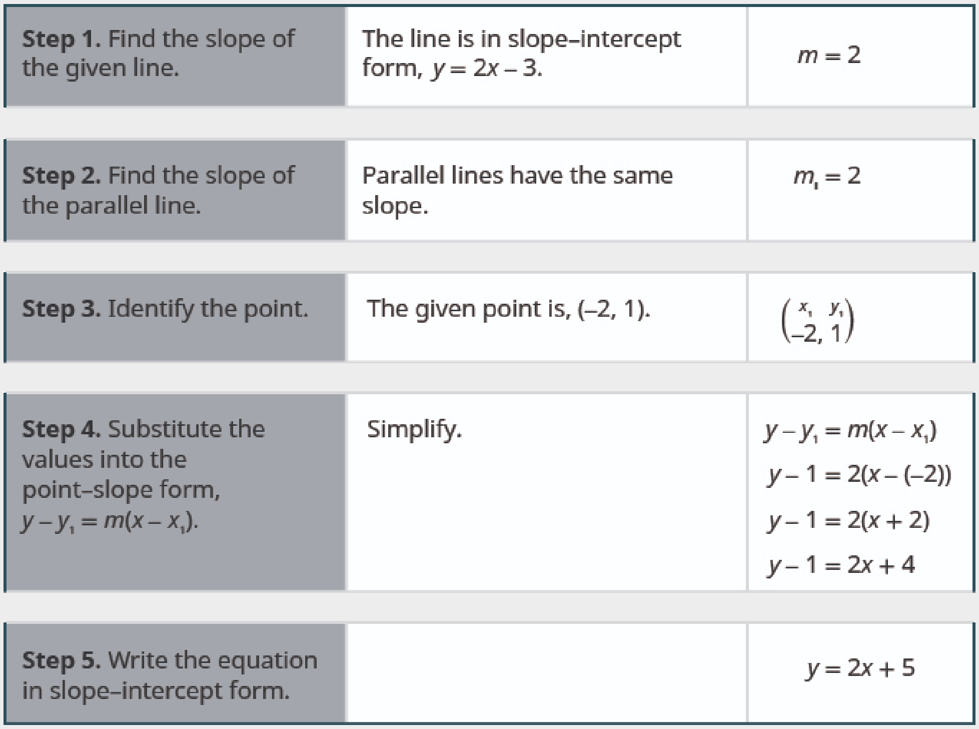
We can use either the slope–intercept form or the point–slope form to find an equation of a line. Here we know one point and can find the slope. So we will use the point–slope form.

Example:
How to Find an Equation of a Line Parallel to a Given Line
Exercise:

Problem:

Find an equation of a line parallel to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope–intercept form.

Solution:
Solution



Step 1. Find the slope of the given line.	The line is in slope–intercept form, $y = 2x - 3$.	$m = 2$
Step 2. Find the slope of the parallel line.	Parallel lines have the same slope.	$m_1 = 2$
Step 3. Identify the point.	The given point is, (–2, 1).	$\overset{x_1,\ \ y_1}{(-2, 1)}$
Step 4. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - 1 = 2(x - (-2))$ $y - 1 = 2(x + 2)$ $y - 1 = 2x + 4$
Step 5. Write the equation in slope–intercept form.		$y = 2x + 5$

Does this equation make sense? What is the y-intercept of the line? What is the slope?

Note:
Exercise:

Problem:

Find an equation of a line parallel to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope–intercept form.

Solution:

$y = 3x - 10$

Note:

Exercise:

Problem: Find an equation of a line parallel to the line $y=\frac{1}{2}x-3$ that contains the point $(6,4)$.

Solution:

$y=\frac{1}{2}x+1$

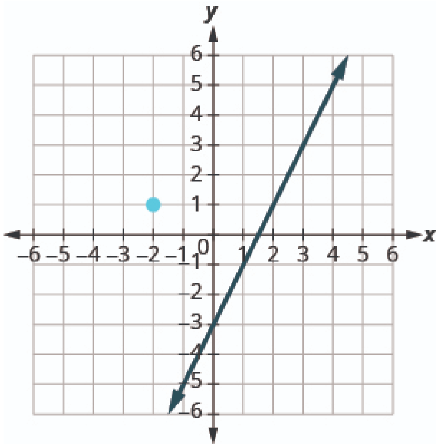
Note:
Find an equation of a line parallel to a given line.

Find the slope of the given line.
Find the slope of the parallel line.
Identify the point.
Substitute the values into the point–slope form,$y-y_1=m\left(x-x_1\right)$.
Write the equation in slope–intercept form.

Find an Equation of a Line Perpendicular to a Given Line

Now, let’s consider perpendicular lines. Suppose we need to find a line passing through a specific point and which is perpendicular to a given line. We can use the fact that perpendicular lines have slopes that are negative reciprocals. We will again use the point–slope equation, like we did with parallel lines.

The graph shows the graph of $y=2x-3$. Now, we want to graph a line perpendicular to this line and passing through $(-2,1)$.

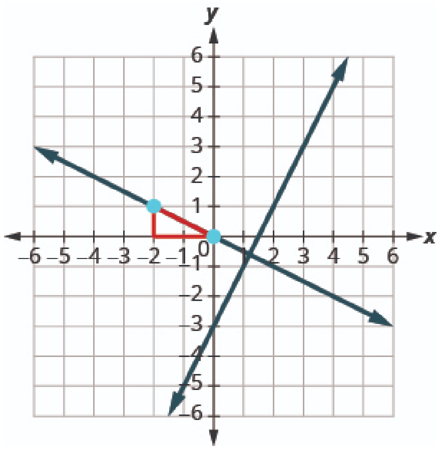


We know that perpendicular lines have slopes that are negative reciprocals. We’ll use the notation $m_{\perp}$ to represent the slope of a line perpendicular to a line with slope m. (Notice that the subscript $\perp$ looks like the right angles made by two perpendicular lines.)
Equation:

$$\begin{array}{ll} y = 2x - 3 & \text{perpendicular line} \\ m = 2 & m_{\perp} = -\frac{1}{2} \end{array}$$

We now know the perpendicular line will pass through $(-2, 1)$ with $m_{\perp} = -\frac{1}{2}$.

To graph the line, we will start at $(-2, 1)$ and count out the rise -1 and the run 2. Then we draw the line.



Do the lines appear perpendicular? Does the second line pass through $(-2, 1)$?

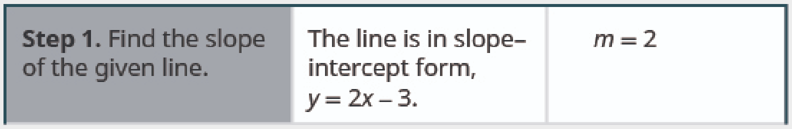
Now, let's see how to do this algebraically. We can use either the slope–intercept form or the point–slope form to find an equation of a line. In this example we know one point, and can find the slope, so we will use the point–slope form.

Example:
How to Find an Equation of a Line Perpendicular to a Given Line
Exercise:

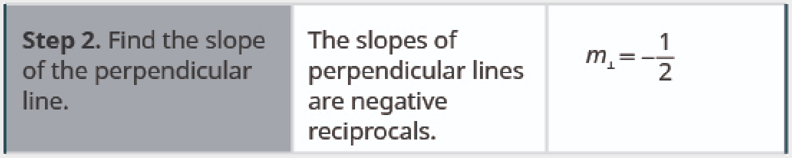
Problem:

Find an equation of a line perpendicular to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope–intercept form.

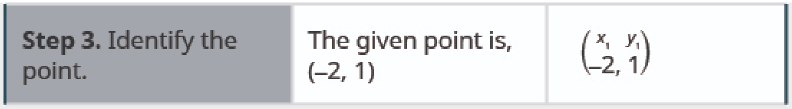
Solution:
Solution



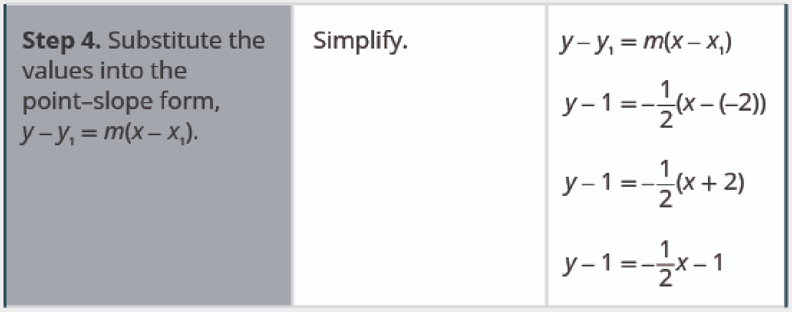
Step 1. Find the slope of the given line.	The line is in slope–intercept form, $y = 2x - 3$.	$m = 2$



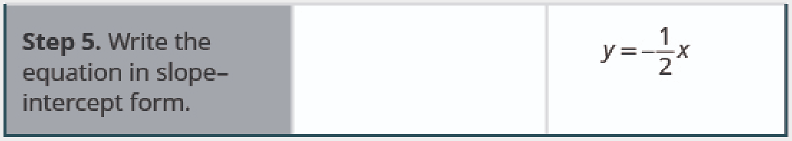
Step 2. Find the slope of the perpendicular line.	The slopes of perpendicular lines are negative reciprocals.	$m_{\perp} = -\frac{1}{2}$



Step 3. Identify the point.	The given point is, (–2, 1)	$\begin{pmatrix} x_1, & y_1 \\ -2, & 1 \end{pmatrix}$



Step 4. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - 1 = -\frac{1}{2}(x - (-2))$ $y - 1 = -\frac{1}{2}(x + 2)$ $y - 1 = -\frac{1}{2}x - 1$



Step 5. Write the equation in slope–intercept form.		$y = -\frac{1}{2}x$

Note:
Exercise:

Problem:

Find an equation of a line perpendicular to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope–intercept form.

Solution:

$y = -\frac{1}{3}x + \frac{10}{3}$

Note:
Exercise:

Problem:

Find an equation of a line perpendicular to the line $y = \frac{1}{2}x - 3$ that contains the point $(6, 4)$.

Solution:

$y = -2x + 16$

Note:
Find an equation of a line perpendicular to a given line.

Find the slope of the given line.
Find the slope of the perpendicular line.
Identify the point.
Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.
Write the equation in slope–intercept form.

Example:
Exercise:

Problem:

Find an equation of a line perpendicular to $x = 5$ that contains the point $(3, -2)$. Write the equation in slope–intercept form.

Solution:
Solution

Again, since we know one point, the point–slope option seems more promising than the slope–intercept option. We need the slope to use this form, and we know the new line will be perpendicular to $x = 5$. This line is vertical, so its perpendicular will be horizontal. This tells us the $m_\perp = 0$.

$$\begin{array}{ll} \text{Identify the point.} & (3, -2) \\ \text{Identify the slope of the perpendicular line.} & m_\perp = 0 \\ \text{Substitute the values into } y - y_1 = m(x - x_1). & y - y_1 = m(x - x_1) \\ & y - (-2) = 0(x - 3) \\ \text{Simplify.} & y + 2 = 0 \\ & y = -2 \end{array}$$

Sketch the graph of both lines. Do they appear to be perpendicular?

Note:
Exercise:

Problem:

Find an equation of a line that is perpendicular to the line $x = 4$ that contains the point $(4, -5)$. Write the equation in slope–intercept form.

Solution:

$y = -5$

Note:
Exercise:

Problem:

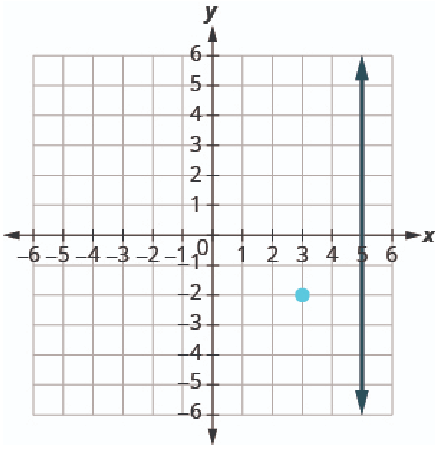
Find an equation of a line that is perpendicular to the line $x = 2$ that contains the point $(2, -1)$. Write the equation in slope–intercept form.

Solution:

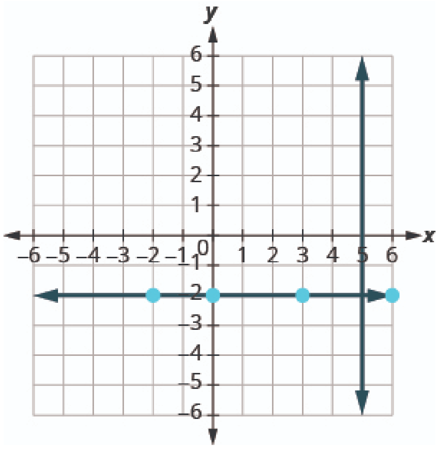
$y = -1$

In [link], we used the point–slope form to find the equation. We could have looked at this in a different way.

We want to find a line that is perpendicular to $x = 5$ that contains the point $(3, -2)$. The graph shows us the line$x = 5$ and the point $(3, -2)$.



We know every line perpendicular to a vetical line is horizontal, so we will sketch the horizontal line through $(3, -2)$.



Do the lines appear perpendicular?

If we look at a few points on this horizontal line, we notice they all have *y*-coordinates of -2. So, the equation of the line perpendicular to the vertical line $x = 5$ is $y = -2$.

Example:
Exercise:

Problem:

Find an equation of a line that is perpendicular to $y = -4$ that contains the point $(-4, 2)$. Write the equation in slope–intercept form.

Solution:
Solution

The line $y = -4$ is a horizontal line. Any line perpendicular to it must be vertical, in the form $x = a$. Since the perpendicular line is vertical and passes through $(-4, 2)$, every point on it has an *x*-coordinate of -4. The equation of the perpendicular line is $x = -4$. You may want to sketch the lines. Do they appear perpendicular?

Note:
Exercise:

Problem:

Find an equation of a line that is perpendicular to the line $y = 1$ that contains the point $(-5, 1)$. Write the equation in slope–intercept form.

Solution:

$x = -5$

Note:
Exercise:

Problem:

Find an equation of a line that is perpendicular to the line $y = -5$ that contains the point $(-4, -5)$.

Solution:

$x = -4$

Note:
Access this online resource for additional instruction and practice with finding the equation of a line.

- Use the Point-Slope Form of an Equation of a Line

Key Concepts

- **To Find an Equation of a Line Given the Slope and a Point**

  Identify the slope.
  Identify the point.
  Substitute the values into the point-slope form,$y - y_1 = m\left(x - x_1\right)$.
  Write the equation in slope-intercept form.

- **To Find an Equation of a Line Given Two Points**

  Find the slope using the given points.
  Choose one point.
  Substitute the values into the point-slope form,$y - y_1 = m\left(x - x_1\right)$.
  Write the equation in slope-intercept form.

- **To Write and Equation of a Line**
  - If given slope and y-intercept, use slope–intercept form $y = mx + b$.
  - If given slope and a point, use point–slope form $y - y_1 = m\left(x - x_1\right)$.
  - If given two points, use point–slope form $y - y_1 = m\left(x - x_1\right)$.

- **To Find an Equation of a Line Parallel to a Given Line**

  Find the slope of the given line.
  Find the slope of the parallel line.
  Identify the point.
  Substitute the values into the point-slope form,$y - y_1 = m\left(x - x_1\right)$.
  Write the equation in slope-intercept form.

- **To Find an Equation of a Line Perpendicular to a Given Line**

  Find the slope of the given line.
  Find the slope of the perpendicular line.
  Identify the point.
  Substitute the values into the point-slope form,$y - y_1 = m\left(x - x_1\right)$.
  Write the equation in slope-intercept form.

Practice Makes Perfect

Find an Equation of the Line Given the Slope and *y*-Intercept

In the following exercises, find the equation of a line with given slope and y-intercept. Write the equation in slope–intercept form.

Exercise:

Problem: slope 3 and *y*-intercept $(0, 5)$

Exercise:

Problem: slope 4 and *y*-intercept $(0, 1)$

Solution:

$y = 4x + 1$

Exercise:

Problem: slope 6 and *y*-intercept $(0, -4)$

Exercise:

Problem: slope 8 and *y*-intercept $(0, -6)$

Solution:

$y = 8x - 6$

Exercise:

Problem: slope -1 and *y*-intercept $(0, 3)$

Exercise:

Problem: slope -1 and *y*-intercept $(0, 7)$

Solution:

$y = -x + 7$

Exercise:

Problem: slope -2 and *y*-intercept $(0, -3)$

Exercise:

Problem: slope -3 and *y*-intercept $(0, -1)$

Solution:

$y = -3x - 1$

Exercise:

Problem: slope $\frac{3}{5}$ and *y*-intercept $(0,-1)$

Exercise:

Problem: slope $\frac{1}{5}$ and *y*-intercept $(0,-5)$

Solution:

$y=\frac{1}{5}x-5$

Exercise:

Problem: slope $-\frac{3}{4}$ and *y*-intercept $(0,-2)$

Exercise:

Problem: slope $-\frac{2}{3}$ and *y*-intercept $(0,-3)$

Solution:

$y=-\frac{2}{3}x-3$

Exercise:

Problem: slope 0 and *y*-intercept $(0,-1)$

Exercise:

Problem: slope 0 and *y*-intercept $(0,2)$

Solution:

$y=2$

Exercise:

Problem: slope -3 and *y*-intercept $(0,0)$

Exercise:

Problem: slope -4 and *y*-intercept $(0,0)$

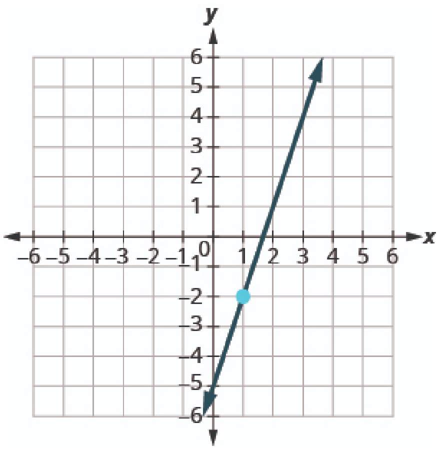
Solution:

$y=-4x$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope–intercept form.

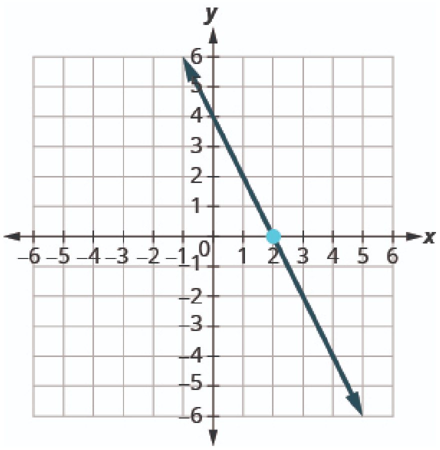
Exercise:

Problem:



Exercise:

Problem:

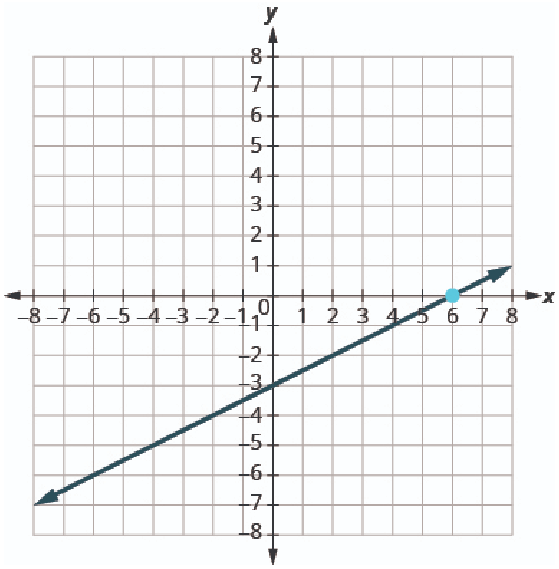


Solution:

$y = -2x + 4$

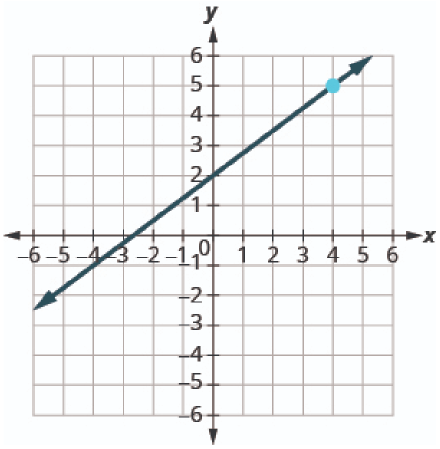
Exercise:

Problem:



Exercise:

Problem:

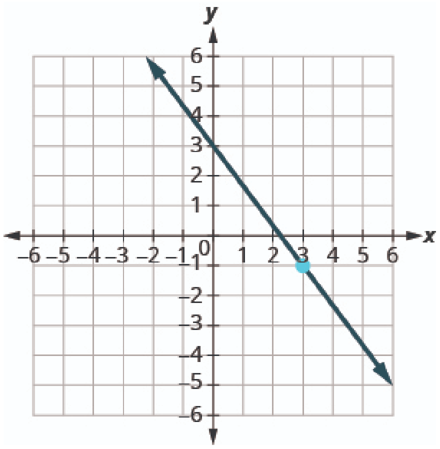


Solution:

$y=\frac{3}{4}x+2$

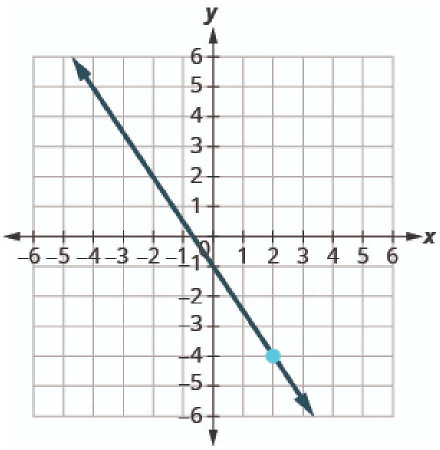
Exercise:

Problem:



Exercise:

Problem:

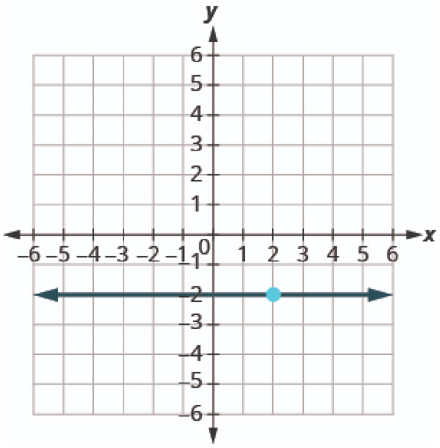


Solution:

$y=-\frac{3}{2}x-1$

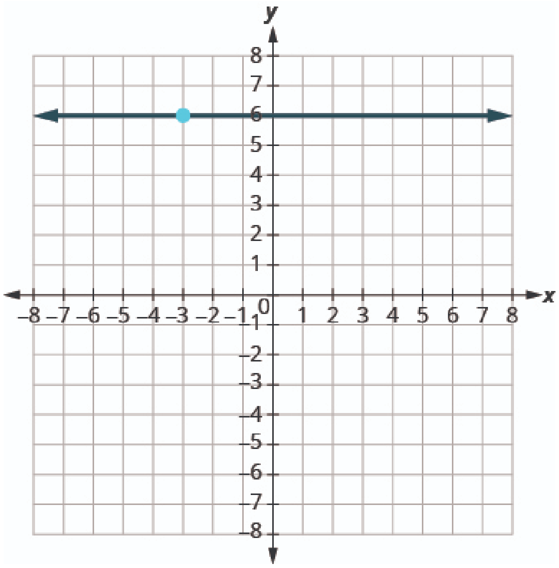
Exercise:

Problem:



Exercise:

Problem:



Solution:

$y=6$

Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope–intercept form.

Exercise:

Problem: $m=\frac{5}{8}$, point $(8,3)$

Exercise:

Problem: $m = \frac{3}{8}$, point $(8, 2)$

Solution:

$y = \frac{3}{8}x - 1$

Exercise:

Problem: $m = \frac{1}{6}$, point $(6, 1)$

Exercise:

Problem: $m = \frac{5}{6}$, point $(6, 7)$

Solution:

$y = \frac{5}{6}x + 2$

Exercise:

Problem: $m = -\frac{3}{4}$, point $(8, -5)$

Exercise:

Problem: $m = -\frac{3}{5}$, point $(10, -5)$

Solution:

$y = -\frac{3}{5}x + 1$

Exercise:

Problem: $m = -\frac{1}{4}$, point $(-12, -6)$

Exercise:

Problem: $m = -\frac{1}{3}$, point $(-9, -8)$

Solution:

$y = -\frac{1}{3}x - 11$

Exercise:

Problem: Horizontal line containing $(-2, 5)$

Exercise:

Problem: Horizontal line containing $(-1, 4)$

Solution:

$y = 4$

Exercise:

Problem: Horizontal line containing $(-2, -3)$

Exercise:

Problem: Horizontal line containing $(-1, -7)$

Solution:

$y = -7$

Exercise:

Problem: $m = -\frac{3}{2}$, point $(-4, -3)$

Exercise:

Problem: $m = -\frac{5}{2}$, point $(-8, -2)$

Solution:

$y = -\frac{5}{2}x - 22$

Exercise:

Problem: $m = -7$, point $(-1, -3)$

Exercise:

Problem: $m = -4$, point $(-2, -3)$

Solution:

$y = -4x - 11$

Exercise:

Problem: Horizontal line containing $(2, -3)$

Exercise:

Problem: Horizontal line containing $(4, -8)$

Solution:

$y = -8$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope–intercept form.

Exercise:

Problem: $(2, 6)$ and $(5, 3)$

Exercise:

Problem: $(3, 1)$ and $(2, 5)$

Solution:

$y = -4x + 13$

Exercise:

Problem: $(4, 3)$ and $(8, 1)$

Exercise:

Problem: $(2, 7)$ and $(3, 8)$

Solution:

$y = x + 5$

Exercise:

Problem: $(-3, -4)$ and $(5 - 2)$

Exercise:

Problem: $(-5, -3)$ and $(4, -6)$

Solution:

$y = -\frac{1}{3}x - \frac{14}{3}$

Exercise:

Problem: $(-1, 3)$ and $(-6, -7)$

Exercise:

Problem: $(-2, 8)$ and $(-4, -6)$

Solution:

$y = 7x + 22$

Exercise:

Problem: $(6, -4)$ and $(-2, 5)$

Exercise:

Problem: $(3, -2)$ and $(-4, 4)$

Solution:

$y = -\frac{6}{7}x + \frac{4}{7}$

Exercise:

Problem: $(0, 4)$ and $(2, -3)$

Exercise:

Problem: $(0, -2)$ and $(-5, -3)$

Solution:

$y = \frac{1}{5}x - 2$

Exercise:

Problem: $(7, 2)$ and $(7, -2)$

Exercise:

Problem: $(4, 2)$ and $(4, -3)$

Solution:

$x = 4$

Exercise:

Problem: $(-7, -1)$ and $(-7, -4)$

Exercise:

Problem: $(-2, 1)$ and $(-2, -4)$

Solution:

$x = -2$

Exercise:

Problem: $(6, 1)$ and $(0, 1)$

Exercise:

Problem: $(6, 2)$ and $(-3, 2)$

Solution:

$y = 2$

Exercise:

Problem: $(3, -4)$ and $(5, -4)$

Exercise:

Problem: $(-6, -3)$ and $(-1, -3)$

Solution:

$y = -3$

Exercise:

Problem: $(4, 3)$ and $(8, 0)$

Exercise:

Problem: $(0, 0)$ and $(1, 4)$

Solution:

$y = 4x$

Exercise:

Problem: $(-2, -3)$ and $(-5, -6)$

Exercise:

Problem: $(-3, 0)$ and $(-7, -2)$

Solution:

$y = \frac{1}{2}x + \frac{3}{2}$

Exercise:

Problem: $(8, -1)$ and $(8, -5)$

Exercise:

Problem: $(3, 5)$ and $(-7, 5)$

Solution:

$y = 5$

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope–intercept form.

Exercise:

Problem: line $y = 4x + 2$, point $(1, 2)$

Exercise:

Problem: line $y = 3x + 4$, point $(2, 5)$

Solution:

$y = 3x - 1$

Exercise:

Problem: line $y = -2x - 3$, point $(-1, 3)$

Exercise:

Problem: line $y = -3x - 1$, point $(2, -3)$

Solution:

$y = -3x + 3$

Exercise:

Problem: line $3x - y = 4$, point $(3, 1)$

Exercise:

Problem: line $2x - y = 6$, point $(3, 0)$

Solution:

$y = 2x - 6$

Exercise:

Problem: line $4x + 3y = 6$, point $(0, -3)$

Exercise:

Problem: line $2x + 3y = 6$, point $(0, 5)$

Solution:

$y = -\frac{2}{3}x + 5$

Exercise:

Problem: line $x = -3$, point $(-2, -1)$

Exercise:

Problem: line $x = -4$, point $(-3, -5)$

Solution:

$x = -3$

Exercise:

Problem: line $x - 2 = 0$, point $(1, -2)$

Exercise:

Problem: line $x - 6 = 0$, point $(4, -3)$

Solution:

$x = 4$

Exercise:

Problem: line $y = 5$, point $(2, -2)$

Exercise:

Problem: line $y = 1$, point $(3, -4)$

Solution:

$y = -4$

Exercise:

Problem: line $y + 2 = 0$, point $(3, -3)$

Exercise:

Problem: line $y + 7 = 0$, point $(1, -1)$

Solution:

$y = -1$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope–intercept form.

Exercise:

Problem: line $y = -2x + 3$, point $(2, 2)$

Exercise:

Problem: line $y = -x + 5$, point $(3, 3)$

Solution:

$y = x$

Exercise:

Problem: line $y = \frac{3}{4}x - 2$, point $(-3, 4)$

Exercise:

Problem: line $y = \frac{2}{3}x - 4$, point $(2, -4)$

Solution:

$y = -\frac{3}{2}x - 1$

Exercise:

Problem: line $2x - 3y = 8$, point $(4, -1)$

Exercise:

Problem: line $4x - 3y = 5$, point $(-3, 2)$

Solution:

$y = -\frac{3}{4}x - \frac{1}{4}$

Exercise:

Problem: line $2x + 5y = 6$, point $(0, 0)$

Exercise:

Problem: line $4x + 5y = -3$, point $(0, 0)$

Solution:

$y = \frac{5}{4}x$

Exercise:

Problem: line $y - 3 = 0$, point $(-2, -4)$

Exercise:

Problem: line $y - 6 = 0$, point $(-5, -3)$

Solution:

$x = -5$

Exercise:

Problem: line *y*-axis, point $(3, 4)$

Exercise:

Problem: line *y*-axis, point $(2, 1)$

Solution:

$y = 1$

Mixed Practice

In the following exercises, find the equation of each line. Write the equation in slope–intercept form.

Exercise:

Problem: Containing the points $(4, 3)$ and $(8, 1)$

Exercise:

Problem: Containing the points $(2, 7)$ and $(3, 8)$

Solution:

$y = x + 5$

Exercise:

Problem: $m = \frac{1}{6}$, containing point $(6, 1)$

Exercise:

Problem: $m = \frac{5}{6}$, containing point $(6, 7)$

Solution:

$y = \frac{5}{6}x + 2$

Exercise:

Problem: Parallel to the line $4x + 3y = 6$, containing point $(0, -3)$

Exercise:

Problem: Parallel to the line $2x + 3y = 6$, containing point $(0, 5)$

Solution:

$y=-\frac{2}{3}x+5$

Exercise:

Problem: $m=-\frac{3}{4}$, containing point $(8,-5)$

Exercise:

Problem: $m=-\frac{3}{5}$, containing point $(10,-5)$

Solution:

$y=-\frac{3}{5}x+1$

Exercise:

Problem: Perpendicular to the line $y-1=0$, point $(-2,6)$

Exercise:

Problem: Perpendicular to the line y-axis, point $(-6,2)$

Solution:

$y=2$

Exercise:

Problem: Containing the points $(4,3)$ and $(8,1)$

Exercise:

Problem: Containing the points $(-2,0)$ and $(-3,-2)$

Solution:

$y=x+2$

Exercise:

Problem: Parallel to the line $x=-3$, containing point $(-2,-1)$

Exercise:

Problem: Parallel to the line $x=-4$, containing point $(-3,-5)$

Solution:

$x=-3$

Exercise:

Problem: Containing the points $(-3,-4)$ and $(2,-5)$

Exercise:

Problem: Containing the points $(-5,-3)$ and $(4,-6)$

Solution:

$y=-\frac{1}{3}x-\frac{14}{3}$

Exercise:

Problem: Perpendicular to the line $x-2y=5$, containing point $(-2,2)$

Exercise:

Problem: Perpendicular to the line $4x+3y=1$, containing point $(0,0)$

Solution:

$y=\frac{3}{4}x$

Everyday Math

Exercise:

Problem:

Cholesterol. The age, x, and LDL cholesterol level, y, of two men are given by the points $(18,68)$ and $(27,122)$. Find a linear equation that models the relationship between age and LDL cholesterol level.

Exercise:

Problem:

Fuel consumption. The city mpg, x, and highway mpg, y, of two cars are given by the points $(29,40)$ and$(19,28)$. Find a linear equation that models the relationship between city mpg and highway mpg.

Solution:

$y=1.2x+5.2$

Writing Exercises

Exercise:

Problem: Why are all horizontal lines parallel?

Exercise:

Problem:

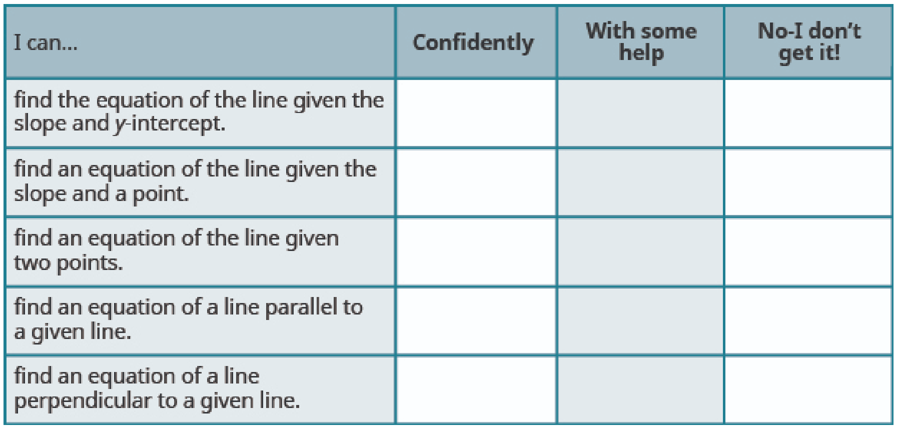
Explain in your own words why the slopes of two perpendicular lines must have opposite signs.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
find the equation of the line given the slope and y-intercept.			
find an equation of the line given the slope and a point.			
find an equation of the line given two points.			
find an equation of a line parallel to a given line.			
find an equation of a line perpendicular to a given line.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

point–slope form

The point–slope form of an equation of a line with slope m and containing the point (x_1, y_1) is $y - y_1 = m\,(x - x_1)$.

Introduction class="introduction"

Designing the number and sizes of windows in a home can pose challenges for an architect.

An architect designing a home may have restrictions on both the area and perimeter of the windows because of energy and structural concerns. The length and width chosen for each window would have to satisfy two equations: one for the area and the other for the perimeter. Similarly, a banker may have a fixed amount of money to put into two investment funds. A restaurant owner may want to increase profits, but in order to do that he will need to hire more staff. A job applicant may compare salary and costs of commuting for two job offers.

In this chapter, we will look at methods to solve situations like these using equations with two variables.

Solve Systems of Equations by Graphing
By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of equations
- Solve a system of linear equations by graphing
- Determine the number of solutions of linear system
- Solve applications of systems of equations by graphing

Note:
Before you get started, take this readiness quiz.

1. For the equation $y=\frac{2}{3}x-4$
ⓐ is $(6,0)$ a solution? ⓑ is $(-3,-2)$ a solution?
If you missed this problem, review [link].
2. Find the slope and *y*-intercept of the line $3x-y=12$.
If you missed this problem, review [link].
3. Find the *x*- and *y*-intercepts of the line $2x-3y=12$.
If you missed this problem, review [link].

Determine Whether an Ordered Pair is a Solution of a System of Equations

In Solving Linear Equations and Inequalities we learned how to solve linear equations with one variable. Remember that the solution of an equation is a value of the variable that makes a true statement when substituted into the equation.

Now we will work with **systems of linear equations**, two or more linear equations grouped together.

Note:
System of Linear Equations
When two or more linear equations are grouped together, they form a system of linear equations.

We will focus our work here on systems of two linear equations in two unknowns. Later, you may solve larger systems of equations.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

Equation:

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

A linear equation in two variables, like $2x + y = 7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line.

To solve a system of two linear equations, we want to find the values of the variables that are solutions to both equations. In other words, we are looking for the ordered pairs (x, y) that make both equations true. These are called the solutions to a system of equations.

Note:
Solutions of a System of Equations
Solutions of a system of equations are the values of the variables that make all the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).

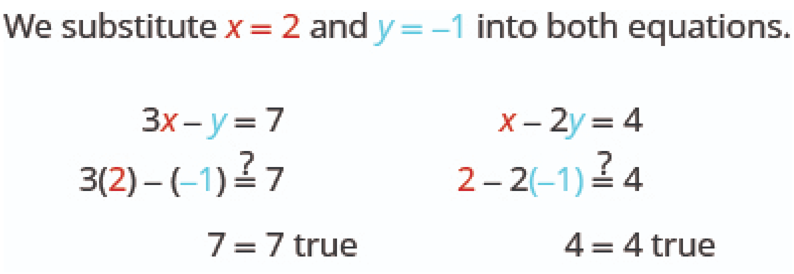
To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

Let’s consider the system below:

Equation:

$$\begin{cases} 3x - y = 7 \\ x - 2y = 4 \end{cases}$$

Is the ordered pair $(2, -1)$ a solution?

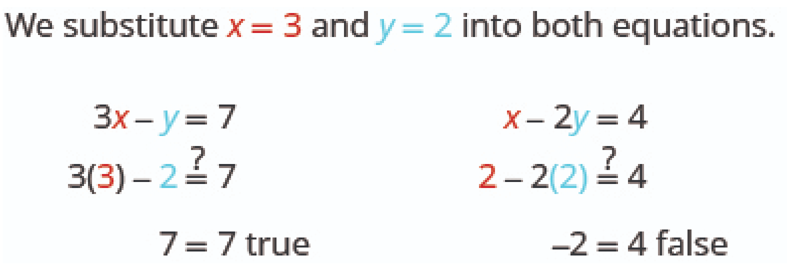


We substitute $x = 2$ and $y = -1$ into both equations.

$$\begin{array}{ll} 3x - y = 7 & x - 2y = 4 \\ 3(2) - (-1) \overset{?}{=} 7 & 2 - 2(-1) \overset{?}{=} 4 \\ 7 = 7 \text{ true} & 4 = 4 \text{ true} \end{array}$$

The ordered pair $(2, -1)$ made both equations true. Therefore $(2, -1)$ is a solution to this system.

Let’s try another ordered pair. Is the ordered pair $(3, 2)$ a solution?



We substitute $x = 3$ and $y = 2$ into both equations.

$$\begin{array}{ll} 3x - y = 7 & x - 2y = 4 \\ 3(3) - 2 \overset{?}{=} 7 & 2 - 2(2) \overset{?}{=} 4 \\ 7 = 7 \text{ true} & -2 = 4 \text{ false} \end{array}$$

The ordered pair $(3, 2)$ made one equation true, but it made the other equation false. Since it is not a solution to **both** equations, it is not a solution to this system.

Example:

Exercise:

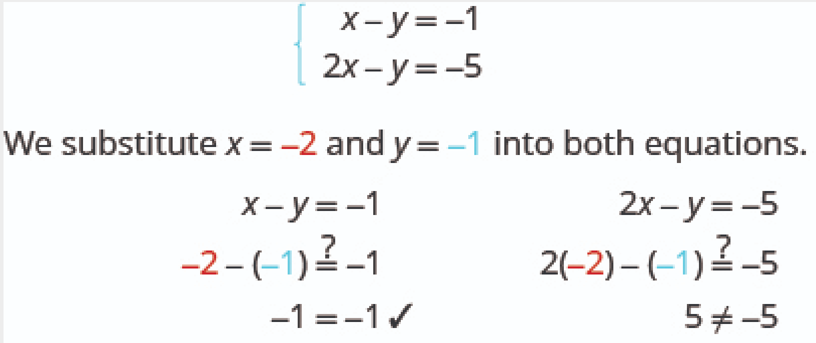
Problem:

Determine whether the ordered pair is a solution to the system:
$$\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$$

ⓐ $(-2, -1)$ ⓑ $(-4, -3)$

Solution:
Solution

ⓐ



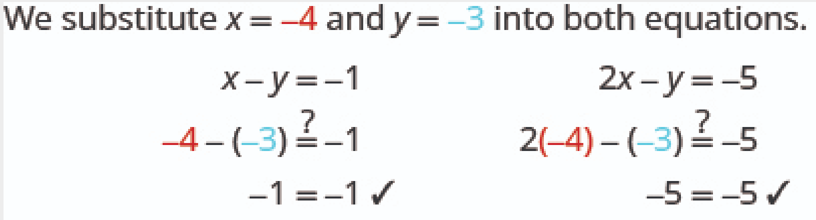
$$\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$$

We substitute $x = -2$ and $y = -1$ into both equations.

$$\begin{array}{rcr} x - y = -1 & \quad & 2x - y = -5 \\ -2 - (-1) \overset{?}{=} -1 & & 2(-2) - (-1) \overset{?}{=} -5 \\ -1 = -1 \checkmark & & 5 \neq -5 \end{array}$$

$(-2, -1)$ does not make both equations true. $(-2, -1)$ is not a solution.

ⓑ



We substitute $x = -4$ and $y = -3$ into both equations.

$$\begin{array}{rcr} x - y = -1 & \quad & 2x - y = -5 \\ -4 - (-3) \overset{?}{=} -1 & & 2(-4) - (-3) \overset{?}{=} -5 \\ -1 = -1 \checkmark & & -5 = -5 \checkmark \end{array}$$

$(-4, -3)$ does make both equations true. $(-4, -3)$ is a solution.

Note:
Exercise:

Problem:

Determine whether the ordered pair is a solution to the system:
$$\begin{cases} 3x + y = 0 \\ x + 2y = -5 \end{cases}$$

ⓐ $(1, -3)$ ⓑ $(0, 0)$

Solution:

ⓐ yes ⓑ no

Note:
Exercise:

Problem:

Determine whether the ordered pair is a solution to the system:
$$\begin{cases} x - 3y = -8 \\ -3x - y = 4 \end{cases}$$

ⓐ $(2, -2)$ ⓑ $(-2, 2)$

Solution:

ⓐ no ⓑ yes

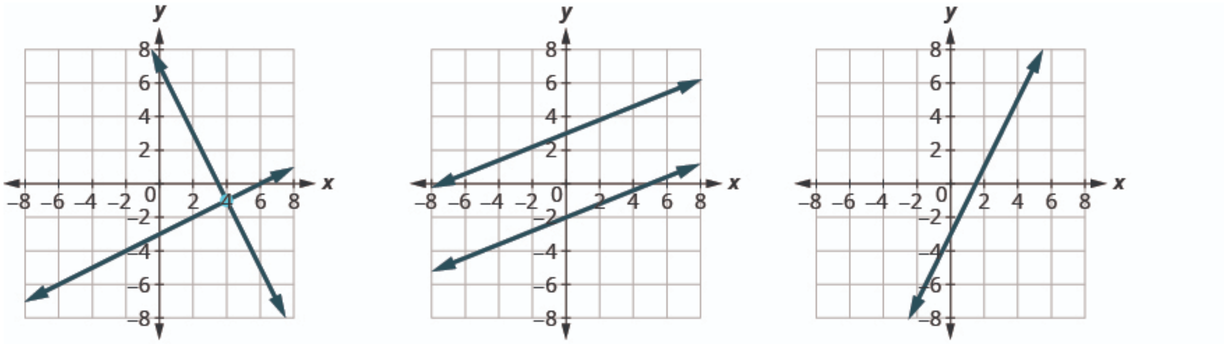
Solve a System of Linear Equations by Graphing

In this chapter we will use three methods to solve a system of linear equations. The first method we'll use is graphing.

The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we’ll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions.

Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown in [link]:



The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.

The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.

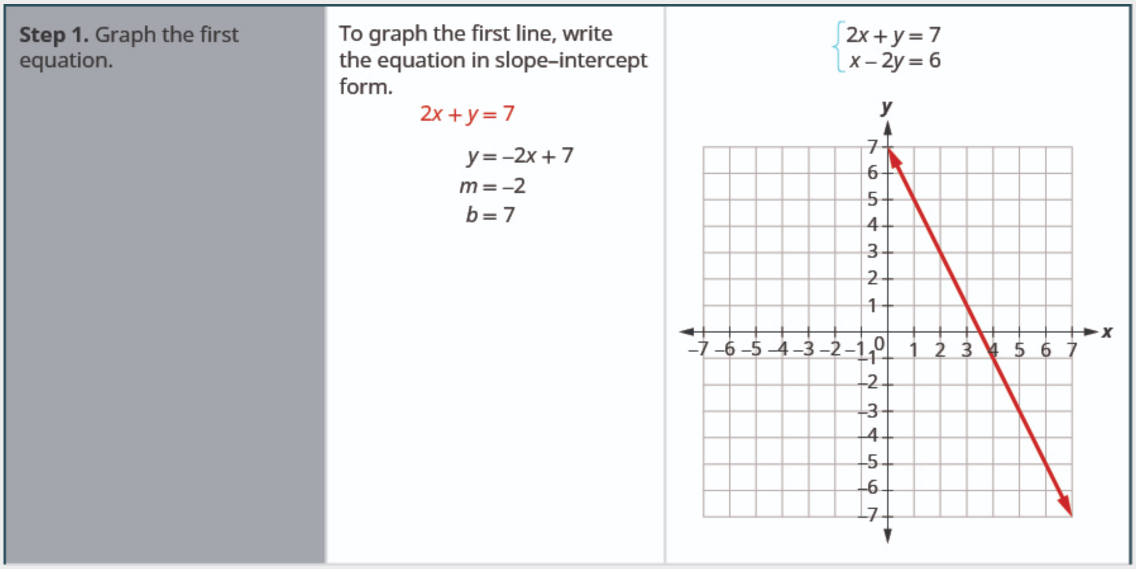
Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

For the first example of solving a system of linear equations in this section and in the next two sections, we will solve the same system of two linear equations. But we’ll use a different method in each section. After seeing the third method, you’ll decide which method was the most convenient way to solve this system.

Example:
How to Solve a System of Linear Equations by Graphing
Exercise:

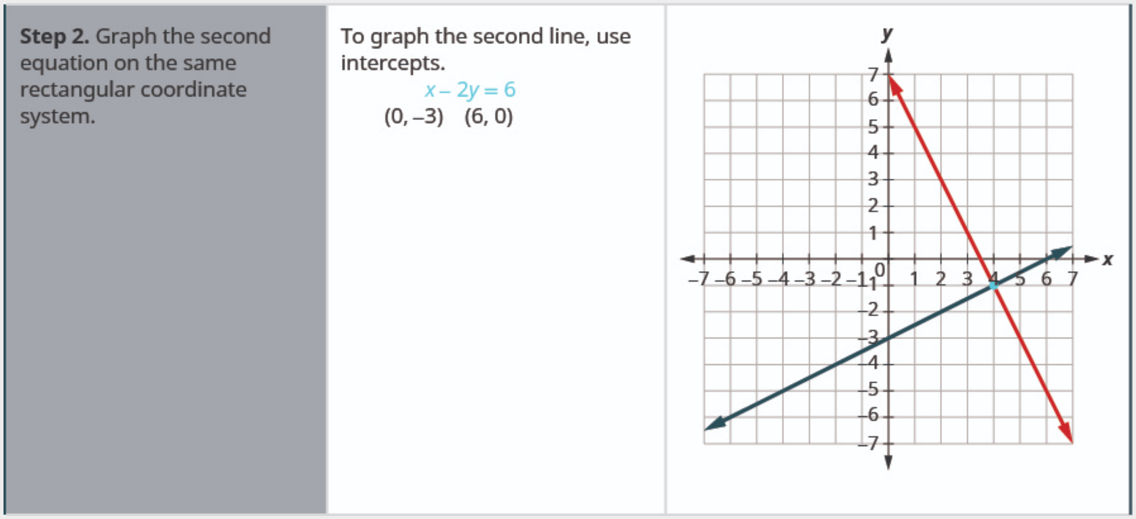
Problem: Solve the system by graphing: $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

Solution:
Solution



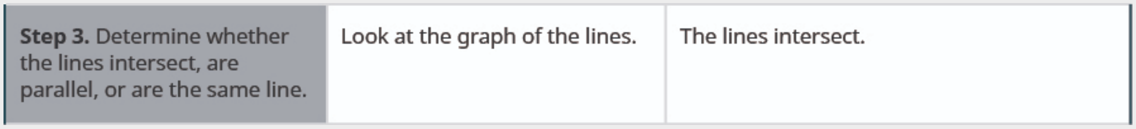
Step 1. Graph the first equation.	To graph the first line, write the equation in slope–intercept form. $2x + y = 7$ $y = -2x + 7$ $m = -2$ $b = 7$	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

y
x
7 6 5 4 3 2 1 0 –1 –2 –3 –4 –5 –6 –7
–7 –6 –5 –4 –3 –2 –1 1 2 3 4 5 6 7

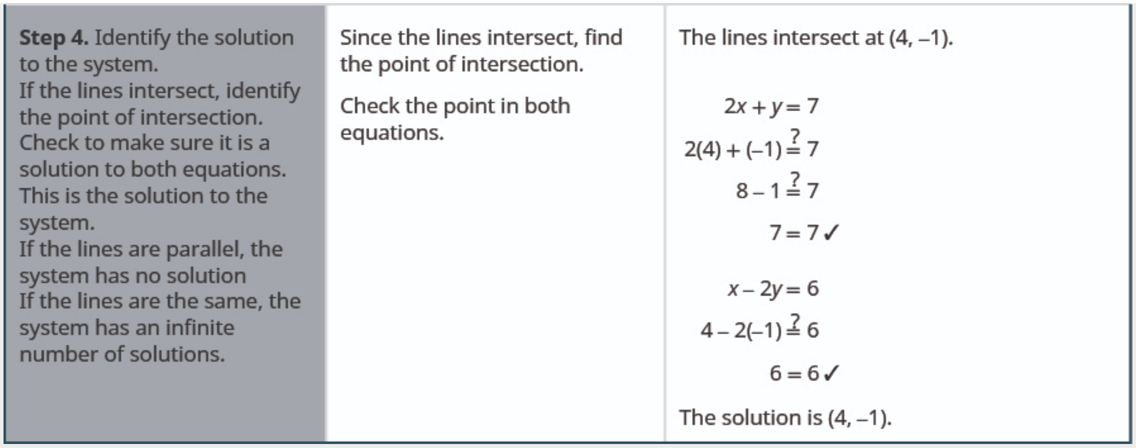


Step 2. Graph the second equation on the same rectangular coordinate system.	To graph the second line, use intercepts. $x - 2y = 6$ $(0, -3)$ $(6, 0)$	

y
x
7 6 5 4 3 2 1 0 –1 –2 –3 –4 –5 –6 –7
–7 –6 –5 –4 –3 –2 –1 1 2 3 4 5 6 7



Step 3. Determine whether the lines intersect, are parallel, or are the same line.	Look at the graph of the lines.	The lines intersect.



Step 4. Identify the solution to the system. If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. If the lines are parallel, the system has no solution If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at (4, –1). $2x + y = 7$ $2(4) + (-1) \overset{?}{=} 7$ $8 - 1 \overset{?}{=} 7$ $7 = 7$ ✓ $x - 2y = 6$ $4 - 2(-1) \overset{?}{=} 6$ $6 = 6$ ✓ The solution is (4, –1).

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} x - 3y = -3 \\ x + y = 5 \end{cases}$

Solution:

$(3, 2)$

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} -x + y = 1 \\ 3x + 2y = 12 \end{cases}$

Solution:

$(2, 3)$

The steps to use to solve a system of linear equations by graphing are shown below.

Note:
To solve a system of linear equations by graphing.

Graph the first equation.
Graph the second equation on the same rectangular coordinate system.
Determine whether the lines intersect, are parallel, or are the same line.
Identify the solution to the system.

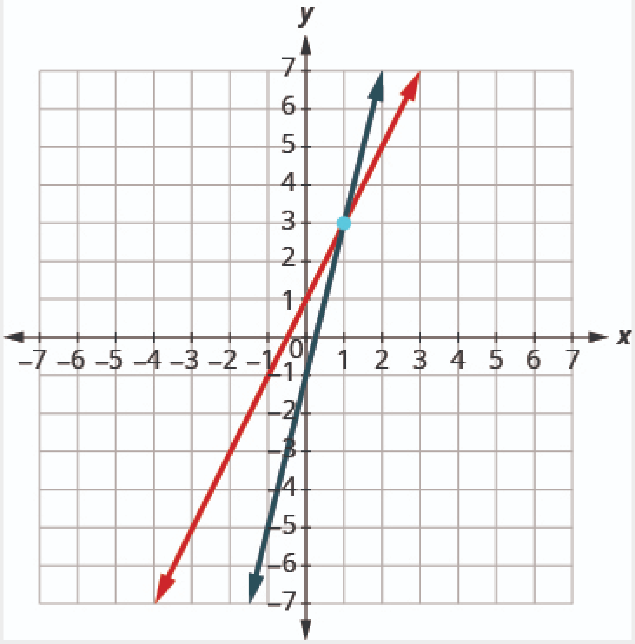
- If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system.
- If the lines are parallel, the system has no solution.
- If the lines are the same, the system has an infinite number of solutions.

Example:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} y = 2x + 1 \\ y = 4x - 1 \end{cases}$

Solution:
Solution

Both of the equations in this system are in slope-intercept form, so we will use their slopes and *y*-intercepts to graph them. $\begin{cases} y = 2x + 1 \\ y = 4x - 1 \end{cases}$

Find the slope and *y*-intercept of the first equation.	$y = 2x + 1$ $m = 2$ $b = 1$
Find the slope and *y*-intercept of the first equation.	$y = 4x - 1$ $m = 4$ $b = -1$
Graph the two lines.	
Determine the point of intersection.	The lines intersect at $(1, 3)$.
	
Check the solution in both equations.	$\begin{aligned} y &= 2x + 1 \\ 3 &\overset{?}{=} 2 \cdot 1 + 1 \\ 3 &= 3 \checkmark \end{aligned}$ $\qquad$ $\begin{aligned} y &= 4x - 1 \\ 3 &\overset{?}{=} 4 \cdot 1 - 1 \\ 3 &= 3 \checkmark \end{aligned}$

The solution is $(1, 3)$.

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} y = 2x + 2 \\ y = -x - 4 \end{cases}$

Solution:

$(-2, -2)$

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} y = 3x + 3 \\ y = -x + 7 \end{cases}$

Solution:

$(1, 6)$

Both equations in [link] were given in slope–intercept form. This made it easy for us to quickly graph the lines. In the next example, we’ll first re-write the equations into slope–intercept form.

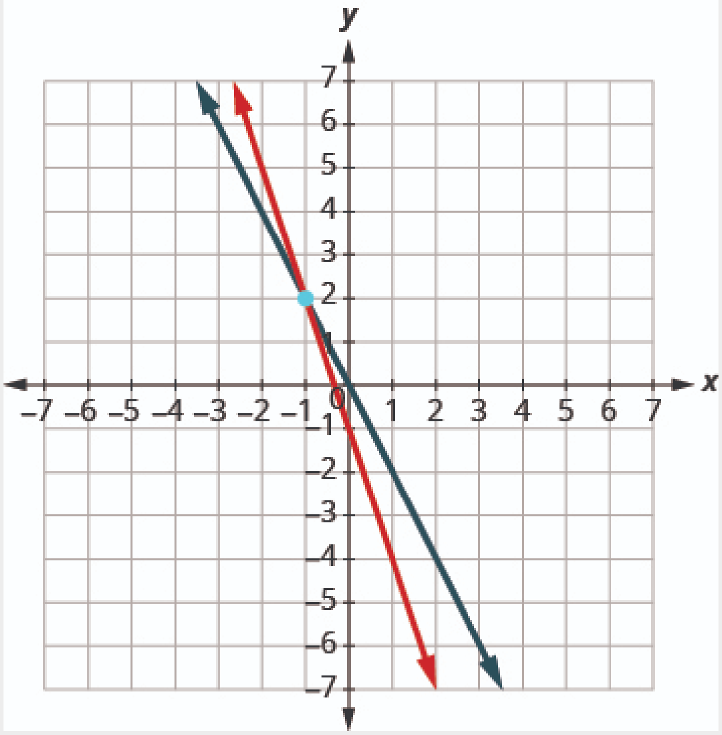
Example:
Exercise:

Problem: Solve the system by graphing: $\begin{cases}3x+y=-1\\2x+y=0\end{cases}$

Solution:
Solution

We’ll solve both of these equations for y so that we can easily graph them using their slopes and y-intercepts. $\begin{cases}3x+y=-1\\2x+y=0\end{cases}$

Solve the first equation for y.	$\begin{aligned}3x+y&=-1\\y&=-3x-1\\m&=-3\\b&=-1\end{aligned}$
Find the slope and y-intercept.	$\begin{aligned}2x+y&=0\\y&=-2x\\m&=-2\\b&=0\end{aligned}$
Solve the second equation for y.	
Find the	

slope and y-intercept.	
Graph the lines.	
Determine the point of intersection.	The lines intersect at $(-1, 2)$.
Check the solution in both equations.	$\begin{array}{rcl} 3x + y & = & -1 \\ 3(-1) + 2 & \overset{?}{=} & -1 \\ -1 & = & -1 \checkmark \end{array}$ $\begin{array}{rcl} 2x + y & = & 0 \\ 2(-1) + 2 & \overset{?}{=} & 0 \\ 0 & = & 0 \checkmark \end{array}$
	The solution is $(-1, 2)$.

Note:

Exercise:

Problem: Solve each system by graphing: $\begin{cases} -x+y=1 \\ 2x+y=10 \end{cases}$

Solution:

$(3,4)$

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} 2x+y=6 \\ x+y=1 \end{cases}$

Solution:

$(5,-4)$

Usually when equations are given in standard form, the most convenient way to graph them is by using the intercepts. We'll do this in [link].

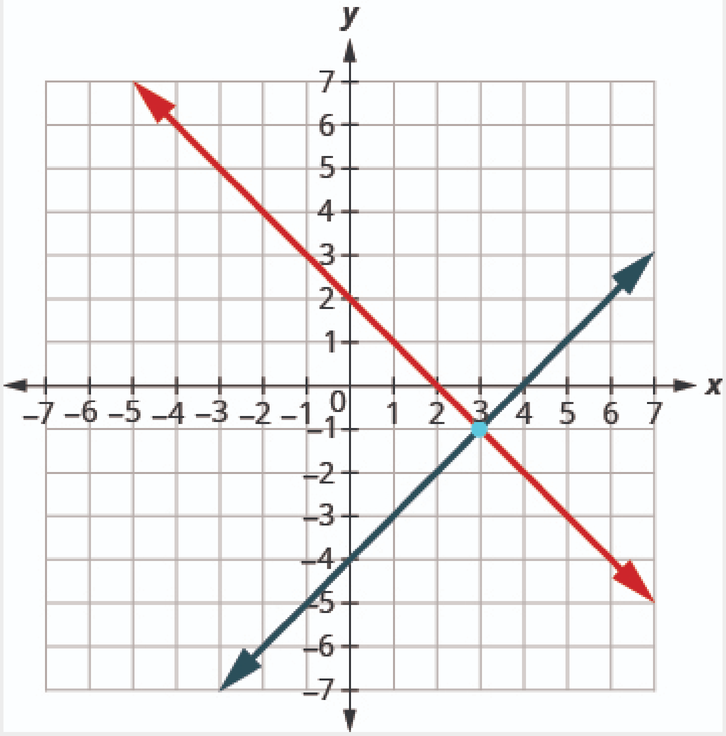
Example:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} x+y=2 \\ x-y=4 \end{cases}$

Solution:
Solution

We will find the x- and y-intercepts of both equations and use them to graph the lines.

	$x+y=2$		
To find the intercepts, let $x=0$ and solve for y, then let $y=0$ and solve for x.	$\begin{aligned} x+y &= 2 \\ 0+y &= 2 \\ y &= 2 \end{aligned}$	$\begin{aligned} x+y &= 2 \\ x+0 &= 2 \\ x &= 2 \end{aligned}$	
	$x-y=4$		
To find the intercepts, let $x=0$ then let $y=0$.	$\begin{aligned} x-y &= 4 \\ 0-y &= 4 \\ -y &= 4 \\ y &= -4 \end{aligned}$	$\begin{aligned} x-y &= 4 \\ x-0 &= 4 \\ x &= 4 \end{aligned}$	
Graph the line.			

		
Determine the point of intersection.	The lines intersect at $(3, -1)$.	
Check the solution in both equations.	$x + y = 2$ $3 + (-1) \overset{?}{=} 2$ $2 = 2 \checkmark$ $x - y = 4$ $3 - (-1) \overset{?}{=} 4$ $4 = 4 \checkmark$ The solution is $(3, -1)$.	

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} x+y=6 \\ x-y=2 \end{cases}$

Solution:

$(4,2)$

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} x+y=2 \\ x-y=-8 \end{cases}$

Solution:

$(-3,5)$

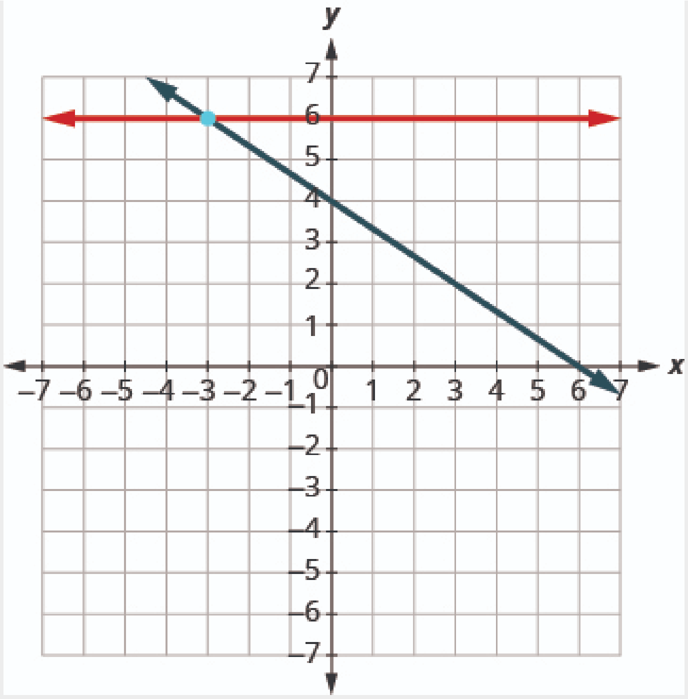
Do you remember how to graph a linear equation with just one variable? It will be either a vertical or a horizontal line.

Example:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} y=6 \\ 2x+3y=12 \end{cases}$

Solution:
Solution

<table>
<tr><td></td><td>$$\begin{cases} y = 6 \\ 2x + 3y = 12 \end{cases}$$</td></tr>
<tr><td>We know the first equation represents a horizontal line whose y-intercept is 6.</td><td>$y = 6$</td></tr>
<tr><td>The second equation is most conveniently graphed using intercepts.</td><td>$2x + 3y = 12$</td></tr>
<tr><td>To find the intercepts, let $x = 0$ and then $y = 0$.</td><td><table><tr><th>x</th><th>y</th></tr><tr><td>0</td><td>4</td></tr><tr><td>6</td><td>0</td></tr></table></td></tr>
<tr><td>Graph the lines.</td><td></td></tr>
</table>

	
Determine the point of intersection.	The lines intersect at $(-3, 6)$.
Check the solution to both equations.	$\begin{array}{rcl} y & = & 6 \\ 6 & \overset{?}{=} & 6\checkmark \\ 2 & = & 2 \end{array}$ $\begin{array}{rcl} 2x+3y & = & 12 \\ 2(-3)+3(6) & \overset{?}{=} & 12 \\ -6+18 & \overset{?}{=} & 12 \\ 12 & = & 12\checkmark \end{array}$
	The solution is $(-3, 6)$.

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} y = -1 \\ x + 3y = 6 \end{cases}$

Solution:

$(9, -1)$

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} x = 4 \\ 3x - 2y = 24 \end{cases}$

Solution:

$(4, -6)$

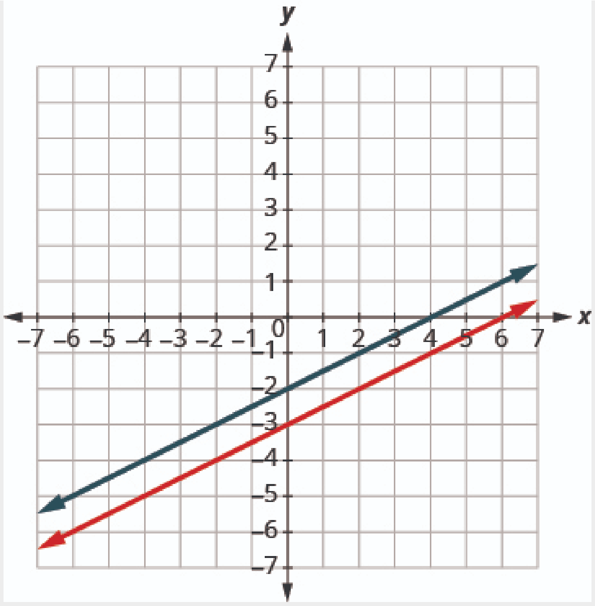
In all the systems of linear equations so far, the lines intersected and the solution was one point. In the next two examples, we’ll look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

Example:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$

Solution:
Solution

	$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$
To graph the first equation, we will use its slope and *y*-intercept.	$y = \frac{1}{2}x - 3$
	$m = \frac{1}{2}$
	$b = -3$
To graph the second equation, we will use the intercepts.	$x - 2y = 4$
	$\begin{array}{|c|c|} \hline x & y \\ \hline 0 & -2 \\ \hline 4 & 0 \\ \hline \end{array}$
Graph the lines.	

	
Determine the point of intersection.	The lines are parallel.
	Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} y = -\frac{1}{4}x + 2 \\ x + 4y = -8 \end{cases}$

Solution:

no solution

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} y = 3x - 1 \\ 6x - 2y = 6 \end{cases}$

Solution:

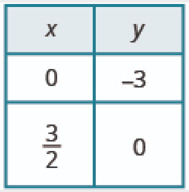
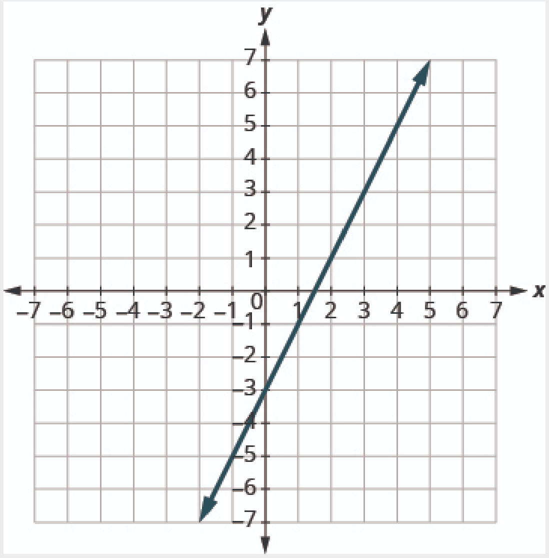
no solution

Example:
Exercise:

Problem: Solve the system by graphing: $\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$

Solution:
Solution

	$\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$
Find the slope and y-intercept of the first equation.	$y = 2x - 3$ $m = 2$ $b = -3$

Find the intercepts of the second equation.	$-6x+3y=-9$
	 $\begin{array}{\|c\|c\|}\hline x & y \\ \hline 0 & -3 \\ \hline \frac{3}{2} & 0 \\ \hline\end{array}$
Graph the lines.	
Determine the point of intersection.	The lines are the same!
	Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true.
	There are infinitely many solutions to this system.

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} y = -3x - 6 \\ 6x + 2y = -12 \end{cases}$

Solution:

infinitely many solutions

Note:
Exercise:

Problem: Solve each system by graphing: $\begin{cases} y = \frac{1}{2}x - 4 \\ 2x - 4y = 16 \end{cases}$

Solution:

infinitely many solutions

If you write the second equation in [link] in slope-intercept form, you may recognize that the equations have the same slope and same y-intercept.

When we graphed the second line in the last example, we drew it right over the first line. We say the two lines are coincident. Coincident lines have the same slope and same y-intercept.

Note:
Coincident Lines
Coincident lines have the same slope and same y-intercept.

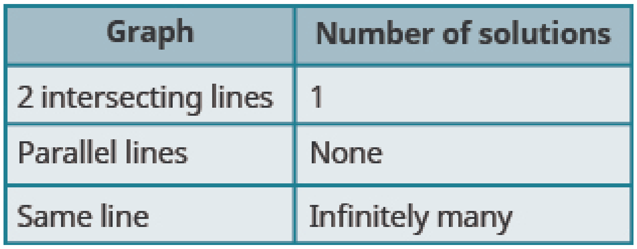
Determine the Number of Solutions of a Linear System

There will be times when we will want to know how many solutions there will be to a system of linear equations, but we might not actually have to find the solution. It will be helpful to determine this without graphing.

We have seen that two lines in the same plane must either intersect or are parallel. The systems of equations in [link] through [link] all had two intersecting lines. Each system had one solution.

A system with parallel lines, like [link], has no solution. What happened in [link]? The equations have coincident lines, and so the system had infinitely many solutions.

We'll organize these results in [link] below:



Graph	Number of solutions
2 intersecting lines	1
Parallel lines	None
Same line	Infinitely many

Parallel lines have the same slope but different y-intercepts. So, if we write both equations in a system of linear equations in slope–intercept form, we can see how many solutions there will be without graphing! Look at the system we solved in [link].

$$\begin{cases} y &= \frac{1}{2}x - 3 \\ x - 2y &= 4 \end{cases}$$

The first line is in slope–intercept form.

$$y = \frac{1}{2}x - 3$$

$$m = \frac{1}{2}, b = -3$$

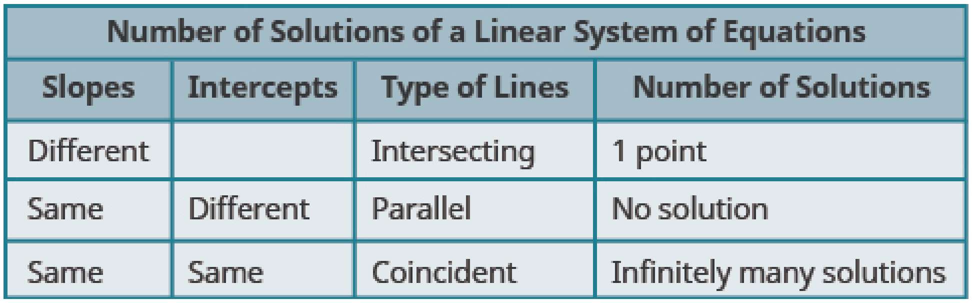
If we solve the second equation for y, we get

$$\begin{aligned} x - 2y &= 4 \\ -2y &= -x + 4 \\ y &= \frac{1}{2}x - 2 \end{aligned}$$

$$m = \frac{1}{2}, b = -2$$

The two lines have the same slope but different *y*-intercepts. They are parallel lines.

[link] shows how to determine the number of solutions of a linear system by looking at the slopes and intercepts.



Number of Solutions of a Linear System of Equations

Slopes	Intercepts	Type of Lines	Number of Solutions
Different		Intersecting	1 point
Same	Different	Parallel	No solution
Same	Same	Coincident	Infinitely many solutions

Let’s take one more look at our equations in [link] that gave us parallel lines.
Equation:

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$$

When both lines were in slope-intercept form we had:
Equation:

$$y = \frac{1}{2}x - 3 \qquad y = \frac{1}{2}x - 2$$

Do you recognize that it is impossible to have a single ordered pair (x, y) that is a solution to both of those equations?

We call a system of equations like this an inconsistent system. It has no solution.

A system of equations that has at least one solution is called a consistent system.

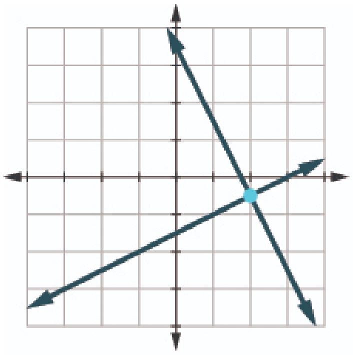
Note:
Consistent and Inconsistent Systems
A **consistent system** of equations is a system of equations with at least one solution.
An **inconsistent system** of equations is a system of equations with no solution.

We also categorize the equations in a system of equations by calling the equations *independent* or *dependent*. If two equations are **independent equations**, they each have their own set of solutions. Intersecting lines and parallel lines are independent.

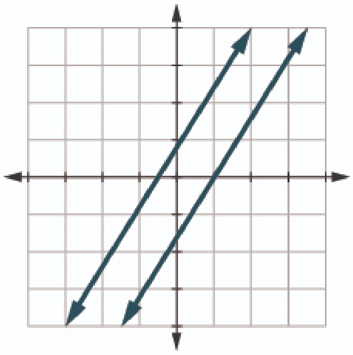
If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two dependent equations, we get coincident lines.

Note:
Independent and Dependent Equations
Two equations are **independent** if they have different solutions.
Two equations are **dependent** if all the solutions of one equation are also solutions of the other equation.

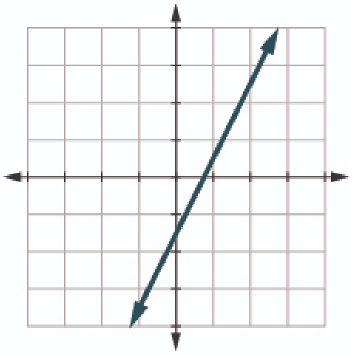
Let’s sum this up by looking at the graphs of the three types of systems. See [link] and [link].



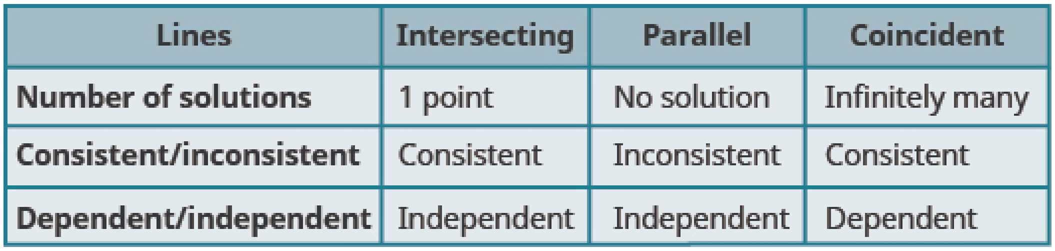
Intersecting



Parallel



Coincident



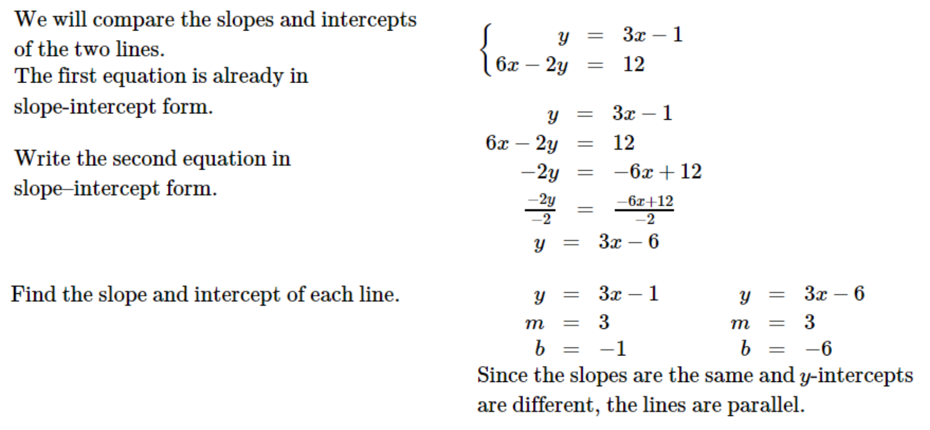
Lines	Intersecting	Parallel	Coincident
Number of solutions	1 point	No solution	Infinitely many
Consistent/inconsistent	Consistent	Inconsistent	Consistent
Dependent/independent	Independent	Independent	Dependent

Example:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations: $\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$

Solution:
Solution



We will compare the slopes and intercepts of the two lines.

$$\begin{cases} y = 3x-1 \\ 6x-2y = 12 \end{cases}$$

The first equation is already in slope-intercept form.

$$\begin{aligned} y &= 3x-1 \end{aligned}$$

Write the second equation in slope–intercept form.

$$\begin{aligned} 6x-2y &= 12 \\ -2y &= -6x+12 \\ \frac{-2y}{-2} &= \frac{-6x+12}{-2} \\ y &= 3x-6 \end{aligned}$$

Find the slope and intercept of each line.

$$\begin{aligned} y &= 3x-1 \\ m &= 3 \\ b &= -1 \end{aligned} \qquad \begin{aligned} y &= 3x-6 \\ m &= 3 \\ b &= -6 \end{aligned}$$

Since the slopes are the same and y-intercepts are different, the lines are parallel.

A system of equations whose graphs are parallel lines has no solution and is inconsistent and independent.

Note:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} y = -2x-4 \\ 4x+2y = 9 \end{cases}$$

Solution:

no solution, inconsistent, independent

Note:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} y = \frac{1}{3}x - 5 \\ x - 3y = 6 \end{cases}$$

Solution:

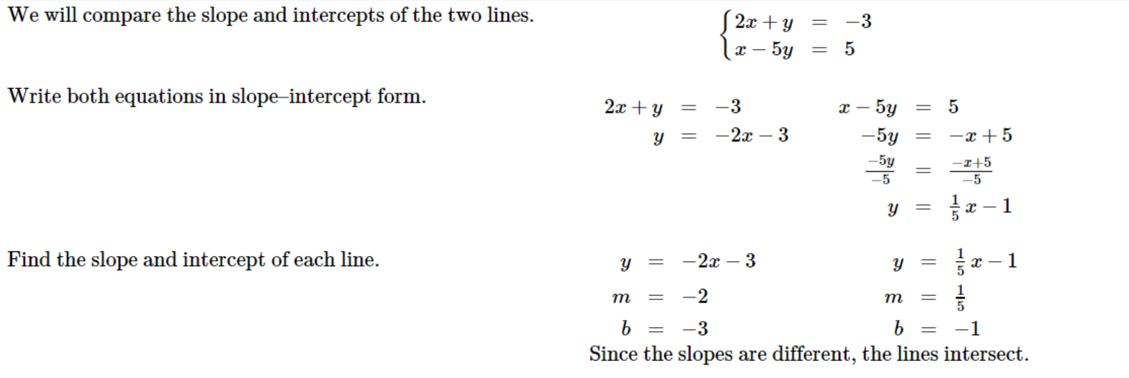
no solution, inconsistent, independent

Example:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations: $\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$

Solution:
Solution



We will compare the slope and intercepts of the two lines.

$$\begin{cases} 2x + y &= -3 \\ x - 5y &= 5 \end{cases}$$

Write both equations in slope–intercept form.

$$\begin{aligned} 2x + y &= -3 \\ y &= -2x - 3 \end{aligned} \qquad \begin{aligned} x - 5y &= 5 \\ -5y &= -x + 5 \\ \frac{-5y}{-5} &= \frac{-x+5}{-5} \\ y &= \frac{1}{5}x - 1 \end{aligned}$$

Find the slope and intercept of each line.

$$\begin{aligned} y &= -2x - 3 \\ m &= -2 \\ b &= -3 \end{aligned} \qquad \begin{aligned} y &= \frac{1}{5}x - 1 \\ m &= \frac{1}{5} \\ b &= -1 \end{aligned}$$

Since the slopes are different, the lines intersect.

A system of equations whose graphs are distinct and intersect has 1 solution and is consistent and independent.

Note:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} 3x + 2y = 2 \\ 2x + y = 1 \end{cases}$$

Solution:

one solution, consistent, independent

Note:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} x + 4y = 12 \\ -x + y = 3 \end{cases}$$

Solution:

one solution, consistent, independent

Example:

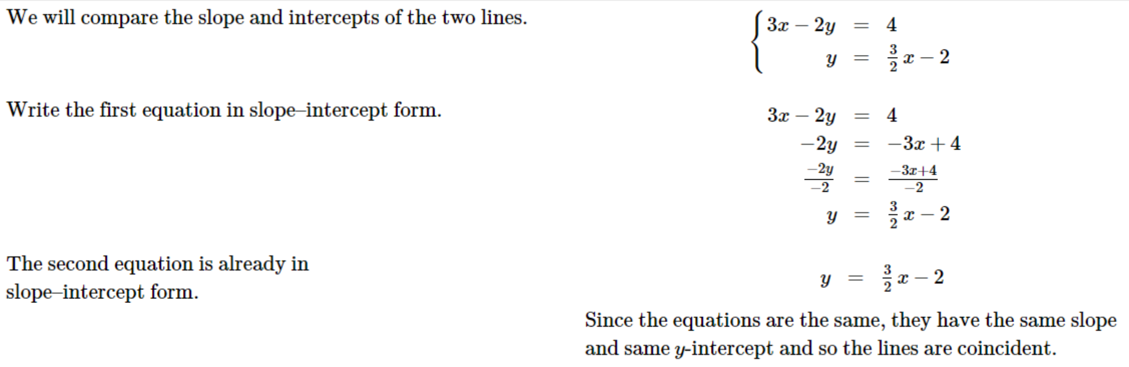
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations. $\begin{cases} 3x - 2y = 4 \\ y = \frac{3}{2}x - 2 \end{cases}$

Solution:

Solution



We will compare the slope and intercepts of the two lines.

$$\begin{cases} 3x - 2y &= 4 \\ y &= \frac{3}{2}x - 2 \end{cases}$$

Write the first equation in slope–intercept form.

$$\begin{aligned} 3x - 2y &= 4 \\ -2y &= -3x + 4 \\ \frac{-2y}{-2} &= \frac{-3x+4}{-2} \\ y &= \frac{3}{2}x - 2 \end{aligned}$$

The second equation is already in slope–intercept form.

$$y = \frac{3}{2}x - 2$$

Since the equations are the same, they have the same slope and same y-intercept and so the lines are coincident.

A system of equations whose graphs are coincident lines has infinitely many solutions and is consistent and dependent.

Note:

Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} 4x - 5y = 20 \\ y = \frac{4}{5}x - 4 \end{cases}$$

Solution:

infinitely many solutions, consistent, dependent

Note:
Exercise:

Problem:

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} -2x - 4y = 8 \\ y = -\frac{1}{2}x - 2 \end{cases}$$

Solution:

infinitely many solutions, consistent, dependent

Solve Applications of Systems of Equations by Graphing

We will use the same problem solving strategy we used in Math Models to set up and solve applications of systems of linear equations. We’ll modify the strategy slightly here to make it appropriate for systems of equations.

Note:
Use a problem solving strategy for systems of linear equations.

Read the problem. Make sure all the words and ideas are understood.
Identify what we are looking for.
Name what we are looking for. Choose variables to represent those quantities.

Translateinto a system of equations.
Solvethe system of equations using good algebra techniques.
Checkthe answer in the problem and make sure it makes sense.
Answerthe question with a complete sentence.

Step 5 is where we will use the method introduced in this section. We will graph the equations and find the solution.

Example:
Exercise:

Problem:

Sondra is making 10 quarts of punch from fruit juice and club soda. The number of quarts of fruit juice is 4 times the number of quarts of club soda. How many quarts of fruit juice and how many quarts of club soda does Sondra need?

Solution:
Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for.

We are looking for the number of quarts of fruit juice and the number of quarts of club soda that Sondra will need.

Step 3. Name what we are looking for. Choose variables to represent those quantities.

Let $f =$ number of quarts of fruit juice.
$c =$ number of quarts of club soda

Step 4. Translate into a system of equations.

$$\underbrace{\text{The number of quarts of fruit juice}}_{f}\ \underbrace{\text{and}}_{+}\ \underbrace{\text{the number of quarts of club soda}}_{c}\ \underbrace{\text{is}}_{=}\ 10$$

$$\underbrace{\text{The number of quarts of fruit juice}}_{f}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{four times the number of quarts of club soda}}_{4c}$$

We now have the system. $\begin{cases} f + c = 10 \\ f = 4c \end{cases}$

Step 5. Solve the system of equations using good algebra techniques.

$f = 4c$

$m = 4$

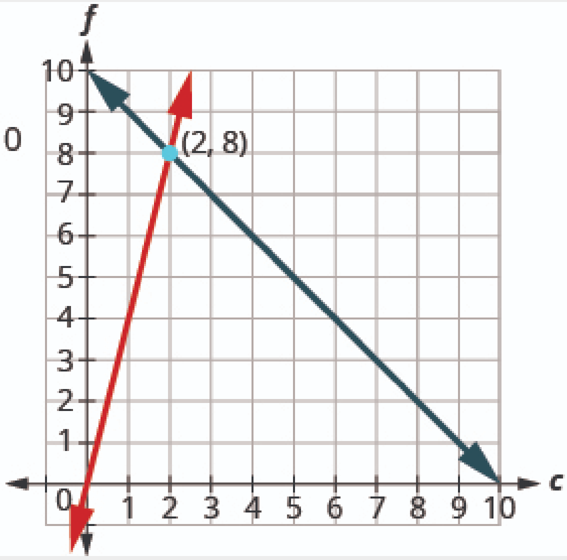
$b = 0$

$f + c = 10$

$f = -c + 10$

$m = -1$

$b = 10$



The point of intersection $(2, 8)$ is the solution. This means Sondra needs 2 quarts of club soda and 8 quarts of fruit juice.

Step 6. Check the answer in the problem and make sure it makes sense.

Does this make sense in the problem?

Yes, the number of quarts of fruit juice, 8 is 4 times the number of quarts of club soda, 2.

Yes, 10 quarts of punch is 8 quarts of fruit juice plus 2 quarts of club soda.

Step 7. Answer the question with a complete sentence.

Sondra needs 8 quarts of fruit juice and 2 quarts of soda.

Note:
Exercise:

Problem:

Manny is making 12 quarts of orange juice from concentrate and water. The number of quarts of water is 3 times the number of quarts of concentrate. How many quarts of concentrate and how many quarts of water does Manny need?

Solution:

Manny needs 3 quarts juice concentrate and 9 quarts water.

Note:
Exercise:

Problem:

Alisha is making an 18 ounce coffee beverage that is made from brewed coffee and milk. The number of ounces of brewed coffee is 5 times greater than the number of ounces of milk. How many ounces of coffee and how many ounces of milk does Alisha need?

Solution:

Alisha needs 15 ounces of coffee and 3 ounces of milk.

Note:
Access these online resources for additional instruction and practice with solving systems of equations by graphing.

- Instructional Video Solving Linear Systems by Graphing
- Instructional Video Solve by Graphing

Key Concepts

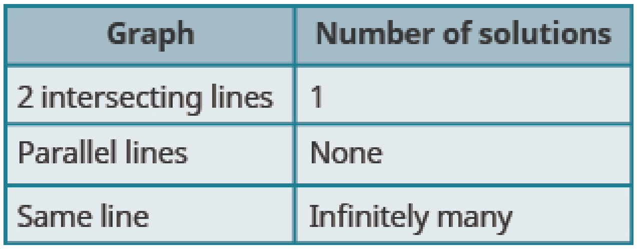
- **To solve a system of linear equations by graphing**

  Graph the first equation.
  Graph the second equation on the same rectangular coordinate system.
  Determine whether the lines intersect, are parallel, or are the same line.

Identify the solution to the system.	If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system.	If the lines are parallel, the system has no solution.	If the lines are the same, the system has an infinite number of solutions.

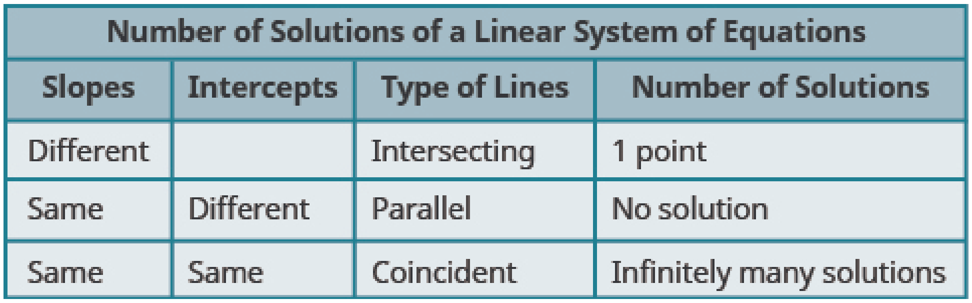
  Check the solution in both equations.

- Determine the number of solutions from the graph of a linear system



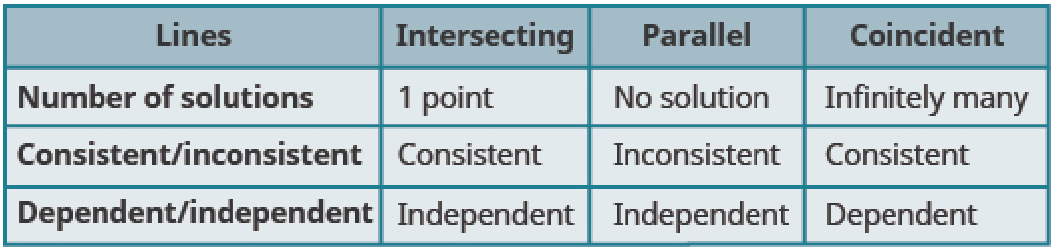
Graph	Number of solutions
2 intersecting lines	1
Parallel lines	None
Same line	Infinitely many

- Determine the number of solutions of a linear system by looking at the slopes and intercepts



Number of Solutions of a Linear System of Equations			
Slopes	**Intercepts**	**Type of Lines**	**Number of Solutions**
Different		Intersecting	1 point
Same	Different	Parallel	No solution
Same	Same	Coincident	Infinitely many solutions

- Determine the number of solutions and how to classify a system of equations



Lines	Intersecting	Parallel	Coincident
Number of solutions	1 point	No solution	Infinitely many
Consistent/inconsistent	Consistent	Inconsistent	Consistent
Dependent/independent	Independent	Independent	Dependent

- **Problem Solving Strategy for Systems of Linear Equations**

  Readthe problem. Make sure all the words and ideas are understood.
  Identifywhat we are looking for.
  Namewhat we are looking for. Choose variables to represent those quantities.
  Translateinto a system of equations.
  Solvethe system of equations using good algebra techniques.
  Checkthe answer in the problem and make sure it makes sense.
  Answerthe question with a complete sentence.

Practice Makes Perfect

Determine Whether an Ordered Pair is a Solution of a System of Equations. In the following exercises, determine if the following points are solutions to the given system of equations.

Exercise:

Problem: $\begin{cases} 2x - 6y = 0 \\ 3x - 4y = 5 \end{cases}$

ⓐ $(3, 1)$ ⓑ $(-3, 4)$

Solution:

ⓐ yes ⓑ no

Exercise:

Problem: $\begin{cases} 7x - 4y = -1 \\ -3x - 2y = 1 \end{cases}$

ⓐ $(-3, 5)$ ⓑ $(1, -2)$

Exercise:

Problem: $\begin{cases} 2x + y = 5 \\ x + y = 1 \end{cases}$

ⓐ $(4, -3)$ ⓑ $(2, 0)$

Solution:

ⓐ yes ⓑ no

Exercise:

Problem: $\begin{cases} -3x + y = 8 \\ -x + 2y = -9 \end{cases}$

ⓐ $(-5, -7)$ ⓑ $(-5, 7)$

Exercise:

Problem: $\begin{cases} x+y=2 \\ y=\frac{3}{4}x \end{cases}$

ⓐ $\left(\frac{8}{7}, \frac{6}{7}\right)$ ⓑ $\left(1, \frac{3}{4}\right)$

Solution:

ⓐ yes ⓑ no

Exercise:

Problem: $\begin{cases} x+y=1 \\ y=\frac{2}{5}x \end{cases}$

ⓐ $\left(\frac{5}{7}, \frac{2}{7}\right)$ ⓑ $(5, 2)$

Exercise:

Problem: $\begin{cases} x+5y=10 \\ y=\frac{3}{5}x+1 \end{cases}$

ⓐ $(-10, 4)$ ⓑ $\left(\frac{5}{4}, \frac{7}{4}\right)$

Solution:

ⓐ no ⓑ yes

Exercise:

Problem: $\begin{cases} x+3y=9 \\ y=\frac{2}{3}x-2 \end{cases}$

ⓐ $(-6, 5)$ ⓑ $\left(5, \frac{4}{3}\right)$

Solve a System of Linear Equations by Graphing In the following exercises, solve the following systems of equations by graphing.

Exercise:

Problem: $\begin{cases} 3x+y=-3 \\ 2x+3y=5 \end{cases}$

Solution:

$(-2,3)$

Exercise:

Problem: $\begin{cases} -x+y=2 \\ 2x+y=-4 \end{cases}$

Exercise:

Problem: $\begin{cases} -3x+y=-1 \\ 2x+y=4 \end{cases}$

Solution:

$(1,2)$

Exercise:

Problem: $\begin{cases} -2x+3y=-3 \\ x+y=4 \end{cases}$

Exercise:

Problem: $\begin{cases} y=x+2 \\ y=-2x+2 \end{cases}$

Solution:

$(0, 2)$

Exercise:

Problem: $\begin{cases} y = x - 2 \\ y = -3x + 2 \end{cases}$

Exercise:

Problem: $\begin{cases} y = \frac{3}{2}x + 1 \\ y = -\frac{1}{2}x + 5 \end{cases}$

Solution:

$(2, 4)$

Exercise:

Problem: $\begin{cases} y = \frac{2}{3}x - 2 \\ y = -\frac{1}{3}x - 5 \end{cases}$

Exercise:

Problem: $\begin{cases} -x + y = -3 \\ 4x + 4y = 4 \end{cases}$

Solution:

$(2, -1)$

Exercise:

Problem: $\begin{cases} x - y = 3 \\ 2x - y = 4 \end{cases}$

Exercise:

Problem: $\begin{cases} -3x + y = -1 \\ 2x + y = 4 \end{cases}$

Solution:

$(1, 2)$

Exercise:

Problem: $\begin{cases} -3x + y = -2 \\ 4x - 2y = 6 \end{cases}$

Exercise:

Problem: $\begin{cases} x + y = 5 \\ 2x - y = 4 \end{cases}$

Solution:

$(3, 2)$

Exercise:

Problem: $\begin{cases} x - y = 2 \\ 2x - y = 6 \end{cases}$

Exercise:

Problem: $\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$

Solution:

$(1, 1)$

Exercise:

Problem: $\begin{cases} x+y=6 \\ x-y=-8 \end{cases}$

Exercise:

Problem: $\begin{cases} x+y=-5 \\ x-y=3 \end{cases}$

Solution:

$(-1,-4)$

Exercise:

Problem: $\begin{cases} x+y=4 \\ x-y=0 \end{cases}$

Exercise:

Problem: $\begin{cases} x+y=-4 \\ -x+2y=-2 \end{cases}$

Solution:

$(-2,-2)$

Exercise:

Problem: $\begin{cases} -x+3y=3 \\ x+3y=3 \end{cases}$

Exercise:

Problem: $\begin{cases} -2x+3y=3 \\ x+3y=12 \end{cases}$

Solution:

$(3, 3)$

Exercise:

Problem: $\begin{cases} 2x - y = 4 \\ 2x + 3y = 12 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x + 3y = 6 \\ y = -2 \end{cases}$

Solution:

$(6, -2)$

Exercise:

Problem: $\begin{cases} -2x + y = 2 \\ y = 4 \end{cases}$

Exercise:

Problem: $\begin{cases} x - 3y = -3 \\ y = 2 \end{cases}$

Solution:

$(3, 2)$

Exercise:

Problem: $\begin{cases} 2x - 2y = 8 \\ y = -3 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x - y = -1 \\ x = 1 \end{cases}$

Solution:

$(1, 3)$

Exercise:

Problem: $\begin{cases} x + 2y = 2 \\ x = -2 \end{cases}$

Exercise:

Problem: $\begin{cases} x - 3y = -6 \\ x = -3 \end{cases}$

Solution:

$(-3, 1)$

Exercise:

Problem: $\begin{cases} x + y = 4 \\ x = 1 \end{cases}$

Exercise:

Problem: $\begin{cases} 4x - 3y = 8 \\ 8x - 6y = 14 \end{cases}$

Solution:

no solution

Exercise:

Problem: $\begin{cases} x+3y=4 \\ -2x-6y=3 \end{cases}$

Exercise:

Problem: $\begin{cases} -2x+4y=4 \\ y=\frac{1}{2}x \end{cases}$

Solution:

no solution

Exercise:

Problem: $\begin{cases} 3x+5y=10 \\ y=-\frac{3}{5}x+1 \end{cases}$

Exercise:

Problem: $\begin{cases} x=-3y+4 \\ 2x+6y=8 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} 4x=3y+7 \\ 8x-6y=14 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x+y=6 \\ -8x-4y=-24 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} 5x + 2y = 7 \\ -10x - 4y = -14 \end{cases}$

Exercise:

Problem: $\begin{cases} x + 3y = -6 \\ 4y = -\frac{4}{3}x - 8 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} -x + 2y = -6 \\ y = -\frac{1}{2}x - 1 \end{cases}$

Exercise:

Problem: $\begin{cases} -3x + 2y = -2 \\ y = -x + 4 \end{cases}$

Solution:

$(2, 2)$

Exercise:

Problem: $\begin{cases} -x + 2y = -2 \\ y = -x - 1 \end{cases}$

Determine the Number of Solutions of a Linear System Without graphing the following systems of equations, determine the number of solutions and then classify the system of equations.

Exercise:

Problem: $\begin{cases} y = \frac{2}{3}x + 1 \\ -2x + 3y = 5 \end{cases}$

Solution:

0 solutions

Exercise:

Problem: $\begin{cases} y = \frac{1}{3}x + 2 \\ x - 3y = 9 \end{cases}$

Exercise:

Problem: $\begin{cases} y = -2x + 1 \\ 4x + 2y = 8 \end{cases}$

Solution:

0 solutions

Exercise:

Problem: $\begin{cases} y = 3x + 4 \\ 9x - 3y = 18 \end{cases}$

Exercise:

Problem: $\begin{cases} y = \frac{2}{3}x + 1 \\ 2x - 3y = 7 \end{cases}$

Solution:

0 solutions

Exercise:

Problem: $\begin{cases} 3x + 4y = 12 \\ y = -3x - 1 \end{cases}$

Exercise:

Problem: $\begin{cases} 4x + 2y = 10 \\ 4x - 2y = -6 \end{cases}$

Solution:

consistent, 1 solution

Exercise:

Problem: $\begin{cases} 5x + 3y = 4 \\ 2x - 3y = 5 \end{cases}$

Exercise:

Problem: $\begin{cases} y = -\frac{1}{2}x + 5 \\ x + 2y = 10 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} y = x + 1 \\ -x + y = 1 \end{cases}$

Exercise:

Problem: $\begin{cases} y = 2x + 3 \\ 2x - y = -3 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} 5x - 2y = 10 \\ y = \frac{5}{2}x - 5 \end{cases}$

Solve Applications of Systems of Equations by Graphing In the following exercises, solve.

Exercise:

Problem:

Molly is making strawberry infused water. For each ounce of strawberry juice, she uses three times as many ounces of water. How many ounces of strawberry juice and how many ounces of water does she need to make 64 ounces of strawberry infused water?

Solution:

Molly needs 16 ounces of strawberry juice and 48 ounces of water.

Exercise:

Problem:

Jamal is making a snack mix that contains only pretzels and nuts. For every ounce of nuts, he will use 2 ounces of pretzels. How many ounces of pretzels and how many ounces of nuts does he need to make 45 ounces of snack mix?

Exercise:

Problem:

Enrique is making a party mix that contains raisins and nuts. For each ounce of nuts, he uses twice the amount of raisins. How many ounces of nuts and how many ounces of raisins does he need to make 24 ounces of party mix?

Solution:

Enrique needs 8 ounces of nuts and 16 ounces of water.

Exercise:

Problem:

Owen is making lemonade from concentrate. The number of quarts of water he needs is 4 times the number of quarts of concentrate. How many quarts of water and how many quarts of concentrate does Owen need to make 100 quarts of lemonade?

Everyday Math

Exercise:

Problem:

Leo is planning his spring flower garden. He wants to plant tulip and daffodil bulbs. He will plant 6 times as many daffodil bulbs as tulip bulbs. If he wants to plant 350 bulbs, how many tulip bulbs and how many daffodil bulbs should he plant?

Solution:

Leo should plant 50 tulips and 300 daffodils.

Exercise:

Problem:

A marketing company surveys 1,200 people. They surveyed twice as many females as males. How many males and females did they survey?

Writing Exercises

Exercise:

Problem:

In a system of linear equations, the two equations have the same slope. Describe the possible solutions to the system.

Solution:

Given that it is only known that the slopes of both linear equations are the same, then there is either no solution if the intercepts are different (the graphs of the equations are parallel) or there are infinitely many solutions if the intercepts are also the same.

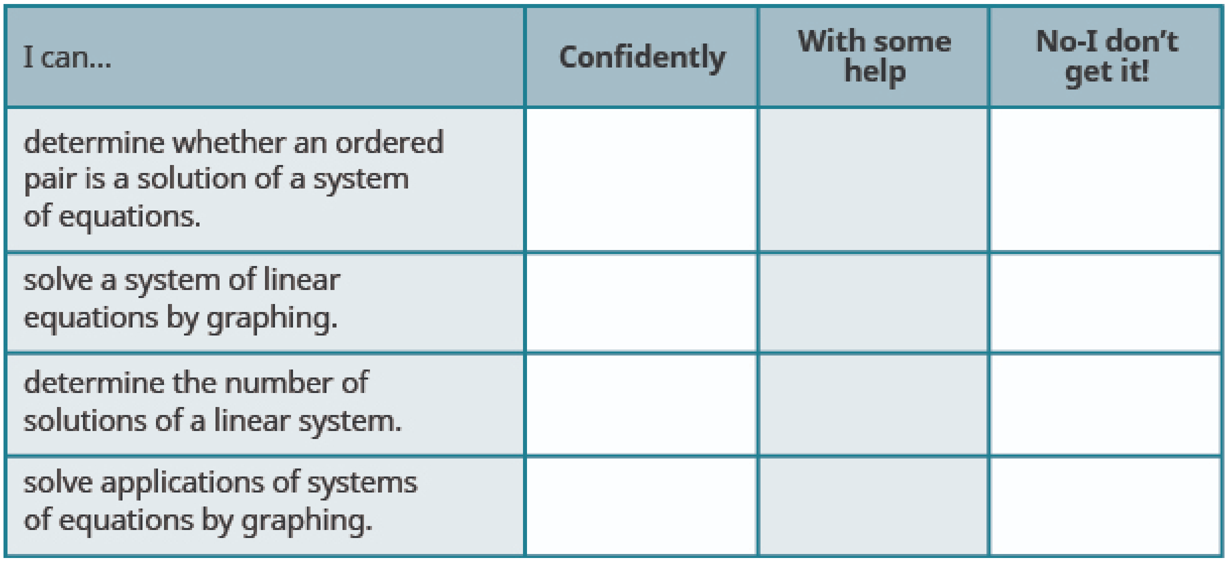
Exercise:

Problem:

In a system of linear equations, the two equations have the same intercepts. Describe the possible solutions to the system.

Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don’t get it!
determine whether an ordered pair is a solution of a system of equations.			
solve a system of linear equations by graphing.			
determine the number of solutions of a linear system.			
solve applications of systems of equations by graphing.			

If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don’t get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

coincident lines

Coincident lines are lines that have the same slope and same *y*-intercept.

consistent system	A consistent system of equations is a system of equations with at least one solution.
dependent equations	Two equations are dependent if all the solutions of one equation are also solutions of the other equation.
inconsistent system	An inconsistent system of equations is a system of equations with no solution.
independent equations	Two equations are independent if they have different solutions.
solutions of a system of equations	Solutions of a system of equations are the values of the variables that make all the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).
system of linear equations	When two or more linear equations are grouped together, they form a system of linear equations.

Solve Systems of Equations by Substitution

By the end of this section, you will be able to:

- Solve a system of equations by substitution
- Solve applications of systems of equations by substitution

Note:
Before you get started, take this readiness quiz.

1. Simplify $-5\left(3-x\right)$.
   If you missed this problem, review [link].
2. Simplify $4-2\left(n+5\right)$.
   If you missed this problem, review [link].
3. Solve for y. $8y-8=32-2y$
   If you missed this problem, review [link].
4. Solve for x. $3x-9y=-3$
   If you missed this problem, review [link].

Solving systems of linear equations by graphing is a good way to visualize the types of solutions that may result. However, there are many cases where solving a system by graphing is inconvenient or imprecise. If the graphs extend beyond the small grid with x and y both between −10 and 10, graphing the lines may be cumbersome. And if the solutions to the system are not integers, it can be hard to read their values precisely from a graph.

In this section, we will solve systems of linear equations by the substitution method.

Solve a System of Equations by Substitution

We will use the same system we used first for graphing.

Equation:

$$\begin{cases}2x+y=7\\ x-2y=6\end{cases}$$

We will first solve one of the equations for either x or y. We can choose either equation and solve for either variable—but we’ll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

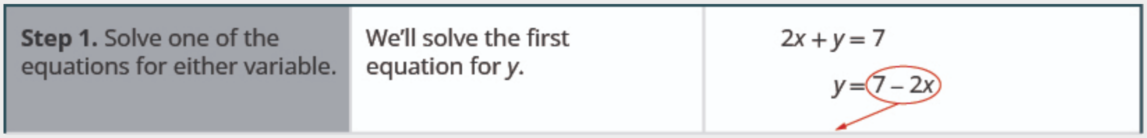
After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true.

We’ll fill in all these steps now in [link].

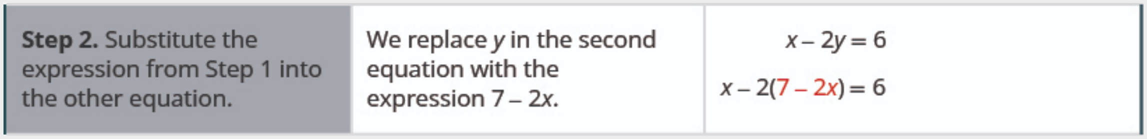
Example:
How to Solve a System of Equations by Substitution
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

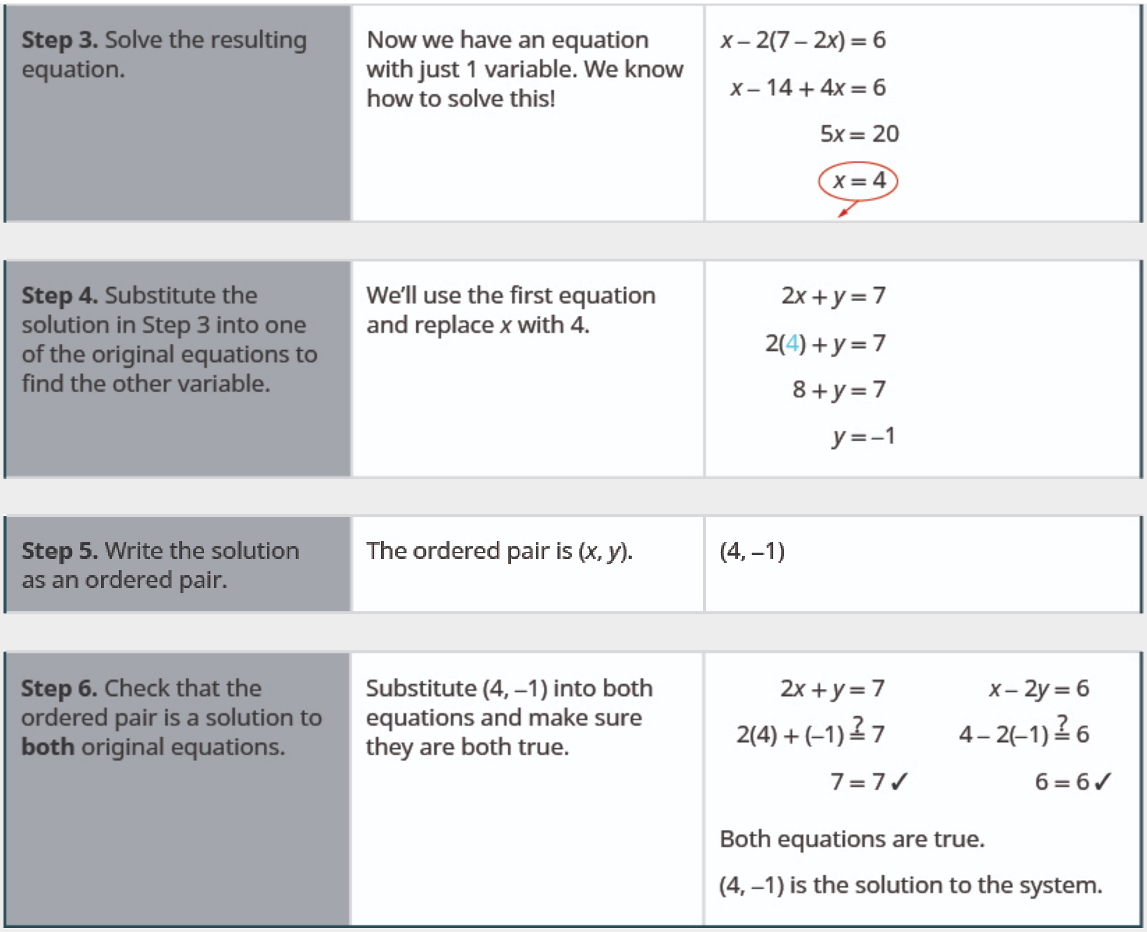
Solution:
Solution



Step 1. Solve one of the equations for either variable.	We’ll solve the first equation for *y*.	$2x + y = 7$ $y = 7 - 2x$



Step 2. Substitute the expression from Step 1 into the other equation.	We replace *y* in the second equation with the expression 7 – 2*x*.	$x - 2y = 6$ $x - 2(7 - 2x) = 6$



Step 3. Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x - 2(7 - 2x) = 6$ $x - 14 + 4x = 6$ $5x = 20$ $x = 4$
Step 4. Substitute the solution in Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace x with 4.	$2x + y = 7$ $2(4) + y = 7$ $8 + y = 7$ $y = -1$
Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(4, -1)$
Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute $(4, -1)$ into both equations and make sure they are both true.	$2x + y = 7$ $x - 2y = 6$ $2(4) + (-1) \stackrel{?}{=} 7$ $4 - 2(-1) \stackrel{?}{=} 6$ $7 = 7$ ✓ $6 = 6$ ✓ Both equations are true. $(4, -1)$ is the solution to the system.

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} -2x + y = -11 \\ x + 3y = 9 \end{cases}$

Solution:

$(6, 1)$

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} x+3y=10 \\ 4x+y=18 \end{cases}$

Solution:

$(4,2)$

Note:
Solve a system of equations by substitution.

Solve one of the equations for either variable.
Substitute the expression from Step 1 into the other equation.
Solve the resulting equation.
Substitute the solution in Step 3 into one of the original equations to find the other variable.
Write the solution as an ordered pair.
Check that the ordered pair is a solution to**both**original equations.

If one of the equations in the system is given in slope–intercept form, Step 1 is already done! We'll see this in [link].

Example:
Exercise:

Problem: Solve the system by substitution.

$$\begin{cases} x+y=-1 \\ y=x+5 \end{cases}$$

Solution:
Solution

The second equation is already solved for y. We will substitute the expression in place of y in the first equation.

	$\begin{cases} x+y=-1 \\ y=x+5 \end{cases}$
The second equation is already solved for y . We will substitute into the first equation.	
Replace the y with $x+5$.	$y=\boxed{x+5}$ $x+y=-1$
Solve the resulting equation for x.	$x+x+5=-1$
	$2x+5=-1$
	$2x=-6$
Substitute $x=-3$ into $y=x+5$ to find y.	$x\boxed{=-3}$ $y=x+5$
	$y=-3+5$
The ordered pair is $(-3,2)$.	

	 $y=2$
Check the ordered pair in both equations: $\begin{array}{rcl} x+y & = & -1 \\ -3+2 & \overset{?}{=} & -1 \\ -1 & = & -1\checkmark \end{array} \qquad \begin{array}{rcl} y & = & x+5 \\ 2 & \overset{?}{=} & -3+5 \\ 2 & = & 2\checkmark \end{array}$	
	The solution is $(-3, 2)$.

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} x+y=6 \\ y=3x-2 \end{cases}$

Solution:

$(2, 4)$

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 2x-y=1 \\ y=-3x-6 \end{cases}$

Solution:

$(-1, -3)$

If the equations are given in standard form, we’ll need to start by solving for one of the variables. In this next example, we’ll solve the first equation for *y*.

Example:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 3x + y = 5 \\ 2x + 4y = -10 \end{cases}$

Solution:
Solution

We need to solve one equation for one variable. Then we will substitute that expression into the other equation.

Solve for y. Substitute into the other equation.	$3x + y = 5$ $y = \boxed{-3x + 5}$ $2x + 4y = -10$
Replace the y with $-3x + 5$.	$2x + 4(-3x + 5) = -10$
Solve the resulting equation for x.	$2x - 12x + 20 = -10$
	$-10x + 20 = -10$ $-10x = -30$
Substitute $y = 3$ into $3x + y = 5$ to find y.	$\boxed{x = 3}$ $3x + y = 5$

	$3(3)+y=5$ $9+y=5$
The ordered pair is $(3,-4)$.	$y=-4$
Check the ordered pair in both equations: $\begin{aligned} 3x+y &= 5 \\ 3\cdot 3+(-4) &\overset{?}{=} 5 \\ 9-4 &\overset{?}{=} 5 \\ 5 &= 5\checkmark \end{aligned}$ $\begin{aligned} 2x+4y &= -10 \\ 2\cdot 3+4(-4) &= -10 \\ 6-16 &\overset{?}{=} -10 \\ -10 &= -10\checkmark \end{aligned}$	
	The solution is $(3,-4)$.

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 4x+y=2 \\ 3x+2y=-1 \end{cases}$

Solution:

$(1,-2)$

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} -x+y=4 \\ 4x-y=2 \end{cases}$

Solution:

$(2,6)$

In [link] it was easiest to solve for y in the first equation because it had a coefficient of 1. In [link] it will be easier to solve for x.

Example:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} x-2y=-2 \\ 3x+2y=34 \end{cases}$

Solution:
Solution

We will solve the first equation for x and then substitute the expression into the second equation.

	$x-2y=-2$
Solve for x. Substitute into the other equation.	$x=\boxed{2y-2}$ $3x+2y=34$
Replace the x with $2y-2$.	$3(2y-2)+2y=34$

Solve the resulting equation for y.	$6y - 6 + 2y = 34$
Substitute $y = 5$ into $x - 2y = -2$ to find x.	$8y - 6 = 34$ $8y = 40$ $y = 5$ $x - 2y = -2$ $x - 2 \cdot 5 = -2$ $x - 10 = -2$ $x = 8$
The ordered pair is $(8, 5)$.	
Check the ordered pair in both equations: $x - 2y = -2$ $3x + 2y = 34$ $8 - 2 \cdot 5 \overset{?}{=} -2$ $3 \cdot 8 + 2 \cdot 5 \overset{?}{=} 34$ $8 - 10 \overset{?}{=} -2$ $24 + 10 \overset{?}{=} 34$ $-2 = -2 \checkmark$ $34 = 34 \checkmark$	
	The solution is $(8, 5)$.

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} x - 5y = 13 \\ 4x - 3y = 1 \end{cases}$

Solution:

$(-2, -3)$

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} x - 6y = -6 \\ 2x - 4y = 4 \end{cases}$

Solution:

$(6, 2)$

When both equations are already solved for the same variable, it is easy to substitute!

Example:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} y = -2x + 5 \\ y = \frac{1}{2}x \end{cases}$

Solution:
Solution

Since both equations are solved for y, we can substitute one into the other.

Substitute $\frac{1}{2}x$ for y in the first equation.	$y = \frac{1}{2}x$ $y = -2x + 5$
Replace the y with $\frac{1}{2}x$.	$\frac{1}{2}x = -2x + 5$
Solve the resulting equation. Start by clearing the fraction.	$2\left(\frac{1}{2}x\right) = 2(-2x + 5)$
Solve for x.	$x = -4x + 10$
	$5x = 10$
Substitute $x = 2$ into $y = \frac{1}{2}x$ to find y.	$x = 2$ $y = \frac{1}{2}x$ $y = \frac{1}{2} \cdot 2$ $y = 1$
The ordered pair is $(2, 1)$.	
Check the ordered pair in both equations:	

$\begin{aligned} y &= \tfrac{1}{2}x \\ 1 &\overset{?}{=} \tfrac{1}{2}\cdot 2 \\ 1 &= 1 \checkmark \end{aligned}$ $\quad\begin{aligned} y &= -2x+5 \\ 1 &\overset{?}{=} -2\cdot 2+5 \\ 1 &= -4+5 \\ 1 &= 1 \checkmark \end{aligned}$	
	The solution is $(2,1)$.

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} y = 3x - 16 \\ y = \frac{1}{3}x \end{cases}$

Solution:

$(6,2)$

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} y = -x + 10 \\ y = \frac{1}{4}x \end{cases}$

Solution:

$(8,2)$

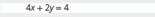
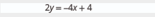
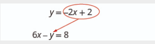
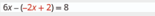
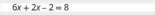
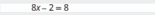
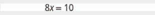
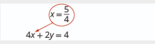
Be very careful with the signs in the next example.

Example:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 4x + 2y = 4 \\ 6x - y = 8 \end{cases}$

Solution:
Solution

We need to solve one equation for one variable. We will solve the first equation for y.

	 $4x + 2y = 4$
Solve the first equation for y.	 $2y = -4x + 4$
Substitute $-2x + 2$ for y in the second equation.	 $y = -2x + 2$ $6x - y = 8$
Replace the y with $-2x + 2$.	 $6x - (-2x + 2) = 8$
Solve the equation for x.	 $6x + 2x - 2 = 8$
	 $8x - 2 = 8$  $8x = 10$
	 $x = \frac{5}{4}$ $4x + 2y = 4$

Substitute $x = \frac{5}{4}$ into $4x + 2y = 4$ to find y.

$$4\left(\frac{5}{4}\right) + 2y = 4$$

$$5 + 2y = 4$$

$$2y = -1$$

$$y = -\frac{1}{2}$$

The ordered pair is $\left(\frac{5}{4}, -\frac{1}{2}\right)$.

Check the ordered pair in both equations.

$$\begin{aligned} 4x + 2y &= 4 \\ 4\left(\frac{5}{4}\right) + 2\left(-\frac{1}{2}\right) &\overset{?}{=} 4 \\ 5 - 1 &\overset{?}{=} 4 \\ 4 &= 4 \checkmark \end{aligned}$$

$$\begin{aligned} 6x - y &= 8 \\ 6\left(\frac{5}{4}\right) - \left(-\frac{1}{2}\right) &\overset{?}{=} 8 \\ \frac{15}{4} - \left(-\frac{1}{2}\right) &\overset{?}{=} 8 \\ \frac{16}{2} &\overset{?}{=} 8 \\ 8 &= 8 \checkmark \end{aligned}$$

The solution is $\left(\frac{5}{4}, -\frac{1}{2}\right)$.

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} x - 4y = -4 \\ -3x + 4y = 0 \end{cases}$

Solution:

$\left(2, \frac{3}{2}\right)$

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 4x - y = 0 \\ 2x - 3y = 5 \end{cases}$

Solution:

$\left(-\frac{1}{2}, -2\right)$

In [link], it will take a little more work to solve one equation for x or y.

Example:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 4x - 3y = 6 \\ 15y - 20x = -30 \end{cases}$

Solution:
Solution

We need to solve one equation for one variable. We will solve the first equation for x.

	$4x - 3y = 6$
Solve the first equation for x.	$4x = 3y + 6$
Substitute $\frac{3}{4}y + \frac{3}{2}$ for x in the second equation.	$x = \frac{3}{4}y + \frac{3}{2}$ $15y - 20x = -30$
Replace the x with $\frac{3}{4}y + \frac{3}{2}$.	$15y - 20\left(\frac{3}{4}y + \frac{3}{2}\right) = -30$
Solve for y.	$15y - 15y - 30 = -30$
	$0 - 30 = -30$
	$0 = 0$

Since 0 = 0 is a true statement, the system is consistent. The equations are dependent. The graphs of these two equations would give the same line. The system has infinitely many solutions.

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 2x - 3y = 12 \\ -12y + 8x = 48 \end{cases}$

Solution:

infinitely many solutions

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 5x + 2y = 12 \\ -4y - 10x = -24 \end{cases}$

Solution:

infinitely many solutions

Look back at the equations in [link]. Is there any way to recognize that they are the same line?

Let’s see what happens in the next example.

Example:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 5x - 2y = -10 \\ y = \frac{5}{2}x \end{cases}$

Solution:
Solution

The second equation is already solved for y, so we can substitute for y in the first equation.

Substitute x for y in the first equation.	$y = \frac{5}{2}x$ $5x - 2y = -10$
Replace the y with $\frac{5}{2}x$.	$5x - 2\left(\frac{5}{2}x\right) = -10$
Solve for x.	$5x - 5x = -10$
	$0 \neq -10$

Since 0 = −10 is a false statement the equations are inconsistent. The graphs of the two equation would be parallel lines. The system has no solutions.

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 3x + 2y = 9 \\ y = -\frac{3}{2}x + 1 \end{cases}$

Solution:

no solution

Note:
Exercise:

Problem: Solve the system by substitution. $\begin{cases} 5x - 3y = 2 \\ y = \frac{5}{3}x - 4 \end{cases}$

Solution:

no solution

Solve Applications of Systems of Equations by Substitution

We’ll copy here the problem solving strategy we used in the Solving Systems of Equations by Graphing section for solving systems of equations. Now that we know how to solve systems by substitution, that’s what we’ll do in Step 5.

Note:
How to use a problem solving strategy for systems of linear equations.

Read the problem. Make sure all the words and ideas are understood.
Identify what we are looking for.
Name what we are looking for. Choose variables to represent those quantities.
Translate into a system of equations.
Solve the system of equations using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

Some people find setting up word problems with two variables easier than setting them up with just one variable. Choosing the variable names is easier when all you need to do is write down two letters. Think about this in the next example—how would you have done it with just one variable?

Example:
Exercise:

Problem:

The sum of two numbers is zero. One number is nine less than the other. Find the numbers.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name what we are looking for.	Let $n =$ the first number Let $m =$ the second number
Step 4. Translate into a system of equations.	The sum of two numbers is zero.
	$n + m = 0$
	One number is nine less than the other.
	$n = m - 9$
The system is:	$\begin{cases} n + m = 0 \\ n = m - 9 \end{cases}$
Step 5. Solve the system of equations. We will use substitution since the second equation is solved for n.	
Substitute $m - 9$ for n in the first equation.	$n = m - 9$ $n + m = 0$

Solve for m.	$m - 9 + m = 0$
	$2m - 9 = 0$
	$2m = 9$
Substitute $m = \frac{9}{2}$ into the second equation and then solve for n.	$m = \frac{9}{2}$ $n = m - 9$
	$m = \frac{9}{2} - 9$
	$m = \frac{9}{2} - \frac{18}{2}$
	$n = -\frac{9}{2}$
Step 6. Check the answer in the problem.	Do these numbers make sense in the problem? We will leave this to you!
Step 7. Answer the question.	The numbers are $\frac{9}{2}$ and $-\frac{9}{2}$.

Note:
Exercise:

Problem:

The sum of two numbers is 10. One number is 4 less than the other. Find the numbers.

Solution:

The numbers are 3 and 7.

Note:
Exercise:

Problem:

The sum of two number is −6. One number is 10 less than the other. Find the numbers.

Solution:

The numbers are 2 and −8.

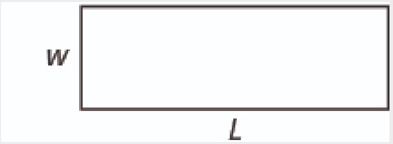
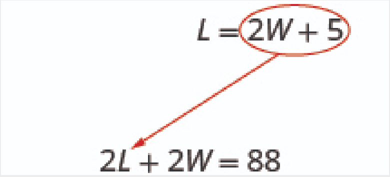
In the [link], we’ll use the formula for the perimeter of a rectangle, $P = 2L + 2W$.

Example:
Exercise:

Problem:

The perimeter of a rectangle is 88. The length is five more than twice the width. Find the length and the width.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what we are looking for.	Let L = the length W = the width
Step 4. Translate into a system of equations.	The perimeter of a rectangle is 88.
	$2L + 2W = P$ $2L + 2W = 88$
	The length is five more than twice the width.
	$L = 2W + 5$
The system is:	$\begin{cases} 2L + 2W = 88 \\ L = 2W + 5 \end{cases}$
Step 5. Solve the system of equations. We will use substitution since the second equation is solved for L. Substitute $2W + 5$ for L in the first equation.	$L = 2W + 5$ $2L + 2W = 88$
Solve for W.	$2(2W + 5) + 2W = 88$

	$4W + 10 + 2W = 88$
	$6W + 10 = 88$
	$6W = 78$
Substitute *W* = 13 into the second equation and then solve for *L*.	$W = \enclose{circle}{13}$ $L = 2W + 5$
	$L = 2 \cdot 13 + 5$
	$L = 31$
Step 6. Check the answer in the problem.	Does a rectangle with length 31 and width 13 have perimeter 88? Yes.
Step 7. Answer the equation.	The length is 31 and the width is 13.

Note:
Exercise:

Problem:

The perimeter of a rectangle is 40. The length is 4 more than the width. Find the length and width of the rectangle.

Solution:

The length is 12 and the width is 8.

Note:
Exercise:

Problem:

The perimeter of a rectangle is 58. The length is 5 more than three times the width. Find the length and width of the rectangle.

Solution:

The length is 23 and the width is 6.

For [link] we need to remember that the sum of the measures of the angles of a triangle is 180 degrees and that a right triangle has one 90 degree angle.

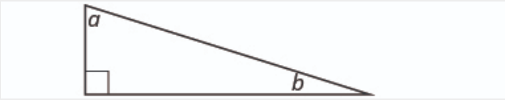
Example:
Exercise:

Problem:

The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle. Find the measures of both angles.

Solution:
Solution

We will draw and label a figure.

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the measures of the angles.
Step 3. Name what we are looking for.	Let a = the measure of the 1st angle b = the measure of the 2nd angle
Step 4. Translate into a system of equations.	The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle.
	$a = 3b + 10$
	The sum of the measures of the angles of a triangle is 180.
	$a + b + 90 = 180$
The system is:	$\begin{cases} a = 3b + 10 \\ a + b + 90 = 180 \end{cases}$
Step 5. Solve the system of equations. We will use substitution since the first equation is solved for a.	$a = \boxed{3b + 10}$ $a + b + 90 = 180$

Substitute $3b + 10$ for a in the second equation.	$(3b + 10) + b + 90 = 180$
Solve for b.	$4b + 100 = 180$
	$4b = 80$
	$b = 20$ $a = 3b + 10$
Substitute $b = 20$ into the first equation and then solve for a.	$a = 3 \cdot 20 + 10$ $a = 70$
Step 6. Check the answer in the problem.	We will leave this to you!
Step 7. Answer the question.	The measures of the small angles are 20 and 70.

Note:
Exercise:

Problem:

The measure of one of the small angles of a right triangle is 2 more than 3 times the measure of the other small angle. Find the measure of both angles.

Solution:

The measure of the angles are 22 degrees and 68 degrees.

Note:
Exercise:

Problem:

The measure of one of the small angles of a right triangle is 18 less than twice the measure of the other small angle. Find the measure of both angles.

Solution:

The measure of the angles are 36 degrees and 54 degrees.

Example:
Exercise:

Problem:

Heather has been offered two options for her salary as a trainer at the gym. Option A would pay her \$25,000 plus \$15 for each training session. Option B would pay her \$10,000 + \$40 for each training session. How many training sessions would make the salary options equal?

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	We are looking for the number of training sessions that would make the pay equal.
Step 3. Name what we are	Let $s =$ Heather’s salary.

looking for.	$n =$ the number of training sessions
Step 4. Translate into a system of equations.	Option A would pay her $25,000 plus $15 for each training session.
	$s = 25{,}000 + 15n$
	Option B would pay her $10,000 + $40 for each training session
	$s = 10{,}000 + 40n$
The system is:	$\begin{cases} s = 25{,}000 + 15n \\ s = 10{,}000 + 40n \end{cases}$
Step 5. Solve the system of equations. We will use substitution.	$s = \boxed{25{,}000 + 15n}$ $s = 10{,}000 + 40n$
Substitute 25,000 + 15*n* for *s* in the second equation.	$25{,}000 + 15n = 10{,}000 + 40n$
Solve for *n*.	$25{,}000 = 10{,}000 + 25n$
	$15{,}000 = 25n$
	$600 = n$

Step 6. Check the answer.	Are 600 training sessions a year reasonable? Are the two options equal when $n = 600$?
Step 7. Answer the question.	The salary options would be equal for 600 training sessions.

Note:
Exercise:

Problem:

Geraldine has been offered positions by two insurance companies. The first company pays a salary of $12,000 plus a commission of $100 for each policy sold. The second pays a salary of $20,000 plus a commission of $50 for each policy sold. How many policies would need to be sold to make the total pay the same?

Solution:

There would need to be 160 policies sold to make the total pay the same.

Note:
Exercise:

Problem:

Kenneth currently sells suits for company A at a salary of $22,000 plus a $10 commission for each suit sold. Company B offers him a position with a salary of $28,000 plus a $4 commission for each suit sold. How many suits would Kenneth need to sell for the options to be equal?

Solution:

Kenneth would need to sell 1,000 suits.

Note:
Access these online resources for additional instruction and practice with solving systems of equations by substitution.

- [Instructional Video-Solve Linear Systems by Substitution]
- [Instructional Video-Solve by Substitution]

Key Concepts

- **Solve a system of equations by substitution**

  Solve one of the equations for either variable.
  Substitute the expression from Step 1 into the other equation.
  Solve the resulting equation.
  Substitute the solution in Step 3 into one of the original equations to find the other variable.
  Write the solution as an ordered pair.
  Check that the ordered pair is a solution to both original equations.

Practice Makes Perfect

Solve a System of Equations by Substitution

In the following exercises, solve the systems of equations by substitution.

Exercise:

Problem: $\begin{cases} 2x + y = -4 \\ 3x - 2y = -6 \end{cases}$

Solution:

$(-2, 0)$

Exercise:

Problem: $\begin{cases} 2x + y = -2 \\ 3x - y = 7 \end{cases}$

Exercise:

Problem: $\begin{cases} x - 2y = -5 \\ 2x - 3y = -4 \end{cases}$

Solution:

$(7, 6)$

Exercise:

Problem: $\begin{cases} x - 3y = -9 \\ 2x + 5y = 4 \end{cases}$

Exercise:

Problem: $\begin{cases} 5x - 2y = -6 \\ y = 3x + 3 \end{cases}$

Solution:

$(0, 3)$

Exercise:

Problem: $\begin{cases} -2x + 2y = 6 \\ y = -3x + 1 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x + 3y = 3 \\ y = -x + 3 \end{cases}$

Solution:

$(6, -3)$

Exercise:

Problem: $\begin{cases} 2x+5y=-14 \\ y=-2x+2 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x+5y=1 \\ y=\frac{1}{3}x-2 \end{cases}$

Solution:

$(3,-1)$

Exercise:

Problem: $\begin{cases} 3x+4y=1 \\ y=-\frac{2}{5}x+2 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x-2y=6 \\ y=\frac{2}{3}x+2 \end{cases}$

Solution:

$(6,6)$

Exercise:

Problem: $\begin{cases} -3x-5y=3 \\ y=\frac{1}{2}x-5 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x+y=10 \\ -x+y=-5 \end{cases}$

Solution:

$(5,0)$

Exercise:

Problem: $\begin{cases} -2x + y = 10 \\ -x + 2y = 16 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x + y = 1 \\ -4x + y = 15 \end{cases}$

Solution:

$(-2, 7)$

Exercise:

Problem: $\begin{cases} x + y = 0 \\ 2x + 3y = -4 \end{cases}$

Exercise:

Problem: $\begin{cases} x + 3y = 1 \\ 3x + 5y = -5 \end{cases}$

Solution:

$(-5, 2)$

Exercise:

Problem: $\begin{cases} x + 2y = -1 \\ 2x + 3y = 1 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x + y = 5 \\ x - 2y = -15 \end{cases}$

Solution:

$(-1, 7)$

Exercise:

Problem: $\begin{cases} 4x + y = 10 \\ x - 2y = -20 \end{cases}$

Exercise:

Problem: $\begin{cases} y = -2x - 1 \\ y = -\frac{1}{3}x + 4 \end{cases}$

Solution:

$(-3, 5)$

Exercise:

Problem: $\begin{cases} y = x - 6 \\ y = -\frac{3}{2}x + 4 \end{cases}$

Exercise:

Problem: $\begin{cases} y = 2x - 8 \\ y = \frac{3}{5}x + 6 \end{cases}$

Solution:

$(10, 12)$

Exercise:

Problem: $\begin{cases} y = -x - 1 \\ y = x + 7 \end{cases}$

Exercise:

Problem: $\begin{cases} 4x + 2y = 8 \\ 8x - y = 1 \end{cases}$

Solution:

$\left(\frac{1}{2}, 3\right)$

Exercise:

Problem: $\begin{cases} -x - 12y = -1 \\ 2x - 8y = -6 \end{cases}$

Exercise:

Problem: $\begin{cases} 15x + 2y = 6 \\ -5x + 2y = -4 \end{cases}$

Solution:

$\left(\frac{1}{2}, -\frac{3}{4}\right)$

Exercise:

Problem: $\begin{cases} 2x - 15y = 7 \\ 12x + 2y = -4 \end{cases}$

Exercise:

Problem: $\begin{cases} y = 3x \\ 6x - 2y = 0 \end{cases}$

Solution:

Infinitely many solutions

Exercise:

Problem: $\begin{cases} x = 2y \\ 4x - 8y = 0 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x + 16y = 8 \\ -x - 8y = -4 \end{cases}$

Solution:

Infinitely many solutions

Exercise:

Problem: $\begin{cases} 15x+4y=6 \\ -30x-8y=-12 \end{cases}$

Exercise:

Problem: $\begin{cases} y=-4x \\ 4x+y=1 \end{cases}$

Solution:

No solution

Exercise:

Problem: $\begin{cases} y=-\frac{1}{4}x \\ x+4y=8 \end{cases}$

Exercise:

Problem: $\begin{cases} y=\frac{7}{8}x+4 \\ -7x+8y=6 \end{cases}$

Solution:

No solution

Exercise:

Problem: $\begin{cases} y=-\frac{2}{3}x+5 \\ 2x+3y=11 \end{cases}$

Solve Applications of Systems of Equations by Substitution

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

The sum of two numbers is 15. One number is 3 less than the other. Find the numbers.

Solution:

The numbers are 6 and 9.

Exercise:

Problem:

The sum of two numbers is 30. One number is 4 less than the other. Find the numbers.

Exercise:

Problem:

The sum of two numbers is −26. One number is 12 less than the other. Find the numbers.

Solution:

The numbers are −7 and −19.

Exercise:

Problem:

The perimeter of a rectangle is 50. The length is 5 more than the width. Find the length and width.

Exercise:

Problem:

The perimeter of a rectangle is 60. The length is 10 more than the width. Find the length and width.

Solution:

The length is 20 and the width is 10.

Exercise:

Problem:

The perimeter of a rectangle is 58. The length is 5 more than three times the width. Find the length and width.

Exercise:

Problem:

The perimeter of a rectangle is 84. The length is 10 more than three times the width. Find the length and width.

Solution:

The length is 34 and the width is 8.

Exercise:

Problem:

The measure of one of the small angles of a right triangle is 14 more than 3 times the measure of the other small angle. Find the measure of both angles.

Exercise:

Problem:

The measure of one of the small angles of a right triangle is 26 more than 3 times the measure of the other small angle. Find the measure of both angles.

Solution:

The measures are 16° and 74°.

Exercise:

Problem:

The measure of one of the small angles of a right triangle is 15 less than twice the measure of the other small angle. Find the measure of both angles.

Exercise:

Problem:

The measure of one of the small angles of a right triangle is 45 less than twice the measure of the other small angle. Find the measure of both angles.

Solution:

The measures are 45° and 45°.

Exercise:

Problem:

Maxim has been offered positions by two car dealers. The first company pays a salary of $10,000 plus a commission of $1,000 for each car sold. The second pays a salary of $20,000 plus a commission of $500 for each car sold. How many cars would need to be sold to make the total pay the same?

Exercise:

Problem:

Jackie has been offered positions by two cable companies. The first company pays a salary of $ 14,000 plus a commission of $100 for each cable package sold. The second pays a salary of $20,000 plus a commission of $25 for each cable package sold. How many cable packages would need to be sold to make the total pay the same?

Solution:

80 cable packages would need to be sold.

Exercise:

Problem:

Amara currently sells televisions for company A at a salary of $17,000 plus a $100 commission for each television she sells. Company B offers her a position with a salary of $29,000 plus a $20 commission for each television she sells. How many televisions would Amara need to sell for the options to be equal?

Exercise:

Problem:

Mitchell currently sells stoves for company A at a salary of $12,000 plus a $150 commission for each stove he sells. Company B offers him a position with a salary of $24,000 plus a $50 commission for each stove he sells. How many stoves would Mitchell need to sell for the options to be equal?

Solution:

Mitchell would need to sell 120 stoves.

Everyday Math

Exercise:

Problem:

When Gloria spent 15 minutes on the elliptical trainer and then did circuit training for 30 minutes, her fitness app said she burned 435 calories. When she spent 30 minutes on the elliptical trainer and 40 minutes circuit training she burned 690 calories. Solve the system $\begin{cases} 15e + 30c = 435 \\ 30e + 40c = 690 \end{cases}$ for e, the number of calories she burns for each minute on the elliptical trainer, and c, the number of calories she burns for each minute of circuit training.

Exercise:

Problem:

Stephanie left Riverside, California, driving her motorhome north on Interstate 15 towards Salt Lake City at a speed of 56 miles per hour. Half an hour later, Tina left Riverside in her car on the same route as Stephanie, driving 70 miles per hour. Solve the system $\begin{cases} 56s = 70t \\ s = t + \frac{1}{2} \end{cases}$.

ⓐ for t to find out how long it will take Tina to catch up to Stephanie.
ⓑ for s to find the number of hours Stephanie will have driven before Tina catches up to her.

Solution:

ⓐ $t = 2$ hours ⓑ $s = 2\frac{1}{2}$ hours

Writing Exercises

Exercise:

Problem: Solve the system of equations $\begin{cases} x + y = 10 \\ x - y = 6 \end{cases}$

ⓐ by graphing.
ⓑ by substitution.
ⓒ Which method do you prefer? Why?

Exercise:

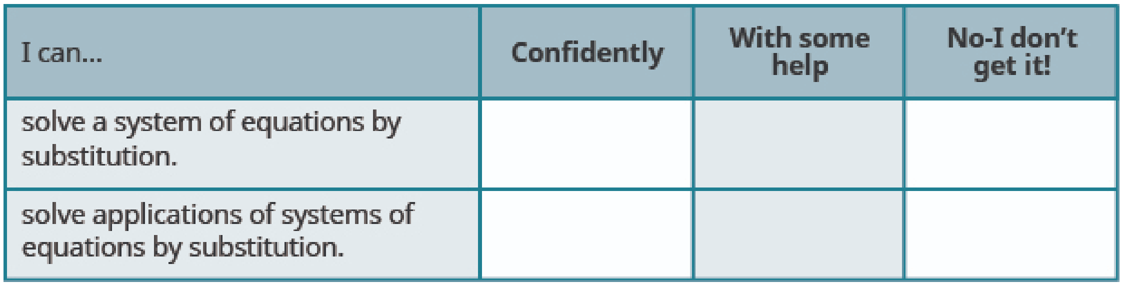
Problem: Solve the system of equations $\begin{cases} 3x + y = 12 \\ x = y - 8 \end{cases}$ by substitution and explain all your steps in words.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
solve a system of equations by substitution.			
solve applications of systems of equations by substitution.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Solve Systems of Equations by Elimination
By the end of this section, you will be able to:

- Solve a system of equations by elimination
- Solve applications of systems of equations by elimination
- Choose the most convenient method to solve a system of linear equations

Note:
Before you get started, take this readiness quiz.

1. Simplify $-5\left(6-3a\right)$.
   If you missed this problem, review [link].
2. Solve the equation $\frac{1}{3}x+\frac{5}{8}=\frac{31}{24}$.
   If you missed this problem, review [link].

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the Elimination Method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is what we'll do with the elimination method, too, but we'll have a different way to get there.

Solve a System of Equations by Elimination

The Elimination Method is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal.

For any expressions a, b, c, and d,
Equation:

$$\begin{array}{lrcl} \text{if} & a & = & b \\ \text{and} & c & = & d \\ \text{then} & a+c & = & b+d \end{array}$$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable.

Notice how that works when we add these two equations together:
Equation:

$$\begin{array}{l} 3x+y=5 \\ \underline{2x-y=0} \\ 5x \qquad\; =5 \end{array}$$

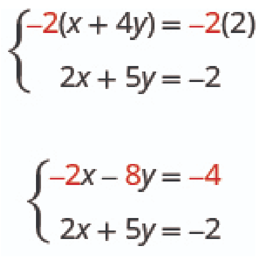
The y’s add to zero and we have one equation with one variable.

Let’s try another one:
Equation:

$$\begin{cases} x+4y=2 \\ 2x+5y=-2 \end{cases}$$

This time we don’t see a variable that can be immediately eliminated if we add the equations.

But if we multiply the first equation by -2, we will make the coefficients of x opposites. We must multiply every term on both sides of the equation by -2.



$$\begin{cases} -2(x+4y)=-2(2) \\ 2x+5y=-2 \end{cases}$$

$$\begin{cases} -2x-8y=-4 \\ 2x+5y=-2 \end{cases}$$

Now we see that the coefficients of the x terms are opposites, so x will be eliminated when we add these two equations.

Add the equations yourself—the result should be $-3y = -6$. And that looks easy to solve, doesn’t it? Here is what it would look like.

$$\begin{cases} -2x - 8y = -4 \\ \underline{\;\;2x + 5y = -2} \end{cases}$$
$$-3y = -6$$

We’ll do one more:
Equation:

$$\begin{cases} 4x - 3y = 10 \\ 3x + 5y = -7 \end{cases}$$

It doesn’t appear that we can get the coefficients of one variable to be opposites by multiplying one of the equations by a constant, unless we use fractions. So instead, we’ll have to multiply both equations by a constant.

We can make the coefficients of x be opposites if we multiply the first equation by 3 and the second by −4, so we get $12x$ and $-12x$.

$$3(4x - 3y) = 3(10)$$
$$-4(3x + 5y) = -4(-7)$$

This gives us these two new equations:
Equation:

$$\begin{cases} 12x - 9y = 30 \\ -12x - 20y = 28 \end{cases}$$

When we add these equations,
Equation:

$$\begin{cases} 12x - 9y = 30 \\ \underline{-12x - 20y = 28} \end{cases}$$
$$-29y = 58$$

the x’s are eliminated and we just have $-29y = 58$.

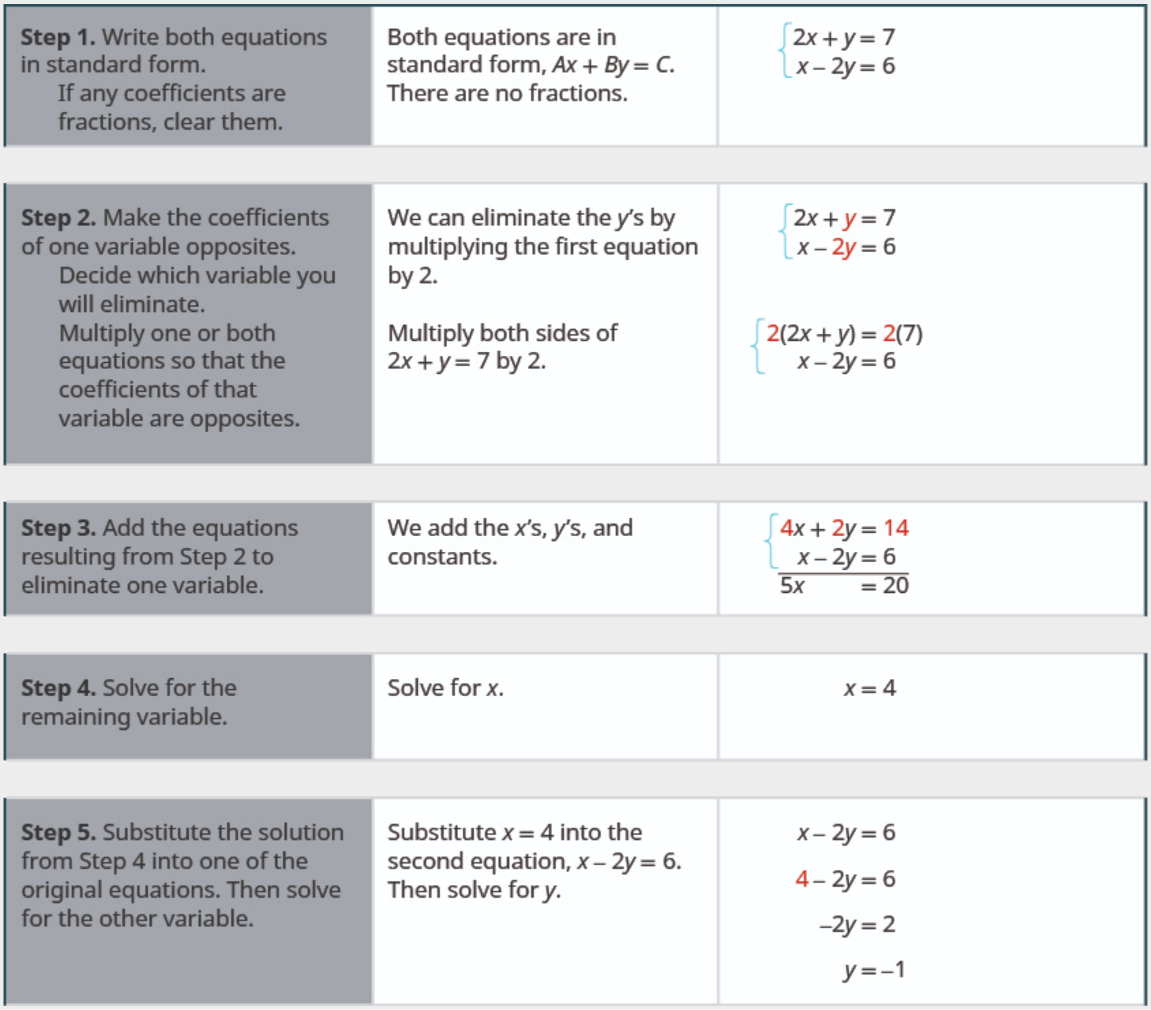
Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

Now we’ll see how to use elimination to solve the same system of equations we solved by graphing and by substitution.

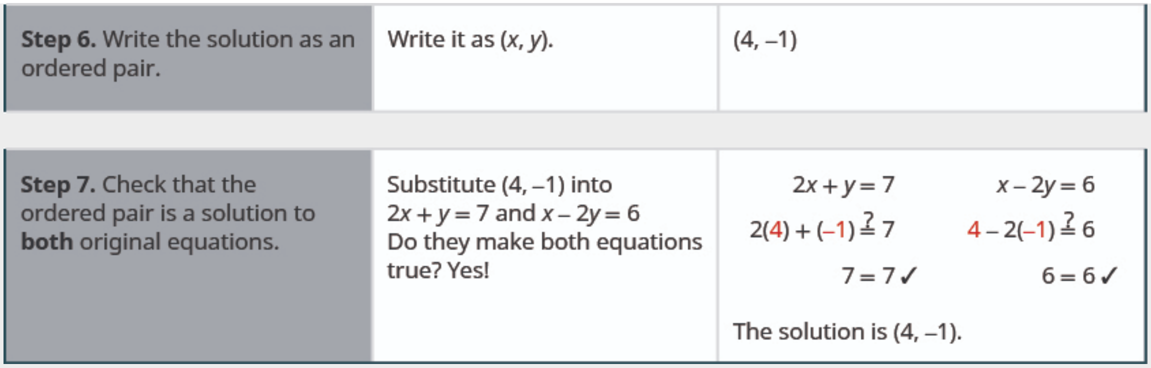
Example:
How to Solve a System of Equations by Elimination
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Solution:
Solution



Step 1. Write both equations in standard form. If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax + By = C$. There are no fractions.	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$
Step 2. Make the coefficients of one variable opposites. Decide which variable you will eliminate. Multiply one or both equations so that the coefficients of that variable are opposites.	We can eliminate the *y*’s by multiplying the first equation by 2. Multiply both sides of $2x + y = 7$ by 2.	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$ $\begin{cases} 2(2x+y)=2(7) \\ x-2y=6 \end{cases}$
Step 3. Add the equations resulting from Step 2 to eliminate one variable.	We add the *x*’s, *y*’s, and constants.	$\begin{cases} 4x+2y=14 \\ x-2y=6 \end{cases}$ $5x \quad = 20$
Step 4. Solve for the remaining variable.	Solve for *x*.	$x = 4$
Step 5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.	Substitute $x = 4$ into the second equation, $x - 2y = 6$. Then solve for *y*.	$x-2y=6$ $4-2y=6$ $-2y=2$ $y=-1$



Step 6. Write the solution as an ordered pair.	Write it as (x, y).	$(4, -1)$
Step 7. Check that the ordered pair is a solution to **both** original equations.	Substitute $(4, -1)$ into $2x + y = 7$ and $x - 2y = 6$ Do they make both equations true? Yes!	$2x + y = 7$ $2(4) + (-1) \overset{?}{=} 7$ $7 = 7$ ✓ $x - 2y = 6$ $4 - 2(-1) \overset{?}{=} 6$ $6 = 6$ ✓ The solution is $(4, -1)$.

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 3x + y = 5 \\ 2x - 3y = 7 \end{cases}$

Solution:

$(2, -1)$

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 4x + y = -5 \\ -2x - 2y = -2 \end{cases}$

Solution:

$(-2, 3)$

The steps are listed below for easy reference.

Note:
How to solve a system of equations by elimination.

Write both equations in standard form. If any coefficients are fractions, clear them.
Make the coefficients of one variable opposites.

- Decide which variable you will eliminate.
- Multiply one or both equations so that the coefficients of that variable are opposites.

Add the equations resulting from Step 2 to eliminate one variable.
Solve for the remaining variable.
Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
Write the solution as an ordered pair.
Check that the ordered pair is a solution to**both**original equations.

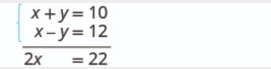
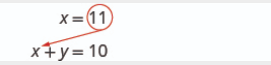
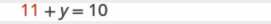
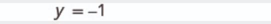
First we'll do an example where we can eliminate one variable right away.

Example:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} x + y = 10 \\ x - y = 12 \end{cases}$

Solution:
Solution

	$\begin{cases} x + y = 10 \\ x - y = 12 \end{cases}$
Both equations are in standard form.	

The coefficients of y are already opposites.	
Add the two equations to eliminate y. The resulting equation has only 1 variable, x.	 $\begin{cases} x+y=10 \\ x-y=12 \end{cases}$ $2x \quad = 22$
Solve for x, the remaining variable. Substitute $x = 11$ into one of the original equations.	 $x = \circled{11}$ $x+y=10$
	 $11 + y = 10$
Solve for the other variable, y.	 $y = -1$
Write the solution as an ordered pair.	The ordered pair is $(11, -1)$.
Check that the ordered pair is a solution to **both** original equations. $x + y = 10$ $\quad$ $x - y = 12$ $11 + (-1) \overset{?}{=} 10$ $\quad$ $11 - (-1) \overset{?}{=} 12$ $10 = 10 \checkmark$ $\quad$ $12 = 12 \checkmark$	
	The solution is $(11, -1)$.

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 2x + y = 5 \\ x - y = 4 \end{cases}$

Solution:

$(3, -1)$

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} x + y = 3 \\ -2x - y = -1 \end{cases}$

Solution:

$(-2, 5)$

In [link], we will be able to make the coefficients of one variable opposites by multiplying one equation by a constant.

Example:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 3x - 2y = -2 \\ 5x - 6y = 10 \end{cases}$

Solution:
Solution

	$\begin{cases} 3x - 2y = -2 \\ 5x - 6y = 10 \end{cases}$
Both equations are in standard form.	
None of the coefficients are opposites.	

We can make the coefficients of y opposites by multiplying the first equation by −3.	$\begin{cases} -3(3x-2y)=-3(-2) \\ 5x-6y=10 \end{cases}$
Simplify.	$\begin{cases} -9x+6y=6 \\ 5x-6y=10 \end{cases}$
Add the two equations to eliminate y.	$\begin{cases} -9x+6y=6 \\ 5x-6y=10 \end{cases}$ $-4x=16$
Solve for the remaining variable, x. Substitute $x=-4$ into one of the original equations.	$x=-4$ $3x-2y=-2$
	$3(-4)-2y=-2$
Solve for y.	$-12-2y=-2$ $-2y=10$ $y=-5$
Write the solution as an ordered pair.	The ordered pair is $(-4,-5)$.
Check that the ordered pair is a solution to both original equations. $\begin{aligned} 3x-2y &= -2 \\ 3(-4)-2(-5) &\overset{?}{=} -2 \\ -12+10 &\overset{?}{=} -2 \\ -2y &= -2 ✓ \end{aligned}$ $\begin{aligned} 5x-6y &= 10 \\ 3(-4)-6(-5) &\overset{?}{=} 10 \\ -20+30 &\overset{?}{=} 10 \\ 10 &= 10 ✓ \end{aligned}$	
	The solution

is
$(-4,-5)$
.

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases}4x-3y=1\\5x-9y=-4\end{cases}$

Solution:

$(1,1)$

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases}3x+2y=2\\6x+5y=8\end{cases}$

Solution:

$(-2,4)$

Now we’ll do an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

Example:
Exercise:

Problem: Solve the system by elimination. $\begin{cases}4x-3y=9\\7x+2y=-6\end{cases}$

Solution:
Solution

In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by a constant to get the opposites.

	$\begin{cases}4x-3y=9\\7x+2y=-6\end{cases}$
Both equations are in standard form. To get opposite coefficients of y, we will multiply the first equation by 2 and the second equation by 3.	$\begin{cases}2(4x-3y)=2(9)\\3(7x+2y)=3(-6)\end{cases}$
Simplify.	$\begin{cases}8x-6y=18\\21x+6y=-18\end{cases}$
Add the two equations to eliminate y.	$\begin{array}{l}\begin{cases}8x-6y=18\\21x+6y=-18\end{cases}\\\hline 39x=0\end{array}$
Solve for x. Substitute $x=0$ into one of the original equations.	$x=0$ $7x+2y=-6$
	$7\cdot 0+2y=-6$
Solve for y.	$2y=-6$
	$y=-3$

Write the solution as an ordered pair.	The ordered pair is $(0,-3)$.
Check that the ordered pair is a solution to **both** original equations. $\begin{array}{rcl} 4x-3y & = & 9 \\ 4(0)-3(-3) & \overset{?}{=} & 9 \\ 9 & = & 9\,\checkmark \end{array} \qquad \begin{array}{rcl} 7x+2y & = & -6 \\ 7(0)+2(-3) & \overset{?}{=} & -6 \\ -6 & = & -6\,\checkmark \end{array}$	
	The solution is $(0,-3)$.

What other constants could we have chosen to eliminate one of the variables? Would the solution be the same?

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 3x-4y=-9 \\ 5x+3y=14 \end{cases}$

Solution:

$(1,3)$

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 7x+8y=4 \\ 3x-5y=27 \end{cases}$

Solution:

$(4, -3)$

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by its LCD.

Example:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} x + \frac{1}{2}y = 6 \\ \frac{3}{2}x + \frac{2}{3}y = \frac{17}{2} \end{cases}$

Solution:
Solution

In this example, both equations have fractions. Our first step will be to multiply each equation by its LCD to clear the fractions.

	$\begin{cases} x + \frac{1}{2}y = 6 \\ \frac{3}{2}x + \frac{2}{3}y = \frac{17}{2} \end{cases}$
To clear the fractions, multiply each equation by its LCD.	$\begin{cases} 2\left(x + \frac{1}{2}y\right) = 2(6) \\ 6\left(\frac{3}{2}x + \frac{2}{3}y\right) = 6\left(\frac{17}{2}\right) \end{cases}$
Simplify.	$\begin{cases} 2x + y = 12 \\ 9x + 4y = 51 \end{cases}$
Now we are ready to eliminate one of the variables.	

Notice that both equations are in standard form.	
We can eliminate y multiplying the top equation by -4.	$\begin{cases} -4(2x+y) = -4(12) \\ 9x+4y=51 \end{cases}$
Simplify and add. Substitute $x=3$ into one of the original equations.	$\begin{array}{l} \begin{cases} -8x-4y=-48 \\ 9x+4y=51 \end{cases} \\ \hline x \quad = \textcircled{3} \\ x+\frac{1}{2}y=6 \end{array}$
Solve for y.	$3+\frac{1}{2}y=6$
	$\frac{1}{2}y=3$
	$y=6$
Write the solution as an ordered pair.	The ordered pair is $(3,6)$.
Check that the ordered pair is a solution to **both** original equations. $\begin{aligned} x+\frac{1}{2}y &= 6 \\ 3+\frac{1}{2}(6) &\stackrel{?}{=} 6 \\ 3+6 &\stackrel{?}{=} 6 \\ 6 &= 6\,\checkmark \end{aligned}$ $\begin{aligned} \frac{3}{2}x+\frac{2}{3}y &= \frac{17}{2} \\ \frac{3}{2}(3)+\frac{2}{3}(6) &\stackrel{?}{=} \frac{17}{2} \\ \frac{9}{2}+4 &\stackrel{?}{=} \frac{17}{2} \\ \frac{9}{2}+\frac{8}{2} &\stackrel{?}{=} \frac{17}{2} \\ \frac{17}{2} &= \frac{17}{2}\,\checkmark \end{aligned}$	
	The solution is $(3,6)$.

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} \frac{1}{3}x - \frac{1}{2}y = 1 \\ \frac{3}{4}x - y = \frac{5}{2} \end{cases}$

Solution:

$(6, 2)$

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} x + \frac{3}{5}y = -\frac{1}{5} \\ -\frac{1}{2}x - \frac{2}{3}y = \frac{5}{6} \end{cases}$

Solution:

$(1, -2)$

In the Solving Systems of Equations by Graphing we saw that not all systems of linear equations have a single ordered pair as a solution. When the two equations were really the same line, there were infinitely many solutions. We called that a consistent system. When the two equations described parallel lines, there was no solution. We called that an inconsistent system.

Example:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 3x + 4y = 12 \\ y = 3 - \frac{3}{4}x \end{cases}$

Solution:
Solution

$$\begin{cases} 3x+4y &= 12 \\ y &= 3-\frac{3}{4}x \end{cases}$$

Write the second equation in standard form.

$$\begin{cases} 3x+4y &= 12 \\ \frac{3}{4}x+y &= 3 \end{cases}$$

Clear the fractions by multiplying the second equation by 4.

$$\begin{cases} 3x+4y &= 12 \\ 4\left(\frac{3}{4}x+y\right) &= 4(3) \end{cases}$$

Simplify.

$$\begin{cases} 3x+4y &= 12 \\ 3x+4y &= 12 \end{cases}$$

To eliminate a variable, we multiply the second equation by -1.

$$\begin{cases} 3x+4y &= 12 \\ -3x-4y &= -12 \end{cases}$$
$$\overline{\qquad 0=0 \qquad}$$

Simplify and add.

This is a true statement. The equations are consistent but dependent. Their graphs would be the same line. The system has infinitely many solutions.

After we cleared the fractions in the second equation, did you notice that the two equations were the same? That means we have coincident lines.

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 5x-3y=15 \\ y=-5+\frac{5}{3}x \end{cases}$

Solution:

infinitely many solutions

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} x + 2y = 6 \\ y = -\frac{1}{2}x + 3 \end{cases}$

Solution:

infinitely many solutions

Example:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} -6x + 15y = 10 \\ 2x - 5y = -5 \end{cases}$

Solution:
Solution

The equations are in standard form.

$$\begin{cases} -6x + 15y & = & 10 \\ 2x - 5y & = & -5 \end{cases}$$

Multiply the second equation by 3 to eliminate a variable.

$$\begin{cases} -6x + 15y & = & 10 \\ 3\left(2x - 5y\right) & = & 3\left(-5\right) \end{cases}$$

Simplify and add.

$$\begin{array}{l} \begin{cases} -6x + 15y & = & 10 \\ 6x - 15y & = & -15 \end{cases} \\ \hline \qquad 0 \neq -5 \end{array}$$

This statement is false. The equations are inconsistent and so their graphs would be parallel lines.

The system does not have a solution.

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} -3x + 2y = 8 \\ 9x - 6y = 13 \end{cases}$

Solution:

no solution

Note:
Exercise:

Problem: Solve the system by elimination. $\begin{cases} 7x - 3y = -2 \\ -14x + 6y = 8 \end{cases}$

Solution:

no solution

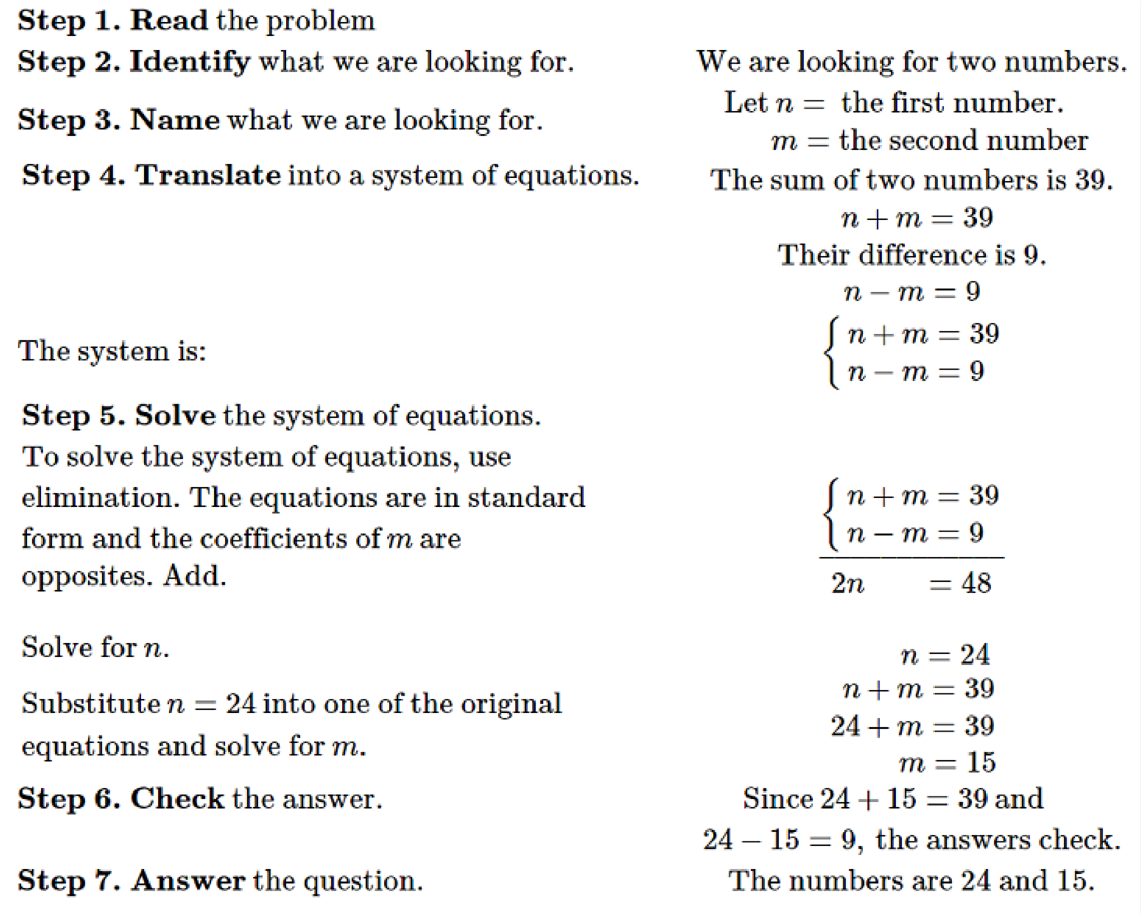
Solve Applications of Systems of Equations by Elimination

Some applications problems translate directly into equations in standard form, so we will use the elimination method to solve them. As before, we use our Problem Solving Strategy to help us stay focused and organized.

Example:
Exercise:

Problem: The sum of two numbers is 39. Their difference is 9. Find the numbers.

Solution:
Solution



Step 1. Read the problem	
Step 2. Identify what we are looking for.	We are looking for two numbers.
Step 3. Name what we are looking for.	Let $n =$ the first number. $m =$ the second number
Step 4. Translate into a system of equations.	The sum of two numbers is 39. $n+m=39$ Their difference is 9. $n-m=9$
The system is:	$\begin{cases} n+m=39 \\ n-m=9 \end{cases}$
Step 5. Solve the system of equations. To solve the system of equations, use elimination. The equations are in standard form and the coefficients of m are opposites. Add.	$\begin{array}{l} \begin{cases} n+m=39 \\ n-m=9 \end{cases} \\ \hline 2n \qquad =48 \end{array}$
Solve for n.	$n=24$
Substitute $n=24$ into one of the original equations and solve for m.	$n+m=39$ $24+m=39$ $m=15$
Step 6. Check the answer.	Since $24+15=39$ and $24-15=9$, the answers check.
Step 7. Answer the question.	The numbers are 24 and 15.

Note:
Exercise:

Problem: The sum of two numbers is 42. Their difference is 8. Find the numbers.

Solution:

The numbers are 25 and 17.

Note:
Exercise:

Problem:

The sum of two numbers is −15. Their difference is −35. Find the numbers.

Solution:

The numbers are −25 and 10.

Example:
Exercise:

Problem:

Joe stops at a burger restaurant every day on his way to work. Monday he had one order of medium fries and two small sodas, which had a total of 620 calories. Tuesday he had two orders of medium fries and one small soda, for a total of 820 calories. How many calories are there in one order of medium fries? How many calories in one small soda?

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of calories in one order of medium fries and in one small soda.
Step 3. Name what we are looking for.	Let f = the number of calories in 1 order of medium fries. s = the number of calories in 1 small soda.
Step 4. Translate into a system of equations:	one medium fries and two small sodas had a

	total of 620 calories
	$f+2s=620$
	two medium fries and one small soda had a total of 820 calories.
	$2f+s=820$
Our system is:	$\begin{cases} f+2s=620 \\ 2f+s=820 \end{cases}$
Step 5. Solve the system of equations. To solve the system of equations, use elimination. The equations are in standard form. To get opposite coefficients of f, multiply the top equation by -2.	$\begin{cases} -2(f+2s)=-2(620) \\ 2f+s=820 \end{cases}$
Simplify and add.	$\begin{array}{r} \begin{cases} -2f-4s=-1240 \\ 2f+s=820 \end{cases} \\ \hline -3s=-420 \end{array}$
Solve for s.	$s=140$
Substitute $s=140$ into one of the original equations and then solve for f.	$f+2s=620$
	$f+2\cdot 140=620$

	$f + 280 = 620$
	$f = 340$
Step 6. Check the answer.	Verify that these numbers make sense in the problem and that they are solutions to both equations. We leave this to you!
Step 7. Answer the question.	The small soda has 140 calories and the fries have 340 calories.

Note:
Exercise:

Problem:

Malik stops at the grocery store to buy a bag of diapers and 2 cans of formula. He spends a total of $37. The next week he stops and buys 2 bags of diapers and 5 cans of formula for a total of $87. How much does a bag of diapers cost? How much is one can of formula?

Solution:

The bag of diapers costs $11 and the can of formula costs $13.

Note:
Exercise:

Problem:

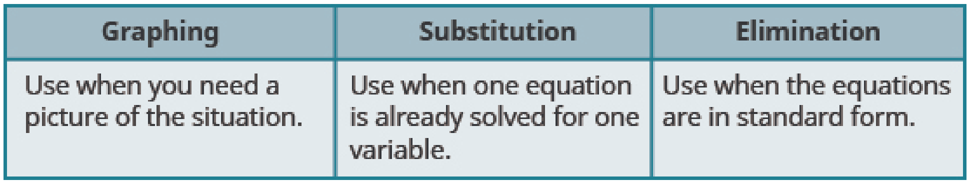
To get her daily intake of fruit for the day, Sasha eats a banana and 8 strawberries on Wednesday for a calorie count of 145. On the following Wednesday, she eats two bananas and 5 strawberries for a total of 235 calories for the fruit. How many calories are there in a banana? How many calories are in a strawberry?

Solution:

There are 105 calories in a banana and 5 calories in a strawberry.

Choose the Most Convenient Method to Solve a System of Linear Equations

When you will have to solve a system of linear equations in a later math class, you will usually not be told which method to use. You will need to make that decision yourself. So you'll want to choose the method that is easiest to do and minimizes your chance of making mistakes.



Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved for one variable.	Use when the equations are in standard form.

Example:
Exercise:

Problem:

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} 3x + 8y = 40 \\ 7x - 4y = -32 \end{cases}$ ⓑ $\begin{cases} 5x + 6y = 12 \\ y = \frac{2}{3}x - 1 \end{cases}$

Solution:
Solution

ⓐ $$\begin{cases} 3x + 8y = 40 \\ 7x - 4y = -32 \end{cases}$$

Since both equations are in standard form, using elimination will be most convenient.

ⓑ $$\begin{cases} 5x + 6y = 12 \\ y = \frac{2}{3}x - 1 \end{cases}$$

Since one equation is already solved for y, using substitution will be most convenient.

Note:
Exercise:

Problem:

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} 4x - 5y = -32 \\ 3x + 2y = -1 \end{cases}$ ⓑ $\begin{cases} x = 2y - 1 \\ 3x - 5y = -7 \end{cases}$

Solution:

ⓐ Since both equations are in standard form, using elimination will be most convenient. ⓑ Since one equation is already solved for x, using substitution will be most convenient.

Note:
Exercise:

Problem:

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

ⓐ $\begin{cases} y = 2x - 1 \\ 3x - 4y = -6 \end{cases}$ ⓑ $\begin{cases} 6x - 2y = 12 \\ 3x + 7y = -13 \end{cases}$

Solution:

ⓐ Since one equation is already solved for y, using substitution will be most convenient; ⓑ Since both equations are in standard form, using elimination will be most convenient.

Note:
Access these online resources for additional instruction and practice with solving systems of linear equations by elimination.

- Instructional Video-Solving Systems of Equations by Elimination
- Instructional Video-Solving by Elimination
- Instructional Video-Solving Systems by Elimination

Key Concepts

- **To Solve a System of Equations by Elimination**

  Write both equations in standard form. If any coefficients are fractions, clear them.
  Make the coefficients of one variable opposites.
    - Decide which variable you will eliminate.
    - Multiply one or both equations so that the coefficients of that variable are opposites.

  Add the equations resulting from Step 2 to eliminate one variable.
  Solve for the remaining variable.
  Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
  Write the solution as an ordered pair.
  Check that the ordered pair is a solution to**both**original equations.

Practice Makes Perfect

Solve a System of Equations by Elimination

In the following exercises, solve the systems of equations by elimination.
Exercise:

Problem: $\begin{cases} 5x + 2y = 2 \\ -3x - y = 0 \end{cases}$

Exercise:

Problem: $\begin{cases} -3x + y = -9 \\ x - 2y = -12 \end{cases}$

Solution:

$(6, 9)$

Exercise:

Problem: $\begin{cases} 6x - 5y = -1 \\ 2x + y = 13 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x - y = -7 \\ 4x + 2y = -6 \end{cases}$

Solution:

$(-2, 1)$

Exercise:

Problem: $\begin{cases} x + y = -1 \\ x - y = -5 \end{cases}$

Exercise:

Problem: $\begin{cases} x + y = -8 \\ x - y = -6 \end{cases}$

Solution:

$(-7, -1)$

Exercise:

Problem: $\begin{cases} 3x - 2y = 1 \\ -x + 2y = 9 \end{cases}$

Exercise:

Problem: $\begin{cases} -7x + 6y = -10 \\ x - 6y = 22 \end{cases}$

Solution:

$(-2, -4)$

Exercise:

Problem: $\begin{cases} 3x + 2y = -3 \\ -x - 2y = -19 \end{cases}$

Exercise:

Problem: $\begin{cases} 5x + 2y = 1 \\ -5x - 4y = -7 \end{cases}$

Solution:

$(-1, 3)$

Exercise:

Problem: $\begin{cases} 6x + 4y = -4 \\ -6x - 5y = 8 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x - 4y = -11 \\ x - 2y = -5 \end{cases}$

Solution:

$(-1, 2)$

Exercise:

Problem: $\begin{cases} 5x - 7y = 29 \\ x + 3y = -3 \end{cases}$

Exercise:

Problem: $\begin{cases} 6x - 5y = -75 \\ -x - 2y = -13 \end{cases}$

Solution:

$(-5, 9)$

Exercise:

Problem: $\begin{cases} -x + 4y = 8 \\ 3x + 5y = 10 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x - 5y = 7 \\ 3x - y = 17 \end{cases}$

Solution:

$(6, 1)$

Exercise:

Problem: $\begin{cases} 5x - 3y = -1 \\ 2x - y = 2 \end{cases}$

Exercise:

Problem: $\begin{cases} 7x + y = -4 \\ 13x + 3y = 4 \end{cases}$

Solution:

$(-2, 10)$

Exercise:

Problem: $\begin{cases} -3x + 5y = -13 \\ 2x + y = -26 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x - 5y = -9 \\ 5x + 2y = 16 \end{cases}$

Solution:

$(2, 3)$

Exercise:

Problem: $\begin{cases} 4x - 3y = 3 \\ 2x + 5y = -31 \end{cases}$

Exercise:

Problem: $\begin{cases} 4x + 7y = 14 \\ -2x + 3y = 32 \end{cases}$

Solution:

$(-7, 6)$

Exercise:

Problem: $\begin{cases} 5x + 2y = 21 \\ 7x - 4y = 9 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x + 8y = -3 \\ 2x + 5y = -3 \end{cases}$

Solution:

$(-9, 3)$

Exercise:

Problem: $\begin{cases} 11x + 9y = -5 \\ 7x + 5y = -1 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x + 8y = 67 \\ 5x + 3y = 60 \end{cases}$

Solution:

$(9, 5)$

Exercise:

Problem: $\begin{cases} 2x + 9y = -4 \\ 3x + 13y = -7 \end{cases}$

Exercise:

Problem: $\begin{cases} \frac{1}{3}x - y = -3 \\ x + \frac{5}{2}y = 2 \end{cases}$

Solution:

$(-3, 2)$

Exercise:

Problem: $\begin{cases} x + \frac{1}{2}y = \frac{3}{2} \\ \frac{1}{5}x - \frac{1}{5}y = 3 \end{cases}$

Exercise:

Problem: $\begin{cases} x + \frac{1}{3}y = -1 \\ \frac{1}{2}x - \frac{1}{3}y = -2 \end{cases}$

Solution:

$(-2, 3)$

Exercise:

Problem: $\begin{cases} \frac{1}{3}x - y = -3 \\ \frac{2}{3}x + \frac{5}{2}y = 3 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x + y = 3 \\ 6x + 3y = 9 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} x - 4y = -1 \\ -3x + 12y = 3 \end{cases}$

Exercise:

Problem: $\begin{cases} -3x - y = 8 \\ 6x + 2y = -16 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} 4x + 3y = 2 \\ 20x + 15y = 10 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x + 2y = 6 \\ -6x - 4y = -12 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} 5x - 8y = 12 \\ 10x - 16y = 20 \end{cases}$

Exercise:

Problem: $\begin{cases} -11x + 12y = 60 \\ -22x + 24y = 90 \end{cases}$

Solution:

inconsistent, no solution

Exercise:

Problem: $\begin{cases} 7x - 9y = 16 \\ -21x + 27y = -24 \end{cases}$

Exercise:

Problem: $\begin{cases} 5x - 3y = 15 \\ y = \frac{5}{3}x - 2 \end{cases}$

Solution:

inconsistent, no solution

Exercise:

Problem: $\begin{cases} 2x + 4y = 7 \\ y = -\frac{1}{2}x - 4 \end{cases}$

Solve Applications of Systems of Equations by Elimination

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

The sum of two numbers is 65. Their difference is 25. Find the numbers.

Solution:

The numbers are 20 and 45.

Exercise:

Problem: The sum of two numbers is 37. Their difference is 9. Find the numbers.

Exercise:

Problem:

The sum of two numbers is −27. Their difference is −59. Find the numbers.

Solution:

The numbers are 16 and −43.

Exercise:

Problem:

The sum of two numbers is −45. Their difference is −89. Find the numbers.

Exercise:

Problem:

Andrea is buying some new shirts and sweaters. She is able to buy 3 shirts and 2 sweaters for $114 or she is able to buy 2 shirts and 4 sweaters for $164. How much does a shirt cost? How much does a sweater cost?

Solution:

A shirt costs $16 and a sweater costs $33.

Exercise:

Problem:

Peter is buying office supplies. He is able to buy 3 packages of paper and 4 staplers for $40 or he is able to buy 5 packages of paper and 6 staplers for $62. How much does a package of paper cost? How much does a stapler cost?

Exercise:

Problem:

The total amount of sodium in 2 hot dogs and 3 cups of cottage cheese is 4720 mg. The total amount of sodium in 5 hot dogs and 2 cups of cottage cheese is 6300 mg. How much sodium is in a hot dog? How much sodium is in a cup of cottage cheese?

Solution:

There are 860 mg in a hot dog. There are 1,000 mg in a cup of cottage cheese.

Exercise:

Problem:

The total number of calories in 2 hot dogs and 3 cups of cottage cheese is 960 calories. The total number of calories in 5 hot dogs and 2 cups of cottage cheese is 1190 calories. How many calories are in a hot dog? How many calories are in a cup of cottage cheese?

Choose the Most Convenient Method to Solve a System of Linear Equations

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

Exercise:

ⓐ $\begin{cases} 8x - 15y = -32 \\ 6x + 3y = -5 \end{cases}$

Problem: ⓑ $\begin{cases} x = 4y - 3 \\ 4x - 2y = -6 \end{cases}$

Solution:

ⓐ elimination ⓑ substitution

Exercise:

ⓐ $\begin{cases} y = 7x - 5 \\ 3x - 2y = 16 \end{cases}$

Problem: ⓑ $\begin{cases} 12x - 5y = -42 \\ 3x + 7y = -15 \end{cases}$

Exercise:

ⓐ $\begin{cases} y = 4x + 9 \\ 5x - 2y = -21 \end{cases}$

Problem: ⓑ $\begin{cases} 9x - 4y = 24 \\ 3x + 5y = -14 \end{cases}$

Solution:

ⓐ substitution ⓑ elimination

Exercise:

ⓐ $\begin{cases} 14x - 15y = -30 \\ 7x + 2y = 10 \end{cases}$

Problem: ⓑ $\begin{cases} x = 9y - 11 \\ 2x - 7y = -27 \end{cases}$

Everyday Math

Exercise:

Problem:

Norris can row 3 miles upstream against the current in the same amount of time it takes him to row 5 miles downstream, with the current. Solve the system.

$\begin{cases} r - c = 3 \\ r + c = 5 \end{cases}$

ⓐ for r, his rowing speed in still water.
ⓑ Then solve for c, the speed of the river current.

Solution:

ⓐ $r = 4$ ⓑ $c = 1$

Exercise:

Problem:

Josie wants to make 10 pounds of trail mix using nuts and raisins, and she wants the total cost of the trail mix to be $54. Nuts cost $6 per pound and raisins cost $3 per pound. Solve the system $\begin{cases} n + r = 10 \\ 6n + 3r = 54 \end{cases}$ to find n, the number of pounds of nuts, and r, the number of pounds of raisins she should use.

Writing Exercises

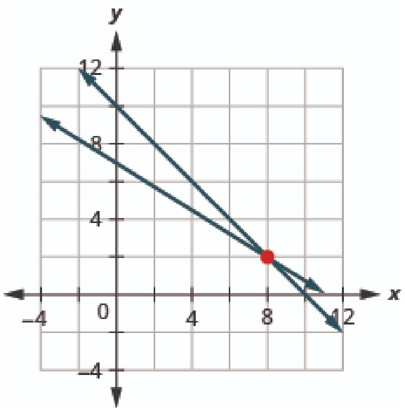
Exercise:

Problem: Solve the system $\begin{cases} x + y = 10 \\ 5x + 8y = 56 \end{cases}$

ⓐ by substitution
ⓑ by graphing
ⓒ Which method do you prefer? Why?

Solution:

ⓐ (8, 2)
ⓑ



ⓒ Answers will vary.

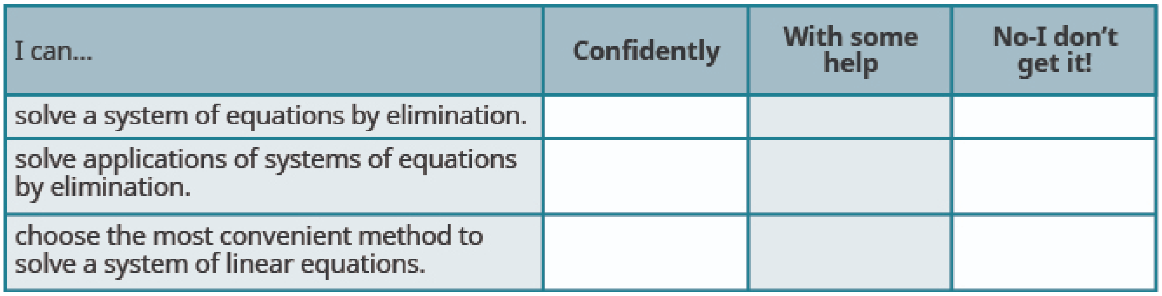
Exercise:

Problem: Solve the system
$$\begin{cases} x + y = -12 \\ y = 4 - \frac{1}{2}x \end{cases}$$

ⓐ by substitution
ⓑ by graphing
ⓒ Which method do you prefer? Why?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
solve a system of equations by elimination.			
solve applications of systems of equations by elimination.			
choose the most convenient method to solve a system of linear equations.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Solve Applications with Systems of Equations

By the end of this section, you will be able to:

- Translate to a system of equations
- Solve direct translation applications
- Solve geometry applications
- Solve uniform motion applications

Note:

Before you get started, take this readiness quiz.

1. The sum of twice a number and nine is 31. Find the number.
   If you missed this problem, review [link].
2. Twins Jon and Ron together earned \$96,000 last year. Ron earned \$8,000 more than three times what Jon earned. How much did each of the twins earn?
   If you missed this problem, review [link].
3. Alessio rides his bike $3\frac{1}{2}$ hours at a rate of 10 miles per hour. How far did he ride?
   If you missed this problem, review [link].

Previously in this chapter we solved several applications with systems of linear equations. In this section, we'll look at some specific types of applications that relate two quantities. We'll translate the words into linear equations, decide which is the most convenient method to use, and then solve them.

We will use our Problem Solving Strategy for Systems of Linear Equations.

Note:

Use a problem solving strategy for systems of linear equations.

Readthe problem. Make sure all the words and ideas are understood.
Identifywhat we are looking for.
Namewhat we are looking for. Choose variables to represent those quantities.
Translateinto a system of equations.
Solvethe system of equations using good algebra techniques.
Checkthe answer in the problem and make sure it makes sense.
Answerthe question with a complete sentence.

Translate to a System of Equations

Many of the problems we solved in earlier applications related two quantities. Here are two of the examples from the chapter on Math Models.

- The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.
- A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

In that chapter we translated each situation into one equation using only one variable. Sometimes it was a bit of a challenge figuring out how to name the two quantities, wasn't it?

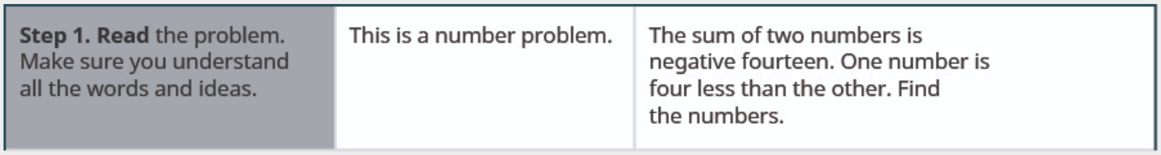
Let's see how we can translate these two problems into a system of equations with two variables. We'll focus on Steps 1 through 4 of our Problem Solving Strategy.

Example:
How to Translate to a System of Equations
Exercise:

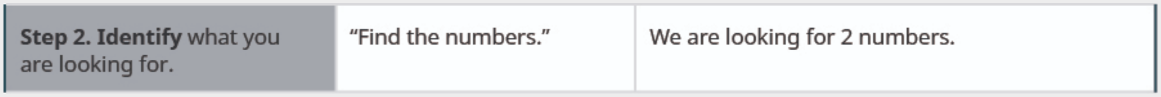
Problem: Translate to a system of equations:

The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.

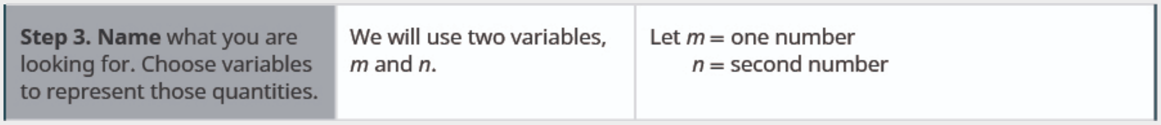
Solution:
Solution



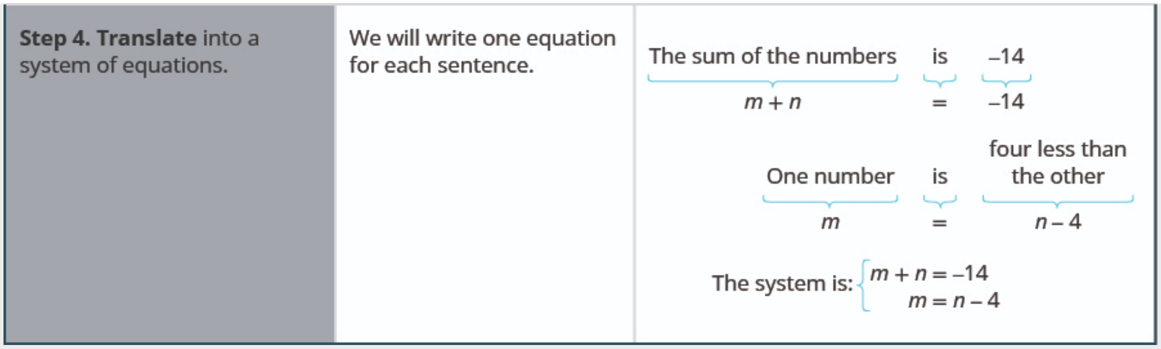
Step 1. Read the problem. Make sure you understand all the words and ideas.	This is a number problem.	The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.



Step 2. Identify what you are looking for.	"Find the numbers."	We are looking for 2 numbers.



Step 3. Name what you are looking for. Choose variables to represent those quantities.	We will use two variables, m and n.	Let m = one number n = second number



Step 4. Translate into a system of equations.	We will write one equation for each sentence.	$\underbrace{\text{The sum of the numbers}}_{m+n}\ \underbrace{\text{is}}_{=}\ \underbrace{-14}_{-14}$ $\underbrace{\text{One number}}_{m}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{four less than the other}}_{n-4}$ The system is: $\begin{cases} m+n=-14 \\ m=n-4 \end{cases}$

Note:
Exercise:

Problem: Translate to a system of equations:

The sum of two numbers is negative twenty-three. One number is 7 less than the other. Find the numbers.

Solution:

$$\begin{cases} m+n=-23 \\ m=n-7 \end{cases}$$

Note:
Exercise:

Problem: Translate to a system of equations:

The sum of two numbers is negative eighteen. One number is 40 more than the other. Find the numbers.

Solution:

$$\begin{cases} m+n=-18 \\ m=n+40 \end{cases}$$

We’ll do another example where we stop after we write the system of equations.

Example:
Exercise:

Problem: Translate to a system of equations:

A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

Solution:
Solution

We are looking for the amount that the husband and wife each earn.

Let h = the amount the husband earns.
w = the amount the wife earns

Translate.

A married couple together earns $110,000.

$$w + h = 110{,}000$$

The wife earns $16,000 less than twice what husband earns.

$$w = 2h - 16{,}000$$

The system of equations is:

$$\begin{cases} w + h = 110{,}000 \\ w = 2h - 16{,}000 \end{cases}$$

Note:
Exercise:

Problem: Translate to a system of equations:

A couple has a total household income of $84,000. The husband earns $18,000 less than twice what the wife earns. How much does the wife earn?

Solution:

$$\begin{cases} w + h = 84{,}000 \\ h = 2w - 18{,}000 \end{cases}$$

Note:
Exercise:

Problem: Translate to a system of equations:

A senior employee makes $5 less than twice what a new employee makes per hour. Together they make $43 per hour. How much does each employee make per hour?

Solution:

$$\begin{cases} s = 2n - 5 \\ s + n = 43 \end{cases}$$

Solve Direct Translation Applications

We set up, but did not solve, the systems of equations in [link] and [link] Now we'll translate a situation to a system of equations and then solve it.

Example:
Exercise:

Problem: Translate to a system of equations and then solve:

Devon is 26 years older than his son Cooper. The sum of their ages is 50. Find their ages.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the ages of Devon and Cooper.
Step 3. Name what we are looking for.	Let $d =$ Devon's age. $c =$ Cooper's age
Step 4. Translate into a system of equations.	Devon is 26 years older than Cooper.
	$d = c + 26$
	The sum of their ages is 50.

	$d + c = 50$
The system is:	$\begin{cases} d = c + 26 \\ d + c = 50 \end{cases}$
Step 5. Solve the system of equations. Solve by substitution.	$\begin{cases} d = \boxed{c + 26} \\ d + c = 50 \end{cases}$ $d + c = 50$
Substitute *c* + 26 into the second equation.	$c + 26 + c = 50$
Solve for *c*.	$2c + 26 = 50$
	$2c = 24$
	$c = \boxed{12}$ $d = c + 26$
Substitute *c* = 12 into the first equation and then solve for *d*.	$d = 12 + 26$
	$d = 38$
Step 6. Check the answer in the problem.	Is Devon’s age 26 more than Cooper’s? Yes, 38 is 26 more than 12. Is the sum of their ages 50? Yes, 38 plus 12 is 50.
Step 7. Answer the question.	Devon is 38 and Cooper is 12 years old.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Ali is 12 years older than his youngest sister, Jameela. The sum of their ages is 40. Find their ages.

Solution:

Ali is 26 and Jameela is 14.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Jake’s dad is 6 more than 3 times Jake’s age. The sum of their ages is 42. Find their ages.

Solution:

Jake is 9 and his dad is 33.

Example:
Exercise:

Problem: Translate to a system of equations and then solve:

When Jenna spent 10 minutes on the elliptical trainer and then did circuit training for 20 minutes, her fitness app said she burned 278 calories. When she spent 20 minutes on the elliptical trainer and 30 minutes circuit training she burned 473 calories. How many calories does she burn for each minute on the elliptical trainer? How many calories does she burn for each minute of circuit training?

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of calories burned each minute on the elliptical trainer and each minute of circuit training.
Step 3. Name what we are looking for.	Let $e =$ number of calories burned per minute on the elliptical trainer. $c =$ number of calories burned per minute while circuit training
Step 4. Translate into a system of equations.	10 minutes on the elliptical and circuit training for 20 minutes, burned 278 calories
	$10e + 20c = 278$
	20 minutes on the elliptical and 30 minutes of circuit training burned 473 calories
	$20e + 30c = 473$
The system is:	$\begin{cases} 10e + 20c = 278 \\ 20e + 30c = 473 \end{cases}$
Step 5. Solve the system of equations.	
Multiply the first equation by −2 to get opposite coefficients of *e*.	$\begin{cases} -2(10e + 20c) = -2(278) \\ 20e + 30c = 473 \end{cases}$
Simplify and add the equations. Solve for *c*.	$\begin{cases} -20e - 40c = -556 \\ 20e + 30c = 473 \end{cases}$ $-10c = -83$ $c = 8.3$
Substitute *c* = 8.3 into one of the original equations to solve for *e*.	$10e + 20c = 278$

	$10e + 20(8.3) = 278$
	$10e + 166 = 278$
	$10e = 112$
	$e = 11.2$
Step 6. Check the answer in the problem.	Check the math on your own.
$\begin{cases} 10(11.2) + 20(8.3) \overset{?}{=} 278 \\ 20(11.2) + 30(8.3) \overset{?}{=} 473 \end{cases}$	
Step 7. Answer the question.	Jenna burns 8.3 calories per minute circuit training and 11.2 calories per minute while on the elliptical trainer.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Mark went to the gym and did 40 minutes of Bikram hot yoga and 10 minutes of jumping jacks. He burned 510 calories. The next time he went to the gym, he did 30 minutes of Bikram hot yoga and 20 minutes of jumping jacks burning 470 calories. How many calories were burned for each minute of yoga? How many calories were burned for each minute of jumping jacks?

Solution:

Mark burned 11 calories for each minute of yoga and 7 calories for each minute of jumping jacks.

Note:

Exercise:

Problem: Translate to a system of equations and then solve:

Erin spent 30 minutes on the rowing machine and 20 minutes lifting weights at the gym and burned 430 calories. During her next visit to the gym she spent 50 minutes on the rowing machine and 10 minutes lifting weights and burned 600 calories. How many calories did she burn for each minutes on the rowing machine? How many calories did she burn for each minute of weight lifting?

Solution:

Erin burned 11 calories for each minute on the rowing machine and 5 calories for each minute of weight lifting.

Solve Geometry Applications

When we learned about Math Models, we solved geometry applications using properties of triangles and rectangles. Now we'll add to our list some properties of angles.

The measures of two complementary angles add to 90 degrees. The measures of two supplementary angles add to 180 degrees.

Note:
Complementary and Supplementary Angles
Two angles are **complementary** if the sum of the measures of their angles is 90 degrees.
Two angles are **supplementary** if the sum of the measures of their angles is 180 degrees.

If two angles are complementary, we say that *one angle is the complement of the other.*

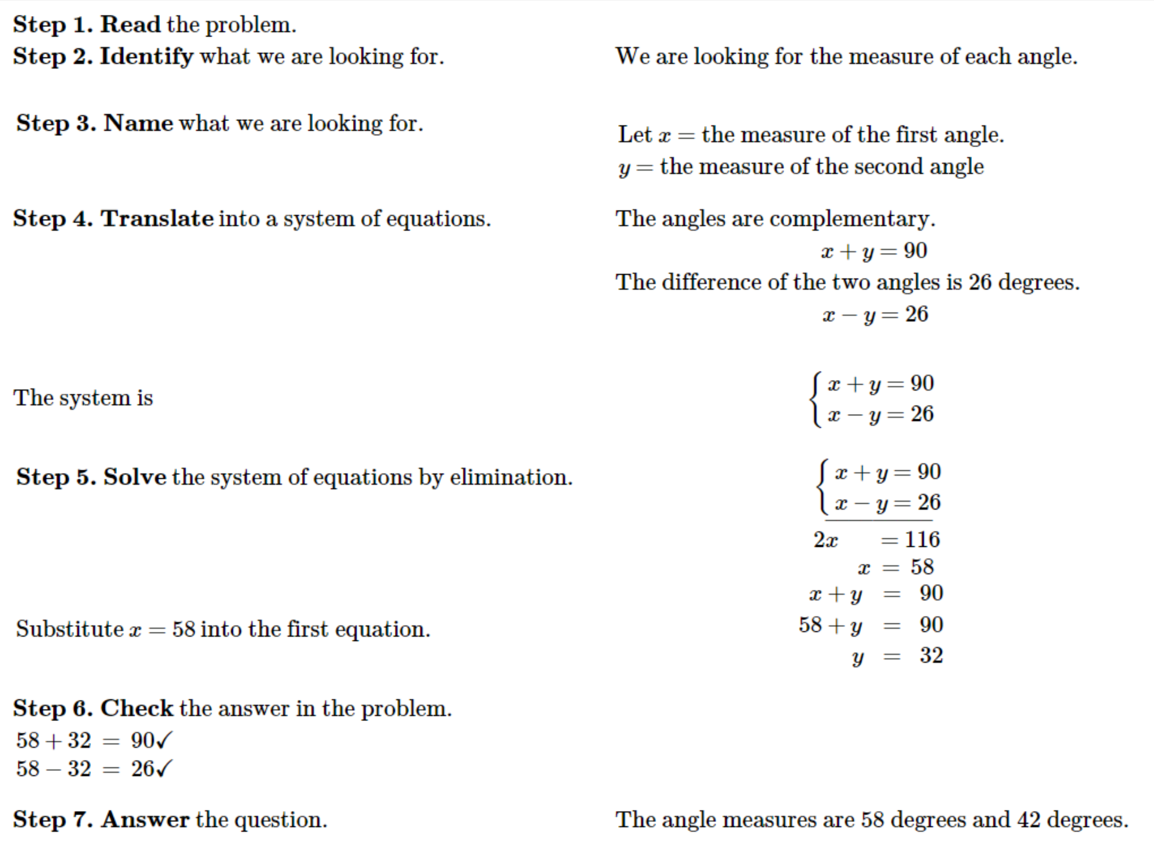
If two angles are supplementary, we say that *one angle is the supplement of the other.*

Example:
Exercise:

Problem: Translate to a system of equations and then solve:

The difference of two complementary angles is 26 degrees. Find the measures of the angles.

Solution:
Solution



Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the measure of each angle.
Step 3. Name what we are looking for.	Let $x =$ the measure of the first angle. $y =$ the measure of the second angle
Step 4. Translate into a system of equations.	The angles are complementary. $x + y = 90$ The difference of the two angles is 26 degrees. $x - y = 26$
The system is	$\begin{cases} x + y = 90 \\ x - y = 26 \end{cases}$
Step 5. Solve the system of equations by elimination.	$\begin{cases} x + y = 90 \\ x - y = 26 \end{cases}$ $2x = 116$ $x = 58$
Substitute $x = 58$ into the first equation.	$x + y = 90$ $58 + y = 90$ $y = 32$
Step 6. Check the answer in the problem. $58 + 32 = 90✓$ $58 - 32 = 26✓$	
Step 7. Answer the question.	The angle measures are 58 degrees and 42 degrees.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

The difference of two complementary angles is 20 degrees. Find the measures of the angles.

Solution:

The angle measures are 55 degrees and 35 degrees.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

The difference of two complementary angles is 80 degrees. Find the measures of the angles.

Solution:

The angle measures are 5 degrees and 85 degrees.

Example:
Exercise:

Problem: Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is twelve degrees less than five times the measure of the smaller angle. Find the measures of both angles.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the measure of each angle.
Step 3. Name what we are looking for.	Let $x =$ the measure of the first angle. $y =$ the measure of the second angle
Step 4. Translate into a system of equations.	The angles are supplementary.
	$x + y = 180$
	The larger angle is twelve less than five times the smaller angle
	$y = 5x - 12$

The system is: **Step 5. Solve** the system of equations substitution.	$\begin{cases} x+y=180 \\ y=\boxed{5x-12} \end{cases}$ $x+y=180$
Substitute 5*x* − 12 for *y* in the first equation.	$x+5x-12=180$
Solve for *x*.	$6x-12=180$
	$6x=192$
	$x=\boxed{32}$ $y=5x-12$
Substitute 32 for in the second equation, then solve for *y*.	$y=5\cdot 32-12$
	$y=160-12$
	$y=148$
Step 6. Check the answer in the problem. $\begin{aligned} 32+158 &= 180\,✓ \\ 5\cdot 32-12 &= 147\,✓ \end{aligned}$	
Step 7. Answer the question.	The angle measures are 148 and 32.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is 12 degrees more than three times the smaller angle. Find the measures of the angles.

Solution:

The angle measures are 42 degrees and 138 degrees.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is 18 less than twice the measure of the smaller angle. Find the measures of the angles.

Solution:

The angle measures are 66 degrees and 114 degrees.

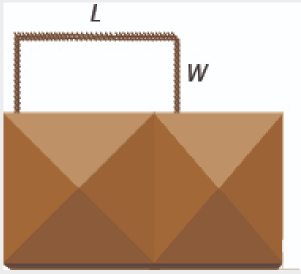
Example:
Exercise:

Problem: Translate to a system of equations and then solve:

Randall has 125 feet of fencing to enclose the rectangular part of his backyard adjacent to his house. He will only need to fence around three sides, because the fourth side will be the wall of the house. He wants the length of the fenced yard (parallel to the house wall) to be 5 feet more than four times as long as the width. Find the length and the width.

Solution:
Solution

Step 1. Read the problem.	

Step 2. Identify what you are looking for.	We are looking for the length and width.
	
Step 3. Name what we are looking for.	Let L = the length of the fenced yard. W = the width of the fenced yard
Step 4. Translate into a system of equations.	One length and two widths equal 125.
	$L+2W=125$
	The length will be 5 feet more than four times the width.
	$L=4W+5$
The system is: **Step 5. Solve** the system of equations by substitution.	$\begin{cases} L+2W=125 \\ L=4W+5 \end{cases}$ $L+2W=125$
Substitute $L = 4W + 5$ into the first equation, then solve for W.	$4W+5+2W=125$
	$6W+5=125$
	$6W=120$
Substitute 20 for W in the second equation, then solve for L.	

	$W = 20$ $L = 4W + 5$
	$L = 4 \cdot 20 + 5$
	$L = 80 + 5$
	$L = 85$
Step 6. Check the answer in the problem. $\begin{aligned} 20 + 28 + 20 &= 125 \checkmark \\ 85 &= 4 \cdot 20 + 5 \checkmark \end{aligned}$	
Step 7. Answer the equation.	The length is 85 feet and the width is 20 feet.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Mario wants to put a rectangular fence around the pool in his backyard. Since one side is adjacent to the house, he will only need to fence three sides. There are two long sides and the one shorter side is parallel to the house. He needs 155 feet of fencing to enclose the pool. The length of the long side is 10 feet less than twice the width. Find the length and width of the pool area to be enclosed.

Solution:

The length is 60 feet and the width is 35 feet.

Note:
Exercise:

Problem: Translate to a system of equations and then solve:

Alexis wants to build a rectangular dog run in her yard adjacent to her neighbor's fence. She will use 136 feet of fencing to completely enclose the rectangular dog run. The length of the dog run along the neighbor's fence will be 16 feet less than twice the width. Find the length and width of the dog run.

Solution:

The length is 60 feet and the width is 38 feet.

Solve Uniform Motion Applications

We used a table to organize the information in uniform motion problems when we introduced them earlier. We'll continue using the table here. The basic equation was $D = rt$ where D is the distance travelled, r is the rate, and t is the time.

Our first example of a uniform motion application will be for a situation similar to some we have already seen, but now we can use two variables and two equations.

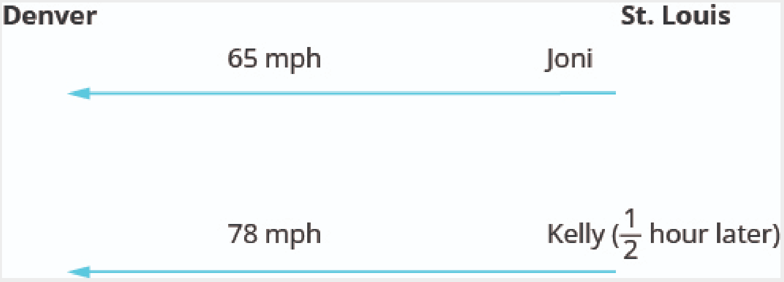
Example:
Exercise:

Problem: Translate to a system of equations and then solve:

Joni left St. Louis on the interstate, driving west towards Denver at a speed of 65 miles per hour. Half an hour later, Kelly left St. Louis on the same route as Joni, driving 78 miles per hour. How long will it take Kelly to catch up to Joni?

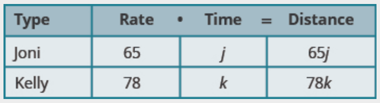
Solution:
Solution

A diagram is useful in helping us visualize the situation.



Identify and name what we are looking for.
A chart will help us organize the data.
We know the rates of both Joni and Kelly, and so we enter them in the chart.

We are looking for the length of time Kelly, *k*, and Joni, *j*, will each drive.
Since $D = r \cdot t$ we can fill in the Distance column.



Type	Rate	•	Time	=	Distance
Joni	65		j		$65j$
Kelly	78		k		$78k$

Translate into a system of equations.
To make the system of equations, we must recognize that Kelly and Joni will drive the same distance. So, $65j = 78k$.

Also, since Kelly left later, her time will be $\frac{1}{2}$ hour less than Joni's time.

So, $k = j - \frac{1}{2}$.

Now we have the system.

$$\begin{cases} k = j - \frac{1}{2} \\ 65j = 78k \end{cases}$$

Solve the system of equations by substitution.

$$65j = 78k$$

Substitute $k = j - \frac{1}{2}$ into the second equation, then solve for *j*.

$$65j = 78\left(j - \frac{1}{2}\right)$$

$$65j = 78j - 39$$

$$-13j = -39$$

$$j = 3$$

To find Kelly’s time, substitute $j = 3$ into the first equation, then solve for k.	$k = j - \frac{1}{2}$
	$k = 3 - \frac{1}{2}$
	$k = \frac{5}{2}$ or $k = 2\frac{1}{2}$
Check the answer in the problem. Joni 3 hours (65 mph) = 195 miles. Kelly $2\frac{1}{2}$ hours (78 mph) = 195 miles. Yes, they will have traveled the same distance when they meet.	
Answer the question.	Kelly will catch up to Joni in $2\frac{1}{2}$ hours. By then, Joni will have traveled 3 hours.

Note:
Exercise:

Problem:

Translate to a system of equations and then solve: Mitchell left Detroit on the interstate driving south towards Orlando at a speed of 60 miles per hour. Clark left Detroit 1 hour later traveling at a speed of 75 miles per hour, following the same route as Mitchell. How long will it take Clark to catch Mitchell?

Solution:

It will take Clark 4 hours to catch Mitchell.

Note:
Exercise:

Problem:

Translate to a system of equations and then solve: Charlie left his mother’s house traveling at an average speed of 36 miles per hour. His sister Sally left 15 minutes (1/4 hour) later traveling the same route at an average speed of 42 miles per hour. How long before Sally catches up to Charlie?

Solution:

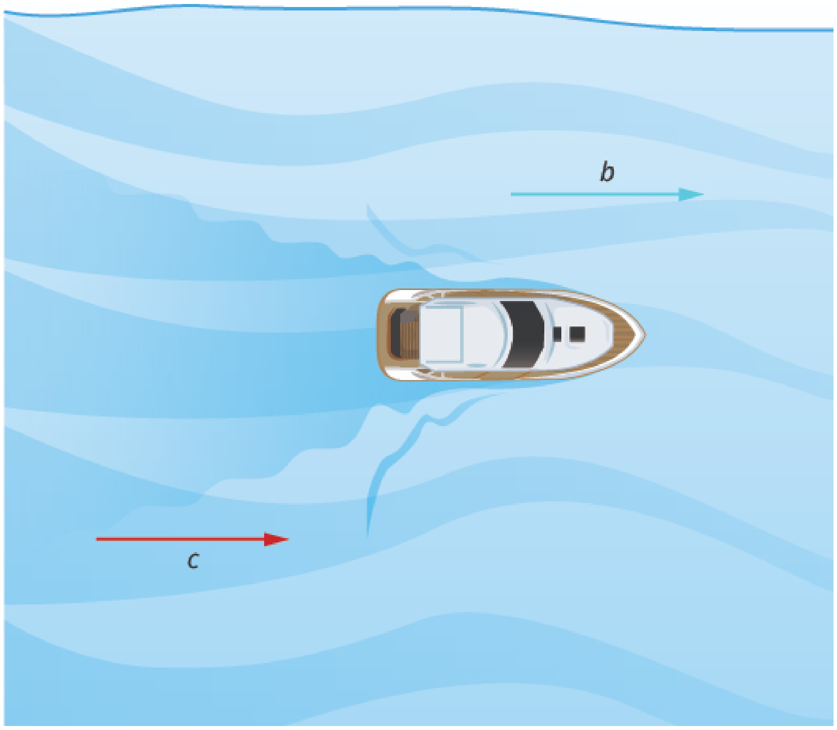
It will take Sally $1\frac{1}{2}$ hours to catch up to Charlie.

Many real-world applications of uniform motion arise because of the effects of currents—of water or air—on the actual speed of a vehicle. Cross-country airplane flights in the United States generally take longer going west than going east because of the prevailing wind currents.

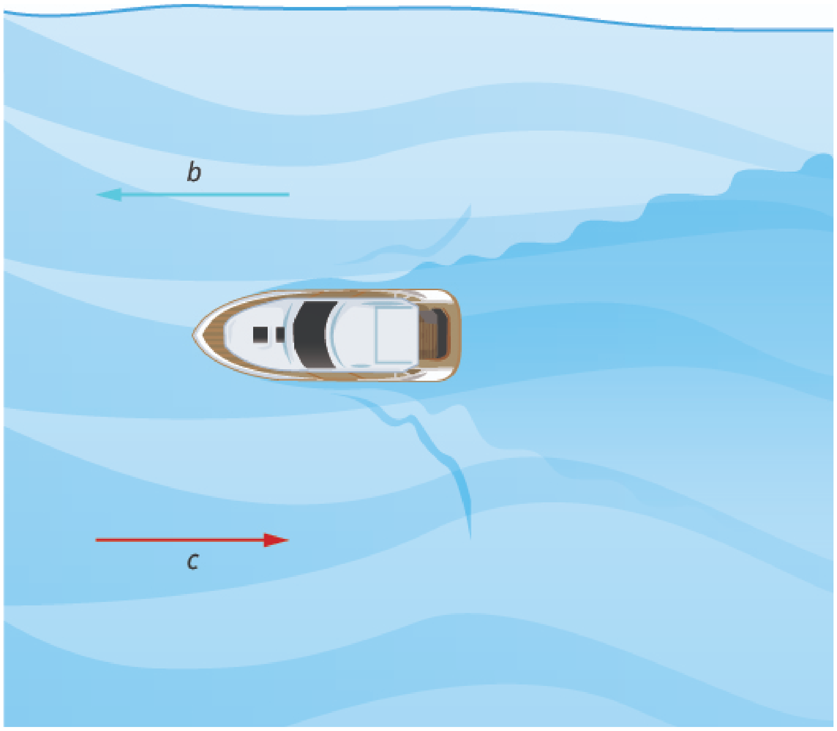
Let’s take a look at a boat travelling on a river. Depending on which way the boat is going, the current of the water is either slowing it down or speeding it up.

[link] and [link] show how a river current affects the speed at which a boat is actually travelling. We’ll call the speed of the boat in still water b and the speed of the river current c.

In [link] the boat is going downstream, in the same direction as the river current. The current helps push the boat, so the boat’s actual speed is faster than its speed in still water. The actual speed at which the boat is moving is $b + c$.



In [link] the boat is going upstream, opposite to the river current. The current is going against the boat, so the boat’s actual speed is slower than its speed in still water. The actual speed of the boat is $b - c$.



We’ll put some numbers to this situation in [link].

Example:
Exercise:

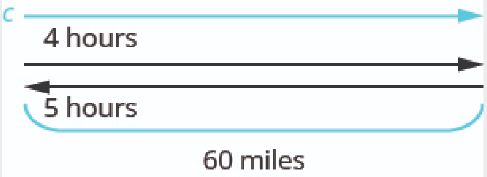
Problem: Translate to a system of equations and then solve:

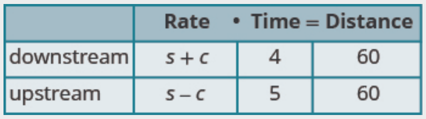
A river cruise ship sailed 60 miles downstream for 4 hours and then took 5 hours sailing upstream to return to the dock. Find the speed of the ship in still water and the speed of the river current.

Solution:
Solution

Read the problem.

This is a uniform motion problem and a picture will help us visualize the situation.



<table>
<tr><td>Identify what we are looking for.</td><td>We are looking for the speed of the ship
in still water and the speed of the current.</td></tr>
<tr><td>Name what we are looking for.</td><td>Let $s =$ the rate of the ship in still water.
$c =$ the rate of the current</td></tr>
<tr><td>A chart will help us organize the information. The ship goes downstream and then upstream. Going downstream, the current helps the ship; therefore, the ship’s actual rate is $s + c$. Going upstream, the current slows the ship; therefore, the actual rate is $s - c$.</td><td>

	Rate	• Time	= Distance
downstream	$s+c$	4	60
upstream	$s-c$	5	60

</td></tr>
<tr><td>Downstream it takes 4 hours.</td><td></td></tr>
</table>

Upstream it takes 5 hours. Each way the distance is 60 miles.	
Translate into a system of equations. Since rate times time is distance, we can write the system of equations.	$\begin{cases} 4(s+c)=60 \\ 5(s-c)=60 \end{cases}$
Solve the system of equations. Distribute to put both equations in standard form, then solve by elimination.	$\begin{cases} 4s+4c=60 \\ 5s-5c=60 \end{cases}$
Multiply the top equation by 5 and the bottom equation by 4. Add the equations, then solve for *s*.	$\begin{cases} 20s+20c=300 \\ 20s-20c=240 \end{cases}$ $40s \quad = 540$
Substitute $s = 13.5$ into one of the original equations.	$s \quad = \boxed{13.5}$ $4(s+c)=60$
	$4(13.5+c)=60$
	$54+4c=60$
	$4c=6$
	$c=1.5$
Check the answer in the problem. The downstream rate would be $13.5 + 1.5 = 15$ mph. In 4 hours the ship would travel $15 \cdot 4 = 60$ miles. The upstream rate would be $13.5 - 1.5 = 12$ mph. In 5 hours the ship would travel $12 \cdot 5 = 60$ miles.	
Answer the question.	The rate of the ship is 13.5 mph and

	the rate of the current is 1.5 mph.

Note:
Exercise:

Problem:

Translate to a system of equations and then solve: A Mississippi river boat cruise sailed 120 miles upstream for 12 hours and then took 10 hours to return to the dock. Find the speed of the river boat in still water and the speed of the river current.

Solution:

The rate of the boat is 11 mph and the rate of the current is 1 mph.

Note:
Exercise:

Problem:

Translate to a system of equations and then solve: Jason paddled his canoe 24 miles upstream for 4 hours. It took him 3 hours to paddle back. Find the speed of the canoe in still water and the speed of the river current.

Solution:

The speed of the canoe is 7 mph and the speed of the current is 1 mph.

Wind currents affect airplane speeds in the same way as water currents affect boat speeds. We’ll see this in [link]. A wind current in the same direction as the plane is flying is called a *tailwind*. A wind current blowing against the direction of the plane is called a *headwind*.

Example:
Exercise:

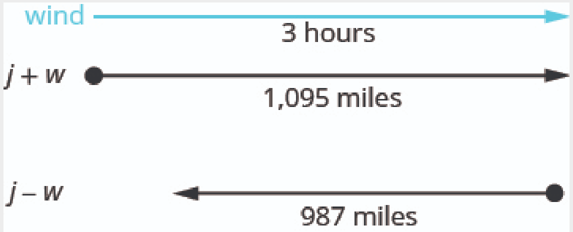
Problem: Translate to a system of equations and then solve:

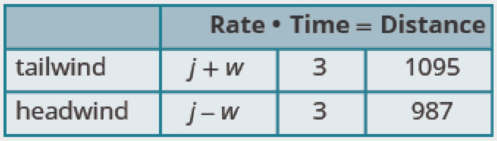
A private jet can fly 1095 miles in three hours with a tailwind but only 987 miles in three hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution:
Solution

Read the problem.

This is a uniform motion problem and a picture will help us visualize.



Identify what we are looking for.	We are looking for the speed of the jet in still air and the speed of the wind.
Name what we are looking for.	Let $j =$ the speed of the jet in still air. $w =$ the speed of the wind
A chart will help us organize the information. The jet makes two trips-one in a tailwind and one in a headwind. In a tailwind, the wind helps the jet and so the rate is $j + w$. In a headwind, the wind slows the jet and so the rate is $j - w$.	
Each trip takes 3 hours. In a tailwind the jet flies 1095 miles. In a headwind the jet flies 987 miles.	
Translate into a system of equations. Since rate times time is distance, we get	$\begin{cases} 3(j + w) = 1095 \\ 3(j - w) = 987 \end{cases}$

	Rate	• Time	= Distance
tailwind	$j + w$	3	1095
headwind	$j - w$	3	987

the system of equations.	
Solve the system of equations. Distribute, then solve by elimination.	$\begin{cases} 3j + 3w = 1095 \\ 3j - 3w = 987 \end{cases}$ $6j = 2082$
Add, and solve for *j*. Substitute $j = 347$ into one of the original equations, then solve for *w*.	$j = 347$ $3(j + w) = 1095$
	$3(347 + w) = 1095$
	$1041 + 3w = 1095$
	$3w = 54$
	$w = 18$
Check the answer in the problem. With the tailwind, the actual rate of the jet would be $347 + 18 = 365$ mph. In 3 hours the jet would travel $365 \cdot 3 = 1095$ miles. Going into the headwind, the jet’s actual rate would be $347 - 18 = 329$ mph. In 3 hours the jet would travel $329 \cdot 3 = 987$ miles.	
Answer the question.	The rate of the jet is 347 mph and the rate of the wind is 18 mph.

Note:
Exercise:

Problem:

Translate to a system of equations and then solve: A small jet can fly 1,325 miles in 5 hours with a tailwind but only 1025 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution:

The speed of the jet is 235 mph and the speed of the wind is 30 mph.

Note:
Exercise:

Problem:

Translate to a system of equations and then solve: A commercial jet can fly 1728 miles in 4 hours with a tailwind but only 1536 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution:

The speed of the jet is 408 mph and the speed of the wind is 24 mph.

Practice Makes Perfect

Translate to a System of Equations

In the following exercises, translate to a system of equations and solve the system.

Exercise:

Problem:

The sum of two numbers is fifteen. One number is three less than the other. Find the numbers.

Solution:

The numbers are 6 and 9.

Exercise:

Problem:

The sum of two numbers is twenty-five. One number is five less than the other. Find the numbers.

Exercise:

Problem:

The sum of two numbers is negative thirty. One number is five times the other. Find the numbers.

Solution:

The numbers are −5 and −25.

Exercise:

Problem:

The sum of two numbers is negative sixteen. One number is seven times the other. Find the numbers.

Exercise:

Problem:

Twice a number plus three times a second number is twenty-two. Three times the first number plus four times the second is thirty-one. Find the numbers.

Solution:

The numbers are 5 and 4.

Exercise:

Problem:

Six times a number plus twice a second number is four. Twice the first number plus four times the second number is eighteen. Find the numbers.

Exercise:

Problem:

Three times a number plus three times a second number is fifteen. Four times the first plus twice the second number is fourteen. Find the numbers.

Solution:

The numbers are 2 and 3.

Exercise:

Problem:

Twice a number plus three times a second number is negative one. The first number plus four times the second number is two. Find the numbers.

Exercise:

Problem:

A married couple together earn \$75,000. The husband earns \$15,000 more than five times what his wife earns. What does the wife earn?

Solution:

\$10,000

Exercise:

Problem:

During two years in college, a student earned \$9,500. The second year she earned \$500 more than twice the amount she earned the first year. How much did she earn the first year?

Exercise:

Problem:

Daniela invested a total of \$50,000, some in a certificate of deposit (CD) and the remainder in bonds. The amount invested in bonds was \$5000 more than twice the amount she put into the CD. How much did she invest in each account?

Solution:

She put \$15,000 into a CD and \$35,000 in bonds.

Exercise:

Problem:

Jorge invested \$28,000 into two accounts. The amount he put in his money market account was \$2,000 less than twice what he put into a CD. How much did he invest in each account?

Exercise:

Problem:

In her last two years in college, Marlene received \$42,000 in loans. The first year she received a loan that was \$6,000 less than three times the amount of the second year’s loan. What was the amount of her loan for each year?

Solution:

The amount of the first year’s loan was $30,000 and the amount of the second year’s loan was $12,000.

Exercise:

Problem:

Jen and David owe $22,000 in loans for their two cars. The amount of the loan for Jen’s car is $2000 less than twice the amount of the loan for David’s car. How much is each car loan?

Solve Direct Translation Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Alyssa is twelve years older than her sister, Bethany. The sum of their ages is forty-four. Find their ages.

Solution:

Bethany is 16 years old and Alyssa is 28 years old.

Exercise:

Problem:

Robert is 15 years older than his sister, Helen. The sum of their ages is sixty-three. Find their ages.

Exercise:

Problem:

The age of Noelle’s dad is six less than three times Noelle’s age. The sum of their ages is seventy-four. Find their ages.

Solution:

Noelle is 20 years old and her dad is 54 years old.

Exercise:

Problem:

The age of Mark’s dad is 4 less than twice Mark’s age. The sum of their ages is ninety-five. Find their ages.

Exercise:

Problem:

Two containers of gasoline hold a total of fifty gallons. The big container can hold ten gallons less than twice the small container. How many gallons does each container hold?

Solution:

The small container holds 20 gallons and the large container holds 30 gallons.

Exercise:

Problem:

June needs 48 gallons of punch for a party and has two different coolers to carry it in. The bigger cooler is five times as large as the smaller cooler. How many gallons can each cooler hold?

Exercise:

Problem:

Shelly spent 10 minutes jogging and 20 minutes cycling and burned 300 calories. The next day, Shelly swapped times, doing 20 minutes of jogging and 10 minutes of cycling and burned the same number of calories. How many calories were burned for each minute of jogging and how many for each minute of cycling?

Solution:

There were 10 calories burned jogging and 10 calories burned cycling.

Exercise:

Problem:

Drew burned 1800 calories Friday playing one hour of basketball and canoeing for two hours. Saturday he spent two hours playing basketball and three hours canoeing and burned 3200 calories. How many calories did he burn per hour when playing basketball?

Exercise:

Problem:

Troy and Lisa were shopping for school supplies. Each purchased different quantities of the same notebook and thumb drive. Troy bought four notebooks and five thumb drives for $116. Lisa bought two notebooks and three thumb dives for $68. Find the cost of each notebook and each thumb drive.

Solution:

Notebooks are $4 and thumb drives are $20.

Exercise:

Problem:

Nancy bought seven pounds of oranges and three pounds of bananas for $17. Her husband later bought three pounds of oranges and six pounds of bananas for $12. What was the cost per pound of the oranges and the bananas?

Solve Geometry Applications In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

The difference of two complementary angles is 30 degrees. Find the measures of the angles.

Solution:

The measures are 60 degrees and 30 degrees.

Exercise:

Problem:

The difference of two complementary angles is 68 degrees. Find the measures of the angles.

Exercise:

Problem:

The difference of two supplementary angles is 70 degrees. Find the measures of the angles.

Solution:

The measures are 125 degrees and 55 degrees.

Exercise:

Problem:

The difference of two supplementary angles is 24 degrees. Find the measure of the angles.

Exercise:

Problem:

The difference of two supplementary angles is 8 degrees. Find the measures of the angles.

Solution:

94 degrees and 86 degrees

Exercise:

Problem:

The difference of two supplementary angles is 88 degrees. Find the measures of the angles.

Exercise:

Problem:

The difference of two complementary angles is 55 degrees. Find the measures of the angles.

Solution:

72.5 degrees and 17.5 degrees

Exercise:

Problem:

The difference of two complementary angles is 17 degrees. Find the measures of the angles.

Exercise:

Problem:

Two angles are supplementary. The measure of the larger angle is four more than three times the measure of the smaller angle. Find the measures of both angles.

Solution:

The measures are 44 degrees and 136 degrees.

Exercise:

Problem:

Two angles are supplementary. The measure of the larger angle is five less than four times the measure of the smaller angle. Find the measures of both angles.

Exercise:

Problem:

Two angles are complementary. The measure of the larger angle is twelve less than twice the measure of the smaller angle. Find the measures of both angles.

Solution:

The measures are 34 degrees and 56 degrees.

Exercise:

Problem:

Two angles are complementary. The measure of the larger angle is ten more than four times the measure of the smaller angle. Find the measures of both angles.

Exercise:

Problem:

Wayne is hanging a string of lights 45 feet long around the three sides of his rectangular patio, which is adjacent to his house. The length of his patio, the side along the house, is five feet longer than twice its width. Find the length and width of the patio.

Solution:

The width is 10 feet and the length is 25 feet.

Exercise:

Problem:

Darrin is hanging 200 feet of Christmas garland on the three sides of fencing that enclose his rectangular front yard. The length, the side along the house, is five feet less than three times the width. Find the length and width of the fencing.

Exercise:

Problem:

A frame around a rectangular family portrait has a perimeter of 60 inches. The length is fifteen less than twice the width. Find the length and width of the frame.

Solution:

The width is 15 feet and the length is 15 feet.

Exercise:

Problem:

The perimeter of a rectangular toddler play area is 100 feet. The length is ten more than three times the width. Find the length and width of the play area.

Solve Uniform Motion Applications In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Sarah left Minneapolis heading east on the interstate at a speed of 60 mph. Her sister followed her on the same route, leaving two hours later and driving at a rate of 70 mph. How long will it take for Sarah's sister to catch up to Sarah?

Solution:

It took Sarah's sister 12 hours.

Exercise:

Problem:

College roommates John and David were driving home to the same town for the holidays. John drove 55 mph, and David, who left an hour later, drove 60 mph. How long will it take for David to catch up to John?

Exercise:

Problem:

At the end of spring break, Lucy left the beach and drove back towards home, driving at a rate of 40 mph. Lucy’s friend left the beach for home 30 minutes (half an hour) later, and drove 50 mph. How long did it take Lucy’s friend to catch up to Lucy?

Solution:

It took Lucy’s friend 2 hours.

Exercise:

Problem:

Felecia left her home to visit her daughter driving 45 mph. Her husband waited for the dog sitter to arrive and left home twenty minutes (1/3 hour) later. He drove 55 mph to catch up to Felecia. How long before he reaches her?

Exercise:

Problem:

The Jones family took a 12 mile canoe ride down the Indian River in two hours. After lunch, the return trip back up the river took three hours. Find the rate of the canoe in still water and the rate of the current.

Solution:

The canoe rate is 5 mph and the current rate is 1 mph.

Exercise:

Problem:

A motor boat travels 60 miles down a river in three hours but takes five hours to return upstream. Find the rate of the boat in still water and the rate of the current.

Exercise:

Problem:

A motor boat traveled 18 miles down a river in two hours but going back upstream, it took 4.5 hours due to the current. Find the rate of the motor boat in still water and the rate of the current. (Round to the nearest hundredth.).

Solution:

The boat rate is 6.5 mph and the current rate is 2.5 mph.

Exercise:

Problem:

A river cruise boat sailed 80 miles down the Mississippi River for four hours. It took five hours to return. Find the rate of the cruise boat in still water and the rate of the current. (Round to the nearest hundredth.).

Exercise:

Problem:

A small jet can fly 1,072 miles in 4 hours with a tailwind but only 848 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution:

The jet rate is 240 mph and the wind speed is 28 mph.

Exercise:

Problem:

A small jet can fly 1,435 miles in 5 hours with a tailwind but only 1215 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Exercise:

Problem:

A commercial jet can fly 868 miles in 2 hours with a tailwind but only 792 miles in 2 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution:

The jet rate is 415 mph and the wind speed is 19 mph.

Exercise:

Problem:

A commercial jet can fly 1,320 miles in 3 hours with a tailwind but only 1,170 miles in 3 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Everyday Math

Exercise:

Problem:

At a school concert, 425 tickets were sold. Student tickets cost $5 each and adult tickets cost $8 each. The total receipts for the concert were $2,851. Solve the system

$$\begin{cases} s + a = 425 \\ 5s + 8a = 2{,}851 \end{cases}$$

to find s, the number of student tickets and a, the number of adult tickets.

Solution:

$s = 183, a = 242$

Exercise:

Problem:

The first graders at one school went on a field trip to the zoo. The total number of children and adults who went on the field trip was 115. The number of adults was $\frac{1}{4}$ the number of children. Solve the system

$$\begin{cases} c + a = 115 \\ a = \frac{1}{4}c \end{cases}$$

to find c, the number of children and a, the number of adults.

Writing Exercises

Exercise:

Problem:

Write an application problem similar to [link] using the ages of two of your friends or family members. Then translate to a system of equations and solve it.

Solution:

Answers will vary.

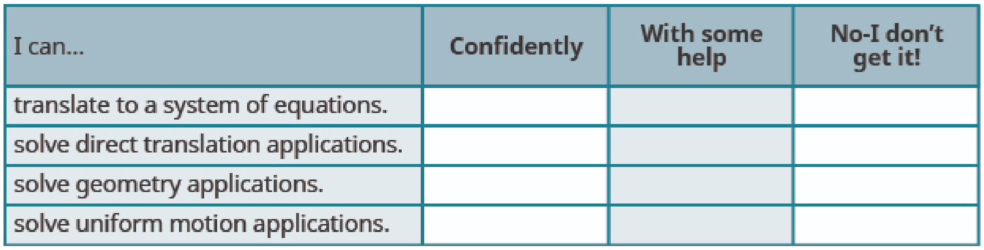
Exercise:

Problem:

Write a uniform motion problem similar to [link] that relates to where you live with your friends or family members. Then translate to a system of equations and solve it.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
translate to a system of equations.			
solve direct translation applications.			
solve geometry applications.			
solve uniform motion applications.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

complementary angles

Two angles are complementary if the sum of the measures of their angles is 90 degrees.

supplementary angles

Two angles are supplementary if the sum of the measures of their angles is 180 degrees.

Solve Mixture Applications with Systems of Equations

By the end of this section, you will be able to:

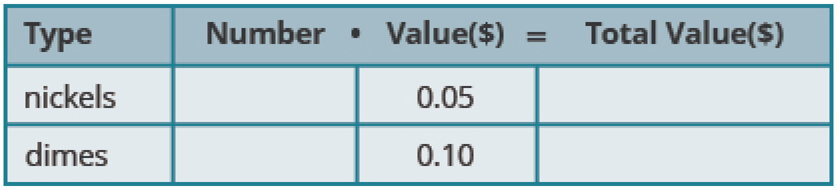
- Solve mixture applications
- Solve interest applications

Note:
Before you get started, take this readiness quiz.

1. Multiply $4.025(1{,}562)$.
   If you missed this problem, review [link].
2. Write 8.2% as a decimal.
   If you missed this problem, review [link].
3. Earl’s dinner bill came to $32.50 and he wanted to leave an 18% tip. How much should the tip be?
   If you missed this problem, review [link].

Solve Mixture Applications

When we solved mixture applications with coins and tickets earlier, we started by creating a table so we could organize the information. For a coin example with nickels and dimes, the table looked like this:



Type	Number •	Value($) =	Total Value($)
nickels		0.05	
dimes		0.10	

Using one variable meant that we had to relate the number of nickels and the number of dimes. We had to decide if we were going to let *n* be the number of nickels and then write the number of dimes in terms of *n*, or if we would let *d* be the number of dimes and write the number of nickels in terms of *d*.

Now that we know how to solve systems of equations with two variables, we'll just let *n* be the number of nickels and *d* be the number of dimes. We'll write one equation based on the total value column, like we did before, and the other equation will come from the number column.

For the first example, we'll do a ticket problem where the ticket prices are in whole dollars, so we won't need to use decimals just yet.

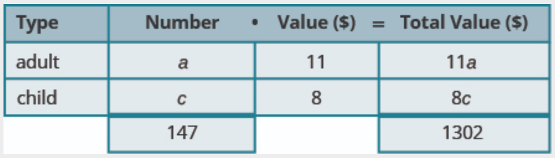
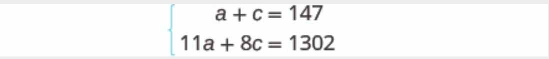
Example:
Exercise:

Problem: Translate to a system of equations and solve:

The box office at a movie theater sold 147 tickets for the evening show, and receipts totaled $1,302. How many $11 adult and how many $8 child tickets were sold?

Solution:
Solution

Step 1. Read the problem.	We will create a table to organize the information.
Step 2. Identify what we are looking for.	We are looking for the number of adult tickets and the number of child tickets sold.
Step 3. Name what we are looking for.	Let $a =$ the number of adult tickets.

<table>
<tr><td></td><td>$c =$ the number of child tickets</td></tr>
<tr><td>A table will help us organize the data.
We have two types of tickets: adult and child.</td><td>Write a and c for the number of tickets.</td></tr>
<tr><td>Write the total number of tickets sold at the bottom of the Number column.</td><td>Altogether 147 were sold.</td></tr>
<tr><td>Write the value of each type of ticket in the Value column.</td><td>The value of each adult ticket is $11.
The value of each child tickets is $8.</td></tr>
<tr><td>The number times the value gives the total value, so the total value of adult tickets is $a \cdot 11 = 11a$, and the total value of child tickets is $c \cdot 8 = 8c$.</td><td>
<table>
<tr><th>Type</th><th>Number</th><th>• Value ($)</th><th>= Total Value ($)</th></tr>
<tr><td>adult</td><td>a</td><td>11</td><td>$11a$</td></tr>
<tr><td>child</td><td>c</td><td>8</td><td>$8c$</td></tr>
<tr><td></td><td>147</td><td></td><td>1302</td></tr>
</table></td></tr>
<tr><td>Altogether the total value of the tickets was $1,302.</td><td>Fill in the Total Value column.</td></tr>
<tr><td>Step 4. Translate into a system of equations.</td><td></td></tr>
<tr><td>The Number column and the Total Value column give us the system of equations.</td><td>
$\begin{cases} a + c = 147 \\ 11a + 8c = 1302 \end{cases}$</td></tr>
</table>

We will use the elimination method to solve this system.	
Multiply the first equation by −8.	$\begin{cases} -8(a + c) = -8(147) \\ 11a + 8c = 1302 \end{cases}$
Simplify and add, then solve for *a*.	$\begin{cases} -8a - 8c = -1176 \\ \underline{11a + 8c = 1302} \end{cases}$ $3a \quad = 126$
	$a \quad = 42$ $a + c = 147$
Substitute *a* = 42 into the first equation, then solve for *c*.	$42 + c = 147$
	$c = 105$
Step 5. Check the answer in the problem. 42 adult tickets at \$11 per ticket makes \$462 105 child tickets at \$8 per ticket makes \$840. The total receipts are \$1,302.✓	
Step 6. Answer the question.	The movie theater sold 42 adult tickets and 105 child tickets.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

The ticket office at the zoo sold 553 tickets one day. The receipts totaled $3,936. How many $9 adult tickets and how many $6 child tickets were sold?

Solution:

There were 206 adult tickets sold and 347 children tickets sold.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

A science center sold 1,363 tickets on a busy weekend. The receipts totaled $12,146. How many $12 adult tickets and how many $7 child tickets were sold?

Solution:

There were 521 adult tickets sold and 842 children tickets sold.

In [link] we’ll solve a coin problem. Now that we know how to work with systems of two variables, naming the variables in the ‘number’ column will be easy.

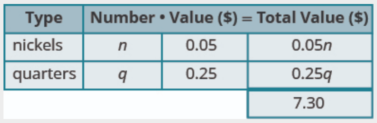
Example:
Exercise:

Problem: Translate to a system of equations and solve:

Priam has a collection of nickels and quarters, with a total value of \$7.30. The number of nickels is six less than three times the number of quarters. How many nickels and how many quarters does he have?

Solution:
Solution

Step 1. Read the problem.	We will create a table to organize the information.
Step 2. Identify what we are looking for.	We are looking for the number of nickels and the number of quarters.
Step 3. Name what we are looking for.	Let $n =$ the number of nickels. $q =$ the number of quarters
A table will help us organize the data. We have two types of coins, nickels and quarters.	Write n and q for the number of each type of coin.
Fill in the Value column with the	The value of each

<table>
<tr><td>value of each
type of coin.</td><td>nickel is $0.05.
The value of each
quarter is $0.25.</td></tr>
<tr><td>The number times the value gives the total value, so, the total value of the nickels is $n\,(0.05) = 0.05n$ and the total value of quarters is $q(0.25) = 0.25q$.
Altogether the total value of the coins is $7.30.</td><td>
<table>
<tr><th>Type</th><th>Number</th><th>• Value ($)</th><th>= Total Value ($)</th></tr>
<tr><td>nickels</td><td>n</td><td>0.05</td><td>$0.05n$</td></tr>
<tr><td>quarters</td><td>q</td><td>0.25</td><td>$0.25q$</td></tr>
<tr><td></td><td></td><td></td><td>7.30</td></tr>
</table></td></tr>
<tr><td>Step 4. Translate into a system of equations.</td><td></td></tr>
<tr><td>The Total value column gives one equation.</td><td>$0.05n + 0.25q = 7.30$</td></tr>
<tr><td>We also know the number of nickels is six less
than three times the number of quarters.
Translate to get the second equation.</td><td>$n = 3q - 6$</td></tr>
<tr><td>Now we have the system to solve.</td><td>$\begin{cases} 0.05n + 0.25q = 7.30 \\ n = 3q - 6 \end{cases}$</td></tr>
<tr><td>Step 5. Solve the system of equations
We will use the substitution method.
Substitute $n = 3q - 6$ into the first equation.
Simplify and solve for q.</td><td>$0.05n + 0.25q = 7.30$</td></tr>
</table>

	$0.05(3q - 6) + 0.25q = 7.3$
	$0.15q - 0.3 + 0.25q = 7.3$
	$0.4q - 0.3 = 7.3$
	$0.4q = 7.6$
	$q = 19$
To find the number of nickels, substitute $q = 19$ into the second equation.	$n = 3q - 6$
	$n = 3 \cdot 19 - 6$
	$n = 51$
Step 6. Check the answer in the problem. $19 \text{ quarters at } \$0.25 = \$4.75$ $51\text{nickels at } \$0.05 = \$2.55$ $\text{Total} = \$7.30 \checkmark$ $3 \cdot 19 - 16 = 51 \checkmark$	
Step 7. Answer the question.	Priam has 19 quarters and 51 nickels.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Matilda has a handful of quarters and dimes, with a total value of \$8.55. The number of quarters is 3 more than twice the number of dimes. How many dimes and how many quarters does she have?

Solution:

Matilda has 13 dimes and 29 quarters.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Juan has a pocketful of nickels and dimes. The total value of the coins is \$8.10. The number of dimes is 9 less than twice the number of nickels. How many nickels and how many dimes does Juan have?

Solution:

Juan has 36 nickels and 63 dimes.

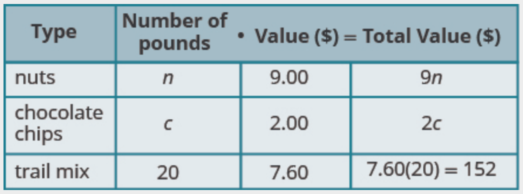
Some mixture applications involve combining foods or drinks. Example situations might include combining raisins and nuts to make a trail mix or using two types of coffee beans to make a blend.

Example:
Exercise:

Problem: Translate to a system of equations and solve:

Carson wants to make 20 pounds of trail mix using nuts and chocolate chips. His budget requires that the trail mix costs him $7.60 per pound. Nuts cost $9.00 per pound and chocolate chips cost $2.00 per pound. How many pounds of nuts and how many pounds of chocolate chips should he use?

Solution:
Solution

Step 1. Read the problem.	We will create a table to organize the information.
Step 2. Identify what we are looking for.	We are looking for the number of pounds of nuts and the number of pounds of chocolate chips.
Step 3. Name what we are looking for.	Let $n =$ the number of pound of nuts. $c =$ the number of pounds of chips
Carson will mix nuts and chocolate chips to get trail mix. Write in *n* and *c* for the number of pounds	

Type	Number of pounds	• Value ($)	= Total Value ($)
nuts	n	9.00	$9n$
chocolate chips	c	2.00	$2c$
trail mix	20	7.60	$7.60(20) = 152$

of nuts and chocolate chips. There will be 20 pounds of trail mix. Put the price per pound of each item in the Value column. Fill in the last column using	
Number · Value = Total Value	
Step 4. Translate into a system of equations. We get the equations from the Number and Total Value columns.	$\begin{cases} n+c=20 \\ 9n+2c=152 \end{cases}$
Step 5. Solve the system of equations We will use elimination to solve the system.	
Multiply the first equation by −2 to eliminate *c*.	$\begin{cases} -2(n+c)=-2(20) \\ 9n+2c=152 \end{cases}$
Simplify and add. Solve for *n*.	$\begin{cases} -2n-2c=-40 \\ \underline{9n+2c=152} \end{cases}$ $7n \quad = 112$
	$n=16$
To find the number of	$n+c=20$

pounds of chocolate chips, substitute *n* = 16 into the first equation, then solve for *c*.	$16 + c = 20$
	$c = 4$
Step 6. Check the answer in the problem. $16 + 4 = 20$ ✓ $9 \cdot 16 + 2 \cdot 4 = 152$ ✓	
Step 7. Answer the question.	Carson should mix 16 pounds of nuts with 4 pounds of chocolate chips to create the trail mix.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Greta wants to make 5 pounds of a nut mix using peanuts and cashews. Her budget requires the mixture to cost her $6 per pound. Peanuts are $4 per pound and cashews are $9 per pound. How many pounds of peanuts and how many pounds of cashews should she use?

Solution:

Greta should use 3 pounds of peanuts and 2 pounds of cashews.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Sammy has most of the ingredients he needs to make a large batch of chili. The only items he lacks are beans and ground beef. He needs a total of 20 pounds combined of beans and ground beef and has a budget of $3 per pound. The price of beans is $1 per pound and the price of ground beef is $5 per pound. How many pounds of beans and how many pounds of ground beef should he purchase?

Solution:

Sammy should purchase 10 pounds of beans and 10 pounds of ground beef.

Another application of mixture problems relates to concentrated cleaning supplies, other chemicals, and mixed drinks. The concentration is given as a percent. For example, a 20% concentrated household cleanser means that 20% of the total amount is cleanser, and the rest is water. To make 35 ounces of a 20% concentration, you mix 7 ounces (20% of 35) of the cleanser with 28 ounces of water.

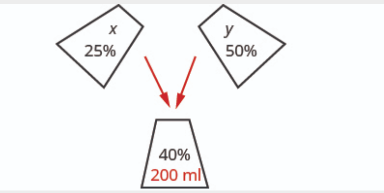
For these kinds of mixture problems, we’ll use percent instead of value for one of the columns in our table.

Example:
Exercise:

Problem: Translate to a system of equations and solve:

Sasheena is a lab assistant at her community college. She needs to make 200 milliliters of a 40% solution of sulfuric acid for a lab experiment. The lab has only 25% and 50% solutions in the storeroom. How much should she mix of the 25% and the 50% solutions to make the 40% solution?

Solution:
Solution

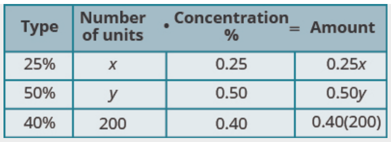
Step 1. Read the problem.	A figure may help us visualize the situation, then we will create a table to organize the information.
Sasheena must mix some of the 25% solution and some of the 50% solution together to get 200 ml of the 40% solution.	
Step 2. Identify what we are looking for.	We are looking for how much of each solution she needs.
Step 3. Name what we are looking for.	Let $x =$ number of ml of 25% solution. $y =$ number of ml of 50% solution

A table will help us organize the data.

She will mix x ml of 25% with y ml of 50% to get 200 ml of 40% solution.

We write the percents as decimals in the chart.

We multiply the number of units times the concentration to get the total amount of sulfuric acid in each solution.



Type	Number of units	· Concentration %	= Amount
25%	x	0.25	$0.25x$
50%	y	0.50	$0.50y$
40%	200	0.40	0.40(200)

Step 4. Translate into a system of equations. We get the equations from the Number column and the Amount column.

Now we have the system.

$$\begin{cases} x + y = 200 \\ 0.25x + 0.50y = 0.40(200) \end{cases}$$

Step 5. Solve the system of equations. We will solve the system by elimination. Multiply the first equation by −0.5 to eliminate y.

$$\begin{cases} -0.5(x + y) = -0.5(200) \\ 0.25x + 0.50y = 80 \end{cases}$$

Simplify and add to solve for x.

	$\begin{cases} -0.5x - 0.5y = -100 \\ 0.25x + 0.5y = 80 \end{cases}$ $-0.25x = -20$ $x = 80$
To solve for *y*, substitute *x* = 80 into the first equation.	$x + y = 200$
	$80 + y = 200$
	$y = 120$
Step 6. Check the answer in the problem. $80 + 120 = 120 \checkmark$ $0.25(80) + 0.50(120) = 80 \checkmark$ Yes!	
Step 7. Answer the question.	Sasheena should mix 80 ml of the 25% solution with 120 ml of the 50% solution to get the 200 ml of the 40% solution.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

LeBron needs 150 milliliters of a 30% solution of sulfuric acid for a lab experiment but only has access to a 25% and a 50% solution. How much of the 25% and how much of the 50% solution should he mix to make the 30% solution?

Solution:

LeBron needs 120 ml of the 25% solution and 30 ml of the 50% solution.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Anatole needs to make 250 milliliters of a 25% solution of hydrochloric acid for a lab experiment. The lab only has a 10% solution and a 40% solution in the storeroom. How much of the 10% and how much of the 40% solutions should he mix to make the 25% solution?

Solution:

Anatole should mix 125 ml of the 10% solution and 125 ml of the 40% solution.

Solve Interest Applications

The formula to model interest applications is $I = Prt$. Interest, I, is the product of the principal, P, the rate, r, and the time, t. In our work here, we will calculate the interest earned in one year, so t will be 1.

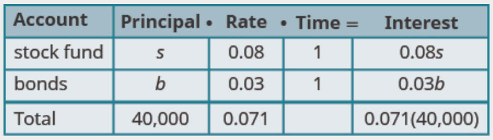
We modify the column titles in the mixture table to show the formula for interest, as you’ll see in [link].

Example:
Exercise:

Problem: Translate to a system of equations and solve:

Adnan has $40,000 to invest and hopes to earn 7.1% interest per year. He will put some of the money into a stock fund that earns 8% per year and the rest into bonds that earns 3% per year. How much money should he put into each fund?

Solution:
Solution

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the amount to invest in each fund.
Step 3. Name what we are looking for.	Let $s =$ the amount invested in stocks. $b =$ the amount invested in bonds.
Write the interest rate as a decimal for each fund.	

Account	Principal •	Rate •	Time =	Interest
stock fund	s	0.08	1	$0.08s$
bonds	b	0.03	1	$0.03b$
Total	40,000	0.071		0.071(40,000)

Multiply: Principal · Rate · Time to get the Interest.	
Step 4. Translate into a system of equations. We get our system of equations from the Principal column and the Interest column.	$\begin{cases} s+b=40{,}000 \\ 0.08s+0.03b=0.071(40{,}000) \end{cases}$
Step 5. Solve the system of equations Solve by elimination. Multiply the top equation by −0.03.	$\begin{cases} -0.03(s+b)=-0.03(40{,}000) \\ 0.08s+0.03b=2{,}840 \end{cases}$
Simplify and add to solve for *s*.	$\begin{array}{r} -0.03s-0.03b=-1{,}200 \\ \underline{0.08s+0.03b=\ \ 2{,}840} \\ 0.05s=\ \ 1{,}640 \end{array}$
	$s=32{,}800$
To find *b*, substitute *s* = 32,800 into the first equation.	$s+b=40{,}000$ $32{,}800+b=40{,}000$
	$b=7{,}200$
Step 6. Check the answer in the problem.	We leave the check to you.

Step 7. Answer the question.	Adnan should invest \$32,800 in stock and \$7,200 in bonds.

Did you notice that the Principal column represents the total amount of money invested while the Interest column represents only the interest earned? Likewise, the first equation in our system, $s + b = 40{,}000$, represents the total amount of money invested and the second equation, $0.08s + 0.03b = 0.071(40{,}000)$, represents the interest earned.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Leon had \$50,000 to invest and hopes to earn 6.2 % interest per year. He will put some of the money into a stock fund that earns 7% per year and the rest in to a savings account that earns 2% per year. How much money should he put into each fund?

Solution:

Leon should put \$42,000 in the stock fund and \$8000 in the savings account.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Julius invested $7,000 into two stock investments. One stock paid 11% interest and the other stock paid 13% interest. He earned 12.5% interest on the total investment. How much money did he put in each stock?

Solution:

Julius invested $1,750 at 11% and $5,250 at 13%.

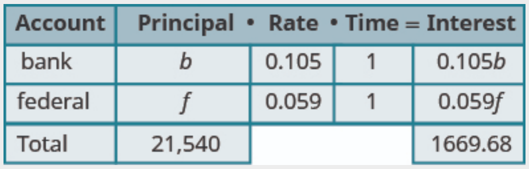
Example:
Exercise:

Problem: Translate to a system of equations and solve:

Rosie owes $21,540 on her two student loans. The interest rate on her bank loan is 10.5% and the interest rate on the federal loan is 5.9%. The total amount of interest she paid last year was $1,669.68. What was the principal for each loan?

Solution:
Solution

Step 1. Read the problem.	A chart will help us organize the information.
Step 2. Identify what we are looking for.	We are looking for the principal of each loan.
Step 3. Name what we are looking for.	Let $b =$ the principal for the bank loan.

<table>
<tr><td></td><td>f = the principal on the federal loan</td></tr>
<tr><td>The total loans are $21,540.</td><td></td></tr>
<tr><td>Record the interest rates as decimals in the chart.</td><td>
<table>
<tr><th>Account</th><th>Principal</th><th>• Rate</th><th>• Time</th><th>= Interest</th></tr>
<tr><td>bank</td><td>b</td><td>0.105</td><td>1</td><td>$0.105b$</td></tr>
<tr><td>federal</td><td>f</td><td>0.059</td><td>1</td><td>$0.059f$</td></tr>
<tr><td>Total</td><td>21,540</td><td></td><td></td><td>1669.68</td></tr>
</table></td></tr>
<tr><td>Multiply using the formula <i>I</i> = Pr <i>t</i> to get the Interest.</td><td></td></tr>
<tr><td>Step 4. Translate into a system of equations.
The system of equations comes from the Principal column and the Interest column.</td><td>$\begin{cases} b + f = 21{,}540 \\ 0.105b + 0.059f = 1669.68 \end{cases}$</td></tr>
<tr><td>Step 5. Solve the system of equations
We will use substitution to solve.
Solve the first equation for <i>b</i>.</td><td>$b + f = 21{,}540$
$b = -f + 21{,}540$</td></tr>
<tr><td>Substitute $b = -f + 21{,}540$ into the second equation.</td><td>$0.105b + 0.059f = 1669.68$
$0.105(-f + 21{,}540) + 0.059f = 1669.68$</td></tr>
<tr><td>Simplify and solve for <i>f</i>.</td><td></td></tr>
</table>

	$-0.105f + 2261.70 + 0.059f = 1669.68$
	$-0.046f + 2261.70 = 1669.68$
	$-0.046f = -592.02$
	$f = 12{,}870$
To find *b*, substitute *f* = 12,870 into the first equation.	$b + f = 21{,}540$ $12{,}870 + f = 21{,}540$
	$f = 8{,}670$
Step 6. Check the answer in the problem.	We leave the check to you.
Step 7. Answer the question.	The principal of the bank loan is \$12,870 and the principal for the federal loan is \$8,670.

Note:

Exercise:

Problem: Translate to a system of equations and solve:

Laura owes $18,000 on her student loans. The interest rate on the bank loan is 2.5% and the interest rate on the federal loan is 6.9 %. The total amount of interest she paid last year was $1,066. What was the principal for each loan?

Solution:

The principal amount for the bank loan was $4,000. The principal amount for the federal loan was $14,000.

Note:
Exercise:

Problem: Translate to a system of equations and solve:

Jill’s Sandwich Shoppe owes $65,200 on two business loans, one at 4.5% interest and the other at 7.2% interest. The total amount of interest owed last year was $3,582. What was the principal for each loan?

Solution:

The principal amount for was $41,200 at 4.5%. The principal amount was, $24,000 at 7.2%.

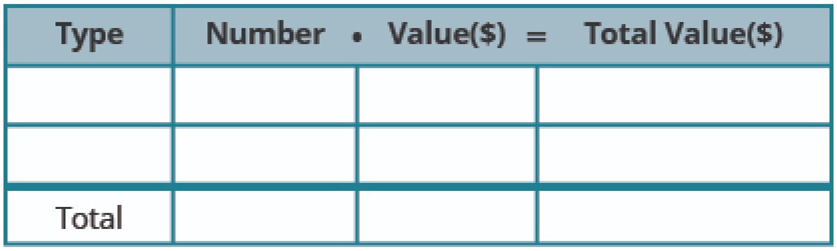
Note:
Access these online resources for additional instruction and practice with solving application problems with systems of linear equations.

- Cost and Mixture Word Problems

- Mixture Problems

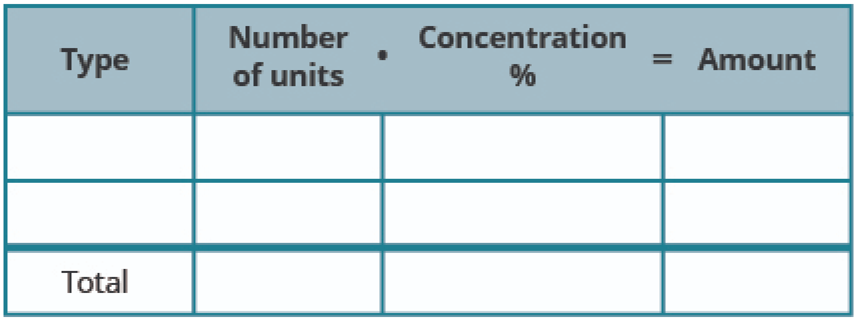
Key Concepts

- **Table for coin and mixture applications**



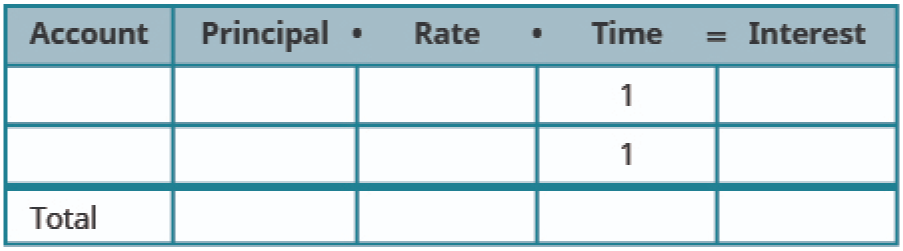
Type	Number •	Value($) =	Total Value($)
Total			

- **Table for concentration applications**



Type	Number of units •	Concentration % =	Amount
Total			

- **Table for interest applications**



Account	Principal •	Rate •	Time =	Interest
			1	
			1	
Total				

Practice Makes Perfect

Solve Mixture Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Tickets to a Broadway show cost $35 for adults and $15 for children. The total receipts for 1650 tickets at one performance were $47,150. How many adult and how many child tickets were sold?

Solution:

There 1120 adult tickets and 530 child tickets sold.

Exercise:

Problem:

Tickets for a show are $70 for adults and $50 for children. One evening performance had a total of 300 tickets sold and the receipts totaled $17,200. How many adult and how many child tickets were sold?

Exercise:

Problem:

Tickets for a train cost $10 for children and $22 for adults. Josie paid $1,200 for a total of 72 tickets. How many children’s tickets and how many adult tickets did Josie buy?

Solution:

Josie bought 40 adult tickets and 32 children tickets.

Exercise:

Problem:

Tickets for a baseball game are $69 for Main Level seats and $39 for Terrace Level seats. A group of sixteen friends went to the game and spent a total of $804 for the tickets. How many of Main Level and how many Terrace Level tickets did they buy?

Exercise:

Problem:

Tickets for a dance recital cost \$15 for adults and \$7 for children. The dance company sold 253 tickets and the total receipts were \$2,771. How many adult tickets and how many child tickets were sold?

Solution:

There were 125 adult tickets and 128 children tickets sold.

Exercise:

Problem:

Tickets for the community fair cost \$12 for adults and \$5 dollars for children. On the first day of the fair, 312 tickets were sold for a total of \$2,204. How many adult tickets and how many child tickets were sold?

Exercise:

Problem:

Brandon has a cup of quarters and dimes with a total value of \$3.80. The number of quarters is four less than twice the number of dimes. How many quarters and how many dimes does Brandon have?

Solution:

Brandon has 12 quarters and 8 dimes.

Exercise:

Problem:

Sherri saves nickels and dimes in a coin purse for her daughter. The total value of the coins in the purse is \$0.95. The number of nickels is two less than five times the number of dimes. How many nickels and how many dimes are in the coin purse?

Exercise:

Problem:

Peter has been saving his loose change for several days. When he counted his quarters and dimes, he found they had a total value $13.10. The number of quarters was fifteen more than three times the number of dimes. How many quarters and how many dimes did Peter have?

Solution:

Peter had 11 dimes and 48 quarters.

Exercise:

Problem:

Lucinda had a pocketful of dimes and quarters with a value of $ $6.20. The number of dimes is eighteen more than three times the number of quarters. How many dimes and how many quarters does Lucinda have?

Exercise:

Problem:

A cashier has 30 bills, all of which are $10 or $20 bills. The total value of the money is $460. How many of each type of bill does the cashier have?

Solution:

The cashier has fourteen $10 bills and sixteen $20 bills.

Exercise:

Problem:

A cashier has 54 bills, all of which are $10 or $20 bills. The total value of the money is $910. How many of each type of bill does the cashier have?

Exercise:

Problem:

Marissa wants to blend candy selling for $1.80 per pound with candy costing $1.20 per pound to get a mixture that costs her $1.40 per pound to make. She wants to make 90 pounds of the candy blend. How many pounds of each type of candy should she use?

Solution:

Marissa should use 60 pounds of the $1.20/lb candy and 30 pounds of the $1.80/lb candy.

Exercise:

Problem:

How many pounds of nuts selling for $6 per pound and raisins selling for $3 per pound should Kurt combine to obtain 120 pounds of trail mix that cost him $5 per pound?

Exercise:

Problem:

Hannah has to make twenty-five gallons of punch for a potluck. The punch is made of soda and fruit drink. The cost of the soda is $1.79 per gallon and the cost of the fruit drink is $2.49 per gallon. Hannah’s budget requires that the punch cost $2.21 per gallon. How many gallons of soda and how many gallons of fruit drink does she need?

Solution:

Hannah needs 10 gallons of soda and 15 gallons of fruit drink.

Exercise:

Problem:

Joseph would like to make 12 pounds of a coffee blend at a cost of $6.25 per pound. He blends Ground Chicory at $4.40 a pound with Jamaican Blue Mountain at $8.84 per pound. How much of each type of coffee should he use?

Exercise:

Problem:

Julia and her husband own a coffee shop. They experimented with mixing a City Roast Columbian coffee that cost $7.80 per pound with French Roast Columbian coffee that cost $8.10 per pound to make a 20 pound blend. Their blend should cost them $7.92 per pound. How much of each type of coffee should they buy?

Solution:

Julia and her husband should buy 12 pounds of City Roast Columbian coffee and 8 pounds of French Roast Columbian coffee.

Exercise:

Problem:

Melody wants to sell bags of mixed candy at her lemonade stand. She will mix chocolate pieces that cost $4.89 per bag with peanut butter pieces that cost $3.79 per bag to get a total of twenty-five bags of mixed candy. Melody wants the bags of mixed candy to cost her $4.23 a bag to make. How many bags of chocolate pieces and how many bags of peanut butter pieces should she use?

Exercise:

Problem:

Jotham needs 70 liters of a 50% alcohol solution. He has a 30% and an 80% solution available. How many liters of the 30% and how many liters of the 80% solutions should he mix to make the 50% solution?

Solution:

Jotham should mix 42 liters of the 30% solution and 28 liters of the 80% solution.

Exercise:

Problem:

Joy is preparing 15 liters of a 25% saline solution. She only has 40% and 10% solution in her lab. How many liters of the 40% and how many liters of the 10% should she mix to make the 25% solution?

Exercise:

Problem:

A scientist needs 65 liters of a 15% alcohol solution. She has available a 25% and a 12% solution. How many liters of the 25% and how many liters of the 12% solutions should she mix to make the 15% solution?

Solution:

The scientist should mix 15 liters of the 25% solution and 50 liters of the 12% solution.

Exercise:

Problem:

A scientist needs 120 liters of a 20% acid solution for an experiment. The lab has available a 25% and a 10% solution. How many liters of the 25% and how many liters of the 10% solutions should the scientist mix to make the 20% solution?

Exercise:

Problem:

A 40% antifreeze solution is to be mixed with a 70% antifreeze solution to get 240 liters of a 50% solution. How many liters of the 40% and how many liters of the 70% solutions will be used?

Solution:

160 liters of the 40% solution and 80 liters of the 70% solution will be used.

Exercise:

Problem:

A 90% antifreeze solution is to be mixed with a 75% antifreeze solution to get 360 liters of a 85% solution. How many liters of the 90% and how many liters of the 75% solutions will be used?

Solve Interest Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Hattie had $3,000 to invest and wants to earn 10.6% interest per year. She will put some of the money into an account that earns 12% per year and the rest into an account that earns 10% per year. How much money should she put into each account?

Solution:

Hattie should invest $900 at 12% and $2,100 at 10%.

Exercise:

Problem:

Carol invested $2,560 into two accounts. One account paid 8% interest and the other paid 6% interest. She earned 7.25% interest on the total investment. How much money did she put in each account?

Exercise:

Problem:

Sam invested $48,000, some at 6% interest and the rest at 10%. How much did he invest at each rate if he received $4,000 in interest in one year?

Solution:

Sam invested $28,000 at 10% and $20,000 at 6%.

Exercise:

Problem:

Arnold invested $64,000, some at 5.5% interest and the rest at 9%. How much did he invest at each rate if he received $4,500 in interest in one year?

Exercise:

Problem:

After four years in college, Josie owes $65,800 in student loans. The interest rate on the federal loans is 4.5% and the rate on the private bank loans is 2%. The total interest she owed for one year was $2,878.50. What is the amount of each loan?

Solution:

The federal loan is $62,500 and the bank loan is $3,300.

Exercise:

Problem:

Mark wants to invest $10,000 to pay for his daughter’s wedding next year. He will invest some of the money in a short term CD that pays 12% interest and the rest in a money market savings account that pays 5% interest. How much should he invest at each rate if he wants to earn $1,095 in interest in one year?

Exercise:

Problem:

A trust fund worth \$25,000 is invested in two different portfolios. This year, one portfolio is expected to earn 5.25% interest and the other is expected to earn 4%. Plans are for the total interest on the fund to be \$1150 in one year. How much money should be invested at each rate?

Solution:

\$12,000 should be invested at 5.25% and \$13,000 should be invested at 4%.

Exercise:

Problem:

A business has two loans totaling \$85,000. One loan has a rate of 6% and the other has a rate of 4.5%. This year, the business expects to pay \$4650 in interest on the two loans. How much is each loan?

Everyday Math

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Laurie was completing the treasurer’s report for her son’s Boy Scout troop at the end of the school year. She didn’t remember how many boys had paid the \$15 full-year registration fee and how many had paid the \$10 partial-year fee. She knew that the number of boys who paid for a full-year was ten more than the number who paid for a partial-year. If \$250 was collected for all the registrations, how many boys had paid the full-year fee and how many had paid the partial-year fee?

Solution:

14 boys paid the full-year fee. 4 boys paid the partial-year fee,

Exercise:

Problem:

As the treasurer of her daughter’s Girl Scout troop, Laney collected money for some girls and adults to go to a three-day camp. Each girl paid \$75 and each adult paid \$30. The total amount of money collected for camp was \$765. If the number of girls is three times the number of adults, how many girls and how many adults paid for camp?

Writing Exercises

Exercise:

Problem:

Take a handful of two types of coins, and write a problem similar to [link] relating the total number of coins and their total value. Set up a system of equations to describe your situation and then solve it.

Solution:

Answers will vary.

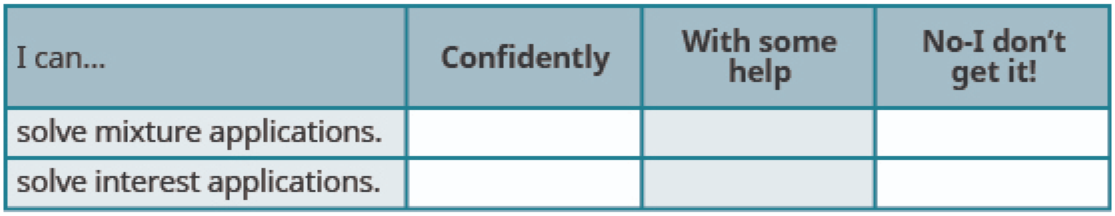
Exercise:

Problem:

In [link] we solved the system of equations $\begin{cases} b+f=21{,}540 \\ 0.105b+0.059f=1669.68 \end{cases}$ by substitution. Would you have used substitution or elimination to solve this system? Why?

Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
solve mixture applications.			
solve interest applications.			

After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Review Exercises

Solve Systems of Equations by Graphing

Determine Whether an Ordered Pair is a Solution of a System of Equations.

In the following exercises, determine if the following points are solutions to the given system of equations.

Exercise:

Problem: $\begin{cases} x+3y=-9 \\ 2x-4y=12 \end{cases}$

ⓐ $(-3,-2)$ ⓑ $(0,-3)$

Solution:

ⓐ no ⓑ yes

Exercise:

Problem: $\begin{cases} x+y=8 \\ y=x-4 \end{cases}$

ⓐ $(6,2)$ ⓑ $(9,-1)$

Solve a System of Linear Equations by Graphing

In the following exercises, solve the following systems of equations by graphing.

Exercise:

Problem: $\begin{cases} 3x+y=6 \\ x+3y=-6 \end{cases}$

Solution:

$(3,-3)$
[missing_resource: CNX_ElemAlg_Figure_05_06_244_img.jpg]

Exercise:

Problem: $\begin{cases} y=x-2 \\ y=-2x-2 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x-y=6 \\ y=4 \end{cases}$

Solution:

$(5,4)$
[missing_resource: CNX_ElemAlg_Figure_05_06_246_img.jpg]

Exercise:

Problem: $\begin{cases} x+4y=-1 \\ x=3 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x-y=5 \\ 4x-2y=10 \end{cases}$

Solution:

coincident lines
[missing_resource: CNX_ElemAlg_Figure_05_06_248_img.jpg]

Exercise:

Problem: $\begin{cases} -x+2y=4 \\ y=\frac{1}{2}x-3 \end{cases}$

Determine the Number of Solutions of a Linear System

In the following exercises, without graphing determine the number of solutions and then classify the system of equations.

Exercise:

Problem: $\begin{cases} y=\frac{2}{5}x+2 \\ -2x+5y=10 \end{cases}$

Solution:

infinitely many solutions, consistent system, dependent equations

Exercise:

Problem: $\begin{cases} 3x+2y=6 \\ y=-3x+4 \end{cases}$

Exercise:

Problem: $\begin{cases} 5x - 4y = 0 \\ y = \frac{5}{4}x - 5 \end{cases}$

Solution:

no solutions, inconsistent system, independent equations

Exercise:

Problem: $\begin{cases} y = -\frac{3}{4}x + 1 \\ 6x + 8y = 8 \end{cases}$

Solve Applications of Systems of Equations by Graphing

Exercise:

Problem:

LaVelle is making a pitcher of caffe mocha. For each ounce of chocolate syrup, she uses five ounces of coffee. How many ounces of chocolate syrup and how many ounces of coffee does she need to make 48 ounces of caffe mocha?

Solution:

LaVelle needs 8 ounces of chocolate syrup and 40 ounces of coffee.

Exercise:

Problem:

Eli is making a party mix that contains pretzels and chex. For each cup of pretzels, he uses three cups of chex. How many cups of pretzels and how many cups of chex does he need to make 12 cups of party mix?

Solve Systems of Equations by Substitution

Solve a System of Equations by Substitution

In the following exercises, solve the systems of equations by substitution.

Exercise:

Problem: $\begin{cases} 3x - y = -5 \\ y = 2x + 4 \end{cases}$

Solution:

$(-1, 2)$

Exercise:

Problem: $\begin{cases} 3x - 2y = 2 \\ y = \frac{1}{2}x + 3 \end{cases}$

Exercise:

Problem: $\begin{cases} x - y = 0 \\ 2x + 5y = -14 \end{cases}$

Solution:

$(-2, -2)$

Exercise:

Problem: $\begin{cases} y = -2x + 7 \\ y = \frac{2}{3}x - 1 \end{cases}$

Exercise:

Problem: $\begin{cases} y = -5x \\ 5x + y = 6 \end{cases}$

Solution:

no solution

Exercise:

Problem: $\begin{cases} y = -\frac{1}{3}x + 2 \\ x + 3y = 6 \end{cases}$

Solve Applications of Systems of Equations by Substitution

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

The sum of two number is 55. One number is 11 less than the other. Find the numbers.

Solution:

The numbers are 22 and 33.

Exercise:

Problem:

The perimeter of a rectangle is 128. The length is 16 more than the width. Find the length and width.

Exercise:

Problem:

The measure of one of the small angles of a right triangle is 2 less than 3 times the measure of the other small angle. Find the measure of both angles.

Solution:

The measures are 23 degrees and 67 degrees.

Exercise:

Problem:

Gabriela works for an insurance company that pays her a salary of \$32,000 plus a commission of \$100 for each policy she sells. She is considering changing jobs to a company that would pay a salary of \$40,000 plus a commission of \$80 for each policy sold. How many policies would Gabriela need to sell to make the total pay the same?

Solve Systems of Equations by Elimination

Solve a System of Equations by Elimination In the following exercises, solve the systems of equations by elimination.

Exercise:

Problem: $\begin{cases} x + y = 12 \\ x - y = -10 \end{cases}$

Solution:

$(1, 11)$

Exercise:

Problem: $\begin{cases} 4x + 2y = 2 \\ -4x - 3y = -9 \end{cases}$

Exercise:

Problem: $\begin{cases} 3x - 8y = 20 \\ x + 3y = 1 \end{cases}$

Solution:

$(4, -1)$

Exercise:

Problem: $\begin{cases} 3x - 2y = 6 \\ 4x + 3y = 8 \end{cases}$

Exercise:

Problem: $\begin{cases} 9x + 4y = 2 \\ 5x + 3y = 5 \end{cases}$

Solution:

$(-2, 5)$

Exercise:

Problem: $\begin{cases} -x + 3y = 8 \\ 2x - 6y = -20 \end{cases}$

Solve Applications of Systems of Equations by Elimination

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

The sum of two numbers is -90. Their difference is 16. Find the numbers.

Solution:

The numbers are -37 and -53.

Exercise:

Problem:

Omar stops at a donut shop every day on his way to work. Last week he had 8 donuts and 5 cappuccinos, which gave him a total of 3,000 calories. This week he had 6 donuts and 3 cappuccinos, which was a total of 2,160 calories. How many calories are in one donut? How many calories are in one cappuccino?

Choose the Most Convenient Method to Solve a System of Linear Equations

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

Exercise:

Problem: $\begin{cases} 6x - 5y = 27 \\ 3x + 10y = -24 \end{cases}$

Solution:

elimination

Exercise:

Problem: $\begin{cases} y = 3x - 9 \\ 4x - 5y = 23 \end{cases}$

Solve Applications with Systems of Equations

Translate to a System of Equations

In the following exercises, translate to a system of equations. Do not solve the system.

Exercise:

Problem:

The sum of two numbers is -32. One number is two less than twice the other. Find the numbers.

Solution:

$$\begin{cases} x + y = -32 \\ x = 2y - 2 \end{cases}$$

Exercise:

Problem:

Four times a number plus three times a second number is -9. Twice the first number plus the second number is three. Find the numbers.

Exercise:

Problem:

Last month Jim and Debbie earned $7,200. Debbie earned $1,600 more than Jim earned. How much did they each earn?

Solution:

$$\begin{cases} j + d = 7200 \\ d = j + 1600 \end{cases}$$

Exercise:

Problem:

Henri has $24,000 invested in stocks and bonds. The amount in stocks is $6,000 more than three times the amount in bonds. How much is each investment?

Solve Direct Translation Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Pam is 3 years older than her sister, Jan. The sum of their ages is 99. Find their ages.

Solution:

Pam is 51 and Jan is 48.

Exercise:

Problem:

Mollie wants to plant 200 bulbs in her garden. She wants all irises and tulips. She wants to plant three times as many tulips as irises. How many irises and how many tulips should she plant?

Solve Geometry Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

The difference of two supplementary angles is 58 degrees. Find the measures of the angles.

Solution:

The measures are 119 degrees and 61 degrees.

Exercise:

Problem:

Two angles are complementary. The measure of the larger angle is five more than four times the measure of the smaller angle. Find the measures of both angles.

Exercise:

Problem:

Becca is hanging a 28 foot floral garland on the two sides and top of a pergola to prepare for a wedding. The height is four feet less than the width. Find the height and width of the pergola.

Solution:

The pergola is 8 feet high and 12 feet wide.

Exercise:

Problem:

The perimeter of a city rectangular park is 1428 feet. The length is 78 feet more than twice the width. Find the length and width of the park.

Solve Uniform Motion Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Sheila and Lenore were driving to their grandmother’s house. Lenore left one hour after Sheila. Sheila drove at a rate of 45 mph, and Lenore drove at a rate of 60 mph. How long will it take for Lenore to catch up to Sheila?

Solution:

It will take Lenore 3 hours.

Exercise:

Problem:

Bob left home, riding his bike at a rate of 10 miles per hour to go to the lake. Cheryl, his wife, left 45 minutes ($\frac{3}{4}$ hour) later, driving her car at a rate of 25 miles per hour. How long will it take Cheryl to catch up to Bob?

Exercise:

Problem:

Marcus can drive his boat 36 miles down the river in three hours but takes four hours to return upstream. Find the rate of the boat in still water and the rate of the current.

Solution:

The rate of the boat is 10.5 mph. The rate of the current is 1.5 mph.

Exercise:

Problem:

A passenger jet can fly 804 miles in 2 hours with a tailwind but only 776 miles in 2 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solve Mixture Applications with Systems of Equations

Solve Mixture Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Lynn paid a total of $2,780 for 261 tickets to the theater. Student tickets cost $10 and adult tickets cost $15. How many student tickets and how many adult tickets did Lynn buy?

Solution:

Lynn bought 227 student tickets and 34 adult tickets.

Exercise:

Problem:

Priam has dimes and pennies in a cup holder in his car. The total value of the coins is $4.21. The number of dimes is three less than four times the number of pennies. How many dimes and how many pennies are in the cup?

Exercise:

Problem:

Yumi wants to make 12 cups of party mix using candies and nuts. Her budget requires the party mix to cost her $1.29 per cup. The candies are $2.49 per cup and the nuts are $0.69 per cup. How many cups of candies and how many cups of nuts should she use?

Solution:

Yumi should use 4 cups of candies and 8 cups of nuts.

Exercise:

Problem:

A scientist needs 70 liters of a 40% solution of alcohol. He has a 30% and a 60% solution available. How many liters of the 30% and how many liters of the 60% solutions should he mix to make the 40% solution?

Solve Interest Applications

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

Jack has $12,000 to invest and wants to earn 7.5% interest per year. He will put some of the money into a savings account that earns 4% per year and the rest into CD account that earns 9% per year. How much money should he put into each account?

Solution:

Jack should put $3600 into savings and $8400 into the CD.

Exercise:

Problem:

When she graduates college, Linda will owe $43,000 in student loans. The interest rate on the federal loans is 4.5% and the rate on the private bank loans is 2%. The total interest she owes for one year was $1585. What is the amount of each loan?

Graphing Systems of Linear Inequalities

Determine Whether an Ordered Pair is a Solution of a System of Linear Inequalities

In the following exercises, determine whether each ordered pair is a solution to the system.

Exercise:

Problem: $\begin{cases} 4x + y > 6 \\ 3x - y \leq 12 \end{cases}$

ⓐ $(2,-1)$ ⓑ $(3,-2)$

Solution:

ⓐ yes ⓑ no

Exercise:

Problem: $\begin{cases} y > \frac{1}{3}x + 2 \\ x - \frac{1}{4}y \le 10 \end{cases}$

ⓐ $(6,5)$ ⓑ $(15,8)$

Solve a System of Linear Inequalities by Graphing

In the following exercises, solve each system by graphing.

Exercise:

Problem: $\begin{cases} y < 3x + 1 \\ y \ge -x - 2 \end{cases}$

Solution: [missing_resource: CNX_ElemAlg_Figure_05_06_250_img.jpg]

Exercise:

Problem: $\begin{cases} x - y > -1 \\ y < \frac{1}{3}x - 2 \end{cases}$

Exercise:

Problem: $\begin{cases} 2x - 3y < 6 \\ 3x + 4y \ge 12 \end{cases}$

Solution: [missing_resource: CNX_ElemAlg_Figure_05_06_252_img.jpg]

Exercise:

Problem: $\begin{cases} y \le -\frac{3}{4}x + 1 \\ x \ge -5 \end{cases}$

Exercise:

Problem: $\begin{cases} x + 3y < 5 \\ y \ge -\frac{1}{3}x + 6 \end{cases}$

Solution:

No solution
[missing_resource: CNX_ElemAlg_Figure_05_06_254_img.jpg]

Exercise:

Problem: $\begin{cases} y \ge 2x - 5 \\ -6x + 3y > -4 \end{cases}$

Solve Applications of Systems of Inequalities

In the following exercises, translate to a system of inequalities and solve.
Exercise:

Problem:

Roxana makes bracelets and necklaces and sells them at the farmers' market. She sells the bracelets for $12 each and the necklaces for $18 each. At the market next weekend she will have room to display no more than 40 pieces, and she needs to sell at least $500 worth in order to earn a profit.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Should she display 26 bracelets and 14 necklaces?
ⓓ Should she display 39 bracelets and 1 necklace?

Solution:

ⓐ $\begin{cases} b+n \le 40 \\ 12b+18n \ge 500 \end{cases}$
ⓑ
[missing_resource: CNX_ElemAlg_Figure_05_06_256_img.jpg]

ⓒ yes
ⓓ no

Exercise:

Problem:

Annie has a budget of $600 to purchase paperback books and hardcover books for her classroom. She wants the number of hardcover to be at least 5 more than three times the number of paperback books. Paperback books cost $4 each and hardcover books cost $15 each.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can she buy 8 paperback books and 40 hardcover books?
ⓓ Can she buy 10 paperback books and 37 hardcover books?

Practice Test

Exercise:

Problem: $\begin{cases} x-4y=-8 \\ 2x+5y=10 \end{cases}$

ⓐ $(0,2)$ ⓑ $(4,3)$

Solution:

ⓐ yes ⓑ no

In the following exercises, solve the following systems by graphing.

Exercise:

Problem: $\begin{cases} x - y = 5 \\ x + 2y = -4 \end{cases}$

Exercise:

Problem: $\begin{cases} x - y > -2 \\ y \leq 3x + 1 \end{cases}$

Solution: [missing_resource: CNX_ElemAlg_Figure_05_06_259_img.jpg]

In the following exercises, solve each system of equations. Use either substitution or elimination.

Exercise:

Problem: $\begin{cases} 3x - 2y = 3 \\ y = 2x - 1 \end{cases}$

Exercise:

Problem: $\begin{cases} x + y = -3 \\ x - y = 11 \end{cases}$

Solution:

$(4, -7)$

Exercise:

Problem: $\begin{cases} 4x-3y=7 \\ 5x-2y=0 \end{cases}$

Exercise:

Problem: $\begin{cases} y=-\frac{4}{5}x+1 \\ 8x+10y=10 \end{cases}$

Solution:

infinitely many solutions

Exercise:

Problem: $\begin{cases} 2x+3y=12 \\ -4x+6y=-16 \end{cases}$

In the following exercises, translate to a system of equations and solve.

Exercise:

Problem:

The sum of two numbers is −24. One number is 104 less than the other. Find the numbers.

Solution:

The numbers are 40 and −64.

Exercise:

Problem:

Ramon wants to plant cucumbers and tomatoes in his garden. He has room for 16 plants, and he wants to plant three times as many cucumbers as tomatoes. How many cucumbers and how many tomatoes should he plant?

Exercise:

Problem:

Two angles are complementary. The measure of the larger angle is six more than twice the measure of the smaller angle. Find the measures of both angles.

Solution:

The measures of the angles are 28 degrees and 62 degrees.

Exercise:

Problem:

On Monday, Lance ran for 30 minutes and swam for 20 minutes. His fitness app told him he had burned 610 calories. On Wednesday, the fitness app told him he burned 695 calories when he ran for 25 minutes and swam for 40 minutes. How many calories did he burn for one minute of running? How many calories did he burn for one minute of swimming?

Exercise:

Problem:

Kathy left home to walk to the mall, walking quickly at a rate of 4 miles per hour. Her sister Abby left home 15 minutes later and rode her bike to the mall at a rate of 10 miles per hour. How long will it take Abby to catch up to Kathy?

Solution:

It will take Abby $\frac{1}{6}$ of an hour (or 10 minutes).

Exercise:

Problem:

It takes $5\frac{1}{2}$ hours for a jet to fly 2,475 miles with a headwind from San Jose, California to Lihue, Hawaii. The return flight from Lihue to San Jose with a tailwind, takes 5 hours. Find the speed of the jet in still air and the speed of the wind.

Exercise:

Problem:

Liz paid $160 for 28 tickets to take the Brownie troop to the science museum. Children’s tickets cost $5 and adult tickets cost $9. How many children’s tickets and how many adult tickets did Liz buy?

Solution:

Liz bought 23 children’s tickets and 5 adult tickets.

Exercise:

Problem:

A pharmacist needs 20 liters of a 2% saline solution. He has a 1% and a 5% solution available. How many liters of the 1% and how many liters of the 5% solutions should she mix to make the 2% solution?

Exercise:

Problem: Translate to a system of inequalities and solve.

Andi wants to spend no more than $50 on Halloween treats. She wants to buy candy bars that cost $1 each and lollipops that cost $0.50 each, and she wants the number of lollipops to be at least three times the number of candy bars.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can she buy 20 candy bars and 70 lollipops?
ⓓ Can she buy 15 candy bars and 65 lollipops?

Solution:

ⓐ $\begin{cases} C + 0.5L \leq 50 \\ L \geq 3C \end{cases}$

ⓑ

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ⓒ No

ⓓ Yes

Introduction
class="introduction"

Architects use polynomials to design curved shapes such as this suspension bridge, the Silver Jubilee bridge in Halton, England.

We have seen that the graphs of linear equations are straight lines. Graphs of other types of equations, called polynomial equations, are curves, like the outline of this suspension bridge. Architects use polynomials to design the shape of a bridge like this and to draw the blueprints for it. Engineers use polynomials to calculate the stress on the bridge's supports to ensure they are strong enough for the intended load. In this chapter, you will explore operations with and properties of polynomials.

Add and Subtract Polynomials

By the end of this section, you will be able to:

- Identify polynomials, monomials, binomials, and trinomials
- Determine the degree of polynomials
- Add and subtract monomials
- Add and subtract polynomials
- Evaluate a polynomial for a given value

Note:
Before you get started, take this readiness quiz.

1. Simplify: $8x+3x$.
   If you missed this problem, review [link].
2. Subtract: $(5n+8)-(2n-1)$.
   If you missed this problem, review [link].
3. Write in expanded form: a^5.
   If you missed this problem, review [link].

Identify Polynomials, Monomials, Binomials and Trinomials

You have learned that a *term* is a constant or the product of a constant and one or more variables. When it is of the form ax^m, where a is a constant and m is a whole number, it is called a monomial. Some examples of monomial are $8, -2x^2, 4y^3, \text{ and } 11z^7$.

Note:
Monomials
A **monomial** is a term of the form ax^m, where a is a constant and m is a positive whole number.

A monomial, or two or more monomials combined by addition or subtraction, is a polynomial. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term. A binomial has exactly two terms, and a trinomial has exactly three terms. There are no special names for polynomials with more than three terms.

Note:
Polynomials
polynomial—A monomial, or two or more monomials combined by addition or subtraction, is a polynomial.

- **monomial**—A polynomial with exactly one term is called a monomial.
- **binomial**—A polynomial with exactly two terms is called a binomial.
- **trinomial**—A polynomial with exactly three terms is called a trinomial.

Here are some examples of polynomials.

Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$	
Monomial	14	$8y^2$	$-9x^3y^5$	$-$
Binomial	$a+7$	$4b-5$	y^2-16	$3a$
Trinomial	$x^2-7x+12$	$9y^2+2y-8$	$6m^4-m^3+8m$	z^4

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the "family" of polynomials and so they have special names. We use the words *monomial*, *binomial*, and *trinomial* when referring to these special polynomials and just call all the rest *polynomials*.

Example:
Exercise:

Problem: Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial.

ⓐ $4y^2-8y-6$
ⓑ $-5a^4b^2$
ⓒ $2x^5-5x^3-9x^2+3x+4$
ⓓ $13-5m^3$
ⓔ q

Solution:
Solution

	Polynomial	Number of terms	Type
(a)	$4y^2-8y-6$	3	Trinomial
(b)	$-5a^4b^2$	1	Monomial
(c)	$2x^5-5x^3-9x^2+3x+4$	5	Polynomial
(d)	$13-5m^3$	2	Binomial
(e)	q	1	Monomial

Note:
Exercise:

Problem: Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial:

ⓐ $5b$ ⓑ $8y^3-7y^2-y-3$ ⓒ $-3x^2-5x+9$ ⓓ $81-4a^2$ ⓔ $-5x^6$

Solution:

ⓐ monomial ⓑ polynomial ⓒ trinomial ⓓ binomial ⓔ monomial

Note:
Exercise:

Problem: Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial:

ⓐ $27z^3 - 8$ ⓑ $12m^3 - 5m^2 - 2m$ ⓒ $\frac{5}{6}$ ⓓ $8x^4 - 7x^2 - 6x - 5$ ⓔ $-n^4$

Solution:

ⓐ binomial ⓑ trinomial ⓒ monomial ⓓ polynomial ⓔ monomial

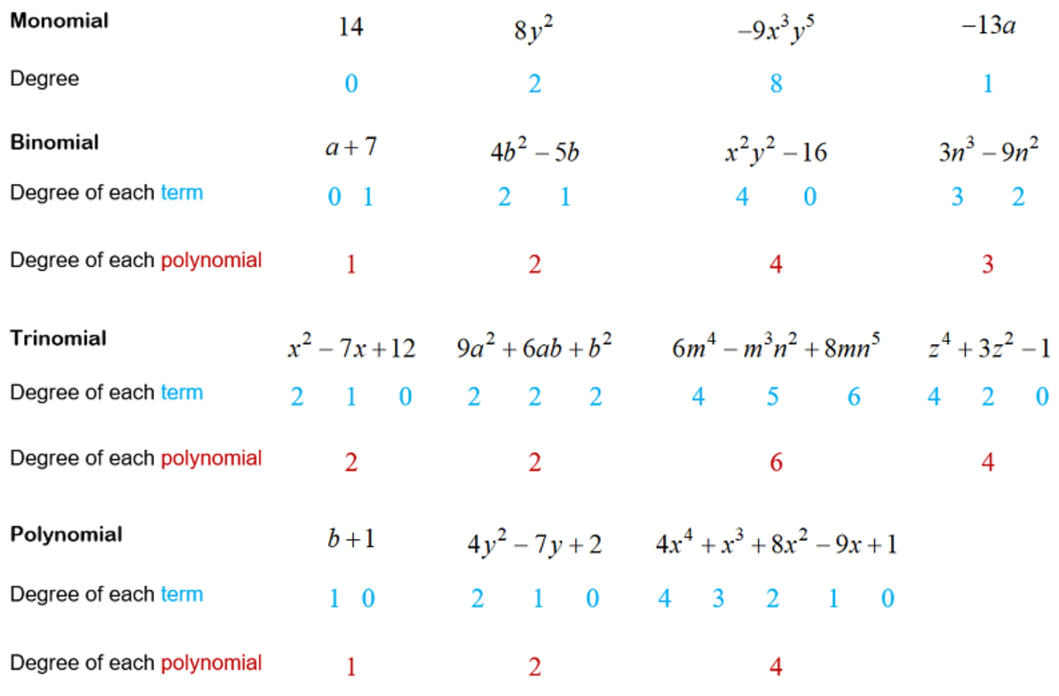
Determine the Degree of Polynomials

The degree of a polynomial and the degree of its terms are determined by the exponents of the variable.

A monomial that has no variable, just a constant, is a special case. The degree of a constant is 0—it has no variable.

Note:
Degree of a Polynomial
The **degree of a term** is the sum of the exponents of its variables.
The **degree of a constant** is 0.
The **degree of a polynomial** is the highest degree of all its terms.

Let’s see how this works by looking at several polynomials. We’ll take it step by step, starting with monomials, and then progressing to polynomials with more terms.



Monomial	14	$8y^2$	$-9x^3y^5$	$-13a$
Degree	0	2	8	1
Binomial	$a+7$	$4b^2-5b$	x^2y^2-16	$3n^3-9n^2$
Degree of each term	0 1	2 1	4 0	3 2
Degree of each polynomial	1	2	4	3
Trinomial	$x^2-7x+12$	$9a^2+6ab+b^2$	$6m^4-m^3n^2+8mn^5$	z^4+3z^2-1
Degree of each term	2 1 0	2 2 2	4 5 6	4 2 0
Degree of each polynomial	2	2	6	4
Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$	
Degree of each term	1 0	2 1 0	4 3 2 1 0	
Degree of each polynomial	1	2	4	

A polynomial is in **standard form** when the terms of a polynomial are written in descending order of degrees. Get in the habit of writing the term with the highest degree first.

Example:
Exercise:

Problem: Find the degree of the following polynomials.

ⓐ $10y$
ⓑ $4x^3 - 7x + 5$
ⓒ -15
ⓓ $-8b^2 + 9b - 2$
ⓔ $8xy^2 + 2y$

Solution:
Solution

ⓐ

	$10y$
The exponent of y is one. $y = y^1$	The degree is 1.

ⓑ

	$4x^3 - 7x + 5$
The highest degree of all the terms is 3.	The degree is 3.

ⓒ

	-15
The degree of a constant is 0.	The degree is 0.

ⓓ

	$-8b^2 + 9b - 2$
The highest degree of all the terms is 2.	The degree is 2.

ⓔ

	$8xy^2 + 2y$
The highest degree of all the terms is 3.	The degree is 3.

Note:
Exercise:

Problem: Find the degree of the following polynomials:

ⓐ $-15b$ ⓑ $10z^4 + 4z^2 - 5$ ⓒ $12c^5d^4 + 9c^3d^9 - 7$ ⓓ $3x^2y - 4x$ ⓔ -9

Solution:

ⓐ 1 ⓑ 4 ⓒ 12 ⓓ 3 ⓔ 0

Note:
Exercise:

Problem: Find the degree of the following polynomials:

ⓐ 52 ⓑ $a^4b - 17a^4$ ⓒ $5x + 6y + 2z$ ⓓ $3x^2 - 5x + 7$ ⓔ $-a^3$

Solution:

ⓐ 0 ⓑ 5 ⓒ 1 ⓓ 2 ⓔ 3

Add and Subtract Monomials

You have learned how to simplify expressions by combining like terms. Remember, like terms must have the same variables with the same exponent. Since monomials are terms, adding and subtracting monomials is the same as combining like terms. If the monomials are like terms, we just combine them by adding or subtracting the coefficient.

Example:
Exercise:

Problem: Add: $25y^2 + 15y^2$.

Solution:
Solution

	$25y^2 + 15y^2$
Combine like terms.	$40y^2$

Note:
Exercise:

Problem: Add: $12q^2 + 9q^2$.

Solution:

$21q^2$

Note:
Exercise:

Problem: Add: $-15c^2 + 8c^2$.

Solution:

$-7c^2$

Example:
Exercise:

Problem: Subtract: $16p-(-7p)$.

Solution:
Solution

	$16p-(-7p)$
Combine like terms.	$23p$

Note:
Exercise:

Problem: Subtract: $8m-(-5m)$.

Solution:

$13m$

Note:
Exercise:

Problem: Subtract: $-15z^3-\left(-5z^3\right)$.

Solution:

$-10z^3$

Remember that like terms must have the same variables with the same exponents.

Example:
Exercise:

Problem: Simplify: $c^2+7d^2-6c^2$.

Solution:
Solution

	$c^2+7d^2-6c^2$
Combine like terms.	$-5c^2+7d^2$

Note:

Exercise:

Problem: Add: $8y^2 + 3z^2 - 3y^2$.

Solution:

$5y^2 + 3z^2$

Note:
Exercise:

Problem: Add: $3m^2 + n^2 - 7m^2$.

Solution:

$-4m^2 + n^2$

Example:
Exercise:

Problem: Simplify: $u^2v + 5u^2 - 3v^2$.

Solution:
Solution

	$u^2v + 5u^2 - 3v^2$
There are no like terms to combine.	$u^2v + 5u^2 - 3v^2$

Note:
Exercise:

Problem: Simplify: $m^2n^2 - 8m^2 + 4n^2$.

Solution:

There are no like terms to combine.

Note:
Exercise:

Problem: Simplify: $pq^2 - 6p - 5q^2$.

Solution:

There are no like terms to combine.

Add and Subtract Polynomials

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

Example:
Exercise:

Problem: Find the sum: $\left(5y^2 - 3y + 15\right) + \left(3y^2 - 4y - 11\right)$.

Solution:
Solution

Identify like terms.	$(\underline{\underline{5y^2}} - \underline{3y} + \underline{1}5) + (\underline{\underline{3y^2}} - \underline{4y} - \underline{1}1)$
Rearrange to get the like terms together.	$\underline{\underline{5y^2 + 3y^2}} - \underline{3y - 4y} + \underline{1}5 - \underline{1}1$
Combine like terms.	$8y^2 - 7y + 4$

Note:
Exercise:

Problem: Find the sum: $\left(7x^2 - 4x + 5\right) + \left(x^2 - 7x + 3\right)$.

Solution:

$8x^2 - 11x + 8$

Note:
Exercise:

Problem: Find the sum: $\left(14y^2 + 6y - 4\right) + \left(3y^2 + 8y + 5\right)$.

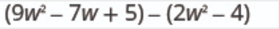
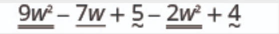
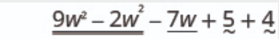
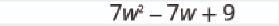
Solution:

$17y^2 + 14y + 1$

Example:
Exercise:

Problem: Find the difference: $\left(9w^2 - 7w + 5\right) - \left(2w^2 - 4\right)$.

Solution:
Solution

	$(9w^2 - 7w + 5) - (2w^2 - 4)$
Distribute and identify like terms.	$\underline{\underline{9w^2}} - \underline{7w} + \underset{\sim}{5} - \underline{\underline{2w^2}} + \underset{\sim}{4}$
Rearrange the terms.	$\underline{\underline{9w^2 - 2w^2}} - \underline{7w} + \underset{\sim}{5} + \underset{\sim}{4}$
Combine like terms.	$7w^2 - 7w + 9$

Note:
Exercise:

Problem: Find the difference: $\left(8x^2 + 3x - 19\right) - \left(7x^2 - 14\right)$.

Solution:

$x^2 + 3x - 5$

Note:
Exercise:

Problem: Find the difference: $\left(9b^2 - 5b - 4\right) - \left(3b^2 - 5b - 7\right)$.

Solution:

$6b^2 + 3$

Example:
Exercise:

Problem: Subtract: $\left(c^2 - 4c + 7\right)$ from $\left(7c^2 - 5c + 3\right)$.

Solution:
Solution

	Subtract $(c^2 - 4c + 7)$ from $(7c^2 - 5c + 3)$.
	$(7c^2 - 5c + 3) - (c^2 - 4c + 7)$
Distribute and identify like terms.	$\underline{7c^2} - \underline{5c} + \underset{\sim}{3} - \underline{c^2} + \underline{4c} - \underset{\sim}{7}$
Rearrange the terms.	$\underline{7c^2 - c^2} - \underline{5c + 4c} + \underset{\sim}{3} - \underset{\sim}{7}$
Combine like terms.	$6c^2 - c - 4$

Note:
Exercise:

Problem: Subtract: $\left(5z^2 - 6z - 2\right)$ from $\left(7z^2 + 6z - 4\right)$.

Solution:

$2z^2 + 12z - 2$

Note:
Exercise:

Problem: Subtract: $\left(x^2 - 5x - 8\right)$ from $\left(6x^2 + 9x - 1\right)$.

Solution:

$5x^2 + 14x + 7$

Example:
Exercise:

Problem: Find the sum: $\left(u^2 - 6uv + 5v^2\right) + \left(3u^2 + 2uv\right)$.

Solution:
Solution

	$\left(u^2 - 6uv + 5v^2\right) + \left(3u^2 + 2uv\right)$
Distribute.	$u^2 - 6uv + 5v^2 + 3u^2 + 2uv$
Rearrange the terms, to put like terms together.	$u^2 + 3u^2 - 6uv + 2uv + 5v^2$
Combine like terms.	$4u^2 - 4uv + 5v^2$

Note:
Exercise:

Problem: Find the sum: $\left(3x^2 - 4xy + 5y^2\right) + \left(2x^2 - xy\right)$.

Solution:

$5x^2 - 5xy + 5y^2$

Note:
Exercise:

Problem: Find the sum: $\left(2x^2 - 3xy - 2y^2\right) + \left(5x^2 - 3xy\right)$.

Solution:

$7x^2 - 6xy - 2y^2$

Example:
Exercise:

Problem: Find the difference: $\left(p^2 + q^2\right) - \left(p^2 + 10pq - 2q^2\right)$.

Solution:
Solution

	$\left(p^2 + q^2\right) - \left(p^2 + 10pq - 2q^2\right)$
Distribute.	$p^2 + q^2 - p^2 - 10pq + 2q^2$
Rearrange the terms, to put like terms together.	$p^2 - p^2 - 10pq + q^2 + 2q^2$
Combine like terms.	$-10pq^2 + 3q^2$

Note:
Exercise:

Problem: Find the difference: $\left(a^2+b^2\right)-\left(a^2+5ab-6b^2\right)$.

Solution:

$-5ab-5b^2$

Note:
Exercise:

Problem: Find the difference: $\left(m^2+n^2\right)-\left(m^2-7mn-3n^2\right)$.

Solution:

$4n^2+7mn$

Example:
Exercise:

Problem: Simplify: $\left(a^3-a^2b\right)-\left(ab^2+b^3\right)+\left(a^2b+ab^2\right)$.

Solution:
Solution

	$\left(a^3-a^2b\right)-\left(ab^2+b^3\right)+\left(a^2b+ab^2\right)$
Distribute.	$a^3-a^2b-ab^2-b^3+a^2b+ab^2$
Rearrange the terms, to put like terms together.	$a^3-a^2b+a^2b-ab^2+ab^2-b^3$
Combine like terms.	a^3-b^3

Note:
Exercise:

Problem: Simplify: $\left(x^3-x^2y\right)-\left(xy^2+y^3\right)+\left(x^2y+xy^2\right)$.

Solution:

x^3-y^3

Note:
Exercise:

Problem: Simplify: $\left(p^3 - p^2q\right) + \left(pq^2 + q^3\right) - \left(p^2q + pq^2\right)$.

Solution:

$p^3 - 2p^2q + q^3$

Evaluate a Polynomial for a Given Value

We have already learned how to evaluate expressions. Since polynomials are expressions, we'll follow the same procedures to evaluate a polynomial. We will substitute the given value for the variable and then simplify using the order of operations.

Example:
Exercise:

Problem: Evaluate $5x^2 - 8x + 4$ when

ⓐ $x = 4$
ⓑ $x = -2$
ⓒ $x = 0$

Solution:
Solution

ⓐ $x = 4$	
	$5x^2 - 8x + 4$
Substitute 4 for *x*.	$5(4)^2 - 8(4) + 4$
Simplify the exponents.	$5 \cdot 16 - 8(4) + 4$
Multiply.	$80 - 32 + 4$
Simplify.	52

ⓑ $x = -2$	
	$5x^2 - 8x + 4$
Substitute -2 for x.	$5(-2)^2 - 8(-2) + 4$
Simplify the exponents.	$5 \cdot 4 - 8(-2) + 4$
Multiply.	$20 + 16 + 4$
Simplify.	40

ⓒ $x = 0$	
	$5x^2 - 8x + 4$
Substitute 0 for x.	$5(0)^2 - 8(0) + 4$
Simplify the exponents.	$5 \cdot 0 - 8(0) + 4$
Multiply.	$0 + 0 + 4$
Simplify.	4

Note:
Exercise:

Problem: Evaluate: $3x^2 + 2x - 15$ when

ⓐ $x = 3$
ⓑ $x = -5$
ⓒ $x = 0$

Solution:

ⓐ 18 ⓑ 50 ⓒ -15

Note:
Exercise:

Problem: Evaluate: $5z^2 - z - 4$ when

ⓐ $z = -2$
ⓑ $z = 0$
ⓒ $z = 2$

Solution:

ⓐ 18 ⓑ -4 ⓒ 14

Example:
Exercise:

Problem:

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250 foot tall building. Find the height after $t = 2$ seconds.

Solution:
Solution

	$-16t^2 + 250$
Substitute $t = 2$.	$-16(2)^2 + 250$
Simplify.	$-16 \cdot 4 + 250$
Simplify.	$-64 + 250$
Simplify.	186
	After 2 seconds the height of the ball is 186 feet.

Note:
Exercise:

Problem:

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 0$ seconds.

Solution:

250

Note:
Exercise:

Problem:

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 3$ seconds.

Solution:

106

Example:
Exercise:

Problem:

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and sides of height y feet. Find the cost of producing a box with $x = 4$ feet and $y = 6$ feet.

Solution:
Solution

	$6x^2 + 15xy$
Substitute $x = 4$, $y = 6$.	$6(4)^2 + 15(4)(6)$
Simplify.	$6 \cdot 16 + 15(4)(6)$
Simplify.	$96 + 360$
Simplify.	456
	The cost of producing the box is \$456.

Note:
Exercise:

Problem:

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and sides of height y feet. Find the cost of producing a box with $x = 6$ feet and $y = 4$ feet.

Solution:

$576

Note:
Exercise:

Problem:

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and sides of height y feet. Find the cost of producing a box with $x = 5$ feet and $y = 8$ feet.

Solution:

$750

Note:
Access these online resources for additional instruction and practice with adding and subtracting polynomials.

- Add and Subtract Polynomials 1
- Add and Subtract Polynomials 2
- Add and Subtract Polynomial 3
- Add and Subtract Polynomial 4

Key Concepts

- **Monomials**
  - A monomial is a term of the form ax^m, where a is a constant and m is a whole number
- **Polynomials**
  - **polynomial**—A monomial, or two or more monomials combined by addition or subtraction is a polynomial.
  - **monomial**—A polynomial with exactly one term is called a monomial.
  - **binomial**—A polynomial with exactly two terms is called a binomial.
  - **trinomial**—A polynomial with exactly three terms is called a trinomial.
- **Degree of a Polynomial**
  - The **degree of a term** is the sum of the exponents of its variables.
  - The **degree of a constant** is 0.
  - The **degree of a polynomial** is the highest degree of all its terms.

Practice Makes Perfect

Identify Polynomials, Monomials, Binomials, and Trinomials

In the following exercises, determine if each of the following polynomials is a monomial, binomial, trinomial, or other polynomial.

Exercise:

ⓐ $81b^5 - 24b^3 + 1$
ⓑ $5c^3 + 11c^2 - c - 8$
ⓒ $\frac{14}{15}y + \frac{1}{7}$
ⓓ 5
Problem: ⓔ $4y + 17$

Solution:

ⓐ trinomial ⓑ polynomial ⓒ binomial ⓓ monomial ⓔ binomial

Exercise:

ⓐ $x^2 - y^2$
ⓑ $-13c^4$
ⓒ $x^2 + 5x - 7$
ⓓ $x^2y^2 - 2xy + 8$
Problem: ⓔ 19

Exercise:

ⓐ $8 - 3x$
ⓑ $z^2 - 5z - 6$
ⓒ $y^3 - 8y^2 + 2y - 16$
ⓓ $81b^5 - 24b^3 + 1$
Problem: ⓔ -18

Solution:

ⓐ binomial ⓑ trinomial ⓒ polynomial ⓓ trinomial ⓔ monomial

Exercise:

ⓐ $11y^2$
ⓑ -73
ⓒ $6x^2 - 3xy + 4x - 2y + y^2$
ⓓ $4y + 17$
Problem: ⓔ $5c^3 + 11c^2 - c - 8$

Determine the Degree of Polynomials

In the following exercises, determine the degree of each polynomial.

Exercise:

ⓐ $6a^2+12a+14$
ⓑ $18xy^2z$
ⓒ $5x+2$
ⓓ $y^3-8y^2+2y-16$
Problem: ⓔ -24

Solution:

ⓐ 2 ⓑ 4 ⓒ 1 ⓓ 3 ⓔ 0

Exercise:

ⓐ $9y^3-10y^2+2y-6$
ⓑ $-12p^4$
ⓒ $a^2+9a+18$
ⓓ $20x^2y^2-10a^2b^2+30$
Problem: ⓔ 17

Exercise:

ⓐ $14-29x$
ⓑ z^2-5z-6
ⓒ $y^3-8y^2+2y-16$
ⓓ $23ab^2-14$
Problem: ⓔ -3

Solution:

ⓐ 1 ⓑ 2 ⓒ 3 ⓓ 3 ⓔ 0

Exercise:

ⓐ $62y^2$
ⓑ 15
ⓒ $6x^2-3xy+4x-2y+y^2$
ⓓ $10-9x$
Problem: ⓔ $m^4+4m^3+6m^2+4m+1$

Add and Subtract Monomials

In the following exercises, add or subtract the monomials.

Exercise:

Problem: $7x^2+5x^2$

Solution:

$12x^2$

Exercise:

Problem: $4\mathrm{y}^3 + 6y^3$

Exercise:

Problem: $-12w + 18w$

Solution:

$6w$

Exercise:

Problem: $-3m + 9m$

Exercise:

Problem: $4\mathrm{a} - 9a$

Solution:

$-5a$

Exercise:

Problem: $-y - 5y$

Exercise:

Problem: $28x - (-12x)$

Solution:

$40x$

Exercise:

Problem: $13z - (-4z)$

Exercise:

Problem: $-5b - 17b$

Solution:

$-22b$

Exercise:

Problem: $-10x - 35x$

Exercise:

Problem: $12a + 5b - 22a$

Solution:

$-10\mathrm{a} + 5b$

Exercise:

Problem: $14\text{x} - 3y - 13x$

Exercise:

Problem: $2a^2 + b^2 - 6a^2$

Solution:

$-4a^2 + b^2$

Exercise:

Problem: $5u^2 + 4v^2 - 6u^2$

Exercise:

Problem: $xy^2 - 5x - 5y^2$

Solution:

$xy^2 - 5x - 5y^2$

Exercise:

Problem: $pq^2 - 4p - 3q^2$

Exercise:

Problem: $a^2b - 4a - 5ab^2$

Solution:

$a^2b - 4a - 5ab^2$

Exercise:

Problem: $x^2y - 3x + 7xy^2$

Exercise:

Problem: $12\text{a} + 8b$

Solution:

$12\text{a} + 8b$

Exercise:

Problem: $19\text{y} + 5z$

Exercise:

Problem: Add: $4a, -3b, -8a$

Solution:

$-4a - 3b$

Exercise:

Problem: Add: $4x, 3y, -3x$

Exercise:

Problem: Subtract $5x^6$ from $-12x^6$.

Solution:

$-17x^6$

Exercise:

Problem: Subtract $2p^4$ from $-7p^4$.

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

Exercise:

Problem: $(5y^2 + 12y + 4) + (6y^2 - 8y + 7)$

Solution:

$11y^2 + 4y + 11$

Exercise:

Problem: $(4y^2 + 10y + 3) + (8y^2 - 6y + 5)$

Exercise:

Problem: $(x^2 + 6x + 8) + (-4x^2 + 11x - 9)$

Solution:

$-3x^2 + 17x - 1$

Exercise:

Problem: $(y^2 + 9y + 4) + (-2y^2 - 5y - 1)$

Exercise:

Problem: $(8x^2 - 5x + 2) + (3x^2 + 3)$

Solution:

$11x^2 - 5x + 5$

Exercise:

Problem: $(7x^2 - 9x + 2) + (6x^2 - 4)$

Exercise:

Problem: $(5a^2+8)+(a^2-4a-9)$

Solution:

$6a^2-4a-1$

Exercise:

Problem: $(p^2-6p-18)+(2p^2+11)$

Exercise:

Problem: $(4m^2-6m-3)-(2m^2+m-7)$

Solution:

$2m^2-7m+4$

Exercise:

Problem: $(3b^2-4b+1)-(5b^2-b-2)$

Exercise:

Problem: $(a^2+8a+5)-(a^2-3a+2)$

Solution:

$11a+3$

Exercise:

Problem: $(b^2-7b+5)-(b^2-2b+9)$

Exercise:

Problem: $(12s^2-15s)-(s-9)$

Solution:

$12s^2-16s+9$

Exercise:

Problem: $(10r^2-20r)-(r-8)$

Exercise:

Problem: Subtract $\left(9x^2+2\right)$ from $\left(12x^2-x+6\right)$.

Solution:

$3x^2-x+4$

Exercise:

Problem: Subtract $\left(5y^2-y+12\right)$ from $\left(10y^2-8y-20\right)$.

Exercise:

Problem: Subtract $\left(7w^2-4w+2\right)$ from $\left(8w^2-w+6\right)$.

Solution:

w^2+3w+4

Exercise:

Problem: Subtract $\left(5x^2-x+12\right)$ from $\left(9x^2-6x-20\right)$.

Exercise:

Problem: Find the sum of $\left(2p^3-8\right)$ and $\left(p^2+9p+18\right)$.

Solution:

$2p^3+p^2+9p+10$

Exercise:

Problem: Find the sum of $\left(q^2+4q+13\right)$ and $\left(7q^3-3\right)$.

Exercise:

Problem: Find the sum of $\left(8a^3-8a\right)$ and $\left(a^2+6a+12\right)$.

Solution:

$8a^3+a^2-2a+12$

Exercise:

Problem: Find the sum of $\left(b^2+5b+13\right)$ and $\left(4b^3-6\right)$.

Exercise:

Problem: Find the difference of $\left(w^2+w-42\right)$ and $\left(w^2-10w+24\right)$.

Solution:

$11w-66$

Exercise:

Problem: Find the difference of $\left(z^2-3z-18\right)$ and $\left(z^2+5z-20\right)$.

Exercise:

Problem: Find the difference of $\left(c^2+4c-33\right)$ and $\left(c^2-8c+12\right)$.

Solution:

$12c-45$

Exercise:

Problem: Find the difference of $\left(t^2-5t-15\right)$ and $\left(t^2+4t-17\right)$.

Exercise:

Problem: $\left(7x^2-2xy+6y^2\right)+\left(3x^2-5xy\right)$

Solution:

$10x^2-7xy+6y^2$

Exercise:

Problem: $\left(-5x^2-4xy-3y^2\right)+\left(2x^2-7xy\right)$

Exercise:

Problem: $\left(7m^2+mn-8n^2\right)+\left(3m^2+2mn\right)$

Solution:

$10m^2+3mn-8n^2$

Exercise:

Problem: $\left(2r^2-3rs-2s^2\right)+\left(5r^2-3rs\right)$

Exercise:

Problem: $\left(a^2-b^2\right)-\left(a^2+3ab-4b^2\right)$

Solution:

$-3ab+3b^2$

Exercise:

Problem: $\left(m^2+2n^2\right)-\left(m^2-8mn-n^2\right)$

Exercise:

Problem: $\left(u^2-v^2\right)-\left(u^2-4uv-3v^2\right)$

Solution:

$4uv+2v^2$

Exercise:

Problem: $\left(j^2-k^2\right)-\left(j^2-8jk-5k^2\right)$

Exercise:

Problem: $\left(p^3-3p^2q\right)+\left(2pq^2+4q^3\right)-\left(3p^2q+pq^2\right)$

Solution:

$p^3 - 6p^2q + pq^2 + 4q^3$

Exercise:

Problem: $\left(a^3 - 2a^2b\right) + \left(ab^2 + b^3\right) - \left(3a^2b + 4ab^2\right)$

Exercise:

Problem: $\left(x^3 - x^2y\right) - \left(4xy^2 - y^3\right) + \left(3x^2y - xy^2\right)$

Solution:

$x^3 + 2x^2y - 5xy^2 + y^3$

Exercise:

Problem: $\left(x^3 - 2x^2y\right) - \left(xy^2 - 3y^3\right) - \left(x^2y - 4xy^2\right)$

Evaluate a Polynomial for a Given Value

In the following exercises, evaluate each polynomial for the given value.

Exercise:

Problem: Evaluate $8y^2 - 3y + 2$ when:

ⓐ $y = 5$
ⓑ $y = -2$
ⓒ $y = 0$

Solution:

ⓐ 187 ⓑ 40 ⓒ 2

Exercise:

Problem: Evaluate $5y^2 - y - 7$ when:

ⓐ $y = -4$
ⓑ $y = 1$
ⓒ $y = 0$

Exercise:

Problem: Evaluate $4 - 36x$ when:

ⓐ $x = 3$
ⓑ $x = 0$
ⓒ $x = -1$

Solution:

ⓐ −104 ⓑ 4 ⓒ 40

Exercise:

Problem: Evaluate $16 - 36x^2$ when:

ⓐ $x = -1$
ⓑ $x = 0$
ⓒ $x = 2$

Exercise:

Problem:

A painter drops a brush from a platform 75 feet high. The polynomial $-16t^2 + 75$ gives the height of the brush t seconds after it was dropped. Find the height after $t = 2$ seconds.

Solution:

11

Exercise:

Problem:

A girl drops a ball off a cliff into the ocean. The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall cliff. Find the height after $t = 2$ seconds.

Exercise:

Problem:

A manufacturer of stereo sound speakers has found that the revenue received from selling the speakers at a cost of p dollars each is given by the polynomial $-4p^2 + 420p$. Find the revenue received when $p = 60$ dollars.

Solution:

$10,800

Exercise:

Problem:

A manufacturer of the latest basketball shoes has found that the revenue received from selling the shoes at a cost of p dollars each is given by the polynomial $-4p^2 + 420p$. Find the revenue received when $p = 90$ dollars.

Everyday Math

Exercise:

Problem:

Fuel Efficiency The fuel efficiency (in miles per gallon) of a car going at a speed of x miles per hour is given by the polynomial $-\frac{1}{150}x^2 + \frac{1}{3}x$. Find the fuel efficiency when $x = 30\,\text{mph}$.

Solution:

4

Exercise:

Problem:

Stopping Distance The number of feet it takes for a car traveling at x miles per hour to stop on dry, level concrete is given by the polynomial $0.06x^2 + 1.1x$. Find the stopping distance when $x = 40\,\text{mph}$.

Exercise:

Problem:

Rental Cost The cost to rent a rug cleaner for d days is given by the polynomial $5.50d + 25$. Find the cost to rent the cleaner for 6 days.

Solution:

$58

Exercise:

Problem:

Height of Projectile The height (in feet) of an object projected upward is given by the polynomial $-16t^2 + 60t + 90$ where t represents time in seconds. Find the height after $t = 2.5$ seconds.

Exercise:

Problem:

Temperature Conversion The temperature in degrees Fahrenheit is given by the polynomial $\frac{9}{5}c + 32$ where c represents the temperature in degrees Celsius. Find the temperature in degrees Fahrenheit when $c = 65^\circ$.

Solution:

149

Writing Exercises

Exercise:

Problem: Using your own words, explain the difference between a monomial, a binomial, and a trinomial.

Exercise:

Problem:

Using your own words, explain the difference between a polynomial with five terms and a polynomial with a degree of 5.

Solution:

Answers will vary.

Exercise:

Problem: Ariana thinks the sum $6y^2 + 5y^4$ is $11y^6$. What is wrong with her reasoning?

Exercise:

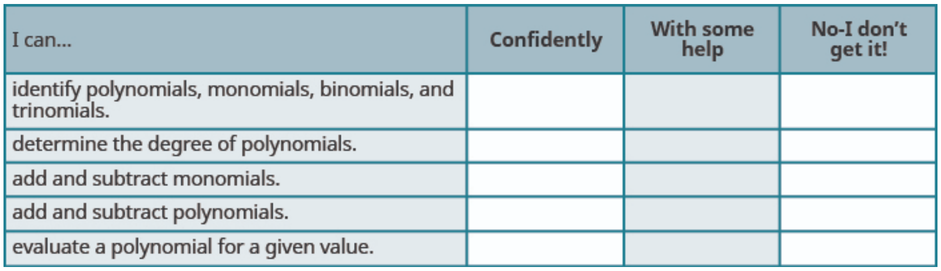
Problem: Jonathan thinks that $\frac{1}{3}$ and $\frac{1}{x}$ are both monomials. What is wrong with his reasoning?

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
identify polynomials, monomials, binomials, and trinomials.			
determine the degree of polynomials.			
add and subtract monomials.			
add and subtract polynomials.			
evaluate a polynomial for a given value.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

binomial
A binomial is a polynomial with exactly two terms.

degree of a constant
The degree of any constant is 0.

degree of a polynomial
The degree of a polynomial is the highest degree of all its terms.

degree of a term
The degree of a term is the sum of the exponents of its variables.

monomial
A monomial is a term of the form ax^m, where a is a constant and m is a whole number; a monomial has exactly one term.

polynomial
A polynomial is a monomial, or two or more monomials combined by addition or subtraction.

standard form
A polynomial is in standard form when the terms of a polynomial are written in descending order of degrees.

trinomial
A trinomial is a polynomial with exactly three terms.

Use Multiplication Properties of Exponents

By the end of this section, you will be able to:

- Simplify expressions with exponents
- Simplify expressions using the Product Property for Exponents
- Simplify expressions using the Power Property for Exponents
- Simplify expressions using the Product to a Power Property
- Simplify expressions by applying several properties
- Multiply monomials

Note:
Before you get started, take this readiness quiz.

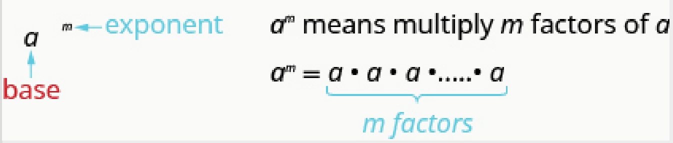
1. Simplify: $\frac{3}{4} \cdot \frac{3}{4}$.
   If you missed this problem, review [link].
2. Simplify: $(-2)\,(-2)\,(-2)$.
   If you missed this problem, review [link].

Simplify Expressions with Exponents

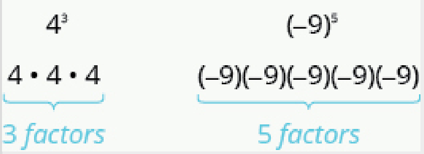
Remember that an exponent indicates repeated multiplication of the same quantity. For example, 2^4 means to multiply 2 by itself 4 times, so 2^4 means $2 \cdot 2 \cdot 2 \cdot 2$.

Let’s review the vocabulary for expressions with exponents.

Note:
Exponential Notation



This is read a to the m^{th} power.
In the expression a^m, the *exponent* m tells us how many times we use the *base* a as a factor.



Before we begin working with variable expressions containing exponents, let's simplify a few expressions involving only numbers.

Example:
Exercise:

Problem: Simplify: ⓐ 4^3 ⓑ 7^1 ⓒ $\left(\frac{5}{6}\right)^2$ ⓓ $(0.63)^2$.

Solution:
Solution

ⓐ

	4^3
Multiply three factors of 4.	$4 \cdot 4 \cdot 4$
Simplify.	64

ⓑ

	7^1
Multiply one factor of 7.	7

ⓒ

	$\left(\frac{5}{6}\right)^2$
Multiply two factors.	$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)$
Simplify.	$\frac{25}{36}$

ⓓ

	$(0.63)^2$
Multiply two factors.	$(0.63)(0.63)$
Simplify.	0.3969

Note:
Exercise:

Problem: Simplify: ⓐ 6^3 ⓑ 15^1 ⓒ $\left(\frac{3}{7}\right)^2$ ⓓ $(0.43)^2$.

Solution:

ⓐ 216 ⓑ 15 ⓒ $\frac{9}{49}$ ⓓ 0.1849

Note:
Exercise:

Problem: Simplify: ⓐ 2^5 ⓑ 21^1 ⓒ $\left(\frac{2}{5}\right)^3$ ⓓ $(0.218)^2$.

Solution:

ⓐ 32 ⓑ 21 ⓒ $\frac{8}{125}$ ⓓ 0.047524

Example:
Exercise:

Problem: Simplify: ⓐ $(-5)^4$ ⓑ -5^4.

Solution:
Solution

ⓐ

	$(-5)^4$
Multiply four factors of −5.	$(-5)(-5)(-5)(-5)$
Simplify.	625

ⓑ

	-5^4
Multiply four factors of 5.	$-(5 \cdot 5 \cdot 5 \cdot 5)$
Simplify.	-625

Note:
Exercise:

Problem: Simplify: ⓐ $(-3)^4$ ⓑ -3^4.

Solution:

ⓐ 81 ⓑ −81

Note:
Exercise:

Problem: Simplify: ⓐ $(-13)^2$ ⓑ -13^2.

Solution:

ⓐ 169 ⓑ -169

Notice the similarities and differences in [link]ⓐ and [link]ⓑ! Why are the answers different? As we follow the order of operations in part ⓐ the parentheses tell us to raise the (-5) to the 4th power. In part ⓑ we raise just the 5 to the 4th power and then take the opposite.

Simplify Expressions Using the Product Property for Exponents

You have seen that when you combine like terms by adding and subtracting, you need to have the same base with the same exponent. But when you multiply and divide, the exponents may be different, and sometimes the bases may be different, too.

We'll derive the properties of exponents by looking for patterns in several examples.

First, we will look at an example that leads to the Product Property.

	$x^2 \cdot x^3$
What does this mean? How many factors altogether?	$\underbrace{\underbrace{x \cdot x}_{2\text{ factors}} \cdot \underbrace{x \cdot x \cdot x}_{3\text{ factors}}}_{5\text{ factors}}$
So, we have	x^5
Notice that 5 is the sum of the exponents, 2 and 3.	$x^2 \cdot x^3$ is x^{2+3}, or x^5

We write:
Equation:

$$\begin{gathered} x^2 \cdot x^3 \\ x^{2+3} \\ x^5 \end{gathered}$$

The base stayed the same and we added the exponents. This leads to the **Product Property for Exponents**.

Note:
Product Property for Exponents
If a is a real number, and m and n are counting numbers, then
Equation:

$$a^m \cdot a^n = a^{m+n}$$

To multiply with like bases, add the exponents.

An example with numbers helps to verify this property.
Equation:

$$\begin{aligned} 2^2 \cdot 2^3 &\overset{?}{=} 2^{2+3} \\ 4 \cdot 8 &\overset{?}{=} 2^5 \\ 32 &= 32 \checkmark \end{aligned}$$

Example:
Exercise:

Problem: Simplify: $y^5 \cdot y^6$.

Solution:
Solution

	$y^5 \cdot y^6$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	y^{5+6}
Simplify.	y^{11}

Note:
Exercise:

Problem: Simplify: $b^9 \cdot b^8$.

Solution:

b^{17}

Note:
Exercise:

Problem: Simplify: $x^{12} \cdot x^4$.

Solution:

x^{16}

Example:
Exercise:

Problem: Simplify: ⓐ $2^5 \cdot 2^9$ ⓑ $3 \cdot 3^4$.

Solution:
Solution

ⓐ

	$2^5 \cdot 2^9$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	2^{5+9}
Simplify.	2^{14}

ⓑ

	$3 \cdot 3^4$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	3^{1+4}
Simplify.	3^5

Note:
Exercise:

Problem: Simplify: ⓐ $5 \cdot 5^5$ ⓑ $4^9 \cdot 4^9$.

Solution:

ⓐ 5^6 ⓑ 4^{18}

Note:
Exercise:

Problem: Simplify: ⓐ $7^6 \cdot 7^8$ ⓑ $10 \cdot 10^{10}$.

Solution:

ⓐ 7^{14} ⓑ 10^{11}

Example:
Exercise:

Problem: Simplify: ⓐ $a^7 \cdot a$ ⓑ $x^{27} \cdot x^{13}$.

Solution:
Solution

ⓐ

	$a^7 \cdot a$
Rewrite, a = a^1.	$a^7 \cdot a^1$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	a^{7+1}
Simplify.	a^8

ⓑ

	$x^{27} \cdot x^{13}$

Notice, the bases are the same, so add the exponents.	x^{27+13}
Simplify.	x^{40}

Note:
Exercise:

Problem: Simplify: ⓐ $p^5 \cdot p$ ⓑ $y^{14} \cdot y^{29}$.

Solution:

ⓐ p^6 ⓑ y^{43}

Note:
Exercise:

Problem: Simplify: ⓐ $z \cdot z^7$ ⓑ $b^{15} \cdot b^{34}$.

Solution:

ⓐ z^8 ⓑ b^{49}

We can extend the Product Property for Exponents to more than two factors.

Example:
Exercise:

Problem: Simplify: $d^4 \cdot d^5 \cdot d^2$.

Solution:
Solution

	$d^4 \cdot d^5 \cdot d^2$
Add the exponents, since bases are the same.	d^{4+5+2}
Simplify.	d^{11}

Note:
Exercise:

Problem: Simplify: $x^6 \cdot x^4 \cdot x^8$.

Solution:

x^{18}

Note:
Exercise:

Problem: Simplify: $b^5 \cdot b^9 \cdot b^5$.

Solution:

b^{19}

Simplify Expressions Using the Power Property for Exponents

Now let’s look at an exponential expression that contains a power raised to a power. See if you can discover a general property.

	$(x^2)^3$
What does this mean? How many factors altogether?	$\underbrace{\overset{x^2}{\underbrace{x \cdot x}_{2\text{ factors}}} \cdot \overset{x^2}{\underbrace{x \cdot x}_{2\text{ factors}}} \cdot \overset{x^2}{\underbrace{x \cdot x}_{2\text{ factors}}}}_{6\text{ factors}}$
So we have	x^6
Notice that 6 is the product of the exponents, 2 and 3.	$(x^2)^3$ is $x^{2\cdot 3}$ or x^6

We write:
Equation:

$$\left(x^2\right)^3$$
$$x^{2\cdot 3}$$
$$x^6$$

We multiplied the exponents. This leads to the **Power Property for Exponents.**

Note:
Power Property for Exponents
If a is a real number, and m and n are whole numbers, then
Equation:

$$(a^m)^n = a^{m\cdot n}$$

To raise a power to a power, multiply the exponents.

An example with numbers helps to verify this property.
Equation:

$$\begin{aligned}(3^2)^3 &\overset{?}{=} 3^{2\cdot 3}\\ (9)^3 &\overset{?}{=} 3^6\\ 729 &= 729\checkmark\end{aligned}$$

Example:
Exercise:

Problem: Simplify: ⓐ $(y^5)^9$ ⓑ $(4^4)^7$.

Solution:
Solution

ⓐ

	$(y^5)^9$
Use the power property, $(a^m)^n = a^{m\cdot n}$.	$y^{5\cdot 9}$
Simplify.	y^{45}

ⓑ

	$(4^4)^7$

Use the power property.	$4^{4\cdot 7}$
Simplify.	4^{28}

Note:
Exercise:

Problem: Simplify: ⓐ $(b^7)^5$ ⓑ $(5^4)^3$.

Solution:

ⓐ b^{35} ⓑ 5^{12}

Note:
Exercise:

Problem: Simplify: ⓐ $(z^6)^9$ ⓑ $(3^7)^7$.

Solution:

ⓐ z^{54} ⓑ 3^{49}

Simplify Expressions Using the Product to a Power Property

We will now look at an expression containing a product that is raised to a power. Can you find this pattern?

	$(2x)^3$
What does this mean?	$2x\cdot 2x\cdot 2x$
We group the like factors together.	$2\cdot 2\cdot 2\cdot x\cdot x\cdot x$
How many factors of 2 and of x?	$2^3\cdot x^3$

Notice that each factor was raised to the power and $(2x)^3$ is $2^3\cdot x^3$.

We write: $(2x)^3$

$2^3 \cdot x^3$

The exponent applies to each of the factors! This leads to the **Product to a Power Property for Exponents.**

Note:

Product to a Power Property for Exponents

If a and b are real numbers and m is a whole number, then

Equation:

$$(ab)^m = a^m b^m$$

To raise a product to a power, raise each factor to that power.

An example with numbers helps to verify this property:

Equation:

$$\begin{array}{rcl} (2 \cdot 3)^2 & \overset{?}{=} & 2^2 \cdot 3^2 \\ 6^2 & \overset{?}{=} & 4 \cdot 9 \\ 36 & = & 36 \checkmark \end{array}$$

Example:

Exercise:

Problem: Simplify: ⓐ $(-9d)^2$ ⓑ $(3mn)^3$.

Solution:

Solution

ⓐ

	$(-9d)^2$

Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(-9)^2 d^2$
Simplify.	$81d^2$

ⓑ

	$(3mn)^3$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(3)^3 m^3 n^3$
Simplify.	$27m^3 n^3$

Note:
Exercise:

Problem: Simplify: ⓐ $(-12y)^2$ ⓑ $(2wx)^5$.

Solution:

ⓐ $144y^2$ ⓑ $32w^5x^5$

Note:
Exercise:

Problem: Simplify: ⓐ $(5wx)^3$ ⓑ $(-3y)^3$.

Solution:

ⓐ $125w^3x^3$ ⓑ $-27y^3$

Simplify Expressions by Applying Several Properties

We now have three properties for multiplying expressions with exponents. Let’s summarize them and then we’ll do some examples that use more than one of the properties.

Note:
Properties of Exponents
If a and b are real numbers, and m and n are whole numbers, then

$$\begin{array}{lrcl} \textbf{Product Property} & a^m \cdot a^n & = & a^{m+n} \\ \textbf{Power Property} & (a^m)^n & = & a^{m \cdot n} \\ \textbf{Product to a Power} & (ab)^m & = & a^m b^m \end{array}$$

All exponent properties hold true for any real numbers m and n. Right now, we only use whole number exponents.

Example:
Exercise:

Problem: Simplify: ⓐ $(y^3)^6(y^5)^4$ ⓑ $(-6x^4y^5)^2$.

Solution:
Solution

ⓐ

	$(y^3)^6(y^5)^4$
Use the Power Property.	$y^{18} \cdot y^{20}$
Add the exponents.	y^{38}

ⓑ

	$(-6x^4y^5)^2$
Use the Product to a Power Property.	$(-6)^2(x^4)^2(y^5)^2$
Use the Power Property.	$(-6)^2(x^8)(y^{10})$
Simplify.	$36x^8y^{10}$

Note:
Exercise:

Problem: Simplify: ⓐ $(a^4)^5(a^7)^4$ ⓑ $\left(-2c^4d^2\right)^3$.

Solution:

ⓐ a^{48} ⓑ $-8c^{12}d^6$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(-3x^6y^7\right)^4$ ⓑ $(q^4)^5(q^3)^3$.

Solution:

ⓐ $81x^{24}y^{28}$ ⓑ q^{29}

Example:
Exercise:

Problem: Simplify: ⓐ $(5m)^2(3m^3)$ ⓑ $(3x^2y)^4(2xy^2)^3$.

Solution:
Solution

ⓐ

	$(5m)^2(3m^3)$
Raise $5m$ to the second power.	$5^2m^2 \cdot 3m^3$
Simplify.	$25m^2 \cdot 3m^3$
Use the Commutative Property.	$25 \cdot 3 \cdot m^2 \cdot m^3$
Multiply the constants and add the exponents.	$75m^5$

ⓑ

	$(3x^2y)^4(2xy^2)^3$
Use the Product to a Power Property.	$(3^4x^8y^4)(2^3x^3y^6)$
Simplify.	$(81x^8y^4)(8x^3y^6)$
Use the Commutative Property.	$81 \cdot 8 \cdot x^8 \cdot x^3 \cdot y^4 \cdot y^6$
Multiply the constants and add the exponents.	$648x^{11}y^{10}$

Note:
Exercise:

Problem: Simplify: ⓐ $(5n)^2(3n^{10})$ ⓑ $(c^4d^2)^5(3cd^5)^4$.

Solution:

ⓐ $75n^{12}$ ⓑ $81c^{24}d^{30}$

Note:
Exercise:

Problem: Simplify: ⓐ $(a^3b^2)^6(4ab^3)^4$ ⓑ $(2x)^3(5x^7)$.

Solution:

ⓐ $256a^{22}b^{24}$ ⓑ $40x^{10}$

Multiply Monomials

Since a monomial is an algebraic expression, we can use the properties of exponents to multiply monomials.

Example:
Exercise:

Problem: Multiply: $\left(3x^2\right)\left(-4x^3\right)$.

Solution:
Solution

	$\left(3x^2\right)\left(-4x^3\right)$
Use the Commutative Property to rearrange the terms.	$3\cdot(-4)\cdot x^2\cdot x^3$
Multiply.	$-12x^5$

Note:
Exercise:

Problem: Multiply: $\left(5y^7\right)\left(-7y^4\right)$.

Solution:

$-35y^{11}$

Note:
Exercise:

Problem: Multiply: $\left(-6b^4\right)\left(-9b^5\right)$.

Solution:

$54b^9$

Example:
Exercise:

Problem: Multiply: $\left(\frac{5}{6}x^3y\right)\left(12xy^2\right)$.

Solution:
Solution

	$\left(\frac{5}{6}x^3y\right)\left(12xy^2\right)$
Use the Commutative Property to rearrange the terms.	$\frac{5}{6}\cdot 12\cdot x^3\cdot x\cdot y\cdot y^2$
Multiply.	$10x^4y^3$

Note:
Exercise:

Problem: Multiply: $\left(\frac{2}{5}a^4b^3\right)\left(15ab^3\right)$.

Solution:

$6a^5b^6$

Note:
Exercise:

Problem: Multiply: $\left(\frac{2}{3}r^5s\right)\left(12r^6s^7\right)$.

Solution:

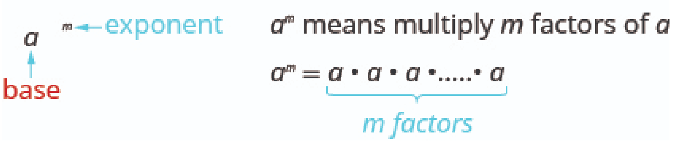
$8r^{11}s^8$

Note:
Access these online resources for additional instruction and practice with using multiplication properties of exponents:

- Multiplication Properties of Exponents

Key Concepts

- **Exponential Notation**



- **Properties of Exponents**
  - If a, b are real numbers and m, n are whole numbers, then

Product Property	$a^m \cdot a^n$	$=$	a^{m+n}
Power Property	$(a^m)^n$	$=$	$a^{m \cdot n}$
Product to a Power	$(ab)^m$	$=$	$a^m b^m$

Practice Makes Perfect

Simplify Expressions with Exponents

In the following exercises, simplify each expression with exponents.

Exercise:

ⓐ 3^5
ⓑ 9^1
ⓒ $\left(\frac{1}{3}\right)^2$
Problem: ⓓ $(0.2)^4$

Exercise:

ⓐ 10^4
ⓑ 17^1
ⓒ $\left(\frac{2}{9}\right)^2$
Problem: ⓓ $(0.5)^3$

Solution:

ⓐ 10,000 ⓑ 17 ⓒ $\frac{4}{81}$ ⓓ 0.125

Exercise:

ⓐ 2^6
ⓑ 14^1
ⓒ $\left(\frac{2}{5}\right)^3$
Problem: ⓓ $(0.7)^2$

Exercise:

ⓐ 8^3
ⓑ 8^1
ⓒ $\left(\frac{3}{4}\right)^3$
Problem: ⓓ $(0.4)^3$

Solution:

ⓐ 512 ⓑ 8 ⓒ $\frac{27}{64}$
ⓓ 0.064

Exercise:

Problem: ⓐ $(-6)^4$
ⓑ -6^4

Exercise:

Problem: ⓐ $(-2)^6$
ⓑ -2^6

Solution:

ⓐ 64 ⓑ -64

Exercise:

Problem: ⓐ $-\left(\frac{1}{4}\right)^4$
ⓑ $\left(-\frac{1}{4}\right)^4$

Exercise:

Problem: ⓐ $-\left(\frac{2}{3}\right)^2$
ⓑ $\left(-\frac{2}{3}\right)^2$

Solution:

ⓐ $-\frac{4}{9}$
ⓑ $\frac{4}{9}$

Exercise:

Problem: ⓐ -0.5^2
ⓑ $(-0.5)^2$

Exercise:

Problem: ⓐ -0.1^4
ⓑ $(-0.1)^4$

Solution:

ⓐ -0.001 ⓑ 0.001

Simplify Expressions Using the Product Property for Exponents

In the following exercises, simplify each expression using the Product Property for Exponents.

Exercise:

Problem: $d^3 \cdot d^6$

Exercise:

Problem: $x^4 \cdot x^2$

Solution:

x^6

Exercise:

Problem: $n^{19} \cdot n^{12}$

Exercise:

Problem: $q^{27} \cdot q^{15}$

Solution:

q^{42}

Exercise:

Problem: ⓐ $4^5 \cdot 4^9$ ⓑ $8^9 \cdot 8$

Exercise:

Problem: ⓐ $3^{10} \cdot 3^6$ ⓑ $5 \cdot 5^4$

Solution:

ⓐ 3^{16} ⓑ 5^5

Exercise:

Problem: ⓐ $y \cdot y^3$ ⓑ $z^{25} \cdot z^8$

Exercise:

Problem: ⓐ $w^5 \cdot w$ ⓑ $u^{41} \cdot u^{53}$

Solution:

ⓐ w^6 ⓑ u^{94}

Exercise:

Problem: $w \cdot w^2 \cdot w^3$

Exercise:

Problem: $y \cdot y^3 \cdot y^5$

Solution:

y^9

Exercise:

Problem: $a^4 \cdot a^3 \cdot a^9$

Exercise:

Problem: $c^5 \cdot c^{11} \cdot c^2$

Solution:

c^{18}

Exercise:

Problem: $m^x \cdot m^3$

Exercise:

Problem: $n^y \cdot n^2$

Solution:

n^{y+2}

Exercise:

Problem: $y^a \cdot y^b$

Exercise:

Problem: $x^p \cdot x^q$

Solution:

x^{p+q}

Simplify Expressions Using the Power Property for Exponents

In the following exercises, simplify each expression using the Power Property for Exponents.

Exercise:

Problem: ⓐ $(m^4)^2$ ⓑ $(10^3)^6$

Exercise:

Problem: ⓐ $(b^2)^7$ ⓑ $(3^8)^2$

Solution:

ⓐ b^{14} ⓑ 3^{16}

Exercise:

Problem: ⓐ $\left(y^3\right)^x$ ⓑ $(5^x)^y$

Exercise:

Problem: ⓐ $\left(x^2\right)^y$ ⓑ $(7^a)^b$

Solution:

ⓐ x^{2y} ⓑ 7^{ab}

Simplify Expressions Using the Product to a Power Property

In the following exercises, simplify each expression using the Product to a Power Property.

Exercise:

Problem: ⓐ $(6a)^2$ ⓑ $(3xy)^2$

Exercise:

Problem: ⓐ $(5x)^2$ ⓑ $(4ab)^2$

Solution:

ⓐ $25x^2$ ⓑ $16a^2b^2$

Exercise:

Problem: ⓐ $(-4m)^3$ ⓑ $(5ab)^3$

Exercise:

Problem: ⓐ $(-7n)^3$ ⓑ $(3xyz)^4$

Solution:

ⓐ $-343n^3$ ⓑ $81x^4y^4z^4$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify each expression.

Exercise:

Problem: ⓐ $(y^2)^4 \cdot (y^3)^2$ ⓑ $(10a^2b)^3$

Exercise:

Problem: ⓐ $(w^4)^3 \cdot (w^5)^2$ ⓑ $(2xy^4)^5$

Solution:

ⓐ w^{22} ⓑ $32x^5y^{20}$

Exercise:

Problem: ⓐ $(-2r^3s^2)^4$ ⓑ $(m^5)^3 \cdot (m^9)^4$

Exercise:

Problem: ⓐ $(-10q^2p^4)^3$ ⓑ $(n^3)^{10} \cdot (n^5)^2$

Solution:

ⓐ $-1000q^6p^{12}$ ⓑ n^{40}

Exercise:

ⓐ $(3x)^2(5x)$
Problem: ⓑ $(5t^2)^3(3t)^2$

Exercise:

ⓐ $(2y)^3(6y)$
Problem: ⓑ $(10k^4)^3(5k^6)^2$

Solution:

ⓐ $48y^4$ ⓑ $25{,}000k^{24}$

Exercise:

ⓐ $(5a)^2(2a)^3$
Problem: ⓑ $(\frac{1}{2}y^2)^3(\frac{2}{3}y)^2$

Exercise:

ⓐ $(4b)^2(3b)^3$
Problem: ⓑ $(\frac{1}{2}j^2)^5(\frac{2}{5}j^3)^2$

Solution:

ⓐ $432b^5$ ⓑ $\frac{1}{200}j^{16}$

Exercise:

ⓐ $\left(\frac{2}{5}x^2y\right)^3$
Problem: ⓑ $\left(\frac{8}{9}xy^4\right)^2$

Exercise:

ⓐ $(2r^2)^3(4r)^2$
Problem: ⓑ $(3x^3)^3(x^5)^4$

Solution:

ⓐ $128r^8$ ⓑ $27x^{29}$

Exercise:

ⓐ $(m^2n)^2(2mn^5)^4$
Problem: ⓑ $(3pq^4)^2(6p^6q)^2$

Multiply Monomials

In the following exercises, multiply the monomials.

Exercise:

Problem: $\left(6y^7\right)\left(-3y^4\right)$

Solution:

$-18y^{11}$

Exercise:

Problem: $\left(-10x^5\right)\left(-3x^3\right)$

Exercise:

Problem: $(-8u^6)(-9u)$

Solution:

$72u^7$

Exercise:

Problem: $(-6c^4)(-12c)$

Exercise:

Problem: $(\frac{1}{5}f^8)(20f^3)$

Solution:

$4f^{11}$

Exercise:

Problem: $(\frac{1}{4}d^5)(36d^2)$

Exercise:

Problem: $(4a^3b)(9a^2b^6)$

Solution:

$36a^5b^7$

Exercise:

Problem: $(6m^4n^3)(7mn^5)$

Exercise:

Problem: $\left(\frac{4}{7}rs^2\right)\left(14rs^3\right)$

Solution:

$8r^2s^5$

Exercise:

Problem: $\left(\frac{5}{8}x^3y\right)\left(24x^5y\right)$

Exercise:

Problem: $\left(\frac{2}{3}x^2y\right)\left(\frac{3}{4}xy^2\right)$

Solution:

$\frac{1}{2}x^3y^3$

Exercise:

Problem: $\left(\frac{3}{5}m^3n^2\right)\left(\frac{5}{9}m^2n^3\right)$

Mixed Practice

In the following exercises, simplify each expression.

Exercise:

Problem: $\left(x^2\right)^4 \cdot \left(x^3\right)^2$

Solution:

x^{14}

Exercise:

Problem: $\left(y^4\right)^3 \cdot \left(y^5\right)^2$

Exercise:

Problem: $\left(a^2\right)^6 \cdot \left(a^3\right)^8$

Solution:

a^{36}

Exercise:

Problem: $\left(b^7\right)^5 \cdot \left(b^2\right)^6$

Exercise:

Problem: $(2m^6)^3$

Solution:

$8m^{18}$

Exercise:

Problem: $(3y^2)^4$

Exercise:

Problem: $(10x^2y)^3$

Solution:

$1000x^6y^3$

Exercise:

Problem: $(2mn^4)^5$

Exercise:

Problem: $(-2a^3b^2)^4$

Solution:

$16a^{12}b^8$

Exercise:

Problem: $(-10u^2v^4)^3$

Exercise:

Problem: $\left(\frac{2}{3}x^2y\right)^3$

Solution:

$\frac{8}{27}x^6y^3$

Exercise:

Problem: $\left(\frac{7}{9}pq^4\right)^2$

Exercise:

Problem: $(8a^3)^2(2a)^4$

Solution:

$1024a^{10}$

Exercise:

Problem: $(5r^2)^3(3r)^2$

Exercise:

Problem: $(10p^4)^3(5p^6)^2$

Solution:

$25000p^{24}$

Exercise:

Problem: $(4x^3)^3(2x^5)^4$

Exercise:

Problem: $\left(\frac{1}{2}x^2y^3\right)^4\left(4x^5y^3\right)^2$

Solution:

$x^{18}y^{18}$

Exercise:

Problem: $\left(\frac{1}{3}m^3n^2\right)^4\left(9m^8n^3\right)^2$

Exercise:

Problem: $(3m^2n)^2(2mn^5)^4$

Solution:

$144m^8n^{22}$

Exercise:

Problem: $(2pq^4)^3(5p^6q)^2$

Everyday Math

Exercise:

Problem:

Email Kate emails a flyer to ten of her friends and tells them to forward it to ten of their friends, who forward it to ten of their friends, and so on. The number of people who receive the email on the second round is 10^2, on the third round is 10^3, as shown in the table below. How many people will receive the email on the sixth round? Simplify the expression to show the number of people who receive the email.

Round	**Number of people**
1	10
2	10^2
3	10^3
…	…
6	?

Solution:

1,000,000

Exercise:

Problem:

Salary Jamal's boss gives him a 3% raise every year on his birthday. This means that each year, Jamal's salary is 1.03 times his last year's salary. If his original salary was \$35,000, his salary after 1 year was $\$35{,}000\,(1.03)$, after 2 years was $\$35{,}000(1.03)^2$, after 3 years was $\$35{,}000(1.03)^3$, as shown in the table below. What will Jamal's salary be after 10 years? Simplify the expression, to show Jamal's salary in dollars.

Year	**Salary**
1	$\$35{,}000\,(1.03)$
2	$\$35{,}000(1.03)^2$
3	$\$35{,}000(1.03)^3$
…	…
10	?

Exercise:

Problem:

Clearance A department store is clearing out merchandise in order to make room for new inventory. The plan is to mark down items by 30% each week. This means that each week the cost of an item is 70% of the previous week's cost. If the original cost of a sofa was $1,000, the cost for the first week would be $\$1,000\,(0.70)$ and the cost of the item during the second week would be $\$1{,}000(0.70)^2$. Complete the table shown below. What will be the cost of the sofa during the eighth week? Simplify the expression, to show the cost in dollars.

Week	**Cost**
1	$\$1,000\,(0.70)$
2	$\$1{,}000(0.70)^2$
3	
…	…
8	?

Solution:

$57.65

Exercise:

Problem:

Depreciation Once a new car is driven away from the dealer, it begins to lose value. Each year, a car loses 10% of its value. This means that each year the value of a car is 90% of the previous year's value. If a new car was purchased for $20,000, the value at the end of the first year would be $\$20,000\,(0.90)$ and the value of the car after the end of the second year would be $\$20{,}000(0.90)^2$. Complete the table shown below. What will be the value of the car at the end of the fifth year? Simplify the expression, to show the value in dollars.

Week	Cost
1	$\$20,000\,(0.90)$
2	$\$20,000(0.90)^2$
3	
4	$\ldots$
5	?

Writing Exercises

Exercise:

Problem: Use the Product Property for Exponents to explain why $x \cdot x = x^2$.

Solution:

Answers will vary.

Exercise:

Problem: Explain why $-5^3 = (-5)^3$ but $-5^4 \neq (-5)^4$.

Exercise:

Problem: Jorge thinks $\left(\frac{1}{2}\right)^2$ is 1. What is wrong with his reasoning?

Solution:

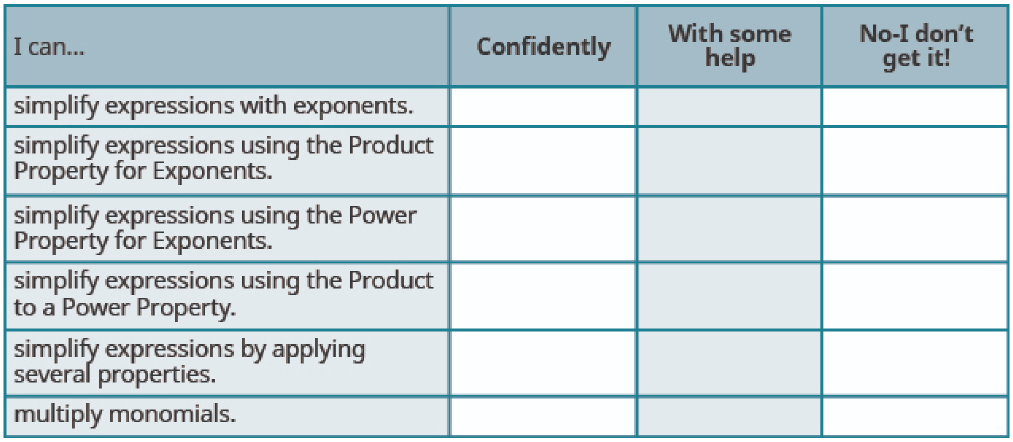
Answers will vary.

Exercise:

Problem: Explain why $x^3 \cdot x^5$ is x^8, and not x^{15}.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
simplify expressions with exponents.			
simplify expressions using the Product Property for Exponents.			
simplify expressions using the Power Property for Exponents.			
simplify expressions using the Product to a Power Property.			
simplify expressions by applying several properties.			
multiply monomials.			

ⓑ After reviewing this checklist, what will you do to become confident for all goals?

Multiply Polynomials
By the end of this section, you will be able to:

- Multiply a polynomial by a monomial
- Multiply a binomial by a binomial
- Multiply a trinomial by a binomial

Note:
Before you get started, take this readiness quiz.

1. Distribute: $2\left(x+3\right)$.
   If you missed this problem, review [link].
2. Combine like terms: $x^2+9x+7x+63$.
   If you missed this problem, review [link].

Multiply a Polynomial by a Monomial

We have used the Distributive Property to simplify expressions like $2(x-3)$. You multiplied both terms in the parentheses, x and 3, by 2, to get $2x-6$. With this chapter's new vocabulary, you can say you were multiplying a binomial, $x-3$, by a monomial, 2.

Multiplying a binomial by a monomial is nothing new for you! Here's an example:

Example:
Exercise:

Problem: Multiply: $4\left(x+3\right)$.

Solution:

Solution

	$4(x+3)$
Distribute.	$4 \cdot x + 4 \cdot 3$
Simplify.	$4x + 12$

Note:
Exercise:

Problem: Multiply: $5\left(x+7\right)$.

Solution:

$5x+35$

Note:
Exercise:

Problem: Multiply: $3\left(y+13\right)$.

Solution:

$3y+39$

Example:
Exercise:

Problem: Multiply: $y\left(y-2\right)$.

Solution:
Solution

	$y(y-2)$
Distribute.	$y\cdot y-y\cdot 2$
Simplify.	y^2-2y

Note:
Exercise:

Problem: Multiply: $x\left(x-7\right)$.

Solution:

x^2-7x

Note:
Exercise:

Problem: Multiply: $d\left(d-11\right)$.

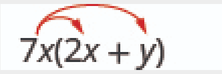
Solution:

d^2-11d

Example:
Exercise:

Problem: Multiply: $7x(2x+y)$.

Solution:
Solution

Distribute.	$7x \cdot 2x + 7x \cdot y$
Simplify.	$14x^2 + 7xy$

Note:
Exercise:

Problem: Multiply: $5x(x+4y)$.

Solution:

$5x^2 + 20xy$

Note:
Exercise:

Problem: Multiply: $2p(6p+r)$.

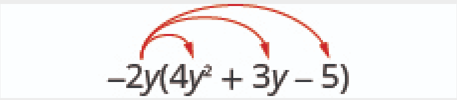
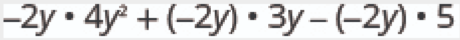
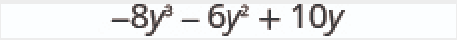
Solution:

$12p^2 + 2pr$

Example:
Exercise:

Problem: Multiply: $-2y\left(4y^2+3y-5\right)$.

Solution:
Solution

	 $-2y(4y^2+3y-5)$
Distribute.	 $-2y \cdot 4y^2 + (-2y) \cdot 3y - (-2y) \cdot 5$
Simplify.	 $-8y^3 - 6y^2 + 10y$

Note:
Exercise:

Problem: Multiply: $-3y\left(5y^2+8y-7\right)$.

Solution:

$-15y^3-24y^2+21y$

Note:
Exercise:

Problem: Multiply: $4x^2(2x^2 - 3x + 5)$.

Solution:

$8x^4 - 12x^3 + 20x^2$

Example:
Exercise:

Problem: Multiply: $2x^3(x^2 - 8x + 1)$.

Solution:
Solution

	$2x^3(x^2 - 8x + 1)$
Distribute.	$2x^3 \cdot x^2 + (2x^3) \cdot (-8x) + (2x^3) \cdot 1$
Simplify.	$2x^5 - 16x^4 + 2x^3$

Note:
Exercise:

Problem: Multiply: $4x\left(3x^2-5x+3\right)$.

Solution:

$12x^3-20x^2+12x$

Note:
Exercise:

Problem: Multiply: $-6a^3(3a^2-2a+6)$.

Solution:

$-18a^5+12a^4-36a^3$

Example:
Exercise:

Problem: Multiply: $(x+3)p$.

Solution:
Solution

The monomial is the second factor.	

	$(x+3)p$
Distribute.	$x \cdot p + 3 \cdot p$
Simplify.	$xp + 3p$

Note:
Exercise:

Problem: Multiply: $(x+8)p$.

Solution:

$xp+8p$

Note:
Exercise:

Problem: Multiply: $(a+4)p$.

Solution:

$ap+4p$

Multiply a Binomial by a Binomial

Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a binomial times a binomial. We will start by using the Distributive Property.

Multiply a Binomial by a Binomial Using the Distributive Property

Look at [link], where we multiplied a binomial by a monomial.

	$(x+3)p$
We distributed the *p* to get:	$xp+3p$
What if we have $(x+7)$ instead of *p*?	$(x+3)(x+7)$
Distribute $(x+7)$.	$x(x+7)+3(x+7)$
Distribute again.	$x^2+7x+3x+21$
Combine like terms.	$x^2+10x+21$

Notice that before combining like terms, you had four terms. You multiplied the two terms of the first binomial by the two terms of the second binomial —four multiplications.

Example:
Exercise:

Problem: Multiply: $(y+5)\,(y+8)$.

Solution:
Solution

	$(y+5)(y+8)$
Distribute $(y + 8)$.	$y(y+8)+5(y+8)$
Distribute again	$y^2+8y+5y+40$
Combine like terms.	$y^2+13y+40$

Note:
Exercise:

Problem: Multiply: $(x+8)(x+9)$.

Solution:

$x^2+17x+72$

Note:
Exercise:

Problem: Multiply: $(5x+9)(4x+3)$.

Solution:

$20x^2+51x+27$

Example:
Exercise:

Problem: Multiply: $(2y+5)(3y+4)$.

Solution:
Solution

	$(2y+5)(3y+4)$
Distribute (3*y* + 4).	$2y(3y+4)+5(3y+4)$
Distribute again	$6y^2+8y+15y+20$
Combine like terms.	$6y^2+23y+20$

Note:
Exercise:

Problem: Multiply: $(3b+5)\,(4b+6)$.

Solution:

$12b^2+38b+30$

Note:
Exercise:

Problem: Multiply: $(a+10)\,(a+7)$.

Solution:

$a^2+17a+70$

Example:
Exercise:

Problem: Multiply: $(4y+3)(2y-5)$.

Solution:
Solution

	$(4y+3)(2y-5)$
Distribute.	$4y(2y-5)+3(2y-5)$
Distribute again.	$8y^2-20y+6y-15$
Combine like terms.	$8y^2-14y-15$

Note:
Exercise:

Problem: Multiply: $(5y+2)\,(6y-3)$.

Solution:

$30y^2-3y-6$

Note:
Exercise:

Problem: Multiply: $(3c+4)\,(5c-2)$.

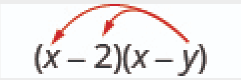
Solution:

$15c^2+14c-8$

Example:
Exercise:

Problem: Multiply: $(x+2)(x-y)$.

Solution:
Solution

Distribute.	$x(x-y)-2(x-y)$
Distribute again.	$x^2-xy-2x+2y$
There are no like terms to combine.	

Note:
Exercise:

Problem: Multiply: $(a+7)(a-b)$.

Solution:

$a^2-ab+7a-7b$

Note:
Exercise:

Problem: Multiply: $(x+5)(x-y)$.

Solution:

$x^2-xy+5x-5y$

Multiply a Binomial by a Binomial Using the FOIL Method

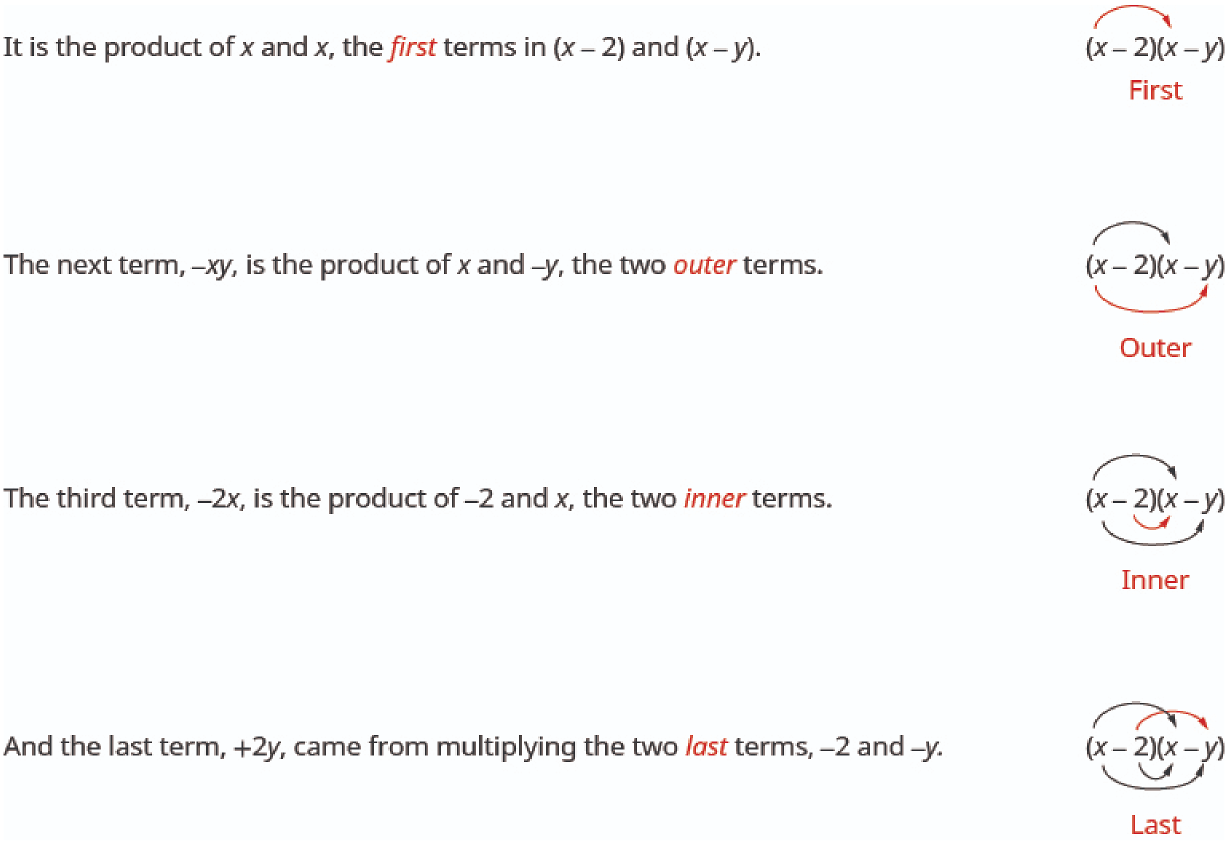
Remember that when you multiply a binomial by a binomial you get four terms. Sometimes you can combine like terms to get a trinomial, but sometimes, like in [link], there are no like terms to combine.

Let’s look at the last example again and pay particular attention to how we got the four terms.

Equation:

$$\begin{gathered}(x-2)(x-y)\\ x^2-xy-2x+2y\end{gathered}$$

Where did the first term, x^2, come from?



It is the product of x and x, the *first* terms in $(x-2)$ and $(x-y)$.

The next term, $-xy$, is the product of x and $-y$, the two *outer* terms.

The third term, $-2x$, is the product of -2 and x, the two *inner* terms.

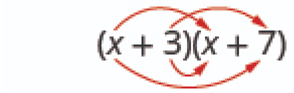
And the last term, $+2y$, came from multiplying the two *last* terms, -2 and $-y$.

We abbreviate “First, Outer, Inner, Last” as FOIL. The letters stand for ‘**F**irst, **O**uter, **I**nner, **L**ast’. The word FOIL is easy to remember and ensures we find all four products.

Equation:

$$(x-2)(x-y)$$
$$\underset{F}{x^2}-\underset{O}{xy}-\underset{I}{2x}+\underset{L}{2y}$$

Let’s look at $(x+3)\,(x+7)$.

Distibutive Property	FOIL
$(x+3)(x+7)$	 $(x+3)(x+7)$
$x(x+7)+3(x+7)$	
$\underset{F}{x^2}+\underset{O}{7x}+\underset{I}{3x}+\underset{L}{21}$	$\underset{F}{x^2}+\underset{O}{7x}+\underset{I}{3x}+\underset{L}{21}$
$x^2+10x+21$	$x^2+10x+21$

Notice how the terms in third line fit the FOIL pattern.

Now we will do an example where we use the FOIL pattern to multiply two binomials.

Example:

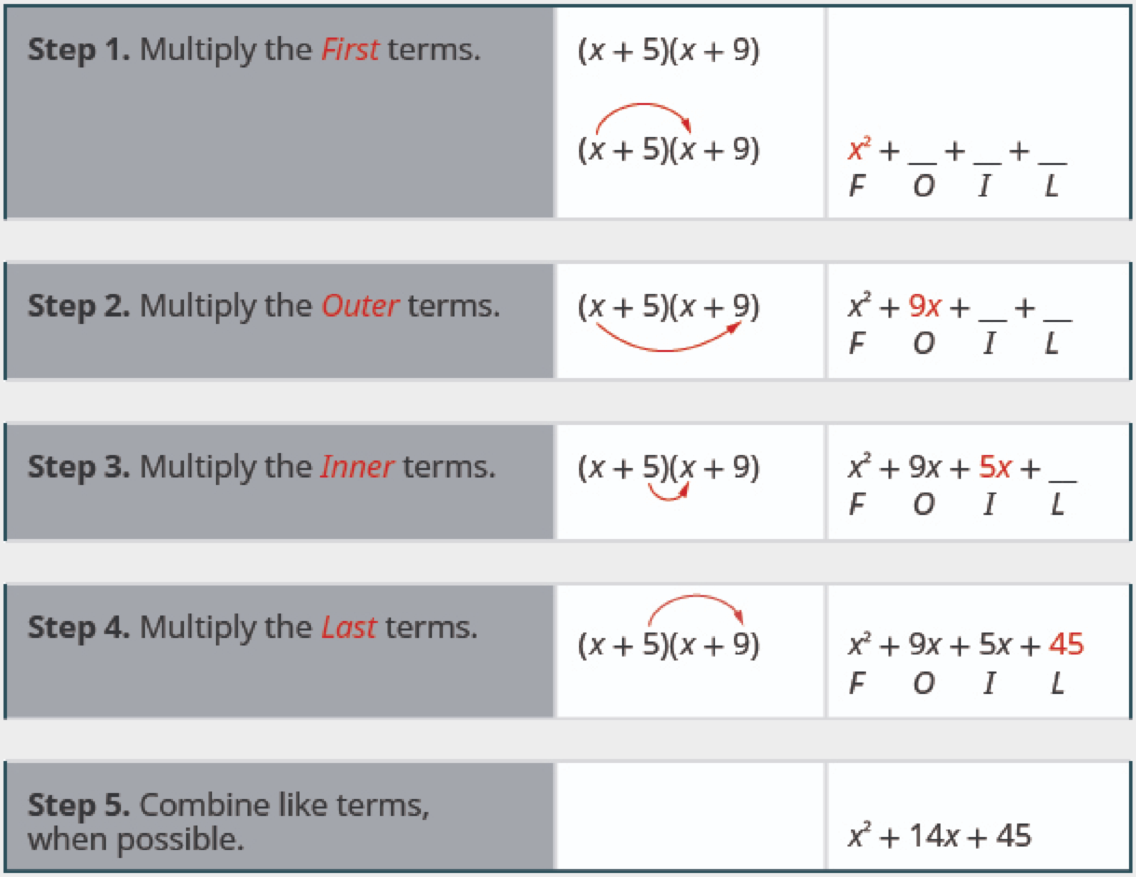
How to Multiply a Binomial by a Binomial using the FOIL Method

Exercise:

Problem: Multiply using the FOIL method: $(x+5)(x+9)$.

Solution:

Solution



Step 1. Multiply the *First* terms.	$(x+5)(x+9)$ $(x+5)(x+9)$	$\underset{F}{x^2}+\underset{O}{\_\_}+\underset{I}{\_\_}+\underset{L}{\_\_}$
Step 2. Multiply the *Outer* terms.	$(x+5)(x+9)$	$\underset{F}{x^2}+\underset{O}{9x}+\underset{I}{\_\_}+\underset{L}{\_\_}$
Step 3. Multiply the *Inner* terms.	$(x+5)(x+9)$	$\underset{F}{x^2}+\underset{O}{9x}+\underset{I}{5x}+\underset{L}{\_\_}$
Step 4. Multiply the *Last* terms.	$(x+5)(x+9)$	$\underset{F}{x^2}+\underset{O}{9x}+\underset{I}{5x}+\underset{L}{45}$
Step 5. Combine like terms, when possible.		$x^2+14x+45$

Note:

Exercise:

Problem: Multiply using the FOIL method: $(x+6)(x+8)$.

Solution:

$x^2 + 14x + 48$

Note:
Exercise:

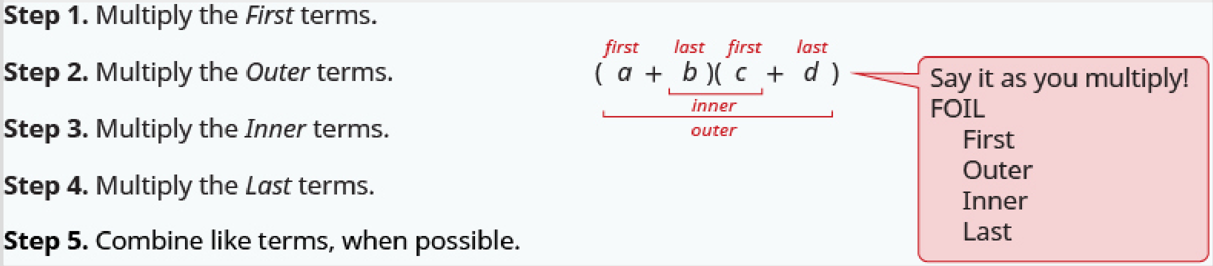
Problem: Multiply using the FOIL method: $(y + 17)(y + 3)$.

Solution:

$y^2 + 20y + 51$

We summarize the steps of the FOIL method below. The FOIL method only applies to multiplying binomials, not other polynomials!

Note:
Multiply two binomials using the FOIL method



Step 1. Multiply the *First* terms.

Step 2. Multiply the *Outer* terms.

Step 3. Multiply the *Inner* terms.

Step 4. Multiply the *Last* terms.

Step 5. Combine like terms, when possible.

first
last
first
last
(a + b)(c + d)
inner
outer

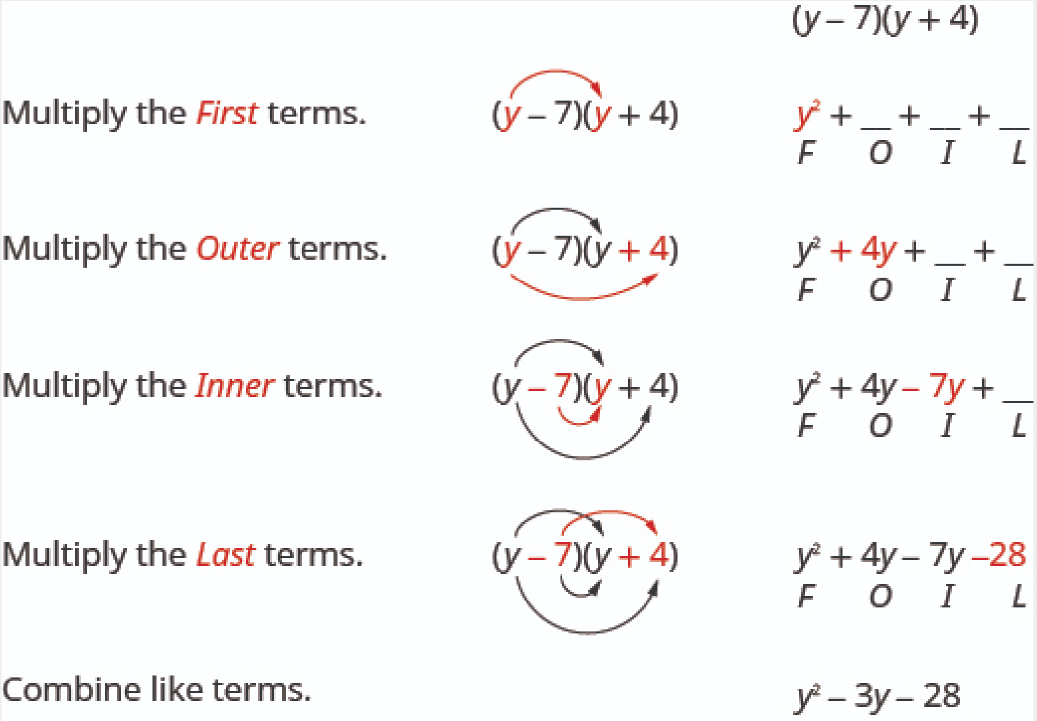
Say it as you multiply!
FOIL
First
Outer
Inner
Last

When you multiply by the FOIL method, drawing the lines will help your brain focus on the pattern and make it easier to apply.

Example:
Exercise:

Problem: Multiply: $(y-7)(y+4)$.

Solution:
Solution



		$(y-7)(y+4)$
Multiply the *First* terms.	$(y-7)(y+4)$	$y^2 + \_\_ + \_\_ + \_\_$ F O I L
Multiply the *Outer* terms.	$(y-7)(y+4)$	$y^2 + 4y + \_\_ + \_\_$ F O I L
Multiply the *Inner* terms.	$(y-7)(y+4)$	$y^2 + 4y - 7y + \_\_$ F O I L
Multiply the *Last* terms.	$(y-7)(y+4)$	$y^2 + 4y - 7y - 28$ F O I L
Combine like terms.		$y^2 - 3y - 28$

Note:
Exercise:

Problem: Multiply: $(x-7)(x+5)$.

Solution:

$x^2 - 2x - 35$

Note:
Exercise:

Problem: Multiply: $(b-3)(b+6)$.

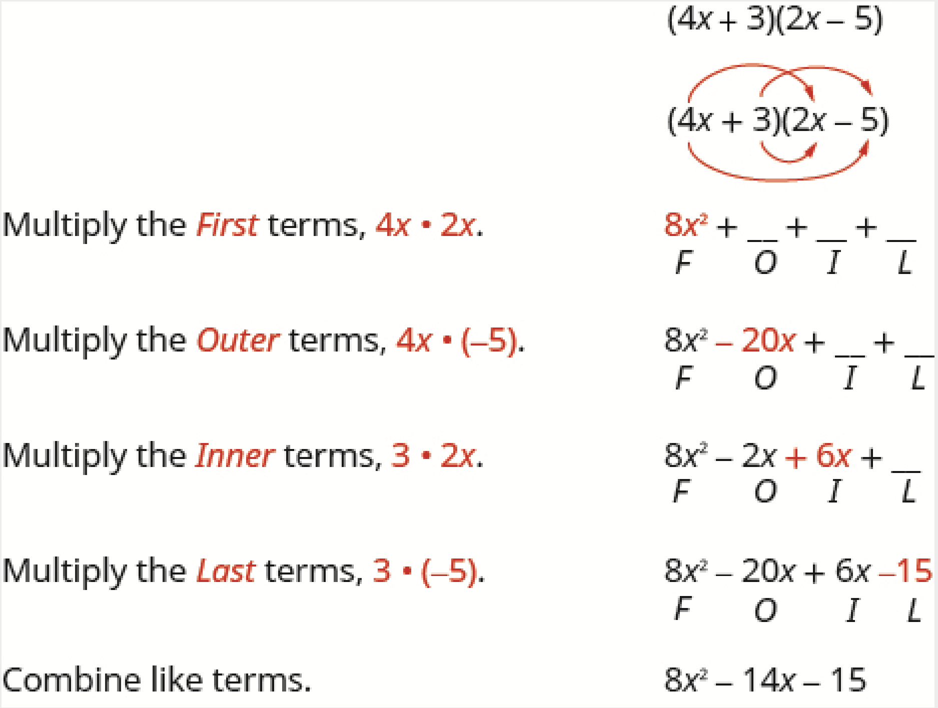
Solution:

$b^2+3b-18$

Example:
Exercise:

Problem: Multiply: $(4x+3)(2x-5)$.

Solution:
Solution



	$(4x+3)(2x-5)$
	$(4x+3)(2x-5)$
Multiply the *First* terms, $4x \cdot 2x$.	$\underset{F}{8x^2}+\underset{O}{\_\_}+\underset{I}{\_\_}+\underset{L}{\_\_}$
Multiply the *Outer* terms, $4x \cdot (-5)$.	$\underset{F}{8x^2}\underset{O}{-20x}+\underset{I}{\_\_}+\underset{L}{\_\_}$
Multiply the *Inner* terms, $3 \cdot 2x$.	$\underset{F}{8x^2}\underset{O}{-2x}\underset{I}{+6x}+\underset{L}{\_\_}$
Multiply the *Last* terms, $3 \cdot (-5)$.	$\underset{F}{8x^2}\underset{O}{-20x}\underset{I}{+6x}\underset{L}{-15}$
Combine like terms.	$8x^2-14x-15$

Note:
Exercise:

Problem: Multiply: $(3x+7)\,(5x-2)$.

Solution:

$15x^2+29x-14$

Note:
Exercise:

Problem: Multiply: $(4y+5)\,(4y-10)$.

Solution:

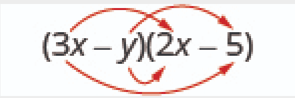
$16y^2-20y-50$

The final products in the last four examples were trinomials because we could combine the two middle terms. This is not always the case.

Example:
Exercise:

Problem: Multiply: $(3x-y)(2x-5)$.

Solution:
Solution

	$(3x-y)(2x-5)$
	 $(3x-y)(2x-5)$
Multiply the *First*.	$\underset{F}{6x^2} + \underset{O}{\_\_} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Outer*.	$\underset{F}{6x^2} \underset{O}{-\,15x} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Inner*.	$\underset{F}{6x^2} \underset{O}{-\,15x} \underset{I}{-\,2xy} + \underset{L}{\_\_}$
Multiply the *Last*.	$\underset{F}{6x^2} \underset{O}{-\,15x} \underset{I}{-\,2xy} \underset{L}{+\,5y}$
Combine like terms—there are none.	$6x^2-15x-2xy+5y$

Note:
Exercise:

Problem: Multiply: $(10c-d)\,(c-6)$.

Solution:

$10c^2 - 60c - cd + 6d$

Note:
Exercise:

Problem: Multiply: $(7x - y)\,(2x - 5)$.

Solution:

$14x^2 - 35x - 2xy + 5y$

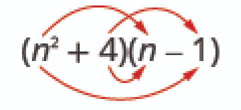
Be careful of the exponents in the next example.

Example:
Exercise:

Problem: Multiply: $\left(n^2 + 4\right)(n - 1)$.

Solution:
Solution

	$(n^2 + 4)(n - 1)$

	 $(n^2+4)(n-1)$
Multiply the *First*.	$\underset{F}{n^3} + \underset{O}{\_\_} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Outer*.	$\underset{F}{n^3} \underset{O}{- n^2} + \underset{I}{\_\_} + \underset{L}{\_\_}$
Multiply the *Inner*.	$\underset{F}{n^3} \underset{O}{- n^2} \underset{I}{+ 4n} + \underset{L}{\_\_}$
Multiply the *Last*.	$\underset{F}{n^3} \underset{O}{- n^2} \underset{I}{+ 4n} \underset{L}{- 4}$
Combine like terms—there are none.	$n^3 - n^2 + 4n - 4$

Note:
Exercise:

Problem: Multiply: $\left(x^2+6\right)(x-8)$.

Solution:

$x^3 - 8x^2 + 6x - 48$

Note:
Exercise:

Problem: Multiply: $\left(y^2 + 7\right)(y - 9)$.

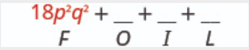
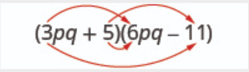
Solution:

$y^3 - 9y^2 + 7y - 63$

Example:
Exercise:

Problem: Multiply: $(3pq + 5)(6pq - 11)$.

Solution:
Solution

	$(3pq + 5)(6pq - 11)$	
Multiply the *First*.	 $\underset{F}{18p^2q^2} + \underset{O}{\_\_} + \underset{I}{\_\_} + \underset{L}{\_\_}$	 $(3pq + 5)(6pq - 11)$
Multiply the *Outer*.		

	$\underset{F}{18p^2q^2}\ \underset{O}{-\ 33pq}\ +\ \underset{I}{\_\_}\ +\ \underset{L}{\_\_}$	
Multiply the *Inner*.	$\underset{F}{18p^2q^2}\ \underset{O}{-\ 33pq}\ \underset{I}{+\ 30pq}\ +\ \underset{L}{\_\_}$	
Multiply the *Last*.	$\underset{F}{18p^2q^2}\ \underset{O}{-\ 33pq}\ \underset{I}{+\ 30pq}\ \underset{L}{-\ 55}$	
Combine like terms—there are none.	$18p^2q^2 - 3pq - 55$	

Note:
Exercise:

Problem: Multiply: $(2ab + 5)\,(4ab - 4)$.

Solution:

$8a^2b^2 + 12ab - 20$

Note:
Exercise:

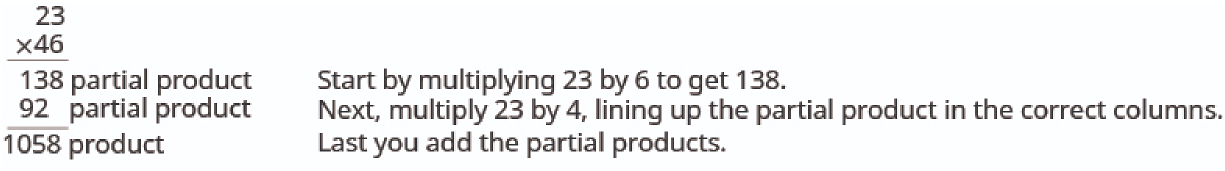
Problem: Multiply: $(2xy + 3)\,(4xy - 5)$.

Solution:

$8x^2y^2 + 2xy - 15$

Multiply a Binomial by a Binomial Using the Vertical Method

The FOIL method is usually the quickest method for multiplying two binomials, but it *only* works for binomials. You can use the Distributive Property to find the product of any two polynomials. Another method that works for all polynomials is the Vertical Method. It is very much like the method you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.



23	
×46	
138 partial product	Start by multiplying 23 by 6 to get 138.
92 partial product	Next, multiply 23 by 4, lining up the partial product in the correct columns.
1058 product	Last you add the partial products.

Now we’ll apply this same method to multiply two binomials.

Example:
Exercise:

Problem: Multiply using the Vertical Method: $(3y-1)\,(2y-6)$.

Solution:
Solution

It does not matter which binomial goes on the top.

Multiply $3y-1$ by -6.
Multiply $3y-1$ by 2y.

Add like terms.

$$\begin{array}{rl} 3y-1 & \\ \times\ 2y-6 & \\ \hline -18y+6 & \text{partial product} \\ 6y^2-2y\phantom{{}+6} & \text{partial product} \\ \hline 6y^2-20y+6 & \text{product} \end{array}$$

Notice the partial products are the same as the terms in the FOIL method.

$$\begin{array}{c} (3y-1)(2y-6) \\ \underline{6y^2-2y}\,\underline{-18y+6} \\ 6y^2-20y+6 \end{array} \qquad \begin{array}{r} 3y-1 \\ \underline{\times\ 2y-6} \\ -18y+6 \\ \underline{6y^2-2y} \\ 6y^2-20x+6 \end{array}$$

Note:
Exercise:

Problem: Multiply using the Vertical Method: $(5m-7)(3m-6)$.

Solution:

$15m^2-51m+42$

Note:
Exercise:

Problem: Multiply using the Vertical Method: $(6b-5)(7b-3)$.

Solution:

$42b^2-53b+15$

We have now used three methods for multiplying binomials. Be sure to practice each method, and try to decide which one you prefer. The methods are listed here all together, to help you remember them.

Note:
Multiplying Two Binomials
To multiply binomials, use the:

- Distributive Property
- FOIL Method
- Vertical Method

Remember, FOIL only works when multiplying two binomials.

Multiply a Trinomial by a Binomial

We have multiplied monomials by monomials, monomials by polynomials, and binomials by binomials. Now we're ready to multiply a trinomial by a binomial. Remember, FOIL will not work in this case, but we can use either the Distributive Property or the Vertical Method. We first look at an example using the Distributive Property.

Example:
Exercise:

Problem:

Multiply using the Distributive Property: $(b+3)(2b^2-5b+8)$.

Solution:
Solution

	$(b+3)(2b^2-5b+8)$
Distribute.	$b(2b^2-5b+8)+3(2b^2-5b+8)$
Multiply.	$2b^3-5b^2+8b+6b^2-15b+24$
Combine like terms.	$2b^3+b^2-7b+24$

Note:
Exercise:

Problem:

Multiply using the Distributive Property: $(y-3)(y^2-5y+2)$.

Solution:

$y^3-8y^2+17y-6$

Note:
Exercise:

Problem:

Multiply using the Distributive Property: $(x+4)(2x^2-3x+5)$.

Solution:

$2x^3+5x^2-7x+20$

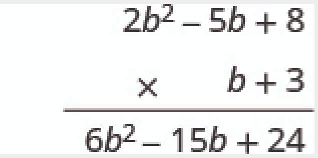
Now let’s do this same multiplication using the Vertical Method.

Example:
Exercise:

Problem: Multiply using the Vertical Method: $(b+3)(2b^2-5b+8)$.

Solution:
Solution

It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

Multiply $(2b^2 - 5b + 8)$ by 3.	 $\begin{array}{r} 2b^2-5b+8 \\ \times \quad b+3 \\ \hline 6b^2-15b+24 \end{array}$

	$\underline{2b^3 - 5b^2 + 8b}$
Multiply $(2b^2 - 5b + 8)$ by b.	$2b^3 + \ b^2 - 7b + 24$
Add like terms.	

Note:
Exercise:

Problem: Multiply using the Vertical Method: $(y-3)(y^2-5y+2)$.

Solution:

$y^3 - 8y^2 + 17y - 6$

Note:
Exercise:

Problem:

Multiply using the Vertical Method: $(x+4)(2x^2-3x+5)$.

Solution:

$2x^3 + 5x^2 - 7x + 20$

We have now seen two methods you can use to multiply a trinomial by a binomial. After you practice each method, you’ll probably find you prefer one way over the other. We list both methods are listed here, for easy reference.

Note:
Multiplying a Trinomial by a Binomial
To multiply a trinomial by a binomial, use the:

- Distributive Property
- Vertical Method

Note:
Access these online resources for additional instruction and practice with multiplying polynomials:

- Multiplying Exponents 1
- Multiplying Exponents 2
- Multiplying Exponents 3

Key Concepts

- **FOIL Method for Multiplying Two Binomials**—To multiply two binomials:

  Multiply the**First**terms.
  Multiply the**Outer**terms.
  Multiply the**Inner**terms.
  Multiply the**Last**terms.
  Combine like terms.

- **Multiplying Two Binomials**—To multiply binomials, use the:
  - Distributive Property ([link])
  - FOIL Method ([link])
  - Vertical Method ([link])
- **Multiplying a Trinomial by a Binomial**—To multiply a trinomial by a binomial, use the:
  - Distributive Property ([link])
  - Vertical Method ([link])

Practice Makes Perfect

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

Exercise:

Problem: $4(w+10)$

Solution:

$4w+40$

Exercise:

Problem: $6(b+8)$

Exercise:

Problem: $-3(a+7)$

Solution:

$-3a-21$

Exercise:

Problem: $-5(p+9)$

Exercise:

Problem: $2(x-7)$

Solution:

$2x-14$

Exercise:

Problem: $7(y-4)$

Exercise:

Problem: $-3(k-4)$

Solution:

$-3k+12$

Exercise:

Problem: $-8(j-5)$

Exercise:

Problem: $q(q+5)$

Solution:

q^2+5q

Exercise:

Problem: $k(k+7)$

Exercise:

Problem: $-b(b+9)$

Solution:

$-b^2-9b$

Exercise:

Problem: $-y(y+3)$

Exercise:

Problem: $-x(x-10)$

Solution:

$-x^2+10x$

Exercise:

Problem: $-p(p-15)$

Exercise:

Problem: $6r(4r+s)$

Solution:

$24r^2+6rs$

Exercise:

Problem: $5c(9c+d)$

Exercise:

Problem: $12x(x-10)$

Solution:

$12x^2-120x$

Exercise:

Problem: $9m(m-11)$

Exercise:

Problem: $-9a(3a+5)$

Solution:

$-27a^2-45a$

Exercise:

Problem: $-4p(2p+7)$

Exercise:

Problem: $3(p^2+10p+25)$

Solution:

$3p^2+30p+75$

Exercise:

Problem: $6(y^2+8y+16)$

Exercise:

Problem: $-8x(x^2+2x-15)$

Solution:

$-8x^3-16x^2+120x$

Exercise:

Problem: $-5t(t^2+3t-18)$

Exercise:

Problem: $5q^3\left(q^3-2q+6\right)$

Solution:

$5q^6-10q^4+30q^3$

Exercise:

Problem: $4x^3\left(x^4-3x+7\right)$

Exercise:

Problem: $-8y(y^2+2y-15)$

Solution:

$-8y^3-16y^2+120y$

Exercise:

Problem: $-5m(m^2+3m-18)$

Exercise:

Problem: $5q^3(q^2-2q+6)$

Solution:

$5q^5 - 10q^4 + 30q^3$

Exercise:

Problem: $9r^3(r^2 - 3r + 5)$

Exercise:

Problem: $-4z^2(3z^2 + 12z - 1)$

Solution:

$-12z^4 - 48z^3 + 4z^2$

Exercise:

Problem: $-3x^2(7x^2 + 10x - 1)$

Exercise:

Problem: $(2m - 9)m$

Solution:

$2m^2 - 9m$

Exercise:

Problem: $(8j - 1)j$

Exercise:

Problem: $(w - 6) \cdot 8$

Solution:

$8w - 48$

Exercise:

Problem: $(k-4)\cdot 5$

Exercise:

Problem: $4(x+10)$

Solution:

$4x+40$

Exercise:

Problem: $6(y+8)$

Exercise:

Problem: $15(r-24)$

Solution:

$15r-360$

Exercise:

Problem: $12(v-30)$

Exercise:

Problem: $-3(m+11)$

Solution:

$-3m-33$

Exercise:

Problem: $-4(p+15)$

Exercise:

Problem: $-8(z-5)$

Solution:

$-8z+40$

Exercise:

Problem: $-3(x-9)$

Exercise:

Problem: $u(u+5)$

Solution:

u^2+5u

Exercise:

Problem: $q(q+7)$

Exercise:

Problem: $n(n^2-3n)$

Solution:

n^3-3n^2

Exercise:

Problem: $s(s^2-6s)$

Exercise:

Problem: $6x(4x+y)$

Solution:

$24x^2+6xy$

Exercise:

Problem: $5a(9a+b)$

Exercise:

Problem: $5p(11p-5q)$

Solution:

$55p^2-25pq$

Exercise:

Problem: $12u(3u-4v)$

Exercise:

Problem: $3(v^2+10v+25)$

Solution:

$3v^2+30v+75$

Exercise:

Problem: $6(x^2+8x+16)$

Exercise:

Problem: $2n(4n^2 - 4n + 1)$

Solution:

$8n^3 - 8n^2 + 2n$

Exercise:

Problem: $3r(2r^2 - 6r + 2)$

Exercise:

Problem: $-8y(y^2 + 2y - 15)$

Solution:

$-8y^3 - 16y^2 + 120y$

Exercise:

Problem: $-5m(m^2 + 3m - 18)$

Exercise:

Problem: $5q^3(q^2 - 2q + 6)$

Solution:

$5q^5 - 10q^4 + 30q^3$

Exercise:

Problem: $9r^3(r^2 - 3r + 5)$

Exercise:

Problem: $-4z^2(3z^2 + 12z - 1)$

Solution:

$-12z^4 - 48z^3 + 4z^2$

Exercise:

Problem: $-3x^2(7x^2 + 10x - 1)$

Exercise:

Problem: $(2y - 9)y$

Solution:

$18y^2 - 9y$

Exercise:

Problem: $(8b - 1)b$

Multiply a Binomial by a Binomial

In the following exercises, multiply the following binomials using: ⓐ the Distributive Property ⓑ the FOIL method ⓒ the Vertical Method.

Exercise:

Problem: $(w + 5)(w + 7)$

Solution:

$w^2 + 12w + 35$

Exercise:

Problem: $(y + 9)(y + 3)$

Exercise:

Problem: $(p + 11)(p - 4)$

Solution:

$p^2 + 7p - 44$

Exercise:

Problem: $(q + 4)(q - 8)$

In the following exercises, multiply the binomials. Use any method.

Exercise:

Problem: $(x + 8)(x + 3)$

Solution:

$x^2 + 11x + 24$

Exercise:

Problem: $(y + 7)(y + 4)$

Exercise:

Problem: $(y - 6)(y - 2)$

Solution:

$y^2 - 8y + 12$

Exercise:

Problem: $(x - 7)(x - 2)$

Exercise:

Problem: $(w-4)(w+7)$

Solution:

$w^2+3w-28$

Exercise:

Problem: $(q-5)(q+8)$

Exercise:

Problem: $(p+12)(p-5)$

Solution:

$p^2+7p-60$

Exercise:

Problem: $(m+11)(m-4)$

Exercise:

Problem: $(6p+5)(p+1)$

Solution:

$6p^2+11p+5$

Exercise:

Problem: $(7m+1)(m+3)$

Exercise:

Problem: $(2t-9)(10t+1)$

Solution:

$20t^2 - 88t - 9$

Exercise:

Problem: $(3r - 8)(11r + 1)$

Exercise:

Problem: $(5x - y)\,(3x - 6)$

Solution:

$15x^2 - 3xy - 30x + 6y$

Exercise:

Problem: $(10a - b)\,(3a - 4)$

Exercise:

Problem: $(a + b)(2a + 3b)$

Solution:

$2a^2 + 5ab + 3b^2$

Exercise:

Problem: $(r + s)(3r + 2s)$

Exercise:

Problem: $(4z - y)(z - 6)$

Solution:

$4z^2-24z-zy+6y$

Exercise:

Problem: $(5x-y)(x-4)$

Exercise:

Problem: $(x^2+3)(x+2)$

Solution:

x^3+2x^2+3x+6

Exercise:

Problem: $(y^2-4)(y+3)$

Exercise:

Problem: $(x^2+8)(x^2-5)$

Solution:

x^4+3x^2-40

Exercise:

Problem: $(y^2-7)(y^2-4)$

Exercise:

Problem: $(5ab-1)(2ab+3)$

Solution:

$10a^2b^2+13ab-3$

Exercise:

Problem: $(2xy+3)(3xy+2)$

Exercise:

Problem: $(6pq-3)(4pq-5)$

Solution:

$24p^2q^2-42pq+15$

Exercise:

Problem: $(3rs-7)(3rs-4)$

Multiply a Trinomial by a Binomial

In the following exercises, multiply using ⓐ the Distributive Property ⓑ the Vertical Method.

Exercise:

Problem: $(x+5)(x^2+4x+3)$

Solution:

$x^3+9x^2+23x+15$

Exercise:

Problem: $(u+4)(u^2+3u+2)$

Exercise:

Problem: $(y+8)(4y^2+y-7)$

Solution:

$4y^3+33y^2+y-56$

Exercise:

Problem: $(a+10)(3a^2+a-5)$

In the following exercises, multiply. Use either method.

Exercise:

Problem: $(w-7)(w^2-9w+10)$

Solution:

$w^3-16w^2+73w-70$

Exercise:

Problem: $(p-4)(p^2-6p+9)$

Exercise:

Problem: $(3q+1)(q^2-4q-5)$

Solution:

$3q^3-11q^2-19q-5$

Exercise:

Problem: $(6r+1)(r^2-7r-9)$

Mixed Practice

Exercise:

Problem: $(10y-6)+(4y-7)$

Solution:

$14y-13$

Exercise:

Problem: $(15p-4)+(3p-5)$

Exercise:

Problem: $\left(x^2-4x-34\right)-\left(x^2+7x-6\right)$

Solution:

$-11x-28$

Exercise:

Problem: $\left(j^2-8j-27\right)-\left(j^2+2j-12\right)$

Exercise:

Problem: $5q(3q^2-6q+11)$

Solution:

$15q^3-30q^2+55q$

Exercise:

Problem: $8t(2t^2-5t+6)$

Exercise:

Problem: $(s-7)\,(s+9)$

Solution:

$s^2+2s-63$

Exercise:

Problem: $(x-5)(x+13)$

Exercise:

Problem: $\left(y^2-2y\right)(y+1)$

Solution:

y^3-y^2-2y

Exercise:

Problem: $\left(a^2-3a\right)(4a+5)$

Exercise:

Problem: $(3n-4)\left(n^2+n-7\right)$

Solution:

$3n^3-n^2-25n+28$

Exercise:

Problem: $(6k-1)\left(k^2+2k-4\right)$

Exercise:

Problem: $(7p+10)(7p-10)$

Solution:

$49p^2-100$

Exercise:

Problem: $(3y+8)(3y-8)$

Exercise:

Problem: $(4m^2 - 3m - 7)m^2$

Solution:

$4m^4 - 3m^3 - 7m^2$

Exercise:

Problem: $(15c^2 - 4c + 5)c^4$

Exercise:

Problem: $(5a + 7b)\,(5a + 7b)$

Solution:

$25a^2 + 70ab + 49b^2$

Exercise:

Problem: $(3x - 11y)\,(3x - 11y)$

Exercise:

Problem: $(4y + 12z)\,(4y - 12z)$

Solution:

$16y^2 - 144z^2$

Everyday Math

Exercise:

Problem:

Mental math You can use binomial multiplication to multiply numbers without a calculator. Say you need to multiply 13 times 15. Think of 13 as $10+3$ and 15 as $10+5$.

ⓐ Multiply $(10+3)(10+5)$ by the FOIL method.
ⓑ Multiply $13\cdot 15$ without using a calculator.
ⓒ Which way is easier for you? Why?

Exercise:

Problem:

Mental math You can use binomial multiplication to multiply numbers without a calculator. Say you need to multiply 18 times 17. Think of 18 as $20-2$ and 17 as $20-3$.

ⓐ Multiply $(20-2)(20-3)$ by the FOIL method.
ⓑ Multiply $18\cdot 17$ without using a calculator.
ⓒ Which way is easier for you? Why?

Solution:

ⓐ 306 ⓑ 306 ⓒ Answers will vary.

Writing Exercises

Exercise:

Problem:

Which method do you prefer to use when multiplying two binomials: the Distributive Property, the FOIL method, or the Vertical Method? Why?

Exercise:

Problem:

Which method do you prefer to use when multiplying a trinomial by a binomial: the Distributive Property or the Vertical Method? Why?

Solution:

Answers will vary.

Exercise:

Problem: Multiply the following:

$(x+2)(x-2)$
$(y+7)(y-7)$
$(w+5)(w-5)$

Explain the pattern that you see in your answers.

Exercise:

Problem: Multiply the following:

$(m-3)(m+3)$
$(n-10)(n+10)$
$(p-8)(p+8)$

Explain the pattern that you see in your answers.

Solution:

Answers may vary.

Exercise:

Problem: Multiply the following:

$(p+3)(p+3)$
$(q+6)(q+6)$
$(r+1)(r+1)$

Explain the pattern that you see in your answers.

Exercise:

Problem: Multiply the following:

$(x-4)(x-4)$
$(y-1)(y-1)$
$(z-7)(z-7)$

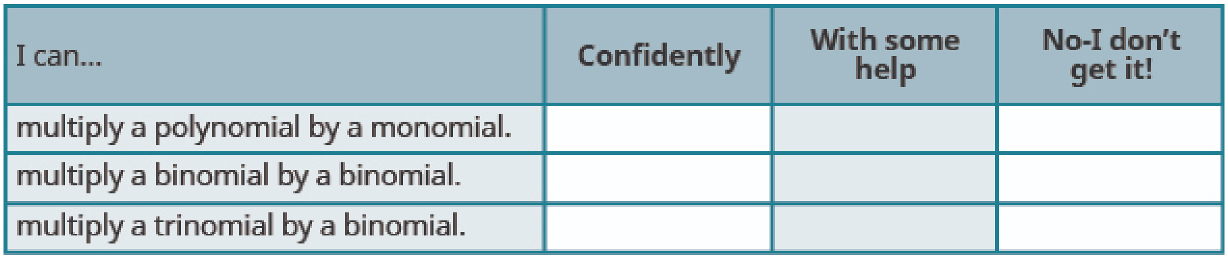
Explain the pattern that you see in your answers.

Solution:

Answers may vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
multiply a polynomial by a monomial.			
multiply a binomial by a binomial.			
multiply a trinomial by a binomial.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Special Products

By the end of this section, you will be able to:

- Square a binomial using the Binomial Squares Pattern
- Multiply conjugates using the Product of Conjugates Pattern
- Recognize and use the appropriate special product pattern

Note:
Before you get started, take this readiness quiz.

1. Simplify: ⓐ 9^2 ⓑ $(-9)^2$ ⓒ -9^2.
   If you missed this problem, review [link].

Square a Binomial Using the Binomial Squares Pattern

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and using the methods of the last section, there is less work to do if you learn to use a pattern.

Let's start by looking at $(x+9)^2$.

What does this mean?	$(x+9)^2$
It means to multiply $(x+9)$ by itself.	$(x+9)(x+9)$
Then, using FOIL, we get:	$x^2+9x+9x+81$
Combining like terms gives:	$x^2+18x+81$

Here's another one:	$(y-7)^2$
Multiply $(y-7)$ by itself.	$(y-7)(y-7)$
Using FOIL, we get:	$y^2-7y-7y+49$
And combining like terms:	$y^2-14y+49$

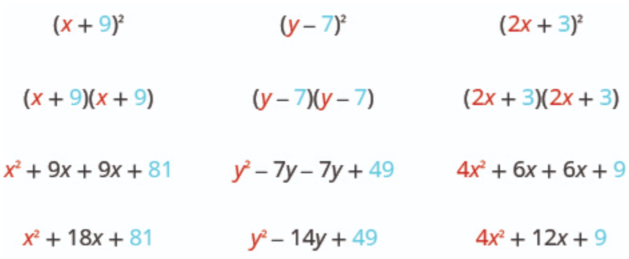
And one more:	$(2x+3)^2$
Multiply.	$(2x+3)(2x+3)$
Use FOIL:	$4x^2+6x+6x+9$
Combine like terms.	$4x^2+12x+9$

Look at these results. Do you see any patterns?

What about the number of terms? In each example we squared a binomial and the result was a trinomial.
Equation:

$$(a+b)^2 = \underline{\quad\quad} + \underline{\quad\quad} + \underline{\quad\quad}$$

Now look at the ***first term*** in each result. Where did it come from?



$(x+9)^2$	$(y-7)^2$	$(2x+3)^2$
$(x+9)(x+9)$	$(y-7)(y-7)$	$(2x+3)(2x+3)$
$x^2+9x+9x+81$	$y^2-7y-7y+49$	$4x^2+6x+6x+9$
$x^2+18x+81$	$y^2-14y+49$	$4x^2+12x+9$

The first term is the product of the first terms of each binomial. Since the binomials are identical, it is just the square of the first term!

Equation:

$$(a+b)^2 = a^2 + \_\_\_\_ + \_\_\_\_$$

To get the ***first term*** of the product, ***square the first term***.

Where did the ***last term*** come from? Look at the examples and find the pattern.

The last term is the product of the last terms, which is the square of the last term.

Equation:

$$(a+b)^2 = \_\_\_\_ + \_\_\_\_ + b^2$$

*To get the **last term** of the product, **square the last term**.*

Finally, look at the ***middle term***. Notice it came from adding the “outer” and the “inner” terms—which are both the same! So the middle term is double the product of the two terms of the binomial.

Equation:

$$(a+b)^2 = \_\_\_\_ + 2ab + \_\_\_\_$$
$$(a-b)^2 = \_\_\_\_ - 2ab + \_\_\_\_$$

*To get the **middle term** of the product, **multiply the terms and double their product**.*

Putting it all together:

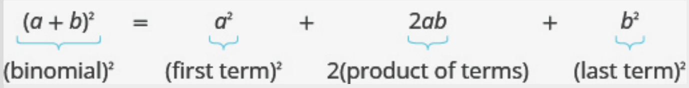
Note:

Binomial Squares Pattern

If a and b are real numbers,

Equation:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

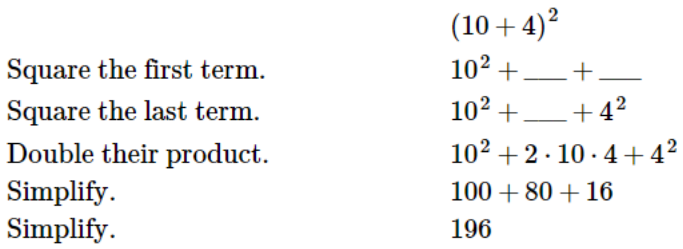


$$\underbrace{(a+b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} + \underbrace{2ab}_{\text{2(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$$

To square a binomial:

- square the first term
- square the last term
- double their product

A number example helps verify the pattern.



	$(10+4)^2$
Square the first term.	$10^2 + \_\_\_ + \_\_\_$
Square the last term.	$10^2 + \_\_\_ + 4^2$
Double their product.	$10^2 + 2 \cdot 10 \cdot 4 + 4^2$
Simplify.	$100 + 80 + 16$
Simplify.	196

To multiply $(10+4)^2$ usually you’d follow the Order of Operations.
Equation:

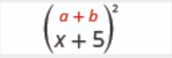
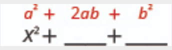
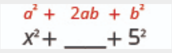
$$\begin{gathered}(10+4)^2\\(14)^2\\196\end{gathered}$$

The pattern works!

Example:
Exercise:

Problem: Multiply: $(x+5)^2$.

Solution:
Solution

	 $\left(\overset{a+b}{x+5}\right)^2$
Square the first term.	 $\overset{a^2}{x^2} \overset{+}{+} \overset{2ab}{\_\_\_\_} \overset{+}{+} \overset{b^2}{\_\_\_\_}$
Square the last term.	 $\overset{a^2}{x^2} \overset{+}{+} \overset{2ab}{\_\_\_\_} \overset{+}{+} \overset{b^2}{5^2}$

Double the product.	$a^2 + 2\cdot a\cdot b + b^2$ $x^2 + 2\cdot x\cdot 5 + 5^2$
Simplify.	$x^2 + 10x + 25$

Note:
Exercise:

Problem: Multiply: $(x+9)^2$.

Solution:

$x^2 + 18x + 81$

Note:
Exercise:

Problem: Multiply: $(y+11)^2$.

Solution:

$y^2 + 22y + 121$

Example:
Exercise:

Problem: Multiply: $(y-3)^2$.

Solution:
Solution

	$\left(\overset{a\ -\ b}{y-3}\right)^2$
Square the first term.	$a^2 - 2ab + b^2$ $y^2 - \_\_\_\_ + \_\_\_\_$

Square the last term.	$a^2 - 2ab + b^2$ $y^2 - \_\_\_\_ + 3^2$
Double the product.	$a^2 - 2 \cdot a \cdot b + b^2$ $y^2 - 2 \cdot y \cdot 3 + 3^2$
Simplify.	$y^2 - 6y + 9$

Note:
Exercise:

Problem: Multiply: $(x - 9)^2$.

Solution:

$x^2 - 18x + 81$

Note:
Exercise:

Problem: Multiply: $(p - 13)^2$.

Solution:

$p^2 - 26p + 169$

Example:
Exercise:

Problem: Multiply: $(4x + 6)^2$.

Solution:
Solution

	$(\overset{a}{4x} + \overset{b}{6})^2$

Use the pattern.	$\overset{a^2}{(4x)^2}+\overset{2\cdot a\cdot b}{2\cdot 4x\cdot 6}+\overset{b^2}{6^2}$
Simplify.	$16x^2+48x+36$

Note:
Exercise:

Problem: Multiply: $(6x+3)^2$.

Solution:

$36x^2+36x+9$

Note:
Exercise:

Problem: Multiply: $(4x+9)^2$.

Solution:

$16x^2+72x+81$

Example:
Exercise:

Problem: Multiply: $(2x-3y)^2$.

Solution:
Solution

	$\left(\overset{a\;-\;b}{2x-3y}\right)^2$
Use the pattern.	$\overset{a^2}{(2x)^2}-\overset{2\cdot a\cdot b}{2\cdot 2x\cdot 3y}+\overset{b^2}{(3y)^2}$

Simplify.	$4x^2 - 12xy + 9y^2$

Note:
Exercise:

Problem: Multiply: $(2c - d)^2$.

Solution:

$4c^2 - 4cd + d^2$

Note:
Exercise:

Problem: Multiply: $(4x - 5y)^2$.

Solution:

$16x^2 - 40xy + 25y^2$

Example:
Exercise:

Problem: Multiply: $\left(4u^3 + 1\right)^2$.

Solution:
Solution

	$\left(\overset{a}{4u^3} \overset{+}{+} \overset{b}{1}\right)^2$
Use the pattern.	$\overset{a^2}{(4u^3)^2} \overset{+ \, 2 \, \cdot}{+ \, 2 \cdot} \overset{a}{4u^3} \overset{\cdot \, b \, +}{\cdot \, 1 +} \overset{b^2}{(1)^2}$
Simplify.	$16u^6 + 8u^3 + 1$

Note:
Exercise:

Problem: Multiply: $\left(2x^2+1\right)^2$.

Solution:

$4x^4+4x^2+1$

Note:
Exercise:

Problem: Multiply: $\left(3y^3+2\right)^2$.

Solution:

$9y^6+12y^3+4$

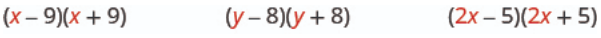
Multiply Conjugates Using the Product of Conjugates Pattern

We just saw a pattern for squaring binomials that we can use to make multiplying some binomials easier. Similarly, there is a pattern for another product of binomials. But before we get to it, we need to introduce some vocabulary.

What do you notice about these pairs of binomials?
Equation:

$$(x-9)\,(x+9) \qquad (y-8)\,(y+8) \qquad (2x-5)\,(2x+5)$$

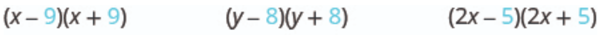
Look at the first term of each binomial in each pair.



$(x-9)(x+9)$ $(y-8)(y+8)$ $(2x-5)(2x+5)$

Notice the first terms are the same in each pair.

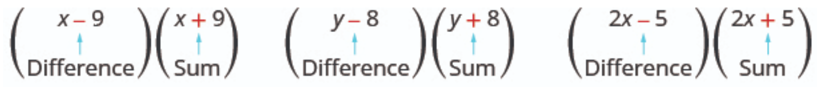
Look at the last terms of each binomial in each pair.



$(x-9)(x+9)$ $(y-8)(y+8)$ $(2x-5)(2x+5)$

Notice the last terms are the same in each pair.

Notice how each pair has one sum and one difference.



$\left(\underset{\text{Difference}}{\underset{\uparrow}{x-9}}\right)\left(\underset{\text{Sum}}{\underset{\uparrow}{x+9}}\right)$ $\left(\underset{\text{Difference}}{\underset{\uparrow}{y-8}}\right)\left(\underset{\text{Sum}}{\underset{\uparrow}{y+8}}\right)$ $\left(\underset{\text{Difference}}{\underset{\uparrow}{2x-5}}\right)\left(\underset{\text{Sum}}{\underset{\uparrow}{2x+5}}\right)$

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference has a special name. It is called a *conjugate pair* and is of the form $(a-b),(a+b)$.

Note:
Conjugate Pair
A **conjugate pair** is two binomials of the form
Equation:

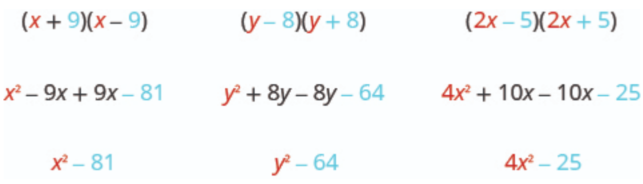
$$(a-b),(a+b).$$

The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

There is a nice pattern for finding the product of conjugates. You could, of course, simply FOIL to get the product, but using the pattern makes your work easier.

Let’s look for the pattern by using FOIL to multiply some conjugate pairs.
Equation:

$$\begin{array}{ccc} (x-9)\,(x+9) & (y-8)\,(y+8) & (2x-5)\,(2x+5) \\ x^2+9x-9x-81 & y^2+8y-8y-64 & 4x^2+10x-10x-25 \\ x^2-81 & y^2-64 & 4x^2-25 \end{array}$$



$$\begin{array}{ccc} (x+9)(x-9) & (y-8)(y+8) & (2x-5)(2x+5) \\ x^2-9x+9x-81 & y^2+8y-8y-64 & 4x^2+10x-10x-25 \\ x^2-81 & y^2-64 & 4x^2-25 \end{array}$$

Each **first term** is the product of the first terms of the binomials, and since they are identical it is the square of the first term.

$$(a+b)(a-b)=a^2-\underline{\quad\quad}$$

$$\text{To get the }\textbf{first term, square the first term}.$$

The **last term** came from multiplying the last terms, the square of the last term.
Equation:

$$(a+b)(a-b)=a^2-b^2$$

$$\text{To get the }\textbf{last term, square the last term}.$$

What do you observe about the products?

The product of the two binomials is also a binomial! Most of the products resulting from FOIL have been trinomials.

Why is there no middle term? Notice the two middle terms you get from FOIL combine to 0 in every case, the result of one addition and one subtraction.

The product of conjugates is always of the form a^2-b^2. This is called a difference of squares.

This leads to the pattern:

Note:
Product of Conjugates Pattern
If a and b are real numbers,

$$(a-b)(a+b)=a^2-b^2 \qquad \underbrace{(a-b)(a+b)}_{\text{conjugates}}=\underset{\text{squares}}{a^2}\overset{\text{difference}}{-}\underset{\text{squares}}{b^2}$$

The product is called a difference of squares.
To multiply conjugates, square the first term, square the last term, and write the product as a difference of squares.

Let’s test this pattern with a numerical example.

	$(10-2)(10+2)$
It is the product of conjugates, so the result will be the difference of two squares.	$\_\_\_\_ - \_\_\_\_$
Square the first term.	$10^2 - \_\_\_\_$
Square the last term.	$10^2 - 2^2$
Simplify.	$100-4$
Simplify.	96
What do you get using the Order of Operations?	
	$(10-2)(10+2)$
	$(8)(12)$
	96

Notice, the result is the same!

Example:
Exercise:

Problem: Multiply: $(x-8)(x+8)$.

Solution:
Solution

First, recognize this as a product of conjugates. The binomials have the same first terms, and the same last terms, and one binomial is a sum and the other is a difference.

It fits the pattern.	$\overset{a\ -\ b}{(x-8)}\overset{a\ +\ b}{(x+8)}$
Square the first term, x.	$\overset{a^2\ -\ b^2}{x^2 - \_\_}$

Square the last term, 8.	$\overset{a^2}{x^2} - \overset{b^2}{8^2}$
The product is a difference of squares.	$\overset{a^2}{x^2} - \overset{b^2}{64}$

Note:
Exercise:

Problem: Multiply: $(x-5)(x+5)$.

Solution:

$x^2 - 25$

Note:
Exercise:

Problem: Multiply: $(w-3)(w+3)$.

Solution:

$w^2 - 9$

Example:
Exercise:

Problem: Multiply: $(2x+5)(2x-5)$.

Solution:
Solution

Are the binomials conjugates?

It is the product of conjugates.	$(\overset{a}{2x} \overset{+}{+} \overset{b}{5})(\overset{a}{2x} \overset{-}{-} \overset{b}{5})$

Square the first term, 2*x*.	$a^2 - b^2$ $(2x)^2 - \_\_$
Square the last term, 5.	$a^2 - b^2$ $(2x)^2 - 5^2$
Simplify. The product is a difference of squares.	$a^2 - b^2$ $4x^2 - 25$

Note:
Exercise:

Problem: Multiply: $(6x + 5)(6x - 5)$.

Solution:

$36x^2 - 25$

Note:
Exercise:

Problem: Multiply: $(2x + 7)(2x - 7)$.

Solution:

$4x^2 - 49$

The binomials in the next example may look backwards – the variable is in the second term. But the two binomials are still conjugates, so we use the same pattern to multiply them.

Example:
Exercise:

Problem: Find the product: $(3 + 5x)(3 - 5x)$.

Solution:
Solution

It is the product of conjugates.	$\overset{a\ -\ b}{(3+5x)}\ \overset{a\ +\ b}{(3-5x)}$
Use the pattern.	$\overset{a^2\ -\ b^2}{3^2-(5x)^2}$
Simplify.	$9-25x^2$

Note:
Exercise:

Problem: Multiply: $(7+4x)(7-4x)$.

Solution:

$49-16x^2$

Note:
Exercise:

Problem: Multiply: $(9-2y)(9+2y)$.

Solution:

$81-4y^2$

Now we'll multiply conjugates that have two variables.

Example:
Exercise:

Problem: Find the product: $(5m-9n)(5m+9n)$.

Solution:
Solution

This fits the pattern.	

	$\left(\overset{a}{5m}\overset{-}{-}\overset{b}{9n}\right)\left(\overset{a}{5m}\overset{+}{+}\overset{b}{9n}\right)$
Use the pattern.	$\overset{a^2}{(5m)^2}\overset{-}{-}\overset{b^2}{(9n)^2}$
Simplify.	$25m^2-81n^2$

Note:
Exercise:

Problem: Find the product: $(4p-7q)(4p+7q)$.

Solution:

$16p^2-49q^2$

Note:
Exercise:

Problem: Find the product: $(3x-y)(3x+y)$.

Solution:

$9x^2-y^2$

Example:
Exercise:

Problem: Find the product: $(cd-8)(cd+8)$.

Solution:
Solution

This fits the pattern.	$\left(\overset{a}{cd}\overset{-}{-}\overset{b}{8}\right)\left(\overset{a}{cd}\overset{+}{+}\overset{b}{8}\right)$
Use the pattern.	

	$\overset{a^2}{(cd)^2} - \overset{b^2}{(8)^2}$
Simplify.	$c^2d^2 - 64$

Note:
Exercise:

Problem: Find the product: $(xy-6)(xy+6)$.

Solution:

x^2y^2-36

Note:
Exercise:

Problem: Find the product: $(ab-9)(ab+9)$.

Solution:

a^2b^2-81

Example:
Exercise:

Problem: Find the product: $\left(6u^2-11v^5\right)\left(6u^2+11v^5\right)$.

Solution:
Solution

This fits the pattern.	$\left(\overset{a}{6u^2} - \overset{b}{11v^5}\right)\left(\overset{a}{6u^2} + \overset{b}{11v^5}\right)$
Use the pattern.	$\overset{a^2}{(6u^2)^2} - \overset{b^2}{(11v^5)^2}$
Simplify.	

	$36u^4 - 121v^{10}$

Note:
Exercise:

Problem: Find the product: $\left(3x^2 - 4y^3\right)\left(3x^2 + 4y^3\right)$.

Solution:

$9x^4 - 16y^6$

Note:
Exercise:

Problem: Find the product: $\left(2m^2 - 5n^3\right)\left(2m^2 + 5n^3\right)$.

Solution:

$4m^4 - 25n^6$

Recognize and Use the Appropriate Special Product Pattern

We just developed special product patterns for Binomial Squares and for the Product of Conjugates. The products look similar, so it is important to recognize when it is appropriate to use each of these patterns and to notice how they differ. Look at the two patterns together and note their similarities and differences.

Note:
Comparing the Special Product Patterns

Binomial Squares	**Product of Conjugates**
$(a + b)^2 = a^2 + 2ab + b^2$	$(a - b)(a + b) = a^2 - b^2$
$(a - b)^2 = a^2 - 2ab + b^2$	
- Squaring a binomial	- Multiplying conjugates
- Product is a **trinomial**	- Product is a **binomial**
- Inner and outer terms with FOIL are **the same.**	- Inner and outer terms with FOIL are **opposit**
- Middle term is **double the product** of the terms.	- There is **no** middle term.

Example:
Exercise:

Problem: Choose the appropriate pattern and use it to find the product:

ⓐ $(2x-3)(2x+3)$ ⓑ $(5x-8)^2$ ⓒ $(6m+7)^2$ ⓓ $(5x-6)(6x+5)$

Solution:
Solution

ⓐ $(2x-3)(2x+3)$ These are conjugates. They have the same first numbers, and the same last numbers, and one binomial is a sum and the other is a difference. It fits the Product of Conjugates pattern.

This fits the pattern.	$(\overset{a}{2x} - \overset{b}{3})(\overset{a}{2x} + \overset{b}{3})$
Use the pattern.	$\overset{a^2}{(2x)^2} - \overset{b^2}{3^2}$
Simplify.	$4x^2 - 9$

ⓑ $(8x-5)^2$ We are asked to square a binomial. It fits the **binomial squares** pattern.

	$(\overset{a}{8x} - \overset{b}{5})^2$
Use the pattern.	$\overset{a^2}{(8x)^2} - \overset{2ab}{2 \cdot 8x \cdot 5} + \overset{b^2}{5^2}$
Simplify.	$64x^2 - 80x + 25$

ⓒ $(6m+7)^2$ Again, we will square a binomial so we use the **binomial squares** pattern.

	$(\overset{a}{6m} + \overset{b}{7})^2$

Use the pattern.	$\overset{a^2}{(6m)^2} \overset{+}{+} \overset{2ab}{2 \cdot 6m \cdot 7} \overset{+}{+} \overset{b^2}{7^2}$
Simplify.	$36m^2 + 84m + 49$

ⓓ $(5x - 6)(6x + 5)$ This product does not fit the patterns, so we will use FOIL.

	$(5x - 6)(6x + 5)$
Use FOIL.	$30x^2 + 25x - 36x - 30$
Simplify.	$30x^2 - 11x - 30$

Note:
Exercise:

Problem: Choose the appropriate pattern and use it to find the product:

ⓐ $(9b - 2)(2b + 9)$ ⓑ $(9p - 4)^2$ ⓒ $(7y + 1)^2$ ⓓ $(4r - 3)(4r + 3)$

Solution:

ⓐ FOIL; $18b^2 + 77b - 18$ ⓑ Binomial Squares; $81p^2 - 72p + 16$ ⓒ Binomial Squares; $49y^2 + 14y + 1$ ⓓ Product of Conjugates; $16r^2 - 9$

Note:
Exercise:

Problem: Choose the appropriate pattern and use it to find the product:

ⓐ $(6x + 7)^2$ ⓑ $(3x - 4)(3x + 4)$ ⓒ $(2x - 5)(5x - 2)$ ⓓ $(6n - 1)^2$

Solution:

ⓐ Binomial Squares; $36x^2 + 84x + 49$ ⓑ Product of Conjugates; $9x^2 - 16$ ⓒ FOIL; $10x^2 - 29x + 10$ ⓓ Binomial Squares; $36n^2 - 12n + 1$

Note:
Access these online resources for additional instruction and practice with special products:

- Special Products

Key Concepts

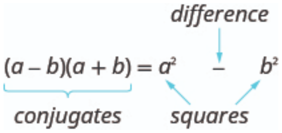
- **Binomial Squares Pattern**

  - If a, b are real numbers,

$$\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2}$$

  - $(a+b)^2 = a^2 + 2ab + b^2$
  - $(a-b)^2 = a^2 - 2ab + b^2$
  - To square a binomial: square the first term, square the last term, double their product.

- **Product of Conjugates Pattern**
  - If a, b are real numbers,



  - $(a-b)(a+b) = a^2 - b^2$
  - The product is called a difference of squares.

- **To multiply conjugates:**
  - **square the first term, square the last term**, write it as a difference of squares

Practice Makes Perfect

Square a Binomial Using the Binomial Squares Pattern

In the following exercises, square each binomial using the Binomial Squares Pattern.

Exercise:

Problem: $(w+4)^2$

Exercise:

Problem: $(q+12)^2$

Solution:

$q^2 + 24q + 144$

Exercise:

Problem: $\left(y+\frac{1}{4}\right)^2$

Exercise:

Problem: $\left(x+\frac{2}{3}\right)^2$

Solution:

$x^2 + \frac{4}{3}x + \frac{4}{9}$

Exercise:

Problem: $(b - 7)^2$

Exercise:

Problem: $(y - 6)^2$

Solution:

$y^2 - 12y + 36$

Exercise:

Problem: $(m - 15)^2$

Exercise:

Problem: $(p - 13)^2$

Solution:

$p^2 - 26p + 169$

Exercise:

Problem: $(3d + 1)^2$

Exercise:

Problem: $(4a + 10)^2$

Solution:

$16a^2 + 80a + 100$

Exercise:

Problem: $(2q + \frac{1}{3})^2$

Exercise:

Problem: $\left(3z + \frac{1}{5}\right)^2$

Solution:

$9z^2 + \frac{6}{5}z + \frac{1}{25}$

Exercise:

Problem: $(3x - y)^2$

Exercise:

Problem: $(2y-3z)^2$

Solution:

$4y^2-12yz+9z^2$

Exercise:

Problem: $\left(\frac{1}{5}x-\frac{1}{7}y\right)^2$

Exercise:

Problem: $\left(\frac{1}{8}x-\frac{1}{9}y\right)^2$

Solution:

$\frac{1}{64}x^2-\frac{1}{36}xy+\frac{1}{81}y^2$

Exercise:

Problem: $(3x^2+2)^2$

Exercise:

Problem: $(5u^2+9)^2$

Solution:

$25u^4+90u^2+81$

Exercise:

Problem: $(4y^3-2)^2$

Exercise:

Problem: $(8p^3-3)^2$

Solution:

$64p^6-48p^3+9$

Multiply Conjugates Using the Product of Conjugates Pattern

In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.

Exercise:

Problem: $(m-7)(m+7)$

Exercise:

Problem: $(c-5)(c+5)$

Solution:

$c^2 - 25$

Exercise:

Problem: $\left(x + \frac{3}{4}\right)\left(x - \frac{3}{4}\right)$

Exercise:

Problem: $\left(b + \frac{6}{7}\right)\left(b - \frac{6}{7}\right)$

Solution:

$b^2 - \frac{36}{49}$

Exercise:

Problem: $(5k + 6)(5k - 6)$

Exercise:

Problem: $(8j + 4)(8j - 4)$

Solution:

$64j^2 - 16$

Exercise:

Problem: $(11k + 4)(11k - 4)$

Exercise:

Problem: $(9c + 5)(9c - 5)$

Solution:

$81c^2 - 25$

Exercise:

Problem: $(11 - b)(11 + b)$

Exercise:

Problem: $(13 - q)(13 + q)$

Solution:

$169 - q^2$

Exercise:

Problem: $(5 - 3x)(5 + 3x)$

Exercise:

Problem: $(4 - 6y)(4 + 6y)$

Solution:

$16-36y^2$

Exercise:

Problem: $(9c-2d)(9c+2d)$

Exercise:

Problem: $(7w+10x)(7w-10x)$

Solution:

$49w^2-100x^2$

Exercise:

Problem: $\left(m+\frac{2}{3}n\right)\left(m-\frac{2}{3}n\right)$

Exercise:

Problem: $\left(p+\frac{4}{5}q\right)\left(p-\frac{4}{5}q\right)$

Solution:

$p^2-\frac{16}{25}q^2$

Exercise:

Problem: $(ab-4)(ab+4)$

Exercise:

Problem: $(xy-9)(xy+9)$

Solution:

x^2y^2-81

Exercise:

Problem: $(uv-\frac{3}{5})(uv+\frac{3}{5})$

Exercise:

Problem: $(rs-\frac{2}{7})(rs+\frac{2}{7})$

Solution:

$r^2s^2-\frac{4}{49}$

Exercise:

Problem: $(2x^2-3y^4)(2x^2+3y^4)$

Exercise:

Problem: $(6m^3-4n^5)(6m^3+4n^5)$

Solution:

$36m^6-16n^{10}$

Exercise:

Problem: $(12p^3-11q^2)(12p^3+11q^2)$

Exercise:

Problem: $(15m^2-8n^4)(15m^2+8n^4)$

Solution:

$225m^4-64n^8$

Recognize and Use the Appropriate Special Product Pattern

In the following exercises, find each product.

Exercise:

ⓐ $(p-3)(p+3)$
ⓑ $(t-9)^2$
ⓒ $(m+n)^2$
Problem: ⓓ $(2x+y)(x-2y)$

Exercise:

Problem:

ⓐ $(2r+12)^2$
ⓑ $(3p+8)(3p-8)$
ⓒ $(7a+b)(a-7b)$
ⓓ $(k-6)^2$

Solution:

ⓐ $4r^2+48r+144$ ⓑ $9p^2-64$ ⓒ $7a^2-48ab-7b^2$ ⓓ $k^2-12k+36$

Exercise:

Problem:

ⓐ $(a^5-7b)^2$
ⓑ $(x^2+8y)(8x-y^2)$
ⓒ $(r^6+s^6)(r^6-s^6)$
ⓓ $(y^4+2z)^2$

Exercise:

Problem:

ⓐ $(x^5 + y^5)(x^5 - y^5)$
ⓑ $(m^3 - 8n)^2$
ⓒ $(9p + 8q)^2$
ⓓ $(r^2 - s^3)(r^3 + s^2)$

Solution:

ⓐ $x^{10} - y^{10}$ ⓑ $m^6 - 16m^3n + 64n^2$ ⓒ $81p^2 + 144pq + 64q^2$ ⓓ $r^5 + r^2s^2 - r^3s^3 - s^5$

Everyday Math

Exercise:

Problem:

Mental math You can use the product of conjugates pattern to multiply numbers without a calculator. Say you need to multiply 47 times 53. Think of 47 as $50 - 3$ and 53 as $50 + 3$.

ⓐ Multiply $(50 - 3)(50 + 3)$ by using the product of conjugates pattern, $(a - b)(a + b) = a^2 - b^2$.
ⓑ Multiply $47 \cdot 53$ without using a calculator.
ⓒ Which way is easier for you? Why?

Exercise:

Problem:

Mental math You can use the binomial squares pattern to multiply numbers without a calculator. Say you need to square 65. Think of 65 as $60 + 5$.

ⓐ Multiply $(60 + 5)^2$ by using the binomial squares pattern, $(a + b)^2 = a^2 + 2ab + b^2$.
ⓑ Square 65 without using a calculator.
ⓒ Which way is easier for you? Why?

Solution:

ⓐ 4,225 ⓑ 4,225 ⓒ Answers will vary.

Writing Exercises

Exercise:

Problem: How do you decide which pattern to use?

Exercise:

Problem: Why does $(a + b)^2$ result in a trinomial, but $(a - b)(a + b)$ result in a binomial?

Solution:

Answers will vary.

Exercise:

Problem: Marta did the following work on her homework paper:
Equation:

$$\begin{array}{c}(3-y)^2 \\ 3^2-y^2 \\ 9-y^2\end{array}$$

Explain what is wrong with Marta’s work.

Exercise:

Problem:

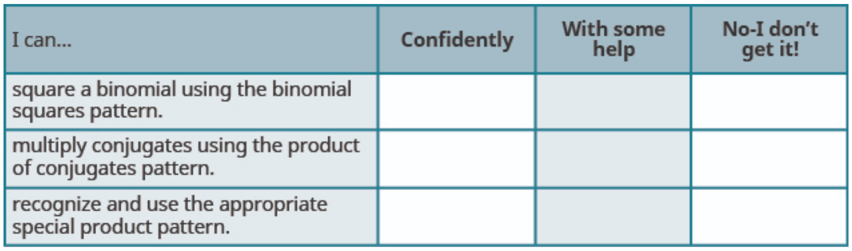
Use the order of operations to show that $(3+5)^2$ is 64, and then use that numerical example to explain why $(a+b)^2 \neq a^2+b^2$.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don’t get it!
square a binomial using the binomial squares pattern.			
multiply conjugates using the product of conjugates pattern.			
recognize and use the appropriate special product pattern.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

conjugate pair
A conjugate pair is two binomials of the form $(a-b), (a+b)$; the pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

Divide Monomials
By the end of this section, you will be able to:

- Simplify expressions using the Quotient Property for Exponents
- Simplify expressions with zero exponents
- Simplify expressions using the quotient to a Power Property
- Simplify expressions by applying several properties
- Divide monomials

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\frac{8}{24}$.
   If you missed this problem, review [link].
2. Simplify: $\left(2m^3\right)^5$.
   If you missed this problem, review [link].
3. Simplify: $\frac{12x}{12y}$.
   If you missed this problem, review [link].

Simplify Expressions Using the Quotient Property for Exponents

Earlier in this chapter, we developed the properties of exponents for multiplication. We summarize these properties below.

Note:
Summary of Exponent Properties for Multiplication
If a and b are real numbers, and m and n are whole numbers, then
Equation:

$$\begin{array}{lrcl} \textbf{Product Property} & a^m \cdot a^n & = & a^{m+n} \\ \textbf{Power Property} & \left(a^m\right)^n & = & a^{m \cdot n} \\ \textbf{Product to a Power} & (ab)^m & = & a^m b^m \end{array}$$

Now we will look at the exponent properties for division. A quick memory refresher may help before we get started. You have learned to simplify fractions by dividing out common factors from the numerator and denominator using the Equivalent Fractions Property. This property will also help you work with algebraic fractions—which are also quotients.

Note:
Equivalent Fractions Property
If a, b, and c are whole numbers where $b \neq 0, c \neq 0$,
Equation:

$$\text{then} \quad \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \quad \text{and} \quad \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

As before, we'll try to discover a property by looking at some examples.

Consider	$\frac{x^5}{x^2}$	and	$\frac{x^2}{x^3}$
What do they mean?	$\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$		$\frac{x \cdot x}{x \cdot x \cdot x}$
Use the Equivalent Fractions Property.	$\frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x}}$		$\frac{\cancel{x} \cdot \cancel{x} \cdot 1}{\cancel{x} \cdot \cancel{x} \cdot x}$
Simplify.	x^3		$\frac{1}{x}$

Notice, in each case the bases were the same and we subtracted exponents.

When the larger exponent was in the numerator, we were left with factors in the numerator.

When the larger exponent was in the denominator, we were left with factors in the denominator—notice the numerator of 1.

We write:
Equation:

$$\begin{array}{cc} \frac{x^5}{x^2} & \frac{x^2}{x^3} \\ x^{5-2} & \frac{1}{x^{3-2}} \\ x^3 & \frac{1}{x} \end{array}$$

This leads to the *Quotient Property for Exponents*.

Note:
Quotient Property for Exponents
If a is a real number, $a \neq 0$, and m and n are whole numbers, then

Equation:

$$\frac{a^m}{a^n} = a^{m-n}, m > n \text{ and } \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, n > m$$

A couple of examples with numbers may help to verify this property.

Equation:

$$\begin{array}{rclcrcl} \frac{3^4}{3^2} & = & 3^{4-2} & \qquad & \frac{5^2}{5^3} & = & \frac{1}{5^{3-2}} \\ \frac{81}{9} & = & 3^2 & & \frac{25}{125} & = & \frac{1}{5^1} \\ 9 & = & 9\checkmark & & \frac{1}{5} & = & \frac{1}{5}\checkmark \end{array}$$

Example:

Exercise:

Problem: Simplify: ⓐ $\frac{x^9}{x^7}$ ⓑ $\frac{3^{10}}{3^2}$

Solution:

Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

ⓐ

Since 9 > 7, there are more factors of *x* in the numerator.	$\frac{x^9}{x^7}$
Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	x^{9-7}
Simplify.	x^2

ⓑ

Since 10 > 2, there are more factors of *x* in the numerator.	$\frac{3^{10}}{3^{2}}$
Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	3^{10-2}
Simplify.	3^{8}

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{x^{15}}{x^{10}}$ ⓑ $\frac{6^{14}}{6^{5}}$

Solution:

ⓐ x^5 ⓑ 6^9

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{y^{43}}{y^{37}}$ ⓑ $\frac{10^{15}}{10^{7}}$

Solution:

ⓐ y^6 ⓑ 10^8

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{b^8}{b^{12}}$ ⓑ $\frac{7^3}{7^5}$

Solution:
Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

ⓐ

Since 12 > 8, there are more factors of *b* in the denominator.	$\frac{b^8}{b^{12}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{b^{12-8}}$
Simplify.	$\frac{1}{b^4}$

ⓑ

Since 5 > 3, there are more factors of 3 in the denominator.	$\frac{7^3}{7^5}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{7^{5-3}}$

Simplify.	$\frac{1}{7^2}$
Simplify.	$\frac{1}{49}$

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{x^{18}}{x^{22}}$ ⓑ $\frac{12^{15}}{12^{30}}$

Solution:

ⓐ $\frac{1}{x^4}$ ⓑ $\frac{1}{12^{15}}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{m^7}{m^{15}}$ ⓑ $\frac{9^8}{9^{19}}$

Solution:

ⓐ $\frac{1}{m^8}$ ⓑ $\frac{1}{9^{11}}$

Notice the difference in the two previous examples:

- If we start with more factors in the numerator, we will end up with factors in the numerator.
- If we start with more factors in the denominator, we will end up with factors in the denominator.

The first step in simplifying an expression using the Quotient Property for Exponents is to determine whether the exponent is larger in the numerator or the denominator.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{a^5}{a^9}$ ⓑ $\frac{x^{11}}{x^7}$

Solution:
Solution

ⓐ Is the exponent of a larger in the numerator or denominator? Since 9 > 5, there are more a′s in the denominator and so we will end up with factors in the denominator.

	$\frac{a^5}{a^9}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{a^{9-5}}$
Simplify.	$\frac{1}{a^4}$

ⓑ Notice there are more factors of x in the numerator, since 11 > 7. So we will end up with factors in the numerator.

	$\frac{x^{11}}{x^7}$

Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	x^{11-7}
Simplify.	x^4

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{b^{19}}{b^{11}}$ ⓑ $\frac{z^5}{z^{11}}$

Solution:

ⓐ b^8 ⓑ $\frac{1}{z^6}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{p^9}{p^{17}}$ ⓑ $\frac{w^{13}}{w^9}$

Solution:

ⓐ $\frac{1}{p^8}$ ⓑ w^4

Simplify Expressions with an Exponent of Zero

A special case of the Quotient Property is when the exponents of the numerator and denominator are equal, such as an expression like $\frac{a^m}{a^m}$. From your earlier work with fractions, you know that:

Equation:

$$\frac{2}{2} = 1 \qquad \frac{17}{17} = 1 \qquad \frac{-43}{-43} = 1$$

In words, a number divided by itself is 1. So, $\frac{x}{x} = 1$, for any x $(x \neq 0)$, since any number divided by itself is 1.

The Quotient Property for Exponents shows us how to simplify $\frac{a^m}{a^n}$ when $m > n$ and when $n < m$ by subtracting exponents. What if $m = n$?

Consider $\frac{8}{8}$, which we know is 1.

	$\frac{8}{8} = 1$
Write 8 as 2^3.	$\frac{2^3}{2^3} = 1$
Subtract exponents.	$2^{3-3} = 1$
Simplify.	$2^0 = 1$

Now we will simplify $\frac{a^m}{a^m}$ in two ways to lead us to the definition of the zero exponent. In general, for $a \neq 0$:

$$\begin{array}{ll}
\frac{a^m}{a^m} & \frac{a^m}{a^m} \\
a^{m-m} & \dfrac{\overbrace{\cancel{a}\cdot\cancel{a}\cdot\ldots\cdot\cancel{a}}^{m\text{ factors}}}{\underbrace{\cancel{a}\cdot\cancel{a}\cdot\ldots\cdot\cancel{a}}_{m\text{ factors}}} \\
a^0 & 1
\end{array}$$

We see $\frac{a^m}{a^m}$ simplifies to a^0 and to 1. So $a^0 = 1$.

Note:
Zero Exponent
If a is a non-zero number, then $a^0 = 1$.
Any nonzero number raised to the zero power is 1.

In this text, we assume any variable that we raise to the zero power is not zero.

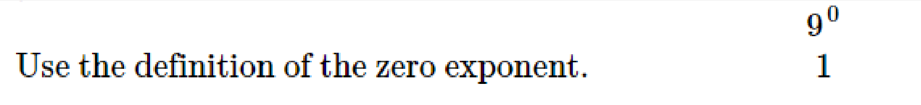
Example:
Exercise:

Problem: Simplify: ⓐ 9^0 ⓑ n^0.

Solution:
Solution

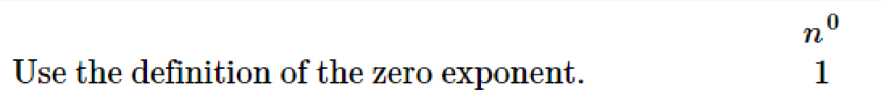
The definition says any non-zero number raised to the zero power is 1.

ⓐ



	9^0
Use the definition of the zero exponent.	1

ⓑ



	n^0
Use the definition of the zero exponent.	1

Note:
Exercise:

Problem: Simplify: ⓐ 15^0 ⓑ m^0.

Solution:

ⓐ 1 ⓑ 1

Note:
Exercise:

Problem: Simplify: ⓐ k^0 ⓑ 29^0.

Solution:

ⓐ 1 ⓑ 1

Now that we have defined the zero exponent, we can expand all the Properties of Exponents to include whole number exponents.

What about raising an expression to the zero power? Let’s look at $(2x)^0$. We can use the product to a power rule to rewrite this expression.

	$(2x)^0$
Use the product to a power rule.	2^0x^0
Use the zero exponent property.	$1 \cdot 1$
Simplify.	1

This tells us that any nonzero expression raised to the zero power is one.

Example:
Exercise:

Problem: Simplify: ⓐ $(5b)^0$ ⓑ $\left(-4a^2b\right)^0$.

Solution:
Solution

ⓐ

	$(5b)^0$
Use the definition of the zero exponent.	1

ⓑ

	$\left(-4a^2b\right)^0$
Use the definition of the zero exponent.	1

Note:
Exercise:

Problem: Simplify: ⓐ $(11z)^0$ ⓑ $\left(-11pq^3\right)^0$.

Solution:

ⓐ 1 ⓑ 1

Note:
Exercise:

Problem: Simplify: ⓐ $(-6d)^0$ ⓑ $\left(-8m^2n^3\right)^0$.

Solution:

ⓐ 1 ⓑ 1

Simplify Expressions Using the Quotient to a Power Property

Now we will look at an example that will lead us to the Quotient to a Power Property.

	$\left(\frac{x}{y}\right)^3$
This means:	$\frac{x}{y}\cdot\frac{x}{y}\cdot\frac{x}{y}$
Multiply the fractions.	$\frac{x\cdot x\cdot x}{y\cdot y\cdot y}$
Write with exponents.	$\frac{x^3}{y^3}$

Notice that the exponent applies to both the numerator and the denominator.

We see that $\left(\frac{x}{y}\right)^3$ is $\frac{x^3}{y^3}$.

We write:	$\left(\frac{x}{y}\right)^3$
	$\frac{x^3}{y^3}$

This leads to the *Quotient to a Power Property for Exponents*.

Note:
Quotient to a Power Property for Exponents
If a and b are real numbers, $b \neq 0$, and m is a counting number, then
Equation:

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

To raise a fraction to a power, raise the numerator and denominator to that power.

An example with numbers may help you understand this property:

Equation:

$$\begin{aligned} \left(\frac{2}{3}\right)^3 &= \frac{2^3}{3^3} \\ \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} &= \frac{8}{27} \\ \frac{8}{27} &= \frac{8}{27}\checkmark \end{aligned}$$

Example:

Exercise:

Problem: Simplify: ⓐ $\left(\frac{3}{7}\right)^2$ ⓑ $\left(\frac{b}{3}\right)^4$ ⓒ $\left(\frac{k}{j}\right)^3$.

Solution:

Solution

ⓐ

	$\left(\frac{3}{7}\right)^2$
Use the Quotient Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{3^2}{7^2}$
Simplify.	$\frac{9}{49}$

ⓑ

	$\left(\frac{b}{3}\right)^4$
Use the Quotient Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{b^4}{3^4}$
Simplify.	$\frac{b^4}{81}$

ⓒ

	$\left(\frac{k}{j}\right)^3$
Raise the numerator and denominator to the third power.	$\frac{k^3}{j^3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\frac{5}{8}\right)^2$ ⓑ $\left(\frac{p}{10}\right)^4$ ⓒ $\left(\frac{m}{n}\right)^7$.

Solution:

ⓐ $\frac{25}{64}$ ⓑ $\frac{p^4}{10{,}000}$ ⓒ $\frac{m^7}{n^7}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\frac{1}{3}\right)^3$ ⓑ $\left(\frac{-2}{q}\right)^3$ ⓒ $\left(\frac{w}{x}\right)^4$.

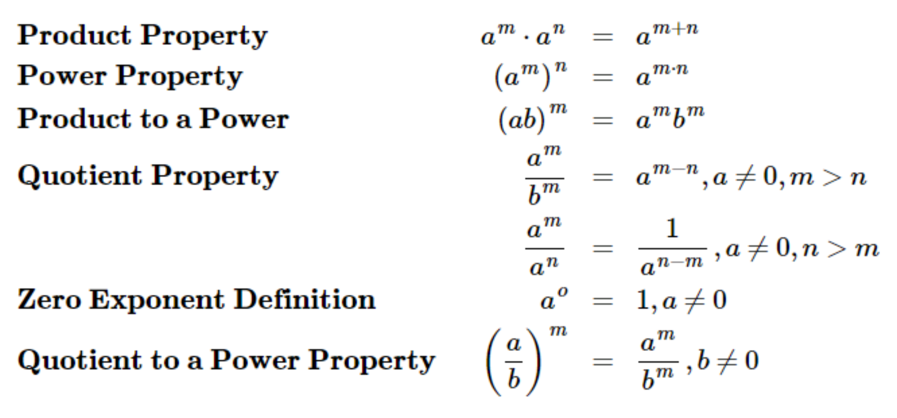
Solution:

ⓐ $\frac{1}{27}$ ⓑ $\frac{-8}{q^3}$ ⓒ $\frac{w^4}{x^4}$

Simplify Expressions by Applying Several Properties

We’ll now summarize all the properties of exponents so they are all together to refer to as we simplify expressions using several properties. Notice that they are now defined for whole number exponents.

Note:
Summary of Exponent Properties
If a and b are real numbers, and m and n are whole numbers, then



Product Property	$a^m \cdot a^n$	$=$	a^{m+n}
Power Property	$(a^m)^n$	$=$	$a^{m \cdot n}$
Product to a Power	$(ab)^m$	$=$	$a^m b^m$
Quotient Property	$\frac{a^m}{b^m}$	$=$	$a^{m-n}, a \neq 0, m > n$
	$\frac{a^m}{a^n}$	$=$	$\frac{1}{a^{n-m}}, a \neq 0, n > m$
Zero Exponent Definition	a^o	$=$	$1, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m$	$=$	$\frac{a^m}{b^m}, b \neq 0$

Example:
Exercise:

Problem: Simplify: $\frac{\left(y^4\right)^2}{y^6}$.

Solution:
Solution

$$\frac{(y^4)^2}{y^6}$$

Multiply the exponents in the numerator. $\frac{y^8}{y^6}$

Subtract the exponents. y^2

Note:
Exercise:

Problem: Simplify: $\frac{(m^5)^4}{m^7}$.

Solution:

m^{13}

Note:
Exercise:

Problem: Simplify: $\frac{(k^2)^6}{k^7}$.

Solution:

k^5

Example:
Exercise:

Problem: Simplify: $\frac{b^{12}}{(b^2)^6}$.

Solution:
Solution

	$\frac{b^{12}}{(b^2)^6}$
Multiply the exponents in the denominator.	$\frac{b^{12}}{b^{12}}$
Subtract the exponents.	b^0
Simplify.	1

Notice that after we simplified the denominator in the first step, the numerator and the denominator were equal. So the final value is equal to 1.

Note:
Exercise:

Problem: Simplify: $\frac{n^{12}}{(n^3)^4}$.

Solution:

1

Note:
Exercise:

Problem: Simplify: $\frac{x^{15}}{(x^3)^5}$.

Solution:

1

Example:
Exercise:

Problem: Simplify: $\left(\frac{y^9}{y^4}\right)^2$.

Solution:
Solution

$$\left(\frac{y^9}{y^4}\right)^2$$

Remember parentheses come before exponents.
Notice the bases are the same, so we can simplify inside the parentheses. Subtract the exponents. $\left(y^5\right)^2$

Multiply the exponents. y^{10}

Note:
Exercise:

Problem: Simplify: $\left(\frac{r^5}{r^3}\right)^4$.

Solution:

r^8

Note:
Exercise:

Problem: Simplify: $\left(\frac{v^6}{v^4}\right)^3$.

Solution:

v^6

Example:
Exercise:

Problem: Simplify: $\left(\frac{j^2}{k^3}\right)^4$.

Solution:
Solution

Here we cannot simplify inside the parentheses first, since the bases are not the same.

$\left(\frac{j^2}{k^3}\right)^4$

Raise the numerator and denominator to the third power using the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$. $\frac{(j^2)^4}{(k^3)^4}$

Use the Power Property and simplify. $\frac{j^8}{k^{12}}$

Note:
Exercise:

Problem: Simplify: $\left(\frac{a^3}{b^2}\right)^4$.

Solution:

$\frac{a^{12}}{b^8}$

Note:
Exercise:

Problem: Simplify: $\left(\frac{q^7}{r^5}\right)^3$.

Solution:

$\frac{q^{21}}{r^{15}}$

Example:
Exercise:

Problem: Simplify: $\left(\frac{2m^2}{5n}\right)^4$.

Solution:
Solution

	$\left(\frac{2m^2}{5n}\right)^4$
Raise the numerator and denominator to the fourth power, using the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{(2m^2)^4}{(5n)^4}$
Raise each factor to the fourth power.	$\frac{2^4(m^2)^4}{5^4n^4}$
Use the Power Property and simplify.	$\frac{16m^8}{625n^4}$

Note:
Exercise:

Problem: Simplify: $\left(\frac{7x^3}{9y}\right)^2$.

Solution:

$\frac{49x^6}{81y^2}$

Note:
Exercise:

Problem: Simplify: $\left(\frac{3x^4}{7y}\right)^2$.

Solution:

$\frac{9x^8}{49y^2}$

Example:
Exercise:

Problem: Simplify: $\frac{(x^3)^4(x^2)^5}{(x^6)^5}$.

Solution:
Solution

	$\frac{\left(x^3\right)^4\left(x^2\right)^5}{\left(x^6\right)^5}$
Use the Power Property, $(a^m)^n = a^{m\cdot n}$.	$\frac{\left(x^{12}\right)\left(x^{10}\right)}{\left(x^{30}\right)}$
Add the exponents in the numerator.	$\frac{x^{22}}{x^{30}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{x^8}$

Note:
Exercise:

Problem: Simplify: $\frac{\left(a^2\right)^3\left(a^2\right)^4}{\left(a^4\right)^5}$.

Solution:

$\frac{1}{a^6}$

Note:
Exercise:

Problem: Simplify: $\frac{\left(p^3\right)^4\left(p^5\right)^3}{\left(p^7\right)^6}$.

Solution:

$\frac{1}{p^{15}}$

Example:
Exercise:

Problem: Simplify: $\frac{\left(10p^3\right)^2}{(5p)^3\left(2p^5\right)^4}$.

Solution:
Solution

	$\frac{(10p^3)^2}{(5p)^3(2p^5)^4}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$\frac{(10)^2(p^3)^2}{(5)^3(p)^3(2)^4(p^5)^4}$
Use the Power Property, $(a^m)^n = a^{m\cdot n}$.	$\frac{100p^6}{125p^3\cdot 16p^{20}}$
Add the exponents in the denominator.	$\frac{100p^6}{125\cdot 16p^{23}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{100}{125\cdot 16p^{17}}$
Simplify.	$\frac{1}{20p^{17}}$

Note:
Exercise:

Problem: Simplify: $\frac{(3r^3)^2(r^3)^7}{(r^3)^3}$.

Solution:

$9r^{18}$

Note:
Exercise:

Problem: Simplify: $\frac{(2x^4)^5}{(4x^3)^2(x^3)^5}$.

Solution:

$\frac{2}{x}$

Divide Monomials

You have now been introduced to all the properties of exponents and used them to simplify expressions. Next, you’ll see how to use these properties to divide monomials. Later, you’ll use them to divide polynomials.

Example:
Exercise:

Problem: Find the quotient: $56x^7 \div 8x^3$.

Solution:
Solution

	$56x^7 \div 8x^3$
Rewrite as a fraction.	$\frac{56x^7}{8x^3}$
Use fraction multiplication.	$\frac{56}{8} \cdot \frac{x^7}{x^3}$
Simplify and use the Quotient Property.	$7x^4$

Note:
Exercise:

Problem: Find the quotient: $42y^9 \div 6y^3$.

Solution:

$7y^6$

Note:
Exercise:

Problem: Find the quotient: $48z^8 \div 8z^2$.

Solution:

$6z^6$

Example:
Exercise:

Problem: Find the quotient: $\frac{45a^2b^3}{-5ab^5}$.

Solution:
Solution

When we divide monomials with more than one variable, we write one fraction for each variable.

	$\frac{45a^2b^3}{-5ab^5}$
Use fraction multiplication.	$\frac{45}{-5}\cdot\frac{a^2}{a}\cdot\frac{b^3}{b^5}$
Simplify and use the Quotient Property.	$-9\cdot a\cdot\frac{1}{b^2}$
Multiply.	$-\frac{9a}{b^2}$

Note:
Exercise:

Problem: Find the quotient: $\frac{-72a^7b^3}{8a^{12}b^4}$.

Solution:

$-\frac{9}{a^5b}$

Note:
Exercise:

Problem: Find the quotient: $\frac{-63c^8d^3}{7c^{12}d^2}$.

Solution:

$\frac{-9d}{c^4}$

Example:
Exercise:

Problem: Find the quotient: $\frac{24a^5b^3}{48ab^4}$.

Solution:

Solution

	$\frac{24a^5b^3}{48ab^4}$
Use fraction multiplication.	$\frac{24}{48} \cdot \frac{a^5}{a} \cdot \frac{b^3}{b^4}$
Simplify and use the Quotient Property.	$\frac{1}{2} \cdot a^4 \cdot \frac{1}{b}$
Multiply.	$\frac{a^4}{2b}$

Note:
Exercise:

Problem: Find the quotient: $\frac{16a^7b^6}{24ab^8}$.

Solution:

$\frac{2a^6}{3b^2}$

Note:
Exercise:

Problem: Find the quotient: $\frac{27p^4q^7}{-45p^{12}q}$.

Solution:

$-\frac{3q^6}{5p^8}$

Once you become familiar with the process and have practiced it step by step several times, you may be able to simplify a fraction in one step.

Example:
Exercise:

Problem: Find the quotient: $\frac{14x^7y^{12}}{21x^{11}y^6}$.

Solution:
Solution

Be very careful to simplify $\frac{14}{21}$ by dividing out a common factor, and to simplify the variables by subtracting their exponents.

$$\frac{14x^7y^{12}}{21x^{11}y^6}$$

Simplify and use the Quotient Property. $\frac{2y^6}{3x^4}$

Note:
Exercise:

Problem: Find the quotient: $\frac{28x^5y^{14}}{49x^9y^{12}}$.

Solution:

$\frac{4y^2}{7x^4}$

Note:
Exercise:

Problem: Find the quotient: $\frac{30m^5n^{11}}{48m^{10}n^{14}}$.

Solution:

$\frac{5}{8m^5n^3}$

In all examples so far, there was no work to do in the numerator or denominator before simplifying the fraction. In the next example, we’ll first find the product of two monomials in the numerator before we simplify the fraction. This follows the order of operations. Remember, a fraction bar is a grouping symbol.

Example:
Exercise:

Problem: Find the quotient: $\frac{\left(6x^2y^3\right)\left(5x^3y^2\right)}{\left(3x^4y^5\right)}$.

Solution:
Solution

	$\frac{\left(6x^2y^3\right)\left(5x^3y^2\right)}{\left(3x^4y^5\right)}$
Simplify the numerator.	$\frac{30x^5y^5}{3x^4y^5}$
Simplify.	$10x$

Note:
Exercise:

Problem: Find the quotient: $\frac{\left(6a^4b^5\right)\left(4a^2b^5\right)}{12a^5b^8}$.

Solution:

$2ab^2$

Note:
Exercise:

Problem: Find the quotient: $\frac{\left(-12x^6y^9\right)\left(-4x^5y^8\right)}{-12x^{10}y^{12}}$.

Solution:

$-4xy^5$

Note:
Access these online resources for additional instruction and practice with dividing monomials:

- Rational Expressions
- Dividing Monomials
- Dividing Monomials 2

Key Concepts

- **Quotient Property for Exponents:**
    - If a is a real number, $a \neq 0$, and m, n are whole numbers, then: $\frac{a^m}{a^n} = a^{m-n}, m > n$ and $\frac{a^m}{a^n} = \frac{1}{a^{m-n}}, n > m$
- **Zero Exponent**
    - If a is a non-zero number, then $a^0 = 1$.
- **Quotient to a Power Property for Exponents**:
    - If a and b are real numbers, $b \neq 0$, and m is a counting number, then: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
    - To raise a fraction to a power, raise the numerator and denominator to that power.
- **Summary of Exponent Properties**
    - If a, b are real numbers and m, n are whole numbers, then

Product Property	$a^m \cdot a^n$	$=$	a^{m+n}
Power Property	$(a^m)^n$	$=$	$a^{m \cdot n}$
Product to a Power	$(ab)^m$	$=$	$a^m b^m$
Quotient Property	$\frac{a^m}{b^m}$	$=$	$a^{m-n}, a \neq 0, m > n$
	$\frac{a^m}{a^n}$	$=$	$\frac{1}{a^{n-m}}, a \neq 0, n > m$
Zero Exponent Definition	a^o	$=$	$1, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m$	$=$	$\frac{a^m}{b^m}, b \neq 0$

Practice Makes Perfect

Simplify Expressions Using the Quotient Property for Exponents

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\frac{x^{18}}{x^3}$ ⓑ $\frac{5^{12}}{5^3}$

Exercise:

Problem: ⓐ $\frac{y^{20}}{y^{10}}$ ⓑ $\frac{7^{16}}{7^2}$

Solution:

ⓐ y^{10} ⓑ 7^{14}

Exercise:

Problem: ⓐ $\frac{p^{21}}{p^7}$ ⓑ $\frac{4^{16}}{4^4}$

Exercise:

Problem: ⓐ $\frac{u^{24}}{u^3}$ ⓑ $\frac{9^{15}}{9^5}$

Solution:

ⓐ u^{21} ⓑ 9^{10}

Exercise:

Problem: ⓐ $\frac{q^{18}}{q^{36}}$ ⓑ $\frac{10^2}{10^3}$

Exercise:

Problem: ⓐ $\frac{t^{10}}{t^{40}}$ ⓑ $\frac{8^3}{8^5}$

Solution:

ⓐ $\frac{1}{t^{30}}$ ⓑ $\frac{1}{64}$

Exercise:

Problem: ⓐ $\frac{b}{b^9}$ ⓑ $\frac{4}{4^6}$

Exercise:

Problem: ⓐ $\frac{x}{x^7}$ ⓑ $\frac{10}{10^3}$

Solution:

ⓐ $\frac{1}{x^6}$ ⓑ $\frac{1}{100}$

Simplify Expressions with Zero Exponents

In the following exercises, simplify.

Exercise:

ⓐ 20^0
Problem: ⓑ b^0

Exercise:

ⓐ 13^0
Problem: ⓑ k^0

Solution:

ⓐ 1 ⓑ 1

Exercise:

ⓐ -27^0
Problem: ⓑ $-\left(27^0\right)$

Exercise:

ⓐ -15^0
Problem: ⓑ $-\left(15^0\right)$

Solution:

ⓐ -1 ⓑ -1

Exercise:

ⓐ $(25x)^0$
Problem: ⓑ $25x^0$

Exercise:

Problem: ⓐ $(6y)^0$
ⓑ $6y^0$

Solution:

ⓐ 1 ⓑ 6

Exercise:

Problem: ⓐ $(12x)^0$
ⓑ $\left(-56p^4q^3\right)^0$

Exercise:

Problem: ⓐ $7y^0(17y)^0$
ⓑ $\left(-93c^7d^{15}\right)^0$

Solution:

ⓐ 7 ⓑ 1

Exercise:

Problem: ⓐ $12n^0-18m^0$
ⓑ $(12n)^0-(18m)^0$

Exercise:

Problem: ⓐ $15r^0-22s^0$
ⓑ $(15r)^0-(22s)^0$

Solution:

ⓐ -7 ⓑ 0

Simplify Expressions Using the Quotient to a Power Property

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\left(\frac{3}{4}\right)^3$ ⓑ $\left(\frac{p}{2}\right)^5$ ⓒ $\left(\frac{x}{y}\right)^6$

Exercise:

Problem: ⓐ $\left(\frac{2}{5}\right)^2$ ⓑ $\left(\frac{x}{3}\right)^4$ ⓒ $\left(\frac{a}{b}\right)^5$

Solution:

ⓐ $\frac{4}{25}$ ⓑ $\frac{x^4}{81}$ ⓒ $\frac{a^5}{b^5}$

Exercise:

Problem: ⓐ $\left(\frac{a}{3b}\right)^4$ ⓑ $\left(\frac{5}{4m}\right)^2$

Exercise:

Problem: ⓐ $\left(\frac{x}{2y}\right)^3$ ⓑ $\left(\frac{10}{3q}\right)^4$

Solution:

ⓐ $\frac{x^3}{8y^3}$ ⓑ $\frac{10{,}000}{81q^4}$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify.

Exercise:

Problem: $\frac{\left(a^2\right)^3}{a^4}$

Exercise:

Problem: $\frac{\left(p^3\right)^4}{p^5}$

Solution:

p^7

Exercise:

Problem: $\frac{\left(y^3\right)^4}{y^{10}}$

Exercise:

Problem: $\frac{\left(x^{4}\right)^{5}}{x^{15}}$

Solution:

x^{5}

Exercise:

Problem: $\frac{u^{6}}{\left(u^{3}\right)^{2}}$

Exercise:

Problem: $\frac{v^{20}}{\left(v^{4}\right)^{5}}$

Solution:

1

Exercise:

Problem: $\frac{m^{12}}{\left(m^{8}\right)^{3}}$

Exercise:

Problem: $\frac{n^{8}}{\left(n^{6}\right)^{4}}$

Solution:

$\frac{1}{n^{16}}$

Exercise:

Problem: $\left(\frac{p^{9}}{p^{3}}\right)^{5}$

Exercise:

Problem: $\left(\frac{q^{8}}{q^{2}}\right)^{3}$

Solution:

q^{18}

Exercise:

Problem: $\left(\frac{r^2}{r^6}\right)^3$

Exercise:

Problem: $\left(\frac{m^4}{m^7}\right)^4$

Solution:

$\frac{1}{m^{12}}$

Exercise:

Problem: $\left(\frac{p}{r^{11}}\right)^2$

Exercise:

Problem: $\left(\frac{a}{b^6}\right)^3$

Solution:

$\frac{a^3}{b^{18}}$

Exercise:

Problem: $\left(\frac{w^5}{x^3}\right)^8$

Exercise:

Problem: $\left(\frac{y^4}{z^{10}}\right)^5$

Solution:

$\frac{y^{20}}{z^{50}}$

Exercise:

Problem: $\left(\frac{2j^3}{3k}\right)^4$

Exercise:

Problem: $\left(\frac{3m^5}{5n}\right)^3$

Solution:

$\frac{27m^{15}}{125n^3}$

Exercise:

Problem: $\left(\frac{3c^2}{4d^6}\right)^3$

Exercise:

Problem: $\left(\frac{5u^7}{2v^3}\right)^4$

Solution:

$\frac{625u^{28}}{16v^{12}}$

Exercise:

Problem: $\left(\frac{k^2k^8}{k^3}\right)^2$

Exercise:

Problem: $\left(\frac{j^2j^5}{j^4}\right)^3$

Solution:

j^9

Exercise:

Problem: $\frac{\left(t^2\right)^5\left(t^4\right)^2}{\left(t^3\right)^7}$

Exercise:

Problem: $\frac{\left(q^3\right)^6\left(q^2\right)^3}{\left(q^4\right)^8}$

Solution:

$\frac{1}{q^8}$

Exercise:

Problem: $\frac{\left(-2p^2\right)^4\left(3p^4\right)^2}{\left(-6p^3\right)^2}$

Exercise:

Problem: $\frac{\left(-2k^3\right)^2\left(6k^2\right)^4}{\left(9k^4\right)^2}$

Solution:

$64k^6$

Exercise:

Problem: $\frac{\left(-4m^3\right)^2\left(5m^4\right)^3}{\left(-10m^6\right)^3}$

Exercise:

Problem: $\frac{\left(-10n^2\right)^3\left(4n^5\right)^2}{\left(2n^8\right)^2}$

Solution:

$-4{,}000$

Divide Monomials

In the following exercises, divide the monomials.

Exercise:

Problem: $56b^8 \div 7b^2$

Exercise:

Problem: $63v^{10} \div 9v^2$

Solution:

$7v^8$

Exercise:

Problem: $-88y^{15} \div 8y^{3}$

Exercise:

Problem: $-72u^{12} \div 12u^{4}$

Solution:

$-6u^{8}$

Exercise:

Problem: $\frac{45a^{6}b^{8}}{-15a^{10}b^{2}}$

Exercise:

Problem: $\frac{54x^{9}y^{3}}{-18x^{6}y^{15}}$

Solution:

$-\frac{3x^{3}}{y^{12}}$

Exercise:

Problem: $\frac{15r^{4}s^{9}}{18r^{9}s^{2}}$

Exercise:

Problem: $\frac{20m^{8}n^{4}}{30m^{5}n^{9}}$

Solution:

$\frac{2m^{3}}{3n^{5}}$

Exercise:

Problem: $\frac{18a^{4}b^{8}}{-27a^{9}b^{5}}$

Exercise:

Problem: $\frac{45x^{5}y^{9}}{-60x^{8}y^{6}}$

Solution:

$\frac{-3y^3}{4x^3}$

Exercise:

Problem: $\frac{64q^{11}r^9s^3}{48q^6r^8s^5}$

Exercise:

Problem: $\frac{65a^{10}b^8c^5}{42a^7b^6c^8}$

Solution:

$\frac{65a^3b^2}{42c^3}$

Exercise:

Problem: $\frac{\left(10m^5n^4\right)\left(5m^3n^6\right)}{25m^7n^5}$

Exercise:

Problem: $\frac{\left(-18p^4q^7\right)\left(-6p^3q^8\right)}{-36p^{12}q^{10}}$

Solution:

$\frac{-3q^5}{p^5}$

Exercise:

Problem: $\frac{\left(6a^4b^3\right)\left(4ab^5\right)}{(12a^2b)(a^3b)}$

Exercise:

Problem: $\frac{\left(4u^2v^5\right)\left(15u^3v\right)}{(12u^3v)(u^4v)}$

Solution:

$\frac{5v^4}{u^2}$

Mixed Practice

Exercise:

ⓐ $24a^5 + 2a^5$
ⓑ $24a^5 - 2a^5$
ⓒ $24a^5 \cdot 2a^5$
Problem: ⓓ $24a^5 \div 2a^5$

Exercise:

ⓐ $15n^{10} + 3n^{10}$
ⓑ $15n^{10} - 3n^{10}$
ⓒ $15n^{10} \cdot 3n^{10}$
Problem: ⓓ $15n^{10} \div 3n^{10}$

Solution:

ⓐ $18n^{10}$
ⓑ $12n^{10}$
ⓒ $45n^{20}$
ⓓ 5

Exercise:

ⓐ $p^4 \cdot p^6$
Problem: ⓑ $\left(p^4\right)^6$

Exercise:

ⓐ $q^5 \cdot q^3$
Problem: ⓑ $\left(q^5\right)^3$

Solution:

ⓐ q^8
ⓑ q^{15}

Exercise:

ⓐ $\frac{y^3}{y}$
Problem: ⓑ $\frac{y}{y^3}$

Exercise:

Problem: ⓐ $\frac{z^6}{z^5}$ ⓑ $\frac{z^5}{z^6}$

Solution:

ⓐ z ⓑ $\frac{1}{z}$

Exercise:

Problem: $\left(8x^5\right)(9x) \div 6x^3$

Exercise:

Problem: $(4y)\left(12y^7\right) \div 8y^2$

Solution:

$6y^6$

Exercise:

Problem: $\frac{27a^7}{3a^3} + \frac{54a^9}{9a^5}$

Exercise:

Problem: $\frac{32c^{11}}{4c^5} + \frac{42c^9}{6c^3}$

Solution:

$15c^6$

Exercise:

Problem: $\frac{32y^5}{8y^2} - \frac{60y^{10}}{5y^7}$

Exercise:

Problem: $\frac{48x^6}{6x^4} - \frac{35x^9}{7x^7}$

Solution:

$3x^2$

Exercise:

Problem: $\frac{63r^6s^3}{9r^4s^2} - \frac{72r^2s^2}{6s}$

Exercise:

Problem: $\frac{56y^4z^5}{7y^3z^3} - \frac{45y^2z^2}{5y}$

Solution:

$-yz^2$

Everyday Math

Exercise:

Problem:

Memory One megabyte is approximately 10^6 bytes. One gigabyte is approximately 10^9 bytes. How many megabytes are in one gigabyte?

Exercise:

Problem:

Memory One gigabyte is approximately 10^9 bytes. One terabyte is approximately 10^{12} bytes. How many gigabytes are in one terabyte?

Solution:

10^3

Writing Exercises

Exercise:

Problem:

Jennifer thinks the quotient $\frac{a^{24}}{a^6}$ simplifies to a^4. What is wrong with her reasoning?

Exercise:

Problem:

Maurice simplifies the quotient $\frac{d^7}{d}$ by writing $\frac{\cancel{d}^7}{\cancel{d}} = 7$. What is wrong with his reasoning?

Solution:

Answers will vary.

Exercise:

Problem:

When Drake simplified -3^0 and $(-3)^0$ he got the same answer. Explain how using the Order of Operations correctly gives different answers.

Exercise:

Problem:

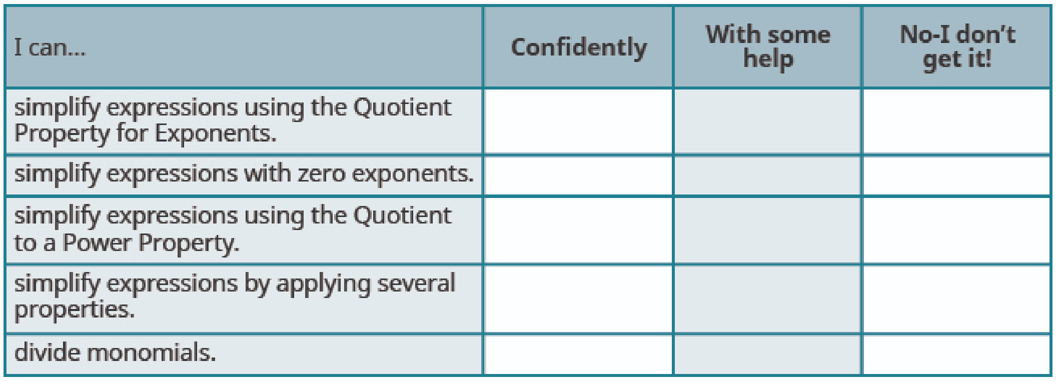
Robert thinks x^0 simplifies to 0. What would you say to convince Robert he is wrong?

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don’t get it!
simplify expressions using the Quotient Property for Exponents.			
simplify expressions with zero exponents.			
simplify expressions using the Quotient to a Power Property.			
simplify expressions by applying several properties.			
divide monomials.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Divide Polynomials
By the end of this section, you will be able to:

- Divide a polynomial by a monomial
- Divide a polynomial by a binomial

Note:
Before you get started, take this readiness quiz.

1. Add: $\frac{3}{d}+\frac{x}{d}$.
   If you missed this problem, review [link].
2. Simplify: $\frac{30xy^3}{5xy}$.
   If you missed this problem, review [link].
3. Combine like terms: $8a^2+12a+1+3a^2-5a+4$.
   If you missed this problem, review [link].

Divide a Polynomial by a Monomial

In the last section, you learned how to divide a monomial by a monomial. As you continue to build up your knowledge of polynomials the next procedure is to divide a polynomial of two or more terms by a monomial.

The method we’ll use to divide a polynomial by a monomial is based on the properties of fraction addition. So we’ll start with an example to review fraction addition.

The sum,	$\frac{y}{5}+\frac{2}{5}$,
simplifies to	$\frac{y+2}{5}$.

Now we will do this in reverse to split a single fraction into separate fractions.

We’ll state the fraction addition property here just as you learned it and in reverse.

Note:
Fraction Addition
If a, b, and c are numbers where $c\neq 0$, then
Equation:

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \quad \text{and} \quad \frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$$

We use the form on the left to add fractions and we use the form on the right to divide a polynomial by a monomial.

For example, $\frac{y+2}{5}$

can be written $\frac{y}{5}+\frac{2}{5}$.

We use this form of fraction addition to divide polynomials by monomials.

Note:
Division of a Polynomial by a Monomial
To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Example:
Exercise:

Problem: Find the quotient: $\frac{7y^2+21}{7}$.

Solution:
Solution

$\frac{7y^2+21}{7}$

Divide each term of the numerator by the denominator. $\frac{7y^2}{7}+\frac{21}{7}$

Simplify each fraction. y^2+3

Note:
Exercise:

Problem: Find the quotient: $\frac{8z^2+24}{4}$.

Solution:

$2z^2+6$

Note:
Exercise:

Problem: Find the quotient: $\frac{18z^2-27}{9}$.

Solution:

$2z^2-3$

Remember that division can be represented as a fraction. When you are asked to divide a polynomial by a monomial and it is not already in fraction form, write a fraction with the polynomial in the numerator and the monomial in the denominator.

Example:
Exercise:

Problem: Find the quotient: $\left(18x^3-36x^2\right)\div 6x$.

Solution:
Solution

	$\left(18x^3-36x^2\right)\div 6x$
Rewrite as a fraction.	$\frac{18x^3-36x^2}{6x}$
Divide each term of the numerator by the denominator.	$\frac{18x^3}{6x}-\frac{36x^2}{6x}$
Simplify.	$3x^2-6x$

Note:
Exercise:

Problem: Find the quotient: $\left(27b^3-33b^2\right)\div 3b$.

Solution:

$9b^2-11b$

Note:
Exercise:

Problem: Find the quotient: $\left(25y^3 - 55y^2\right) \div 5y$.

Solution:

$5y^2 - 11y$

When we divide by a negative, we must be extra careful with the signs.

Example:
Exercise:

Problem: Find the quotient: $\frac{12d^2-16d}{-4}$.

Solution:
Solution

$\frac{12d^2-16d}{-4}$

Divide each term of the numerator by the denominator. $\frac{12d^2}{-4} - \frac{16d}{-4}$

Simplify. Remember, subtracting a negative is like adding a positive! $-3d^2 + 4d$

Note:
Exercise:

Problem: Find the quotient: $\frac{25y^2-15y}{-5}$.

Solution:

$-5y^2 + 3y$

Note:
Exercise:

Problem: Find the quotient: $\frac{42b^2-18b}{-6}$.

Solution:

$-7b^2+3b$

Example:
Exercise:

Problem: Find the quotient: $\frac{105y^5+75y^3}{5y^2}$.

Solution:
Solution

	$\frac{105y^5+75y^3}{5y^2}$
Separate the terms.	$\frac{105y^5}{5y^2}+\frac{75y^3}{5y^2}$
Simplify.	$21y^3+15y$

Note:
Exercise:

Problem: Find the quotient: $\frac{60d^7+24d^5}{4d^3}$.

Solution:

$15d^4+6d^2$

Note:
Exercise:

Problem: Find the quotient: $\frac{216p^7-48p^5}{6p^3}$.

Solution:

$36p^4-8p^2$

Example:
Exercise:

Problem: Find the quotient: $\left(15x^3y - 35xy^2\right) \div (-5xy)$.

Solution:
Solution

	$\left(15x^3y - 35xy^2\right) \div (-5xy)$
Rewrite as a fraction.	$\frac{15x^3y-35xy^2}{-5xy}$
Separate the terms. Be careful with the signs!	$\frac{15x^3y}{-5xy} - \frac{35xy^2}{-5xy}$
Simplify.	$-3x^2 + 7y$

Note:
Exercise:

Problem: Find the quotient: $\left(32a^2b - 16ab^2\right) \div (-8ab)$.

Solution:

$-4a + 2b$

Note:
Exercise:

Problem: Find the quotient: $\left(-48a^8b^4 - 36a^6b^5\right) \div \left(-6a^3b^3\right)$.

Solution:

$8a^5b + 6a^3b^2$

Example:
Exercise:

Problem: Find the quotient: $\frac{36x^3y^2+27x^2y^2-9x^2y^3}{9x^2y}$.

Solution:
Solution

	$\frac{36x^3y^2+27x^2y^2-9x^2y^3}{9x^2y}$
Separate the terms.	$\frac{36x^3y^2}{9x^2y}+\frac{27x^2y^2}{9x^2y}-\frac{9x^2y^3}{9x^2y}$
Simplify.	$4xy+3y-y^2$

Note:
Exercise:

Problem: Find the quotient: $\frac{40x^3y^2+24x^2y^2-16x^2y^3}{8x^2y}$.

Solution:

$5xy+3y-2y^2$

Note:
Exercise:

Problem: Find the quotient: $\frac{35a^4b^2+14a^4b^3-42a^2b^4}{7a^2b^2}$.

Solution:

$5a^2+2a^2b-6b^2$

Example:
Exercise:

Problem: Find the quotient: $\frac{10x^2+5x-20}{5x}$.

Solution:
Solution

	$\frac{10x^2+5x-20}{5x}$
Separate the terms.	$\frac{10x^2}{5x}+\frac{5x}{5x}-\frac{20}{5x}$
Simplify.	$2x+1+\frac{4}{x}$

Note:
Exercise:

Problem: Find the quotient: $\frac{18c^2+6c-9}{6c}$.

Solution:

$3c+1-\frac{3}{2c}$

Note:
Exercise:

Problem: Find the quotient: $\frac{10d^2-5d-2}{5d}$.

Solution:

$2d-1-\frac{2}{5d}$

Key Concepts

- **Fraction Addition**
    - If a, b, and c are numbers where $c \neq 0$, then

      $$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \text{ and } \frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$$
- **Division of a Polynomial by a Monomial**
    - To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Practice Makes Perfect

In the following exercises, divide each polynomial by the monomial.

Exercise:

Problem: $\frac{45y+36}{9}$

Exercise:

Problem: $\frac{30b+75}{5}$

Solution:

$6b+15$

Exercise:

Problem: $\frac{8d^2-4d}{2}$

Exercise:

Problem: $\frac{42x^2-14x}{7}$

Solution:

$6x^2-2x$

Exercise:

Problem: $\left(16y^2-20y\right)\div 4y$

Exercise:

Problem: $\left(55w^2-10w\right)\div 5w$

Solution:

$11w-2$

Exercise:

Problem: $\left(9n^4+6n^3\right)\div 3n$

Exercise:

Problem: $\left(8x^3+6x^2\right)\div 2x$

Solution:

$4x^2 + 3x$

Exercise:

Problem: $\frac{18y^2-12y}{-6}$

Exercise:

Problem: $\frac{20b^2-12b}{-4}$

Solution:

$-5b^2 + 3b$

Exercise:

Problem: $\frac{35a^4+65a^2}{-5}$

Exercise:

Problem: $\frac{51m^4+72m^3}{-3}$

Solution:

$-17m^4 - 24m^3$

Exercise:

Problem: $\frac{310y^4-200y^3}{5y^2}$

Exercise:

Problem: $\frac{412z^8-48z^5}{4z^3}$

Solution:

$103z^5 - 12z^2$

Exercise:

Problem: $\frac{46x^3+38x^2}{2x^2}$

Exercise:

Problem: $\frac{51y^4+42y^2}{3y^2}$

Solution:

$17y^2+14$

Exercise:

Problem: $\left(24p^2-33p\right)\div(-3p)$

Exercise:

Problem: $\left(35x^4-21x\right)\div(-7x)$

Solution:

$-5x^3+3$

Exercise:

Problem: $\left(63m^4-42m^3\right)\div\left(-7m^2\right)$

Exercise:

Problem: $\left(48y^4-24y^3\right)\div\left(-8y^2\right)$

Solution:

$-6y^2+3y$

Exercise:

Problem: $\left(63a^2b^3+72ab^4\right)\div(9ab)$

Exercise:

Problem: $\left(45x^3y^4+60xy^2\right)\div(5xy)$

Solution:

$9x^2y^3+12y$

Exercise:

Problem: $\frac{52p^5q^4+36p^4q^3-64p^3q^2}{4p^2q}$

Exercise:

Problem: $\frac{49c^2d^2-70c^3d^3-35c^2d^4}{7cd^2}$

Solution:

$7c-10c^2d-5cd^2$

Exercise:

Problem: $\frac{66x^3y^2-110x^2y^3-44x^4y^3}{11x^2y^2}$

Exercise:

Problem: $\frac{72r^5s^2+132r^4s^3-96r^3s^5}{12r^2s^2}$

Solution:

$6r^3+11r^2s-8rs^3$

Exercise:

Problem: $\frac{4w^2+2w-5}{2w}$

Exercise:

Problem: $\frac{12q^2+3q-1}{3q}$

Solution:

$4q+1-\frac{1}{3q}$

Exercise:

Problem: $\frac{10x^2+5x-4}{-5x}$

Exercise:

Problem: $\frac{20y^2+12y-1}{-4y}$

Solution:

$-5y-3+\frac{1}{4y}$

Exercise:

Problem: $\frac{36p^3+18p^2-12p}{6p^2}$

Exercise:

Problem: $\frac{63a^3-108a^2+99a}{9a^2}$

Solution:

$7a-12+\frac{11}{a}$

Everyday Math

Exercise:

Problem:

Average cost Pictures Plus produces digital albums. The company's average cost (in dollars) to make x albums is given by the expression $\frac{7x+500}{x}$.

ⓐ Find the quotient by dividing the numerator by the denominator.
ⓑ What will the average cost (in dollars) be to produce 20 albums?

Exercise:

Problem:

Handshakes At a company meeting, every employee shakes hands with every other employee. The number of handshakes is given by the expression $\frac{n^2-n}{2}$, where n represents the number of employees. How many handshakes will there be if there are 10 employees at the meeting?

Solution:

45

Writing Exercises

Exercise:

Problem:

James divides $48y+6$ by 6 this way: $\frac{48y+\cancel{6}}{\cancel{6}}=48y$. What is wrong with his reasoning?

Exercise:

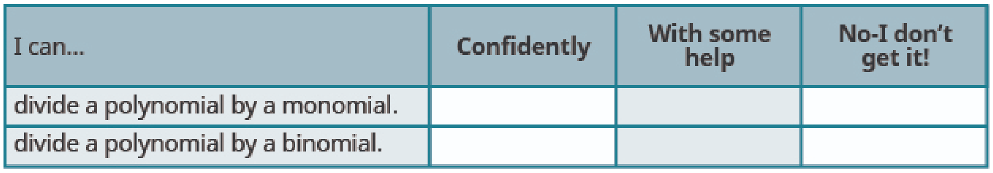
Problem: Divide $\frac{10x^2+x-12}{2x}$ and explain with words how you get each term of the quotient.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
divide a polynomial by a monomial.			
divide a polynomial by a binomial.			

ⓑ After reviewing this checklist, what will you do to become confident for all goals?

Integer Exponents and Scientific Notation
By the end of this section, you will be able to:

- Use the definition of a negative exponent
- Simplify expressions with integer exponents
- Convert from decimal notation to scientific notation
- Convert scientific notation to decimal form
- Multiply and divide using scientific notation

Note:
Before you get started, take this readiness quiz.

1. What is the place value of the 6 in the number $64,891$?
   If you missed this problem, review [link].
2. Name the decimal: 0.0012.
   If you missed this problem, review [link].
3. Subtract: $5-(-3)$.
   If you missed this problem, review [link].

Use the Definition of a Negative Exponent

We saw that the Quotient Property for Exponents introduced earlier in this chapter, has two forms depending on whether the exponent is larger in the numerator or the denominator.

Note:
Quotient Property for Exponents
If a is a real number, $a\ne 0$, and m and n are whole numbers, then
Equation:

$$\frac{a^m}{a^n}=a^{m-n}, m>n \quad \text{and} \quad \frac{a^m}{a^n}=\frac{1}{a^{n-m}}, n>m$$

What if we just subtract exponents regardless of which is larger?

Let’s consider $\frac{x^2}{x^5}$.

We subtract the exponent in the denominator from the exponent in the numerator.
Equation:

$$\frac{x^2}{x^5}$$

$$x^{2-5}$$

$$x^{-3}$$

We can also simplify $\frac{x^2}{x^5}$ by dividing out common factors:

$$\frac{\cancel{x}\cdot\cancel{x}}{\cancel{x}\cdot\cancel{x}\cdot x\cdot x\cdot x}$$

$$\frac{1}{x^3}$$

This implies that $x^{-3} = \frac{1}{x^3}$ and it leads us to the definition of a *negative exponent*.

Note:
Negative Exponent
If n is an integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$.

The negative exponent tells us we can re-write the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in simplest form. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression x^{-3}, we will take one more step and write $\frac{1}{x^3}$. The answer is considered to be in simplest form when it has only positive exponents.

Example:
Exercise:

Problem: Simplify: ⓐ 4^{-2} ⓑ 10^{-3}.

Solution:
Solution

ⓐ

	4^{-2}
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{4^2}$
Simplify.	$\frac{1}{16}$

ⓑ

	10^{-3}
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{10^3}$
Simplify.	$\frac{1}{1000}$

Note:

Exercise:

Problem: Simplify: ⓐ 2^{-3} ⓑ 10^{-7}.

Solution:

ⓐ $\frac{1}{8}$ ⓑ $\frac{1}{10^7}$

Note:
Exercise:

Problem: Simplify: ⓐ 3^{-2} ⓑ 10^{-4}.

Solution:

ⓐ $\frac{1}{9}$ ⓑ $\frac{1}{10{,}000}$

In [link] we raised an integer to a negative exponent. What happens when we raise a fraction to a negative exponent? We'll start by looking at what happens to a fraction whose numerator is one and whose denominator is an integer raised to a negative exponent.

	$\frac{1}{a^{-n}}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\frac{1}{a^n}}$
Simplify the complex fraction.	$1 \cdot \frac{a^n}{1}$
Multiply.	a^n

This leads to the Property of Negative Exponents.

Note:
Property of Negative Exponents
If n is an integer and $a \neq 0$, then $\frac{1}{a^{-n}} = a^n$.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{1}{y^{-4}}$ ⓑ $\frac{1}{3^{-2}}$.

Solution:
Solution

ⓐ

	$\frac{1}{y^{-4}}$
Use the property of a negative exponent, $\frac{1}{a^{-n}} = a^n$.	y^4

ⓑ

	$\frac{1}{3^{-2}}$
Use the property of a negative exponent, $\frac{1}{a^{-n}} = a^n$.	3^2
Simplify.	9

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{1}{p^{-8}}$ ⓑ $\frac{1}{4^{-3}}$.

Solution:

ⓐ p^8 ⓑ 64

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{1}{q^{-7}}$ ⓑ $\frac{1}{2^{-4}}$.

Solution:

ⓐ q^7 ⓑ 16

Suppose now we have a fraction raised to a negative exponent. Let’s use our definition of negative exponents to lead us to a new property.

	$\left(\frac{3}{4}\right)^{-2}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\left(\frac{3}{4}\right)^2}$
Simplify the denominator.	$\frac{1}{\frac{9}{16}}$
Simplify the complex fraction.	$\frac{16}{9}$
But we know that $\frac{16}{9}$ is $\left(\frac{4}{3}\right)^2$.	
This tells us that:	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the *Quotient to a Negative Power Property*.

Note:
Quotient to a Negative Exponent Property
If a and b are real numbers, $a \neq 0, b \neq 0$, and n is an integer, then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(\frac{5}{7}\right)^{-2}$ ⓑ $\left(-\frac{2x}{y}\right)^{-3}$.

Solution:
Solution

ⓐ

	$\left(\frac{5}{7}\right)^{-2}$
Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$\left(\frac{7}{5}\right)^{2}$
Simplify.	$\frac{49}{25}$

ⓑ

	$\left(-\frac{2x}{y}\right)^{-3}$
Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$\left(-\frac{y}{2x}\right)^{3}$
Simplify.	$-\frac{y^3}{8x^3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\frac{2}{3}\right)^{-4}$ ⓑ $\left(-\frac{6m}{n}\right)^{-2}$.

Solution:

ⓐ $\frac{81}{16}$ ⓑ $\frac{n^2}{36m^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\frac{3}{5}\right)^{-3}$ ⓑ $\left(-\frac{a}{2b}\right)^{-4}$.

Solution:

ⓐ $\frac{125}{27}$ ⓑ $\frac{16b^4}{a^4}$

When simplifying an expression with exponents, we must be careful to correctly identify the base.

Example:
Exercise:

Problem: Simplify: ⓐ $(-3)^{-2}$ ⓑ -3^{-2} ⓒ $\left(-\frac{1}{3}\right)^{-2}$ ⓓ $-\left(\frac{1}{3}\right)^{-2}$.

Solution:
Solution

ⓐ Here the exponent applies to the base -3.

	$(-3)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$\frac{1}{(-3)^{-2}}$
Simplify.	$\frac{1}{9}$

ⓑ The expression -3^{-2} means “find the opposite of 3^{-2}”. Here the exponent applies to the base 3.

	-3^{-2}
Rewrite as a product with -1.	$-1 \cdot 3^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$-1 \cdot \frac{1}{3^2}$
Simplify.	$-\frac{1}{9}$

ⓒ Here the exponent applies to the base $\left(-\frac{1}{3}\right)$.

	$\left(-\frac{1}{3}\right)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$\left(-\frac{3}{1}\right)^{2}$
Simplify.	9

ⓓ The expression $-\left(\frac{1}{3}\right)^{-2}$ means “find the opposite of $\left(\frac{1}{3}\right)^{-2}$”. Here the exponent applies to the base $\left(\frac{1}{3}\right)$.

	$-\left(\frac{1}{3}\right)^{-2}$
Rewrite as a product with -1.	$-1 \cdot \left(\frac{1}{3}\right)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$-1 \cdot \left(\frac{3}{1}\right)^{2}$
Simplify.	-9

Note:
Exercise:

Problem: Simplify: ⓐ $(-5)^{-2}$ ⓑ -5^{-2} ⓒ $\left(-\frac{1}{5}\right)^{-2}$ ⓓ $-\left(\frac{1}{5}\right)^{-2}$.

Solution:

ⓐ $\frac{1}{25}$ ⓑ $-\frac{1}{25}$ ⓒ 25 ⓓ -25

Note:
Exercise:

Problem: Simplify: ⓐ $(-7)^{-2}$ ⓑ -7^{-2}, ⓒ $\left(-\frac{1}{7}\right)^{-2}$ ⓓ $-\left(\frac{1}{7}\right)^{-2}$.

Solution:

ⓐ $\frac{1}{49}$ ⓑ $-\frac{1}{49}$ ⓒ 49 ⓓ -49

We must be careful to follow the Order of Operations. In the next example, parts (a) and (b) look similar, but the results are different.

Example:
Exercise:

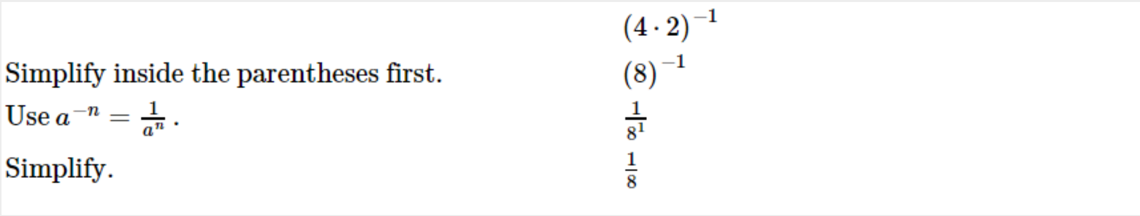
Problem: Simplify: ⓐ $4 \cdot 2^{-1}$ ⓑ $(4 \cdot 2)^{-1}$.

Solution:
Solution

ⓐ

Do exponents before multiplication.	$4 \cdot 2^{-1}$
Use $a^{-n} = \frac{1}{a^n}$.	$4 \cdot \frac{1}{2^1}$
Simplify.	2

ⓑ



	$(4 \cdot 2)^{-1}$
Simplify inside the parentheses first.	$(8)^{-1}$
Use $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{8^1}$
Simplify.	$\frac{1}{8}$

Note:
Exercise:

Problem: Simplify: ⓐ $6\cdot 3^{-1}$ ⓑ $(6\cdot 3)^{-1}$.

Solution:

ⓐ 2 ⓑ $\frac{1}{18}$

Note:
Exercise:

Problem: Simplify: ⓐ $8\cdot 2^{-2}$ ⓑ $(8\cdot 2)^{-2}$.

Solution:

ⓐ 2 ⓑ $\frac{1}{16}$

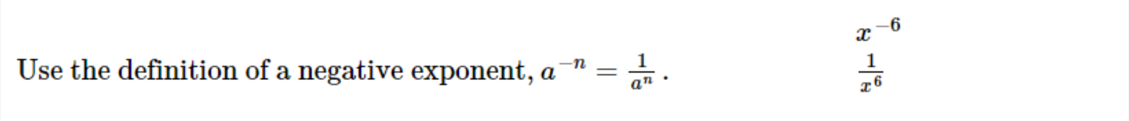
When a variable is raised to a negative exponent, we apply the definition the same way we did with numbers. We will assume all variables are non-zero.

Example:
Exercise:

Problem: Simplify: ⓐ x^{-6} ⓑ $\left(u^{4}\right)^{-3}$.

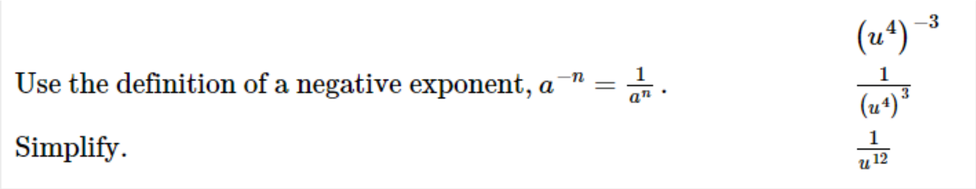
Solution:
Solution

ⓐ



	x^{-6}
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^{n}}$.	$\frac{1}{x^{6}}$

ⓑ



	$\left(u^4\right)^{-3}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\left(u^4\right)^3}$
Simplify.	$\frac{1}{u^{12}}$

Note:
Exercise:

Problem: Simplify: ⓐ y^{-7} ⓑ $\left(z^3\right)^{-5}$.

Solution:

ⓐ $\frac{1}{y^7}$ ⓑ $\frac{1}{z^{15}}$

Note:
Exercise:

Problem: Simplify: ⓐ p^{-9} ⓑ $\left(q^4\right)^{-6}$.

Solution:

ⓐ $\frac{1}{p^9}$ ⓑ $\frac{1}{q^{24}}$

When there is a product and an exponent we have to be careful to apply the exponent to the correct quantity. According to the Order of Operations, we simplify expressions in parentheses before applying exponents. We'll see how this works in the next example.

Example:
Exercise:

Problem: Simplify: ⓐ $5y^{-1}$ ⓑ $(5y)^{-1}$ ⓒ $(-5y)^{-1}$.

Solution:
Solution

ⓐ

Notice the exponent applies to just the base y.	$5y^{-1}$
Take the reciprocal of y and change the sign of the exponent.	$5 \cdot \frac{1}{y^1}$
Simplify.	$\frac{5}{y}$

ⓑ

	$(5y)^{-1}$
Here the parentheses make the exponent apply to the base $5y$.	
Take the reciprocal of $5y$ and change the sign of the exponent.	$\frac{1}{(5y)^1}$
Simplify.	$\frac{1}{5y}$

ⓒ

	$(-5y)^{-1}$
The base here is $-5y$.	
Take the reciprocal of $-5y$ and change the sign of the exponent.	$\frac{1}{(-5y)^1}$
Simplify.	$\frac{1}{-5y}$
Use $\frac{a}{-b} = -\frac{a}{b}$.	$-\frac{1}{5y}$

Note:
Exercise:

Problem: Simplify: ⓐ $8p^{-1}$ ⓑ $(8p)^{-1}$ ⓒ $(-8p)^{-1}$.

Solution:

ⓐ $\frac{8}{p}$ ⓑ $\frac{1}{8p}$ ⓒ $-\frac{1}{8p}$

Note:
Exercise:

Problem: Simplify: ⓐ $11q^{-1}$ ⓑ $(11q)^{-1}$ ⓒ $(-11q)^{-1}$.

Solution:

ⓐ $\frac{1}{11q}$ ⓑ $\frac{1}{11q}$ ⓒ $-\frac{1}{11q}$

With negative exponents, the Quotient Rule needs only one form $\frac{a^m}{a^n} = a^{m-n}$, for $a \neq 0$. When the exponent in the denominator is larger than the exponent in the numerator, the exponent of the quotient will be negative.

Simplify Expressions with Integer Exponents

All of the exponent properties we developed earlier in the chapter with whole number exponents apply to integer exponents, too. We restate them here for reference.

Note:

Summary of Exponent Properties

If a and b are real numbers, and m and n are integers, then

Equation:

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Zero Exponent Property	$a^0 = 1, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0$
Properties of Negative Exponents	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Example:

Exercise:

Problem: Simplify: ⓐ $x^{-4} \cdot x^6$ ⓑ $y^{-6} \cdot y^4$ ⓒ $z^{-5} \cdot z^{-3}$.

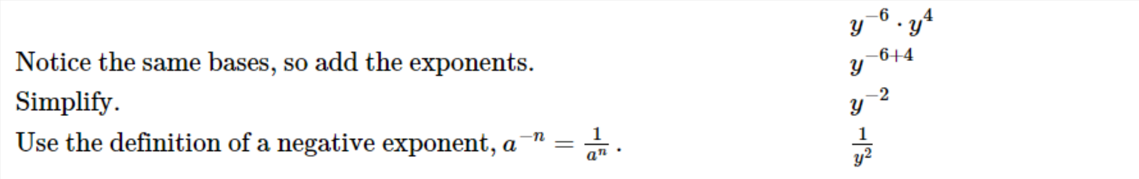
Solution:

Solution

ⓐ

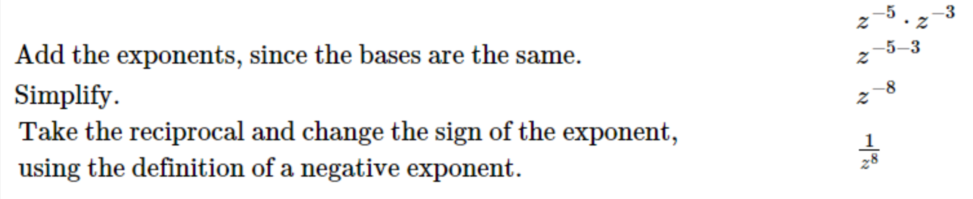
	$x^{-4} \cdot x^6$
Use the Product Property, $a^m \cdot a^n = a^{m+n}$.	x^{-4+6}
Simplify.	x^2

ⓑ



	$y^{-6} \cdot y^4$
Notice the same bases, so add the exponents.	y^{-6+4}
Simplify.	y^{-2}
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{y^2}$

ⓒ



	$z^{-5} \cdot z^{-3}$
Add the exponents, since the bases are the same.	z^{-5-3}
Simplify.	z^{-8}
Take the reciprocal and change the sign of the exponent, using the definition of a negative exponent.	$\frac{1}{z^8}$

Note:
Exercise:

Problem: Simplify: ⓐ $x^{-3} \cdot x^{7}$ ⓑ $y^{-7} \cdot y^{2}$ ⓒ $z^{-4} \cdot z^{-5}$.

Solution:

ⓐ x^{4} ⓑ $\frac{1}{y^{5}}$ ⓒ $\frac{1}{z^{9}}$

Note:
Exercise:

Problem: Simplify: ⓐ $a^{-1} \cdot a^{6}$ ⓑ $b^{-8} \cdot b^{4}$ ⓒ $c^{-8} \cdot c^{-7}$.

Solution:

ⓐ a^{5} ⓑ $\frac{1}{b^{4}}$ ⓒ $\frac{1}{c^{15}}$

In the next two examples, we'll start by using the Commutative Property to group the same variables together. This makes it easier to identify the like bases before using the Product Property.

Example:
Exercise:

Problem: Simplify: $\left(m^{4}n^{-3}\right)\left(m^{-5}n^{-2}\right)$.

Solution:
Solution

	$\left(m^{4}n^{-3}\right)\left(m^{-5}n^{-2}\right)$
Use the Commutative Property to get like bases together.	$m^{4}m^{-5} \cdot n^{-2}n^{-3}$
Add the exponents for each base.	$m^{-1} \cdot n^{-5}$
Take reciprocals and change the signs of the exponents.	$\frac{1}{m^{1}} \cdot \frac{1}{n^{5}}$
Simplify.	$\frac{1}{mn^{5}}$

Note:
Exercise:

Problem: Simplify: $\left(p^{6}q^{-2}\right)\left(p^{-9}q^{-1}\right)$.

Solution:

$\frac{1}{p^3q^3}$

Note:
Exercise:

Problem: Simplify: $\left(r^5s^{-3}\right)\left(r^{-7}s^{-5}\right)$.

Solution:

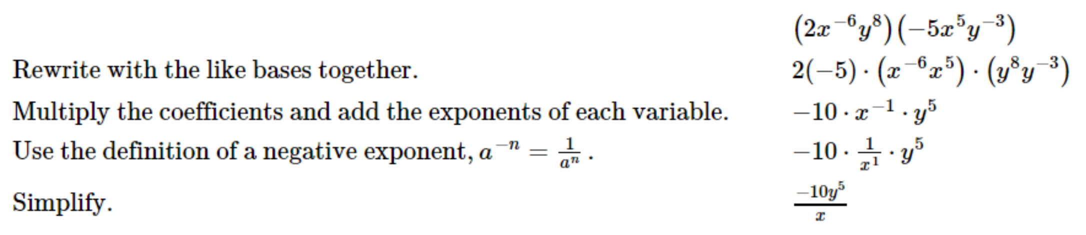
$\frac{1}{r^2s^8}$

If the monomials have numerical coefficients, we multiply the coefficients, just like we did earlier.

Example:
Exercise:

Problem: Simplify: $\left(2x^{-6}y^8\right)\left(-5x^5y^{-3}\right)$.

Solution:
Solution



	$\left(2x^{-6}y^8\right)\left(-5x^5y^{-3}\right)$
Rewrite with the like bases together.	$2(-5)\cdot\left(x^{-6}x^5\right)\cdot\left(y^8y^{-3}\right)$
Multiply the coefficients and add the exponents of each variable.	$-10\cdot x^{-1}\cdot y^5$
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$.	$-10\cdot\frac{1}{x^1}\cdot y^5$
Simplify.	$\frac{-10y^5}{x}$

Note:
Exercise:

Problem: Simplify: $\left(3u^{-5}v^7\right)\left(-4u^4v^{-2}\right)$.

Solution:

$-\frac{12v^5}{u}$

Note:
Exercise:

Problem: Simplify: $\left(-6c^{-6}d^{4}\right)\left(-5c^{-2}d^{-1}\right)$.

Solution:

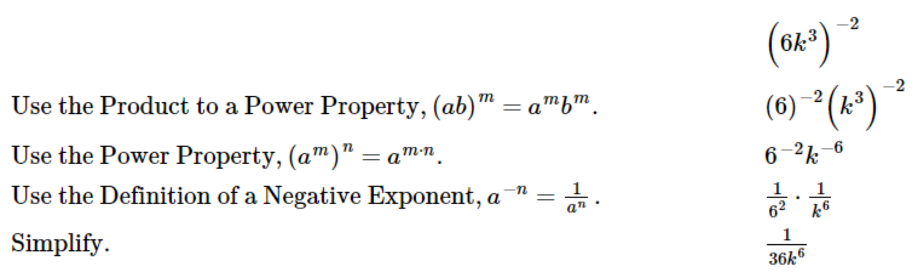
$\frac{30d^3}{c^8}$

In the next two examples, we’ll use the Power Property and the Product to a Power Property.

Example:
Exercise:

Problem: Simplify: $\left(6k^{3}\right)^{-2}$.

Solution:
Solution



	$\left(6k^{3}\right)^{-2}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$(6)^{-2}\left(k^{3}\right)^{-2}$
Use the Power Property, $(a^m)^n = a^{m\cdot n}$.	$6^{-2}k^{-6}$
Use the Definition of a Negative Exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{6^2}\cdot\frac{1}{k^6}$
Simplify.	$\frac{1}{36k^6}$

Note:
Exercise:

Problem: Simplify: $\left(-4x^{4}\right)^{-2}$.

Solution:

$\frac{1}{16x^8}$

Note:
Exercise:

Problem: Simplify: $\left(2b^{3}\right)^{-4}$.

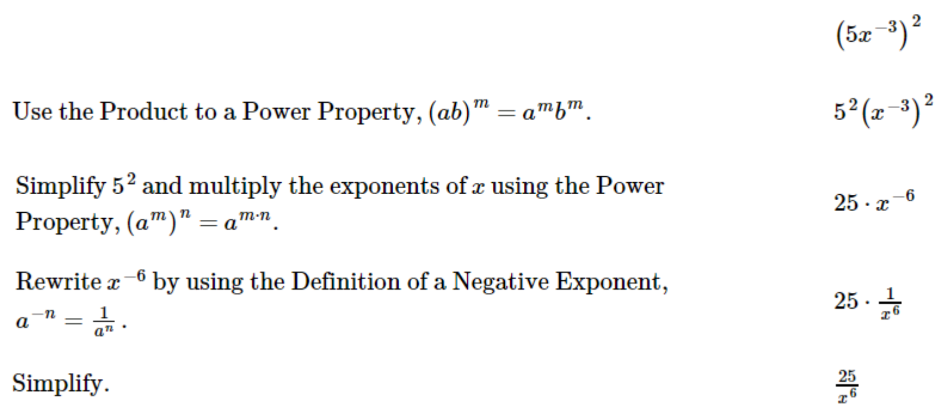
Solution:

$\frac{1}{16b^{12}}$

Example:
Exercise:

Problem: Simplify: $(5x^{-3})^2$.

Solution:
Solution



	$(5x^{-3})^2$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$5^2(x^{-3})^2$
Simplify 5^2 and multiply the exponents of x using the Power Property, $(a^m)^n = a^{m\cdot n}$.	$25 \cdot x^{-6}$
Rewrite x^{-6} by using the Definition of a Negative Exponent, $a^{-n} = \frac{1}{a^n}$.	$25 \cdot \frac{1}{x^6}$
Simplify.	$\frac{25}{x^6}$

Note:
Exercise:

Problem: Simplify: $(8a^{-4})^2$.

Solution:

$\frac{64}{a^8}$

Note:
Exercise:

Problem: Simplify: $(2c^{-4})^3$.

Solution:

$\frac{8}{c^{12}}$

To simplify a fraction, we use the Quotient Property and subtract the exponents.

Example:
Exercise:

Problem: Simplify: $\frac{r^5}{r^{-4}}$.

Solution:
Solution

	$\frac{r^5}{r^{-4}}$
Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	$r^{5-(-4)}$
Simplify.	r^9

Note:
Exercise:

Problem: Simplify: $\frac{x^8}{x^{-3}}$.

Solution:

x^{11}

Note:
Exercise:

Problem: Simplify: $\frac{y^8}{y^{-6}}$.

Solution:

y^{14}

Convert from Decimal Notation to Scientific Notation

Remember working with place value for whole numbers and decimals? Our number system is based on powers of 10. We use tens, hundreds, thousands, and so on. Our decimal numbers are also based on powers of tens—tenths, hundredths, thousandths, and so on. Consider the numbers 4,000 and 0.004. We know that 4,000 means $4 \times 1{,}000$ and 0.004 means $4 \times \frac{1}{1{,}000}$.

If we write the 1000 as a power of ten in exponential form, we can rewrite these numbers in this way:
Equation:

$$\begin{array}{ll} 4{,}000 & 0.004 \\ 4\ \times\ 1{,}000 & 4\ \times\ \frac{1}{1{,}000} \\ 4\ \times\ 10^3 & 4\ \times\ \frac{1}{10^3} \\ & 4\ \times\ 10^{-3} \end{array}$$

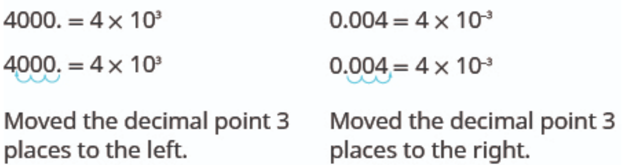
When a number is written as a product of two numbers, where the first factor is a number greater than or equal to one but less than 10, and the second factor is a power of 10 written in exponential form, it is said to be in *scientific notation.*

Note:
Scientific Notation
A number is expressed in **scientific notation** when it is of the form
Equation:

$$a\ \times\ 10^n \text{ where } 1 \le a < 10 \text{ and } n \text{ is an integer}$$

It is customary in scientific notation to use as the $\times$ multiplication sign, even though we avoid using this sign elsewhere in algebra.

If we look at what happened to the decimal point, we can see a method to easily convert from decimal notation to scientific notation.



$4000. = 4 \times 10^3$

$4\underbrace{000.} = 4 \times 10^3$

Moved the decimal point 3 places to the left.

$0.004 = 4 \times 10^{-3}$

$0.\underbrace{004} = 4 \times 10^{-3}$

Moved the decimal point 3 places to the right.

In both cases, the decimal was moved 3 places to get the first factor between 1 and 10.

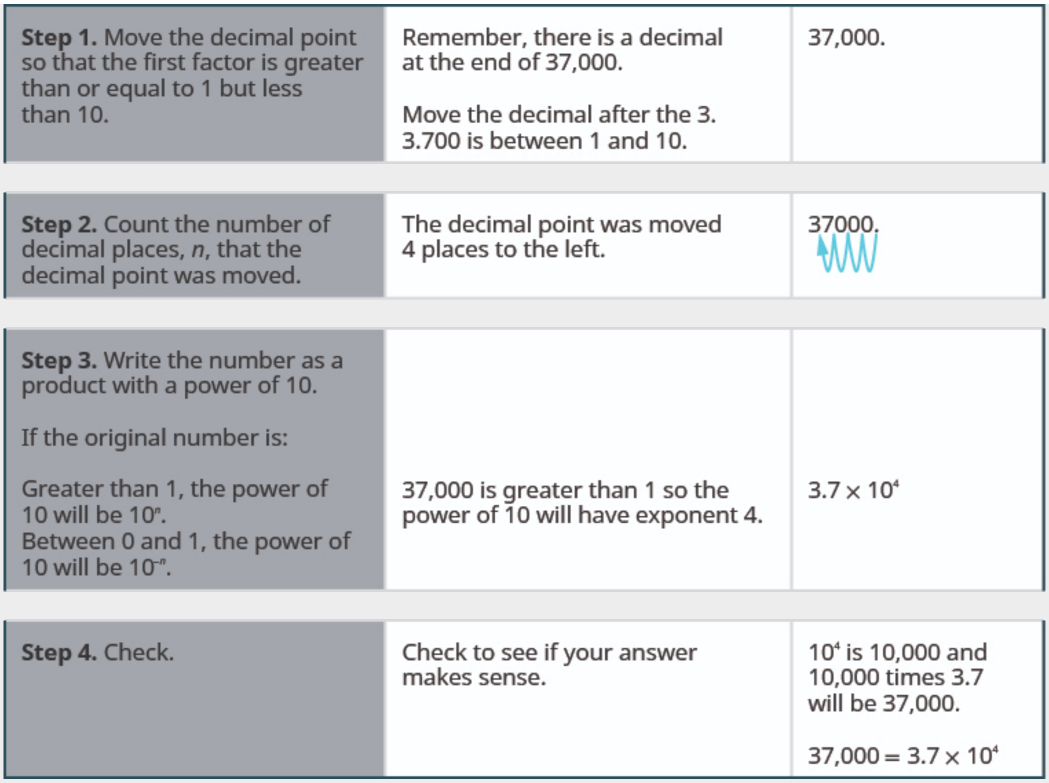
The power of 10 is positive when the number is larger than 1: $4{,}000 = 4\ \times\ 10^3$

The power of 10 is negative when the number is between 0 and 1: $0.004 = 4\ \times\ 10^{-3}$

Example:
How to Convert from Decimal Notation to Scientific Notation
Exercise:

Problem: Write in scientific notation: 37,000.

Solution:
Solution



Step 1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.	Remember, there is a decimal at the end of 37,000. Move the decimal after the 3. 3.700 is between 1 and 10.	37,000.
Step 2. Count the number of decimal places, *n*, that the decimal point was moved.	The decimal point was moved 4 places to the left.	37000.
Step 3. Write the number as a product with a power of 10. If the original number is: Greater than 1, the power of 10 will be 10^n. Between 0 and 1, the power of 10 will be 10^{-n}.	37,000 is greater than 1 so the power of 10 will have exponent 4.	3.7×10^4
Step 4. Check.	Check to see if your answer makes sense.	10^4 is 10,000 and 10,000 times 3.7 will be 37,000. $37{,}000 = 3.7 \times 10^4$

Note:
Exercise:

Problem: Write in scientific notation: 96,000.

Solution:

9.6×10^4

Note:
Exercise:

Problem: Write in scientific notation: 48,300.

Solution:

4.83×10^4

Note:
Convert from decimal notation to scientific notation

Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
Count the number of decimal places,*n*, that the decimal point was moved.
Write the number as a product with a power of 10. If the original number is:

- greater than 1, the power of 10 will be 10^n.
- between 0 and 1, the power of 10 will be 10^{-n}.

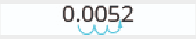
Check.

Example:
Exercise:

Problem: Write in scientific notation: 0.0052.

Solution:
Solution

The original number, 0.0052, is between 0 and 1 so we will have a negative power of 10.

	0.0052
Move the decimal point to get 5.2, a number between 1 and 10.	 0.0052
Count the number of decimal places the point was moved.	3 places
Write as a product with a power of 10.	5.2×10^{-3}
Check.	
5.2×10^{-3} $5.2 \times \frac{1}{10^3}$ $5.2 \times \frac{1}{1000}$ 5.2×0.001	
0.0052	$0.0052 = 5.2 \times 10^{-3}$

Note:
Exercise:

Problem: Write in scientific notation: 0.0078.

Solution:

7.8×10^{-3}

Note:
Exercise:

Problem: Write in scientific notation: 0.0129.

Solution:

1.29×10^{-2}

Convert Scientific Notation to Decimal Form

How can we convert from scientific notation to decimal form? Let’s look at two numbers written in scientific notation and see.

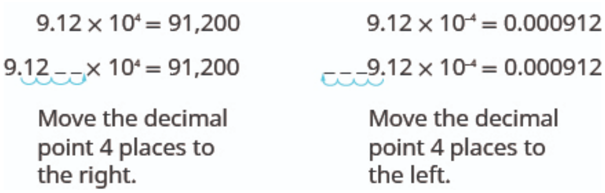
Equation:

$$\begin{array}{cc} 9.12 \times 10^{4} & 9.12 \times 10^{-4} \\ 9.12 \times 10{,}000 & 9.12 \times 0.0001 \\ 91{,}200 & 0.000912 \end{array}$$

If we look at the location of the decimal point, we can see an easy method to convert a number from scientific notation to decimal form.

Equation:

$$9.12 \times 10^{4} = 91{,}200 \qquad 9.12 \times 10^{-4} = 0.000912$$



$9.12 \times 10^{4} = 91{,}200$
$9.12\_\_ \times 10^{4} = 91{,}200$
$9.12 \times 10^{-4} = 0.000912$
$\_\_\_9.12 \times 10^{-4} = 0.000912$

Move the decimal point 4 places to the right.

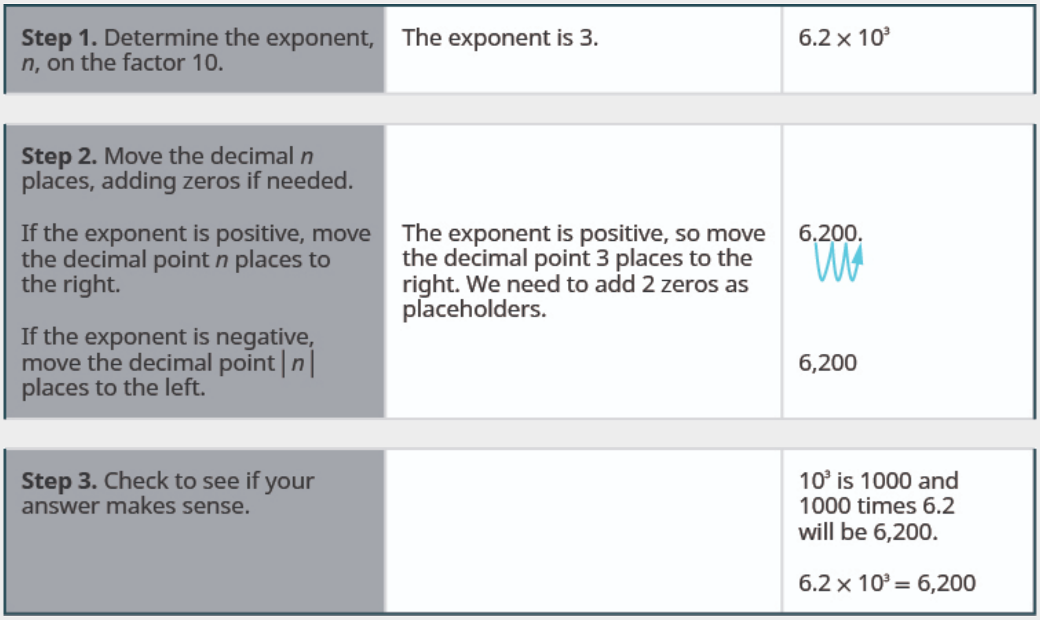
Move the decimal point 4 places to the left.

In both cases the decimal point moved 4 places. When the exponent was positive, the decimal moved to the right. When the exponent was negative, the decimal point moved to the left.

Example:
How to Convert Scientific Notation to Decimal Form
Exercise:

Problem: Convert to decimal form: 6.2×10^3.

Solution:
Solution



Step 1. Determine the exponent, *n*, on the factor 10.	The exponent is 3.	6.2×10^3
Step 2. Move the decimal *n* places, adding zeros if needed. If the exponent is positive, move the decimal point *n* places to the right. If the exponent is negative, move the decimal point $\lvert n \rvert$ places to the left.	The exponent is positive, so move the decimal point 3 places to the right. We need to add 2 zeros as placeholders.	6.200. 6,200
Step 3. Check to see if your answer makes sense.		10^3 is 1000 and 1000 times 6.2 will be 6,200. $6.2 \times 10^3 = 6{,}200$

Note:
Exercise:

Problem: Convert to decimal form: 1.3×10^3.

Solution:

1,300

Note:
Exercise:

Problem: Convert to decimal form: 9.25×10^4.

Solution:

92,500

The steps are summarized below.

Note:
Convert scientific notation to decimal form.
To convert scientific notation to decimal form:

Determine the exponent, n, on the factor 10.
Move the decimal n places, adding zeros if needed.

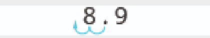
- If the exponent is positive, move the decimal point n places to the right.
- If the exponent is negative, move the decimal point $|n|$ places to the left.

Check.

Example:
Exercise:

Problem: Convert to decimal form: 8.9×10^{-2}.

Solution:
Solution

	8.9×10^{-2}
Determine the exponent, *n*, on the factor 10.	The exponent is −2.
Since the exponent is negative, move the decimal point 2 places to the left.	
Add zeros as needed for placeholders.	$8.9 \times 10^{-2} = 0.089$

Note:
Exercise:

Problem: Convert to decimal form: 1.2×10^{-4}.

Solution:

0.00012

Note:
Exercise:

Problem: Convert to decimal form: 7.5×10^{-2}.

Solution:

0.075

Multiply and Divide Using Scientific Notation

Astronomers use very large numbers to describe distances in the universe and ages of stars and planets. Chemists use very small numbers to describe the size of an atom or the charge on an electron. When scientists perform calculations with very large or very small numbers, they use scientific notation. Scientific notation provides a way for the calculations to be done without writing a lot of zeros. We will see how the Properties of Exponents are used to multiply and divide numbers in scientific notation.

Example:
Exercise:

Problem: Multiply. Write answers in decimal form: $\left(4 \times 10^{5}\right)\left(2 \times 10^{-7}\right)$.

Solution:
Solution

	$\left(4 \times 10^{5}\right)\left(2 \times 10^{-7}\right)$
Use the Commutative Property to rearrange the factors.	$4 \cdot 2 \cdot 10^{5} \cdot 10^{-7}$
Multiply.	8×10^{-2}
Change to decimal form by moving the decimal two places left.	0.08

Note:
Exercise:

Problem: Multiply $\left(3 \times 10^{6}\right)\left(2 \times 10^{-8}\right)$. Write answers in decimal form.

Solution:

0.06

Note:
Exercise:

Problem: Multiply $\left(3 \times 10^{-2}\right)\left(3 \times 10^{-1}\right)$. Write answers in decimal form.

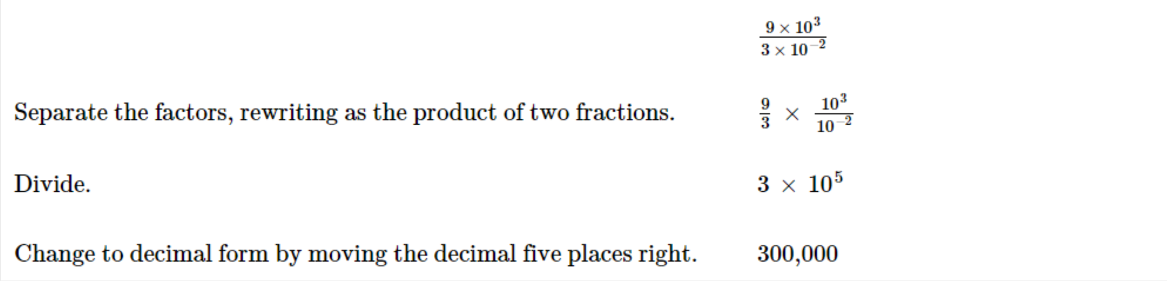
Solution:

0.009

Example:
Exercise:

Problem: Divide. Write answers in decimal form: $\frac{9 \times 10^{3}}{3 \times 10^{-2}}$.

Solution:
Solution



	$\frac{9 \times 10^{3}}{3 \times 10^{-2}}$
Separate the factors, rewriting as the product of two fractions.	$\frac{9}{3} \times \frac{10^{3}}{10^{-2}}$
Divide.	3×10^{5}
Change to decimal form by moving the decimal five places right.	300,000

Note:
Exercise:

Problem: Divide $\frac{8 \times 10^{4}}{2 \times 10^{-1}}$. Write answers in decimal form.

Solution:

400,000

Note:
Exercise:

Problem: Divide $\frac{8 \times 10^{2}}{4 \times 10^{-2}}$. Write answers in decimal form.

Solution:

20,000

Note:
Access these online resources for additional instruction and practice with integer exponents and scientific notation:

- Negative Exponents
- Scientific Notation
- Scientific Notation 2

Key Concepts

- **Property of Negative Exponents**
  - If n is a positive integer and $a \neq 0$, then $\frac{1}{a^{-n}} = a^n$
- **Quotient to a Negative Exponent**
  - If a, b are real numbers, $b \neq 0$ and n is an integer , then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$
- **To convert a decimal to scientific notation:**

  Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
  Count the number of decimal places, n, that the decimal point was moved.
  Write the number as a product with a power of 10. If the original number is:
    - greater than 1, the power of 10 will be 10^n
    - between 0 and 1, the power of 10 will be 10^{-n}

  Check.
- **To convert scientific notation to decimal form:**

  Determine the exponent, n, on the factor 10.
  Move the decimal n places, adding zeros if needed.
    - If the exponent is positive, move the decimal point n places to the right.
    - If the exponent is negative, move the decimal point $|n|$ places to the left.

  Check.

Practice Makes Perfect

Use the Definition of a Negative Exponent

In the following exercises, simplify.

Exercise:

ⓐ 4^{-2}
Problem: ⓑ 10^{-3}

Exercise:

ⓐ 3^{-4}
Problem: ⓑ 10^{-2}

Solution:

ⓐ $\frac{1}{81}$ ⓑ $\frac{1}{100}$

Exercise:

ⓐ 5^{-3}
Problem: ⓑ 10^{-5}

Exercise:

ⓐ 2^{-8}
Problem: ⓑ 10^{-2}

Solution:

ⓐ $\frac{1}{256}$ ⓑ $\frac{1}{100}$

Exercise:

ⓐ $\frac{1}{c^{-5}}$
Problem: ⓑ $\frac{1}{3^{-2}}$

Exercise:

ⓐ $\frac{1}{c^{-5}}$
Problem: ⓑ $\frac{1}{5^{-2}}$

Solution:

ⓐ c^5 ⓑ 25

Exercise:

ⓐ $\frac{1}{q^{-10}}$
Problem: ⓑ $\frac{1}{10^{-3}}$

Exercise:

ⓐ $\frac{1}{t^{-9}}$
Problem: ⓑ $\frac{1}{10^{-4}}$

Solution:

ⓐ t^9 ⓑ 10000

Exercise:

ⓐ $\left(\frac{5}{8}\right)^{-2}$
Problem: ⓑ $\left(-\frac{3m}{n}\right)^{-2}$

Exercise:

ⓐ $\left(\frac{3}{10}\right)^{-2}$
Problem: ⓑ $\left(-\frac{2}{cd}\right)^{-3}$

Solution:

ⓐ $\frac{100}{9}$ ⓑ $-\frac{c^3d^3}{8}$

Exercise:

ⓐ $\left(\frac{4}{9}\right)^{-3}$
Problem: ⓑ $\left(-\frac{u^2}{2v}\right)^{-5}$

Exercise:

ⓐ $\left(\frac{7}{2}\right)^{-3}$
Problem: ⓑ $\left(-\frac{3}{xy^2}\right)^{-3}$

Solution:

ⓐ $\frac{8}{343}$ ⓑ $-\frac{x^3y^6}{27}$

Exercise:

ⓐ $(-5)^{-2}$
ⓑ -5^{-2}
ⓒ $\left(-\frac{1}{5}\right)^{-2}$
Problem: ⓓ $-\left(\frac{1}{5}\right)^{-2}$

Exercise:

ⓐ $(-7)^{-2}$
ⓑ-7^{-2}
ⓒ $\left(-\frac{1}{7}\right)^{-2}$
Problem: ⓓ $-\left(\frac{1}{7}\right)^{-2}$

Solution:

ⓐ $\frac{1}{49}$ ⓑ $-\frac{1}{49}$ ⓒ 49 ⓓ -49

Exercise:

ⓐ -3^{-3}
ⓑ $\left(-\frac{1}{3}\right)^{-3}$
ⓒ $-\left(\frac{1}{3}\right)^{-3}$
Problem: ⓓ $(-3)^{-3}$

Exercise:

ⓐ -5^{-3}
ⓑ $\left(-\frac{1}{5}\right)^{-3}$
ⓒ $-\left(\frac{1}{5}\right)^{-3}$
Problem: ⓓ $(-5)^{-3}$

Solution:

ⓐ $-\frac{1}{125}$ ⓑ -125 ⓒ -125 ⓓ $-\frac{1}{125}$

Exercise:

ⓐ $3 \cdot 5^{-1}$
Problem: ⓑ $(3 \cdot 5)^{-1}$

Exercise:

ⓐ $2 \cdot 5^{-1}$
Problem: ⓑ $(2 \cdot 5)^{-1}$

Solution:

ⓐ $\frac{2}{5}$ ⓑ $\frac{1}{10}$

Exercise:

ⓐ $4 \cdot 5^{-2}$

Problem: ⓑ $(4 \cdot 5)^{-2}$

Exercise:

ⓐ $3 \cdot 4^{-2}$

Problem: ⓑ $(3 \cdot 4)^{-2}$

Solution:

ⓐ $\frac{3}{16}$ ⓑ $\frac{1}{144}$

Exercise:

ⓐ m^{-4}

Problem: ⓑ $\left(x^{3}\right)^{-4}$

Exercise:

ⓐ b^{-5}

Problem: ⓑ $\left(k^{2}\right)^{-5}$

Solution:

ⓐ $\frac{1}{b^{5}}$ ⓑ $\frac{1}{k^{10}}$

Exercise:

ⓐ p^{-10}

Problem: ⓑ $\left(q^{6}\right)^{-8}$

Exercise:

ⓐ s^{-8}

Problem: ⓑ $\left(a^{9}\right)^{-10}$

Solution:

ⓐ $\frac{1}{s^{8}}$ ⓑ $\frac{1}{a^{90}}$

Exercise:

ⓐ $7n^{-1}$

ⓑ $(7n)^{-1}$

Problem: ⓒ $(-7n)^{-1}$

Exercise:

Problem: ⓐ $6r^{-1}$
ⓑ $(6r)^{-1}$
ⓒ $(-6r)^{-1}$

Solution:

ⓐ $\frac{6}{r}$ ⓑ $\frac{1}{6r}$ ⓒ $-\frac{1}{6r}$

Exercise:

Problem: ⓐ $(3p)^{-2}$
ⓑ $3p^{-2}$
ⓒ $-3p^{-2}$

Exercise:

Problem: ⓐ $(2q)^{-4}$
ⓑ $2q^{-4}$
ⓒ $-2q^{-4}$

Solution:

ⓐ $\frac{1}{16q^4}$ ⓑ $\frac{2}{q^4}$ ⓒ $-\frac{2}{q^4}$

Simplify Expressions with Integer Exponents

In the following exercises, simplify.

Exercise:

Problem: ⓐ b^4b^{-8}
ⓑ $r^{-2}r^5$
ⓒ $x^{-7}x^{-3}$

Exercise:

Problem: ⓐ $s^3 \cdot s^{-7}$
ⓑ $q^{-8} \cdot q^3$
ⓒ $y^{-2} \cdot y^{-5}$

Solution:

ⓐ $\frac{1}{s^4}$ ⓑ $\frac{1}{q^5}$ ⓒ $\frac{1}{y^7}$

Exercise:

Ⓐ $a^3 \cdot a^{-3}$
Ⓑ $a \cdot a^3$
Problem: Ⓒ $a \cdot a^{-3}$

Exercise:

Ⓐ $y^5 \cdot y^{-5}$
Ⓑ $y \cdot y^5$
Problem: Ⓒ $y \cdot y^{-5}$

Solution:

Ⓐ 1 Ⓑ y^6 Ⓒ $\frac{1}{y^4}$

Exercise:

Problem: $p^5 \cdot p^{-2} \cdot p^{-4}$

Exercise:

Problem: $x^4 \cdot x^{-2} \cdot x^{-3}$

Solution:

$\frac{1}{x}$

Exercise:

Problem: $\left(w^4 x^{-5}\right)\left(w^{-2} x^{-4}\right)$

Exercise:

Problem: $\left(m^3 n^{-3}\right)\left(m^{-5} n^{-1}\right)$

Solution:

$\frac{1}{m^2 n^4}$

Exercise:

Problem: $\left(u v^{-2}\right)\left(u^{-5} v^{-3}\right)$

Exercise:

Problem: $\left(p q^{-4}\right)\left(p^{-6} q^{-3}\right)$

Solution:

$\frac{1}{p^5 q^7}$

Exercise:

Problem: $\left(-6c^{-3}d^{9}\right)\left(2c^{4}d^{-5}\right)$

Exercise:

Problem: $\left(-2j^{-5}k^{8}\right)\left(7j^{2}k^{-3}\right)$

Solution:

$-\frac{14k^{5}}{j^{3}}$

Exercise:

Problem: $\left(-4r^{-2}s^{-8}\right)\left(9r^{4}s^{3}\right)$

Exercise:

Problem: $\left(-5m^{4}n^{6}\right)\left(8m^{-5}n^{-3}\right)$

Solution:

$-\frac{40n^{3}}{m}$

Exercise:

Problem: $\left(5x^{2}\right)^{-2}$

Exercise:

Problem: $\left(4y^{3}\right)^{-3}$

Solution:

$\frac{1}{64y^{9}}$

Exercise:

Problem: $\left(3z^{-3}\right)^{2}$

Exercise:

Problem: $\left(2p^{-5}\right)^{2}$

Solution:

$\frac{4}{p^{10}}$

Exercise:

Problem: $\frac{t^{9}}{t^{-3}}$

Exercise:

Problem: $\frac{n^5}{n^{-2}}$

Solution:

n^7

Exercise:

Problem: $\frac{x^{-7}}{x^{-3}}$

Exercise:

Problem: $\frac{y^{-5}}{y^{-10}}$

Solution:

y^5

Convert from Decimal Notation to Scientific Notation

In the following exercises, write each number in scientific notation.

Exercise:

Problem: 57,000

Exercise:

Problem: 340,000

Solution:

3.4×10^5

Exercise:

Problem: 8,750,000

Exercise:

Problem: 1,290,000

Solution:

1.29×10^6

Exercise:

Problem: 0.026

Exercise:

Problem: 0.041

Solution:

4.1×10^{-2}

Exercise:

Problem: 0.00000871

Exercise:

Problem: 0.00000103

Solution:

1.03×10^{-6}

Convert Scientific Notation to Decimal Form

In the following exercises, convert each number to decimal form.

Exercise:

Problem: 5.2×10^{2}

Exercise:

Problem: 8.3×10^{2}

Solution:

830

Exercise:

Problem: 7.5×10^{6}

Exercise:

Problem: 1.6×10^{10}

Solution:

16,000,000,000

Exercise:

Problem: 2.5×10^{-2}

Exercise:

Problem: 3.8×10^{-2}

Solution:

0.038

Exercise:

Problem: 4.13×10^{-5}

Exercise:

Problem: 1.93×10^{-5}

Solution:

0.0000193

Multiply and Divide Using Scientific Notation

In the following exercises, multiply. Write your answer in decimal form.

Exercise:

Problem: $\left(3 \times 10^{-5}\right)\left(3 \times 10^{9}\right)$

Exercise:

Problem: $\left(2 \times 10^{2}\right)\left(1 \times 10^{-4}\right)$

Solution:

0.02

Exercise:

Problem: $\left(7.1 \times 10^{-2}\right)\left(2.4 \times 10^{-4}\right)$

Exercise:

Problem: $\left(3.5 \times 10^{-4}\right)\left(1.6 \times 10^{-2}\right)$

Solution:

5.6×10^{-6}

In the following exercises, divide. Write your answer in decimal form.

Exercise:

Problem: $\frac{7 \times 10^{-3}}{1 \times 10^{-7}}$

Exercise:

Problem: $\frac{5 \times 10^{-2}}{1 \times 10^{-10}}$

Solution:

500,000,000

Exercise:

Problem: $\frac{6 \times 10^4}{3 \times 10^{-2}}$

Exercise:

Problem: $\frac{8 \times 10^6}{4 \times 10^{-1}}$

Solution:

20,000,000

Everyday Math

Exercise:

Problem:

The population of the United States on July 4, 2010 was almost 310,000,000. Write the number in scientific notation.

Exercise:

Problem:

The population of the world on July 4, 2010 was more than 6,850,000,000. Write the number in scientific notation

---

Solution:

6.85×10^9.

Exercise:

Problem:

The average width of a human hair is 0.0018 centimeters. Write the number in scientific notation.

Exercise:

Problem:

The probability of winning the 2010 Megamillions lottery was about 0.0000000057. Write the number in scientific notation.

Solution:

5.7×10^{-9}

Exercise:

Problem:

In 2010, the number of Facebook users each day who changed their status to ‘engaged’ was 2×10^4. Convert this number to decimal form.

Exercise:

Problem:

At the start of 2012, the US federal budget had a deficit of more than $\$1.5 \times 10^{13}$. Convert this number to decimal form.

Solution:

15,000,000,000,000

Exercise:

Problem:

The concentration of carbon dioxide in the atmosphere is 3.9×10^{-4}. Convert this number to decimal form.

Exercise:

Problem:

The width of a proton is 1×10^{-5} of the width of an atom. Convert this number to decimal form.

Solution:

0.00001

Exercise:

Problem:

Health care costs The Centers for Medicare and Medicaid projects that consumers will spend more than $4 trillion on health care by 2017.

ⓐ Write 4 trillion in decimal notation.
ⓑ Write 4 trillion in scientific notation.

Exercise:

Problem:

Coin production In 1942, the U.S. Mint produced 154,500,000 nickels. Write 154,500,000 in scientific notation.

Solution:

1.545×10^{8}

Exercise:

Problem:

Distance The distance between Earth and one of the brightest stars in the night sky is 33.7 light years. One light year is about 6,000,000,000,000 (6 trillion), miles.

ⓐ Write the number of miles in one light year in scientific notation.
ⓑUse scientific notation to find the distance between Earth and the star in miles. Write the answer in scientific notation.

Exercise:

Problem:

Debt At the end of fiscal year 2015 the gross United States federal government debt was estimated to be approximately \$18,600,000,000,000 (\$18.6 trillion), according to the Federal Budget. The population of the United States was approximately 300,000,000 people at the end of fiscal year 2015.

ⓐ Write the debt in scientific notation.
ⓑ Write the population in scientific notation.
ⓒ Find the amount of debt per person by using scientific notation to divide the debt by the population. Write the answer in scientific notation.

Solution:

ⓐ 1.86×10^{13} ⓑ 3×10^{8} ⓒ 6.2×10^{4}

Writing Exercises

Exercise:

Problem:

ⓐ Explain the meaning of the exponent in the expression 2^{3}.
ⓑ Explain the meaning of the exponent in the expression 2^{-3}.

Exercise:

Problem:

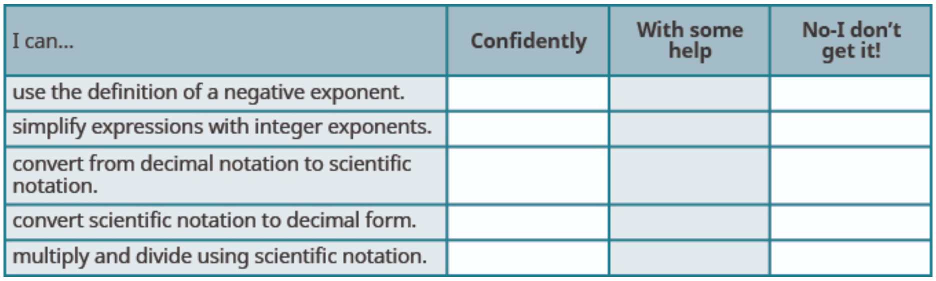
When you convert a number from decimal notation to scientific notation, how do you know if the exponent will be positive or negative?

Solution:

answers will vary

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
use the definition of a negative exponent.			
simplify expressions with integer exponents.			
convert from decimal notation to scientific notation.			
convert scientific notation to decimal form.			
multiply and divide using scientific notation.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Chapter Review Exercises

Add and Subtract Polynomials

Identify Polynomials, Monomials, Binomials and Trinomials

In the following exercises, determine if each of the following polynomials is a monomial, binomial, trinomial, or other polynomial.

Exercise:

ⓐ $11c^4 - 23c^2 + 1$
ⓑ $9p^3 + 6p^2 - p - 5$
ⓒ $\frac{3}{7}x + \frac{5}{14}$
ⓓ 10
Problem: ⓔ $2y - 12$

Exercise:

ⓐ $a^2 - b^2$
ⓑ $24d^3$
ⓒ $x^2 + 8x - 10$
ⓓ $m^2n^2 - 2mn + 6$
Problem: ⓔ $7y^3 + y^2 - 2y - 4$

Solution:

ⓐ binomial ⓑ monomial ⓒ trinomial ⓓ trinomial ⓔ other polynomial

Determine the Degree of Polynomials

In the following exercises, determine the degree of each polynomial.

Exercise:

Problem:

ⓐ $3x^2 + 9x + 10$
ⓑ $14a^2bc$
ⓒ $6y + 1$
ⓓ $n^3 - 4n^2 + 2n - 8$
ⓔ -19

Exercise:

Problem:

ⓐ $5p^3 - 8p^2 + 10p - 4$
ⓑ $-20q^4$
ⓒ $x^2 + 6x + 12$
ⓓ $23r^2s^2 - 4rs + 5$
ⓔ 100

Solution:

ⓐ 3 ⓑ 4 ⓒ 2 ⓓ 4 ⓔ 0

Add and Subtract Monomials

In the following exercises, add or subtract the monomials.

Exercise:

Problem: $5\mathrm{y}^3 + 8y^3$

Exercise:

Problem: $-14k + 19k$

Solution:

$5k$

Exercise:

Problem: $12q - (-6q)$

Exercise:

Problem: $-9c - 18c$

Solution:

$-27c$

Exercise:

Problem: $12\mathrm{x} - 4y - 9x$

Exercise:

Problem: $3m^2 + 7n^2 - 3m^2$

Solution:

$7n^2$

Exercise:

Problem: $6x^2y - 4x + 8xy^2$

Exercise:

Problem: $13\mathrm{a} + b$

Solution:

$13\mathrm{a} + b$

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

Exercise:

Problem: $(5x^2+12x+1)+(6x^2-8x+3)$

Exercise:

Problem: $(9p^2-5p+3)+(4p^2-4)$

Solution:

$13p^2-5p-1$

Exercise:

Problem: $(10m^2-8m-1)-(5m^2+m-2)$

Exercise:

Problem: $(7y^2-8y)-(y-4)$

Solution:

$7y^2-9y+4$

Exercise:

Problem: Subtract $\left(3s^2+10\right)$ from $\left(15s^2-2s+8\right)$

Exercise:

Problem: Find the sum of $\left(a^2+6a+9\right)$ and $\left(5a^3-7\right)$

Solution:

$5a^3+a^2+6a+2$

Evaluate a Polynomial for a Given Value of the Variable

In the following exercises, evaluate each polynomial for the given value.

Exercise:

Problem: Evaluate $3y^2-y+1$ when:

ⓐ $y=5$
ⓑ $y=-1$
ⓒ $y=0$

Exercise:

Problem: Evaluate $10-12x$ when:

ⓐ $x=3$
ⓑ $x=0$

ⓒ $x = -1$

Solution:

ⓐ -26 ⓑ 10 ⓒ 22

Exercise:

Problem:

Randee drops a stone off the 200 foot high cliff into the ocean. The polynomial $-16t^2 + 200$ gives the height of a stone t seconds after it is dropped from the cliff. Find the height after $t = 3$ seconds.

Exercise:

Problem:

A manufacturer of stereo sound speakers has found that the revenue received from selling the speakers at a cost of p dollars each is given by the polynomial $-4p^2 + 460p$. Find the revenue received when $p = 75$ dollars.

Solution:

12,000

Use Multiplication Properties of Exponents

Simplify Expressions with Exponents

In the following exercises, simplify.

Exercise:

Problem: 10^4

Exercise:

Problem: 17^1

Solution:

17

Exercise:

Problem: $\left(\frac{2}{9}\right)^2$

Exercise:

Problem: $(0.5)^3$

Solution:

0.125

Exercise:

Problem: $(-2)^6$

Exercise:

Problem: -2^6

Solution:

-64

Simplify Expressions Using the Product Property for Exponents

In the following exercises, simplify each expression.

Exercise:

Problem: $x^4 \cdot x^3$

Exercise:

Problem: $p^{15} \cdot p^{16}$

Solution:

p^{31}

Exercise:

Problem: $4^{10} \cdot 4^6$

Exercise:

Problem: $8 \cdot 8^5$

Solution:

8^6

Exercise:

Problem: $n \cdot n^2 \cdot n^4$

Exercise:

Problem: $y^c \cdot y^3$

Solution:

y^{c+3}

Simplify Expressions Using the Power Property for Exponents

In the following exercises, simplify each expression.

Exercise:

Problem: $(m^3)^5$

Exercise:

Problem: $(5^3)^2$

Solution:

5^6

Exercise:

Problem: $\left(y^4\right)^x$

Exercise:

Problem: $(3^r)^s$

Solution:

3^{rs}

Simplify Expressions Using the Product to a Power Property

In the following exercises, simplify each expression.

Exercise:

Problem: $(4a)^2$

Exercise:

Problem: $(-5y)^3$

Solution:

$-125y^3$

Exercise:

Problem: $(2mn)^5$

Exercise:

Problem: $(10xyz)^3$

Solution:

$1000x^3y^3z^3$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify each expression.

Exercise:

Problem: $\left(p^2\right)^5 \cdot \left(p^3\right)^6$

Exercise:

Problem: $(4a^3b^2)^3$

Solution:

$64a^9b^6$

Exercise:

Problem: $(5x)^2(7x)$

Exercise:

Problem: $(2q^3)^4(3q)^2$

Solution:

$144q^{14}$

Exercise:

Problem: $\left(\frac{1}{3}x^2\right)^2\left(\frac{1}{2}x\right)^3$

Exercise:

Problem: $\left(\frac{2}{5}m^2n\right)^3$

Solution:

$\frac{8}{125}m^6n^3$

Multiply Monomials

In the following exercises, multiply the monomials.

Exercise:

Problem: $\left(-15x^2\right)\left(6x^4\right)$

Exercise:

Problem: $(-9n^7)(-16n)$

Solution:

$144n^8$

Exercise:

Problem: $(7p^5q^3)(8pq^9)$

Exercise:

Problem: $\left(\frac{5}{9}ab^2\right)\left(27ab^3\right)$

Solution:

$15a^2b^5$

Multiply Polynomials

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

Exercise:

Problem: $7(a+9)$

Exercise:

Problem: $-4(y+13)$

Solution:

$-4y-52$

Exercise:

Problem: $-5(r-2)$

Exercise:

Problem: $p(p+3)$

Solution:

p^2+3p

Exercise:

Problem: $-m(m+15)$

Exercise:

Problem: $-6u(2u+7)$

Solution:

$-12u^2-42u$

Exercise:

Problem: $9(b^2+6b+8)$

Exercise:

Problem: $3q^2(q^2 - 7q + 6)$

Solution:

$3q^4 - 21q^3 + 18q^2$

Exercise:

Problem: $(5z - 1)z$

Exercise:

Problem: $(b - 4) \cdot 11$

Solution:

$11b - 44$

Multiply a Binomial by a Binomial

In the following exercises, multiply the binomials using: ⓐ the Distributive Property, ⓑ the FOIL method, ⓒ the Vertical Method.

Exercise:

Problem: $(x - 4)\,(x + 10)$

Exercise:

Problem: $(6y - 7)\,(2y - 5)$

Solution:

ⓐ $12y^2 - 44y + 35$ ⓑ $12y^2 - 44y + 35$ ⓒ $12y^2 - 44y + 35$

In the following exercises, multiply the binomials. Use any method.

Exercise:

Problem: $(x + 3)(x + 9)$

Exercise:

Problem: $(y - 4)(y - 8)$

Solution:

$y^2 - 12y + 32$

Exercise:

Problem: $(p - 7)(p + 4)$

Exercise:

Problem: $(q + 16)(q - 3)$

Solution:

$q^2 + 13q - 48$

Exercise:

Problem: $(5m - 8)(12m + 1)$

Exercise:

Problem: $(u^2 + 6)(u^2 - 5)$

Solution:

$u^4 + u^2 - 30$

Exercise:

Problem: $(9x - y)\,(6x - 5)$

Exercise:

Problem: $(8mn + 3)(2mn - 1)$

Solution:

$16m^2n^2 - 2mn - 3$

Multiply a Trinomial by a Binomial

In the following exercises, multiply using ⓐ the Distributive Property, ⓑ the Vertical Method.

Exercise:

Problem: $(n + 1)\left(n^2 + 5n - 2\right)$

Exercise:

Problem: $(3x - 4)\left(6x^2 + x - 10\right)$

Solution:

ⓐ $18x^3 - 21x^2 - 34x + 40$ ⓑ $18x^3 - 21x^2 - 34x + 40$

In the following exercises, multiply. Use either method.

Exercise:

Problem: $(y - 2)(y^2 - 8y + 9)$

Exercise:

Problem: $(7m + 1)(m^2 - 10m - 3)$

Solution:

$7m^3 - 69m^2 - 31m - 3$

Special Products

Square a Binomial Using the Binomial Squares Pattern

In the following exercises, square each binomial using the Binomial Squares Pattern.
Exercise:

Problem: $(c+11)^2$

Exercise:

Problem: $(q-15)^2$

Solution:

$q^2 - 30q + 225$

Exercise:

Problem: $\left(x + \frac{1}{3}\right)^2$

Exercise:

Problem: $(8u+1)^2$

Solution:

$64u^2 + 16u + 1$

Exercise:

Problem: $(3n^3 - 2)^2$

Exercise:

Problem: $(4a - 3b)^2$

Solution:

$16a^2 - 24ab + 9b^2$

Multiply Conjugates Using the Product of Conjugates Pattern

In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.
Exercise:

Problem: $(s-7)(s+7)$

Exercise:

Problem: $\left(y+\frac{2}{5}\right)\left(y-\frac{2}{5}\right)$

Solution:

$y^2-\frac{4}{25}$

Exercise:

Problem: $(12c+13)(12c-13)$

Exercise:

Problem: $(6-r)(6+r)$

Solution:

$36-r^2$

Exercise:

Problem: $\left(u+\frac{3}{4}v\right)\left(u-\frac{3}{4}v\right)$

Exercise:

Problem: $(5p^4-4q^3)(5p^4+4q^3)$

Solution:

$25p^8-16q^6$

Recognize and Use the Appropriate Special Product Pattern

In the following exercises, find each product.

Exercise:

Problem: $(3m+10)^2$

Exercise:

Problem: $(6a+11)(6a-11)$

Solution:

$36a^2-121$

Exercise:

Problem: $(5x+y)(x-5y)$

Exercise:

Problem: $(c^4+9d)^2$

Solution:

$c^8 + 18c^4d + 81d^2$

Exercise:

Problem: $(p^5 + q^5)(p^5 - q^5)$

Exercise:

Problem: $(a^2 + 4b)(4a - b^2)$

Solution:

$4a^3 - a^2b^2 + 16ab - 4b^3$

Divide Monomials

Simplify Expressions Using the Quotient Property for Exponents

In the following exercises, simplify.

Exercise:

Problem: $\frac{u^{24}}{u^6}$

Exercise:

Problem: $\frac{10^{25}}{10^5}$

Solution:

10^{20}

Exercise:

Problem: $\frac{3^4}{3^6}$

Exercise:

Problem: $\frac{v^{12}}{v^{48}}$

Solution:

$\frac{1}{v^{36}}$

Exercise:

Problem: $\frac{x}{x^5}$

Exercise:

Problem: $\frac{5}{5^8}$

Solution:

$\frac{1}{5^7}$

Simplify Expressions with Zero Exponents

In the following exercises, simplify.

Exercise:

Problem: 75^0

Exercise:

Problem: x^0

Solution:

1

Exercise:

Problem: -12^0

Exercise:

Problem: $\left(-12^0\right)(-12)^0$

Solution:

1

Exercise:

Problem: $25x^0$

Exercise:

Problem: $(25x)^0$

Solution:

1

Exercise:

Problem: $19n^0 - 25m^0$

Exercise:

Problem: $(19n)^0 - (25m)^0$

Solution:

0

Simplify Expressions Using the Quotient to a Power Property

In the following exercises, simplify.

Exercise:

Problem: $\left(\frac{2}{5}\right)^3$

Exercise:

Problem: $\left(\frac{m}{3}\right)^4$

Solution:

$\frac{m^4}{81}$

Exercise:

Problem: $\left(\frac{r}{s}\right)^8$

Exercise:

Problem: $\left(\frac{x}{2y}\right)^6$

Solution:

$\frac{x^6}{64y^6}$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify.

Exercise:

Problem: $\frac{\left(x^3\right)^5}{x^9}$

Exercise:

Problem: $\frac{n^{10}}{\left(n^5\right)^2}$

Solution:

1

Exercise:

Problem: $\left(\frac{q^6}{q^8}\right)^3$

Exercise:

Problem: $\left(\frac{r^8}{r^3}\right)^4$

Solution:

r^{20}

Exercise:

Problem: $\left(\frac{c^2}{d^5}\right)^9$

Exercise:

Problem: $\left(\frac{3x^4}{2y^2}\right)^5$

Solution:

$\frac{243x^{20}}{32y^{10}}$

Exercise:

Problem: $\left(\frac{v^3v^9}{v^6}\right)^4$

Exercise:

Problem: $\frac{\left(3n^2\right)^4\left(-5n^4\right)^3}{\left(-2n^5\right)^2}$

Solution:

$-\frac{10{,}125n^{10}}{4}$

Divide Monomials

In the following exercises, divide the monomials.

Exercise:

Problem: $-65y^{14} \div 5y^2$

Exercise:

Problem: $\frac{64a^5b^9}{-16a^{10}b^3}$

Solution:

$-\frac{4b^6}{a^5}$

Exercise:

Problem: $\frac{144x^{15}y^8z^3}{18x^{10}y^2z^{12}}$

Exercise:

Problem: $\frac{(8p^6q^2)(9p^3q^5)}{16p^8q^7}$

Solution:

$\frac{9p}{2}$

[Divide Polynomials]

Divide a Polynomial by a Monomial

In the following exercises, divide each polynomial by the monomial.

Exercise:

Problem: $\frac{42z^2-18z}{6}$

Exercise:

Problem: $(35x^2-75x)\div 5x$

Solution:

$7x-15$

Exercise:

Problem: $\frac{81n^4+105n^2}{-3}$

Exercise:

Problem: $\frac{550p^6-300p^4}{10p^3}$

Solution:

$55p^3-30p$

Exercise:

Problem: $(63xy^3+56x^2y^4)\div(7xy)$

Exercise:

Problem: $\frac{96a^5b^2-48a^4b^3-56a^2b^4}{8ab^2}$

Solution:

$12a^4-6a^3b-7ab^2$

Exercise:

Problem: $\frac{57m^2-12m+1}{-3m}$

Exercise:

Problem: $\frac{105y^5+50y^3-5y}{5y^3}$

Solution:

$21y^2+10-\frac{1}{y^2}$

Divide a Polynomial by a Binomial

In the following exercises, divide each polynomial by the binomial.

Exercise:

Problem: $\left(k^2-2k-99\right)\div(k+9)$

Exercise:

Problem: $\left(v^2-16v+64\right)\div(v-8)$

Solution:

$v-8$

Exercise:

Problem: $\left(3x^2-8x-35\right)\div(x-5)$

Exercise:

Problem: $\left(n^2-3n-14\right)\div(n+3)$

Solution:

$n-6+\frac{4}{n+3}$

Exercise:

Problem: $\left(4m^3+m-5\right)\div(m-1)$

Exercise:

Problem: $\left(u^3-8\right)\div(u-2)$

Solution:

u^2+2u+4

Integer Exponents and Scientific Notation

Use the Definition of a Negative Exponent

In the following exercises, simplify.

Exercise:

Problem: 9^{-2}

Exercise:

Problem: $(-5)^{-3}$

Solution:

$-\frac{1}{125}$

Exercise:

Problem: $3 \cdot 4^{-3}$

Exercise:

Problem: $(6u)^{-3}$

Solution:

$\frac{1}{216u^3}$

Exercise:

Problem: $\left(\frac{2}{5}\right)^{-1}$

Exercise:

Problem: $\left(\frac{3}{4}\right)^{-2}$

Solution:

$\frac{16}{9}$

Simplify Expressions with Integer Exponents

In the following exercises, simplify.

Exercise:

Problem: $p^{-2} \cdot p^8$

Exercise:

Problem: $q^{-6} \cdot q^{-5}$

Solution:

$\frac{1}{q^{11}}$

Exercise:

Problem: $\left(c^{-2}d\right)\left(c^{-3}d^{-2}\right)$

Exercise:

Problem: $\left(y^{8}\right)^{-1}$

Solution:

$\frac{1}{y^{8}}$

Exercise:

Problem: $\left(q^{-4}\right)^{-3}$

Exercise:

Problem: $\frac{a^{8}}{a^{12}}$

Solution:

$\frac{1}{a^{4}}$

Exercise:

Problem: $\frac{n^{5}}{n^{-4}}$

Exercise:

Problem: $\frac{r^{-2}}{r^{-3}}$

Solution:

r

Convert from Decimal Notation to Scientific Notation

In the following exercises, write each number in scientific notation.

Exercise:

Problem: 8,500,000

Exercise:

Problem: 0.00429

Solution:

4.29×10^{-3}

Exercise:

Problem: The thickness of a dime is about 0.053 inches.

Exercise:

Problem: In 2015, the population of the world was about 7,200,000,000 people.

Solution:

7.2×10^{9}

Convert Scientific Notation to Decimal Form

In the following exercises, convert each number to decimal form.

Exercise:

Problem: 3.8×10^{5}

Exercise:

Problem: 1.5×10^{10}

Solution:

15,000,000,000

Exercise:

Problem: 9.1×10^{-7}

Exercise:

Problem: 5.5×10^{-1}

Solution:

0.55

Multiply and Divide Using Scientific Notation

In the following exercises, multiply and write your answer in decimal form.

Exercise:

Problem: $\left(2 \times 10^{5}\right)\left(4 \times 10^{-3}\right)$

Exercise:

Problem: $\left(3.5 \times 10^{-2}\right)\left(6.2 \times 10^{-1}\right)$

Solution:

0.0217

In the following exercises, divide and write your answer in decimal form.

Exercise:

Problem: $\frac{8 \times 10^5}{4 \times 10^{-1}}$

Exercise:

Problem: $\frac{9 \times 10^{-5}}{3 \times 10^2}$

Solution:

0.0000003

Chapter Practice Test

Exercise:

Problem: For the polynomial $10x^4 + 9y^2 - 1$
ⓐ Is it a monomial, binomial, or trinomial?
ⓑ What is its degree?

In the following exercises, simplify each expression.

Exercise:

Problem: $\left(12a^2 - 7a + 4\right) + \left(3a^2 + 8a - 10\right)$

Solution:

$15a^2 + a - 6$

Exercise:

Problem: $\left(9p^2 - 5p + 1\right) - \left(2p^2 - 6\right)$

Exercise:

Problem: $\left(-\frac{2}{5}\right)^3$

Solution:

$-\frac{8}{125}$

Exercise:

Problem: $u \cdot u^4$

Exercise:

Problem: $\left(4a^3b^5\right)^2$

Solution:

$16a^6b^{10}$

Exercise:

Problem: $\left(-9r^4s^5\right)\left(4rs^7\right)$

Exercise:

Problem: $3k\left(k^2-7k+13\right)$

Solution:

$3k^3-21k^2+39k$

Exercise:

Problem: $(m+6)(m+12)$

Exercise:

Problem: $(v-9)(9v-5)$

Solution:

$9v^2-86v+45$

Exercise:

Problem: $(4c-11)(3c-8)$

Exercise:

Problem: $(n-6)\left(n^2-5n+4\right)$

Solution:

$n^3-11n^2+34n-24$

Exercise:

Problem: $(2x-15y)(5x+7y)$

Exercise:

Problem: $(7p-5)(7p+5)$

Solution:

$49p^2-25$

Exercise:

Problem: $(9v-2)^2$

Exercise:

Problem: $\frac{3^8}{3^{10}}$

Solution:

$\frac{1}{9}$

Exercise:

Problem: $\left(\frac{m^4 \cdot m}{m^3}\right)^6$

Exercise:

Problem: $\left(87x^{15}y^3z^{22}\right)^0$

Solution:

1

Exercise:

Problem: $\frac{80c^8d^2}{16cd^{10}}$

Exercise:

Problem: $\frac{12x^2+42x-6}{2x}$

Solution:

$6x + 21 - \frac{3}{x}$

Exercise:

Problem: $\left(70xy^4 + 95x^3y\right) \div 5xy$

Exercise:

Problem: $\frac{64x^3-1}{4x-1}$

Solution:

$16x^2 + 4x + 1$

Exercise:

Problem: $\left(y^2 - 5y - 18\right) \div (y + 3)$

Exercise:

Problem: 5^{-2}

Solution:

$\frac{1}{25}$

Exercise:

Problem: $(4m)^{-3}$

Exercise:

Problem: $q^{-4} \cdot q^{-5}$

Solution:

$\frac{1}{q^9}$

Exercise:

Problem: $\frac{n^{-2}}{n^{-10}}$

Exercise:

Problem: Convert 83,000,000 to scientific notation.

Solution:

8.3×10^7

Exercise:

Problem: Convert 6.91×10^{-5} to decimal form.

In the following exercises, simplify, and write your answer in decimal form.

Exercise:

Problem: $\left(3.4 \times 10^9\right)\left(2.2 \times 10^{-5}\right)$

Solution:

74,800

Exercise:

Problem: $\frac{8.4 \times 10^{-3}}{4 \times 10^3}$

Exercise:

Problem:

A helicopter flying at an altitude of 1000 feet drops a rescue package. The polynomial $-16t^2 + 1000$ gives the height of the package t seconds a after it was dropped. Find the height when $t = 6$ seconds.

Solution:

424 feet

Glossary

negative exponent

If n is a positive integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$.

scientific notation

A number is expressed in scientific notation when it is of the form $a \times 10^n$ where $a \geq 1$ and $a < 10$ and n is an integer.

Introduction class="introduction"

The Sydney Harbor Bridge is one of Australia’s most photographed landmarks. It is the world’s largest steel arch bridge with the top of the bridge standing 134 meters above the harbor. Can you see why it is known by the locals as the “Coathanger” ?

Quadratic expressions may be used to model physical properties of a large bridge, the trajectory of a baseball or rocket, and revenue and profit of a business. By factoring these expressions, specific characteristics of the model can be identified. In this chapter, you will explore the process of factoring expressions and see how factoring is used to solve certain types of equations.

Greatest Common Factor and Factor by Grouping
By the end of this section, you will be able to:

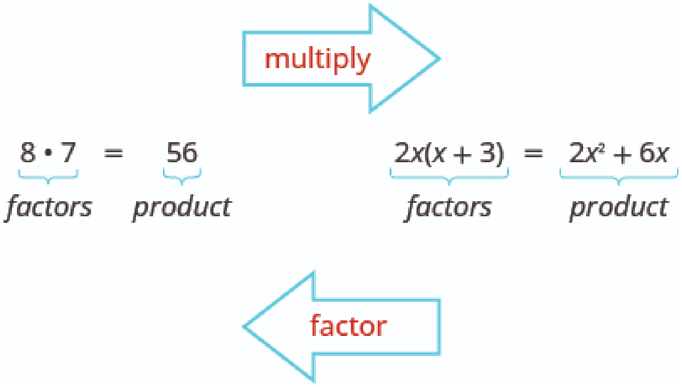
- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

Note:
Before you get started, take this readiness quiz.

1. Factor 56 into primes.
   If you missed this problem, review [link].
2. Find the least common multiple of 18 and 24.
   If you missed this problem, review [link].
3. Simplify $-3(6a+11)$.
   If you missed this problem, review [link].

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a product. Now, we will be reversing this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.


multiply
$\underbrace{8 \cdot 7}_{\text{factors}} = \underbrace{56}_{\text{product}}$
$\underbrace{2x(x+3)}_{\text{factors}} = \underbrace{2x^2+6x}_{\text{product}}$
factor

We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.

Note:

Greatest Common Factor

The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

First we’ll find the GCF of two numbers.

Example:

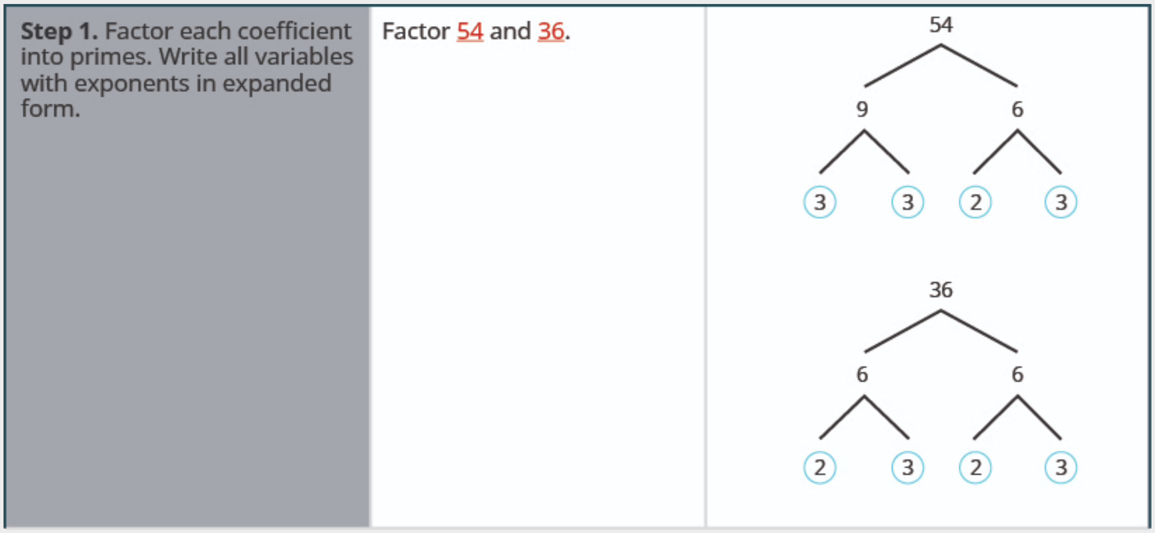
How to Find the Greatest Common Factor of Two or More Expressions

Exercise:

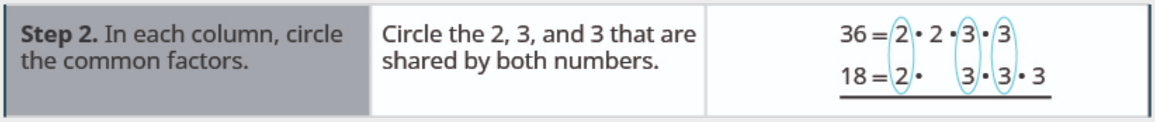
Problem: Find the GCF of 54 and 36.

Solution:

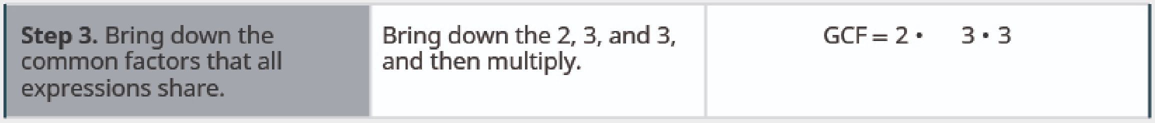
Solution



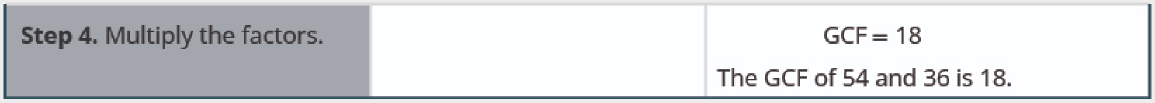
Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.	Factor 54 and 36.	



Step 2. In each column, circle the common factors.	Circle the 2, 3, and 3 that are shared by both numbers.	$36 = 2 \cdot 2 \cdot 3 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3 \cdot 3$



Step 3. Bring down the common factors that all expressions share.	Bring down the 2, 3, and 3, and then multiply.	$\text{GCF} = 2 \cdot \quad 3 \cdot 3$



Step 4. Multiply the factors.		$\text{GCF} = 18$ The GCF of 54 and 36 is 18.

Notice that, because the GCF is a factor of both numbers, 54 and 36 can be written as multiples of 18.

Equation:

$$54 = 18 \cdot 3$$
$$36 = 18 \cdot 2$$

Note:

Exercise:

Problem: Find the GCF of 48 and 80.

Solution:

16

Note:

Exercise:

Problem: Find the GCF of 18 and 40.

Solution:

2

We summarize the steps we use to find the GCF below.

Note:

Find the Greatest Common Factor (GCF) of two expressions.

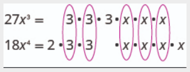
Factor each coefficient into primes. Write all variables with exponents in expanded form.
List all factors—matching common factors in a column. In each column, circle the common factors.
Bring down the common factors that all expressions share.
Multiply the factors.

In the first example, the GCF was a constant. In the next two examples, we will get variables in the greatest common factor.

Example:
Exercise:

Problem: Find the greatest common factor of $27x^3$ and $18x^4$.

Solution:
Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	 $27x^3 = 3 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x$ $18x^4 = 2 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x \cdot x$
Bring down the common factors.	$\text{GCF} = 3 \cdot 3 \cdot x \cdot x \cdot x$
Multiply the factors.	$\text{GCF} = 9x^3$
	The GCF of $27x^3$ and $18x^4$ is $9x^3$.

Note:
Exercise:

Problem: Find the GCF: $12x^2$, $18x^3$.

Solution:

$6x^2$

Note:
Exercise:

Problem: Find the GCF: $16y^2, 24y^3$.

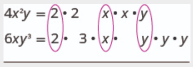
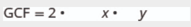
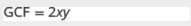
Solution:

$8y^2$

Example:
Exercise:

Problem: Find the GCF of $4x^2y, 6xy^3$.

Solution:
Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	 $4x^2y = 2 \cdot 2 \quad x \cdot x \cdot y$ $6xy^3 = 2 \cdot 3 \cdot x \cdot \quad y \cdot y \cdot y$
Bring down the common factors.	 $\text{GCF} = 2 \cdot x \cdot y$
Multiply the factors.	 $\text{GCF} = 2xy$
	The GCF of $4x^2y$ and $6xy^3$ is 2 xy.

Note:
Exercise:

Problem: Find the GCF: $6ab^4, 8a^2b$.

Solution:

$2ab$

Note:
Exercise:

Problem: Find the GCF: $9m^5n^2, 12m^3n$.

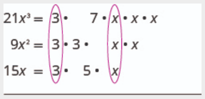
Solution:

$3m^3n$

Example:
Exercise:

Problem: Find the GCF of: $21x^3, 9x^2, 15x$.

Solution:
Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	 $21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot x \cdot x$ $15x = 3 \cdot 5 \cdot x$

Bring down the common factors.	$\text{GCF} = 3 \cdot \quad x$
Multiply the factors.	$\text{GCF} = 3x$
	The GCF of $21x^3$, $9x^2$ and $15x$ is $3x$.

Note:
Exercise:

Problem: Find the greatest common factor: $25m^4, 35m^3, 20m^2$.

Solution:

$5m^2$

Note:
Exercise:

Problem: Find the greatest common factor: $14x^3, 70x^2, 105x$.

Solution:

$7x$

Factor the Greatest Common Factor from a Polynomial

Just like in arithmetic, where it is sometimes useful to represent a number in factored form (for example, 12 as $2 \cdot 6$ or $3 \cdot 4$), in algebra, it can be useful to represent a polynomial in factored form. One way to do this is by finding the GCF of all the terms. Remember, we multiply a polynomial by a monomial as follows:
Equation:

$$\begin{array}{cl} 2\left(x+7\right) & \text{factors} \\ 2\cdot x+2\cdot 7 & \\ 2x+14 & \text{product} \end{array}$$

Now we will start with a product, like $2x+14$, and end with its factors, $2\left(x+7\right)$. To do this we apply the Distributive Property "in reverse."

We state the Distributive Property here just as you saw it in earlier chapters and "in reverse."

Note:
Distributive Property
If a,b,c are real numbers, then
Equation:

$$a\left(b+c\right)=ab+ac\qquad\text{and}\qquad ab+ac=a\left(b+c\right)$$

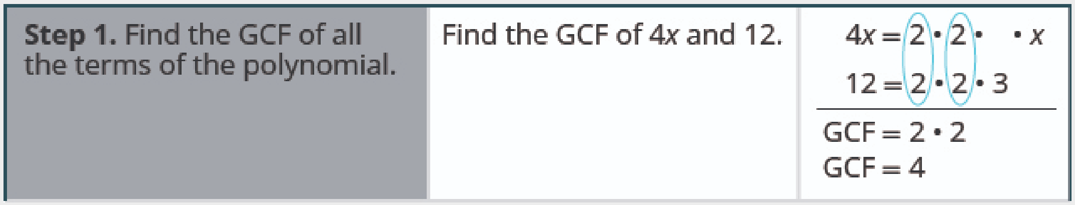
The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

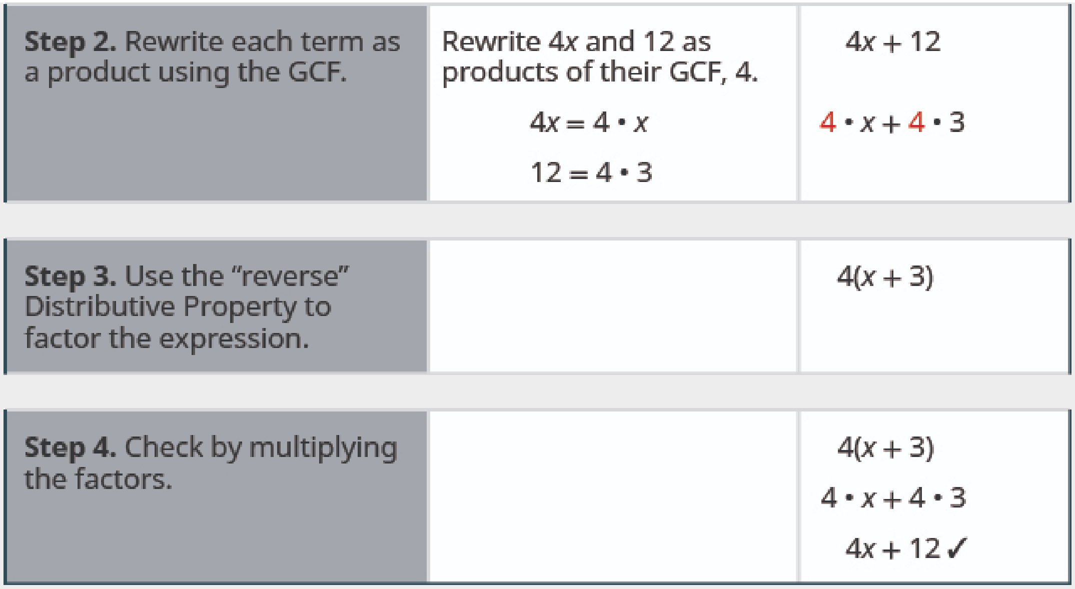
Example:
How to Factor the Greatest Common Factor from a Polynomial
Exercise:

Problem: Factor: $4x+12$.

Solution:
Solution



Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $4x$ and 12.	$4x=2\cdot 2\cdot\quad\cdot x$ $12=2\cdot 2\cdot 3$ $\text{GCF}=2\cdot 2$ $\text{GCF}=4$



Step 2. Rewrite each term as a product using the GCF.	Rewrite $4x$ and 12 as products of their GCF, 4. $4x = 4 \cdot x$ $12 = 4 \cdot 3$	$4x + 12$ $4 \cdot x + 4 \cdot 3$
Step 3. Use the "reverse" Distributive Property to factor the expression.		$4(x + 3)$
Step 4. Check by multiplying the factors.		$4(x + 3)$ $4 \cdot x + 4 \cdot 3$ $4x + 12$ ✓

Note:
Exercise:

Problem: Factor: $6a + 24$.

Solution:

$6\left(a + 4\right)$

Note:
Exercise:

Problem: Factor: $2b + 14$.

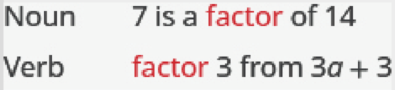
Solution:

$2\left(b + 7\right)$

Note:
Factor the greatest common factor from a polynomial.

Find the GCF of all the terms of the polynomial.
Rewrite each term as a product using the GCF.
Use the “reverse” Distributive Property to factor the expression.
Check by multiplying the factors.

Note:
Factor as a Noun and a Verb
We use “factor” as both a noun and a verb.



Noun	7 is a factor of 14
Verb	factor 3 from $3a + 3$

Example:
Exercise:

Problem: Factor: $5a + 5$.

Solution:
Solution

Find the GCF of 5*a* and 5.	$5a = 5 \cdot a$ $5 = 5$ $\text{GCF} = 5$
	$5a + 5$
Rewrite each term as a product using the GCF.	$5 \cdot a + 5 \cdot 1$

Use the Distributive Property "in reverse" to factor the GCF.	$5(a+1)$
Check by mulitplying the factors to get the orginal polynomial.	
$5(a+1)$	
$5 \cdot a + 5 \cdot 1$	
$5a + 5$✓	

Note:
Exercise:

Problem: Factor: $14x + 14$.

Solution:

$14\left(x+1\right)$

Note:
Exercise:

Problem: Factor: $12p + 12$.

Solution:

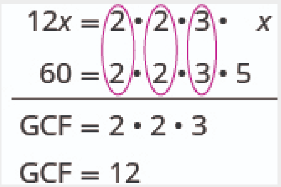
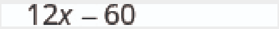
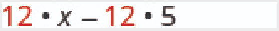
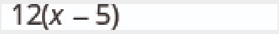
$12\left(p+1\right)$

The expressions in the next example have several factors in common. Remember to write the GCF as the product of all the common factors.

Example:
Exercise:

Problem: Factor: $12x - 60$.

Solution:
Solution

Find the GCF of 12*x* and 60.	 $12x = 2 \cdot 2 \cdot 3 \cdot x$ $60 = 2 \cdot 2 \cdot 3 \cdot 5$ $\text{GCF} = 2 \cdot 2 \cdot 3$ $\text{GCF} = 12$
	 $12x - 60$
Rewrite each term as a product using the GCF.	 $12 \cdot x - 12 \cdot 5$
Factor the GCF.	 $12(x - 5)$
Check by mulitplying the factors.	
$12(x - 5)$	
$12 \cdot x - 12 \cdot 5$	
$12x - 60$✓	

Note:
Exercise:

Problem: Factor: $18u - 36$.

Solution:

$18\left(u-2\right)$

Note:
Exercise:

Problem: Factor: $30y - 60$.

Solution:

$30\left(y-2\right)$

Now we'll factor the greatest common factor from a trinomial. We start by finding the GCF of all three terms.

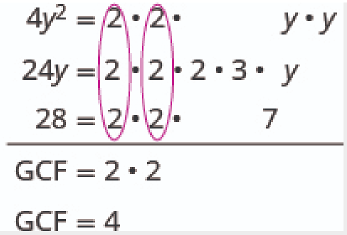
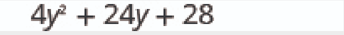
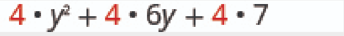
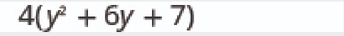
Example:
Exercise:

Problem: Factor: $4y^2 + 24y + 28$.

Solution:
Solution

We start by finding the GCF of all three terms.

Find the GCF of $4y^2$, $24y$ and 28.	

	 $4y^2 = 2 \cdot 2 \cdot y \cdot y$ $24y = 2 \cdot 2 \cdot 2 \cdot 3 \cdot y$ $28 = 2 \cdot 2 \cdot 7$ $\text{GCF} = 2 \cdot 2$ $\text{GCF} = 4$
	 $4y^2 + 24y + 28$
Rewrite each term as a product using the GCF.	 $4 \cdot y^2 + 4 \cdot 6y + 4 \cdot 7$
Factor the GCF.	 $4(y^2 + 6y + 7)$
Check by mulitplying.	
$4\left(y^2+6y+7\right)$	
$4 \cdot y^2 + 4 \cdot 6y + 4 \cdot 7$	
$4y^2+24y+28$✓	

Note:
Exercise:

Problem: Factor: $5x^2 - 25x + 15$.

Solution:

$5\left(x^2 - 5x + 3\right)$

Note:
Exercise:

Problem: Factor: $3y^2 - 12y + 27$.

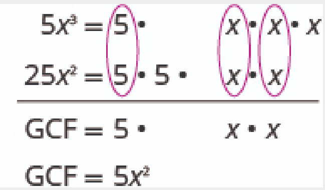
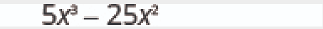
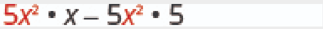
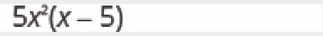
Solution:

$3\left(y^2 - 4y + 9\right)$

Example:
Exercise:

Problem: Factor: $5x^3 - 25x^2$.

Solution:
Solution

Find the GCF of $5x^3$ and $25x^2$.	$5x^3 = 5 \cdot x \cdot x \cdot x$ $25x^2 = 5 \cdot 5 \cdot x \cdot x$ $\text{GCF} = 5 \cdot x \cdot x$ $\text{GCF} = 5x^2$
	$5x^3 - 25x^2$
Rewrite each term.	$5x^2 \cdot x - 5x^2 \cdot 5$
Factor the GCF.	$5x^2(x - 5)$

Check.	
$5x^2(x-5)$	
$5x^2 \cdot x - 5x^2 \cdot 5$	
$5x^3 - 25x^2 \checkmark$	

Note:
Exercise:

Problem: Factor: $2x^3 + 12x^2$.

Solution:

$2x^2\,(x+6)$

Note:
Exercise:

Problem: Factor: $6y^3 - 15y^2$.

Solution:

$3y^2\,(2y-5)$

Example:
Exercise:

Problem: Factor: $21x^3 - 9x^2 + 15x$.

Solution:
Solution

In a previous example we found the GCF of $21x^3, 9x^2, 15x$ to be $3x$.

	$21x^3 - 9x^2 + 15x$
Rewrite each term using the GCF, 3*x*.	$3x \cdot 7x^2 - 3x \cdot 3x + 3x \cdot 5$
Factor the GCF.	$3x(7x^2 - 3x + 5)$
Check.	
$3x\left(7x^2 - 3x + 5\right)$	
$3x \cdot 7x^2 - 3x \cdot 3x + 3x \cdot 5$	
$21x^3 - 9x^2 + 15x$✓	

Note:
Exercise:

Problem: Factor: $20x^3 - 10x^2 + 14x$.

Solution:

$2x\left(10x^2 - 5x + 7\right)$

Note:
Exercise:

Problem: Factor: $24y^3 - 12y^2 - 20y$.

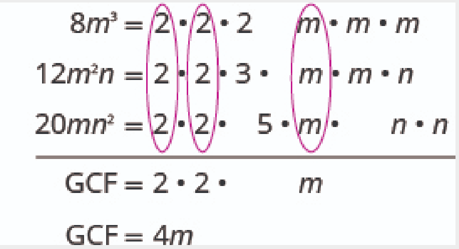
Solution:

$4y\left(6y^2 - 3y - 5\right)$

Example:
Exercise:

Problem: Factor: $8m^3 - 12m^2n + 20mn^2$.

Solution:
Solution

Find the GCF of $8m^3$, $12m^2n$, $20mn^2$.	 $8m^3 = 2 \cdot 2 \cdot 2 \quad m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot \quad m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot \quad 5 \cdot m \cdot \quad n \cdot n$ $\text{GCF} = 2 \cdot 2 \cdot \quad m$ $\text{GCF} = 4m$
	$8m^3 - 12m^2n + 20mn^2$
Rewrite each term.	$4m \cdot 2m^2 - 4m \cdot 3m\,n + 4m \cdot 5n^2$
Factor the GCF.	$4m(2m^2 - 3m\,n + 5n^2)$
Check.	
$4m\left(2m^2 - 3mn + 5n^2\right)$	
$4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$	
$8m^3 - 12m^2n + 20mn^2$✓	

Note:
Exercise:

Problem: Factor: $9xy^2 + 6x^2y^2 + 21y^3$.

Solution:

$3y^2\left(3x + 2x^2 + 7y\right)$

Note:
Exercise:

Problem: Factor: $3p^3 - 6p^2q + 9pq^3$.

Solution:

$3p\left(p^2 - 2pq + 3q^2\right)$

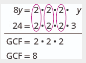
When the leading coefficient is negative, we factor the negative out as part of the GCF.

Example:
Exercise:

Problem: Factor: $-8y - 24$.

Solution:
Solution

When the leading coefficient is negative, the GCF will be negative.

Ignoring the signs of the terms, we first find the GCF of 8*y* and 24 is 8. Since the expression −8*y* − 24 has a negative leading coefficient,	 8y = 2 • 2 • 2 • y 24 = 2 • 2 • 2 • 3 GCF = 2 • 2 • 2 GCF = 8

we use −8 as the GCF.	
Rewrite each term using the GCF.	$-8y - 24$ $-8 \cdot y + (-8) \cdot 3$
Factor the GCF.	$-8(y + 3)$
Check.	
$-8(y+3)$	
$-8 \cdot y + (-8) \cdot 3$	
$-8y - 24$✓	

Note:
Exercise:

Problem: Factor: $-16z - 64$.

Solution:

$-16\,(z + 4)$

Note:
Exercise:

Problem: Factor: $-9y - 27$.

Solution:

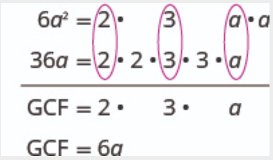
$-9\,(y + 3)$

Example:
Exercise:

Problem: Factor: $-6a^2+36a$.

Solution:
Solution

The leading coefficient is negative, so the GCF will be negative.

Since the leading coefficient is negative, the GCF is negative, $-6a$.	 $6a^2 = 2 \cdot 3 \cdot a \cdot a$ $36a = 2 \cdot 2 \cdot 3 \cdot 3 \cdot a$ $\text{GCF} = 2 \cdot 3 \cdot a$ $\text{GCF} = 6a$ $-6a^2+36a$
Rewrite each term using the GCF.	$-6a \cdot a - (-6a) \cdot 6$
Factor the GCF.	$-6a(a-6)$
Check.	
$-6a(a-6)$	
$-6a \cdot a + (-6a)(-6)$	
$-6a^2+36a$✓	

Note:
Exercise:

Problem: Factor: $-4b^2 + 16b$.

Solution:

$-4b\,(b-4)$

Note:
Exercise:

Problem: Factor: $-7a^2 + 21a$.

Solution:

$-7a\,(a-3)$

Example:
Exercise:

Problem: Factor: $5q\,(q+7) - 6\,(q+7)$.

Solution:
Solution

The GCF is the binomial $q+7$.

	$5q(q+7) - 6(q+7)$
Factor the GCF, (q + 7).	$(q+7)(5q-6)$

Check on your own by multiplying.	

Note:
Exercise:

Problem: Factor: $4m\left(m+3\right)-7\left(m+3\right)$.

Solution:

$\left(m+3\right)\left(4m-7\right)$

Note:
Exercise:

Problem: Factor: $8n\left(n-4\right)+5\left(n-4\right)$.

Solution:

$\left(n-4\right)\left(8n+5\right)$

Factor by Grouping

When there is no common factor of all the terms of a polynomial, look for a common factor in just some of the terms. When there are four terms, a good way to start is by separating the polynomial into two parts with two terms in each part. Then look for the GCF in each part. If the polynomial can be factored, you will find a common factor emerges from both parts.

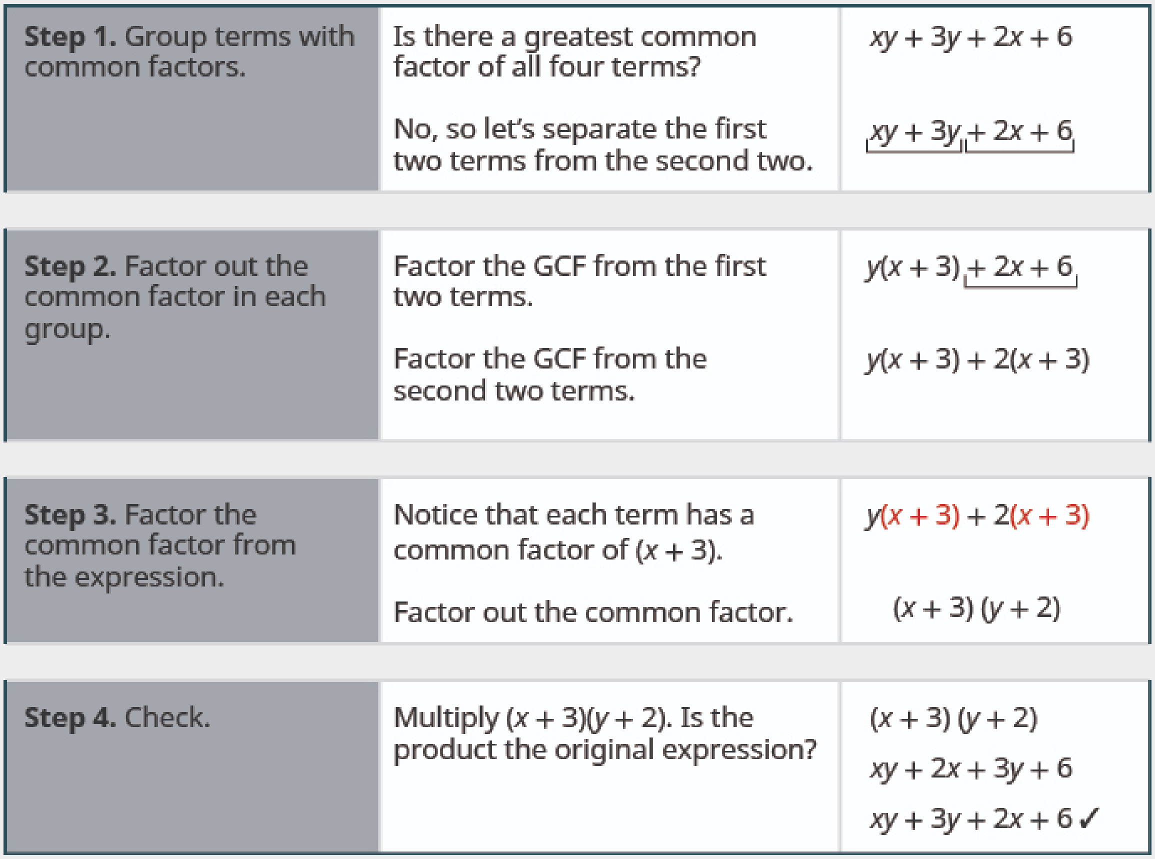
(Not all polynomials can be factored. Just like some numbers are prime, some polynomials are prime.)

Example:
How to Factor by Grouping

Exercise:

Problem: Factor: $xy+3y+2x+6$.

Solution:
Solution



Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms? No, so let's separate the first two terms from the second two.	$xy+3y+2x+6$ $\underbrace{xy+3y}+\underbrace{2x+6}$
Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms. Factor the GCF from the second two terms.	$y(x+3)+\underbrace{2x+6}$ $y(x+3)+2(x+3)$
Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x+3)$. Factor out the common factor.	$y(x+3)+2(x+3)$ $(x+3)(y+2)$
Step 4. Check.	Multiply $(x+3)(y+2)$. Is the product the original expression?	$(x+3)(y+2)$ $xy+2x+3y+6$ $xy+3y+2x+6$ ✓

Note:
Exercise:

Problem: Factor: $xy+8y+3x+24$.

Solution:

$(x+8)(y+3)$

Note:
Exercise:

Problem: Factor: $ab + 7b + 8a + 56$.

Solution:

$(a + 7)(b + 8)$

Note:
Factor by grouping.

Group terms with common factors.
Factor out the common factor in each group.
Factor the common factor from the expression.
Check by multiplying the factors.

Example:
Exercise:

Problem: Factor: $x^2 + 3x - 2x - 6$.

Solution:
Solution

There is no GCF in all four terms.	$x^2 + 3x \ \ -2x - 6$
Separate into two parts.	$x^2 + 3x \ \ -2x - 6$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$x(x + 3) - 2(x + 3)$ $(x + 3)(x - 2)$

Check on your own by multiplying.

Note:

Exercise:

Problem: Factor: $x^2 + 2x - 5x - 10$.

Solution:

$(x - 5)\,(x + 2)$

Note:
Exercise:

Problem: Factor: $y^2 + 4y - 7y - 28$.

Solution:

$(y + 4)\,(y - 7)$

Note:
Access these online resources for additional instruction and practice with greatest common factors (GFCs) and factoring by grouping.

- Greatest Common Factor (GCF)
- Factoring Out the GCF of a Binomial
- Greatest Common Factor (GCF) of Polynomials

Key Concepts

- **Finding the Greatest Common Factor (GCF):** To find the GCF of two expressions:

  Factor each coefficient into primes. Write all variables with exponents in expanded form.
  List all factors—matching common factors in a column. In each column, circle the common factors.
  Bring down the common factors that all expressions share.
  Multiply the factors as in[link].

- **Factor the Greatest Common Factor from a Polynomial:** To factor a greatest common factor from a polynomial:

  Find the GCF of all the terms of the polynomial.
  Rewrite each term as a product using the GCF.
  Use the 'reverse' Distributive Property to factor the expression.
  Check by multiplying the factors as in[link].

- **Factor by Grouping:** To factor a polynomial with 4 four or more terms

  Group terms with common factors.
  Factor out the common factor in each group.
  Factor the common factor from the expression.
  Check by multiplying the factors as in[link].

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

Exercise:

Problem: 8, 18

Solution:

2

Exercise:

Problem: 24, 40

Exercise:

Problem: 72, 162

Solution:

18

Exercise:

Problem: 150, 275

Exercise:

Problem: $10a$, 50

Solution:

10

Exercise:

Problem: $5b$, 30

Exercise:

Problem: $3x, 10x^2$

Solution:

x

Exercise:

Problem: $21b^2, 14b$

Exercise:

Problem: $8w^2, 24w^3$

Solution:

$8w^2$

Exercise:

Problem: $30x^2, 18x^3$

Exercise:

Problem: $10p^3q, 12pq^2$

Solution:

$2pq$

Exercise:

Problem: $8a^2b^3, 10ab^2$

Exercise:

Problem: $12m^2n^3, 30m^5n^3$

Solution:

$6m^2n^3$

Exercise:

Problem: $28x^2y^4, 42x^4y^4$

Exercise:

Problem: $10a^3, 12a^2, 14a$

Solution:

$2a$

Exercise:

Problem: $20y^3, 28y^2, 40y$

Exercise:

Problem: $35x^3, 10x^4, 5x^5$

Solution:

$5x^3$

Exercise:

Problem: $27p^2, 45p^3, 9p^4$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

Exercise:

Problem: $4x + 20$

Solution:

$4\left(x+5\right)$

Exercise:

Problem: $8y+16$

Exercise:

Problem: $6m+9$

Solution:

$3\left(2m+3\right)$

Exercise:

Problem: $14p+35$

Exercise:

Problem: $9q+9$

Solution:

$9\left(q+1\right)$

Exercise:

Problem: $7r+7$

Exercise:

Problem: $8m-8$

Solution:

$8\left(m-1\right)$

Exercise:

Problem: $4n-4$

Exercise:

Problem: $9n-63$

Solution:

$9\,(n-7)$

Exercise:

Problem: $45b-18$

Exercise:

Problem: $3x^2+6x-9$

Solution:

$3(x^2+2x-3)$

Exercise:

Problem: $4y^2+8y-4$

Exercise:

Problem: $8p^2+4p+2$

Solution:

$2(4p^2+2p+1)$

Exercise:

Problem: $10q^2+14q+20$

Exercise:

Problem: $8y^3+16y^2$

Solution:

$8y^2(y+2)$

Exercise:

Problem: $12x^3-10x$

Exercise:

Problem: $5x^3-15x^2+20x$

Solution:

$5x(x^2 - 3x + 4)$

Exercise:

Problem: $8m^2 - 40m + 16$

Exercise:

Problem: $12xy^2 + 18x^2y^2 - 30y^3$

Solution:

$6y^2(2x + 3x^2 - 5y)$

Exercise:

Problem: $21pq^2 + 35p^2q^2 - 28q^3$

Exercise:

Problem: $-2x - 4$

Solution:

$-2\left(x + 4\right)$

Exercise:

Problem: $-3b + 12$

Exercise:

Problem: $5x\left(x + 1\right) + 3\left(x + 1\right)$

Solution:

$\left(x + 1\right)\left(5x + 3\right)$

Exercise:

Problem: $2x\left(x - 1\right) + 9\left(x - 1\right)$

Exercise:

Problem: $3b\left(b-2\right)-13\left(b-2\right)$

Solution:

$\left(b-2\right)\left(3b-13\right)$

Exercise:

Problem: $6m\left(m-5\right)-7\left(m-5\right)$

Factor by Grouping

In the following exercises, factor by grouping.

Exercise:

Problem: $xy+2y+3x+6$

Solution:

$(y+3)(x+2)$

Exercise:

Problem: $mn+4n+6m+24$

Exercise:

Problem: $uv-9u+2v-18$

Solution:

$(u+2)(v-9)$

Exercise:

Problem: $pq-10p+8q-80$

Exercise:

Problem: $b^2+5b-4b-20$

Solution:

$(b-4)(b+5)$

Exercise:

Problem: $m^2 + 6m - 12m - 72$

Exercise:

Problem: $p^2 + 4p - 9p - 36$

Solution:

$(p - 9)(p + 4)$

Exercise:

Problem: $x^2 + 5x - 3x - 15$

Mixed Practice

In the following exercises, factor.

Exercise:

Problem: $-20x - 10$

Solution:

$-10\,(2x + 1)$

Exercise:

Problem: $5x^3 - x^2 + x$

Exercise:

Problem: $3x^3 - 7x^2 + 6x - 14$

Solution:

$(x^2 + 2)(3x - 7)$

Exercise:

Problem: $x^3 + x^2 - x - 1$

Exercise:

Problem: $x^2 + xy + 5x + 5y$

Solution:

$(x + y)(x + 5)$

Exercise:

Problem: $5x^3 - 3x^2 - 5x - 3$

Everyday Math

Exercise:

Problem:

Area of a rectangle The area of a rectangle with length 6 less than the width is given by the expression $w^2 - 6w$, where $w =$ width. Factor the greatest common factor from the polynomial.

Solution:

$w(w - 6)$

Exercise:

Problem:

Height of a baseball The height of a baseball *t* seconds after it is hit is given by the expression $-16t^2 + 80t + 4$. Factor the greatest common factor from the polynomial.

Writing Exercises

Exercise:

Problem: The greatest common factor of 36 and 60 is 12. Explain what this means.

Solution:

Answers will vary.

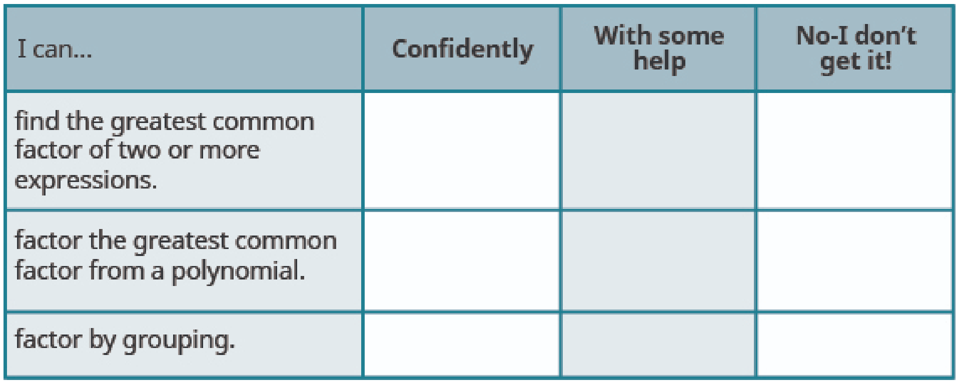
Exercise:

Problem:

What is the GCF of y^4, y^5, and y^{10}? Write a general rule that tells you how to find the GCF of y^a, y^b, andy^c.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
find the greatest common factor of two or more expressions.			
factor the greatest common factor from a polynomial.			
factor by grouping.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential—every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

factoring

Factoring is splitting a product into factors; in other words, it is the reverse process of multiplying.

greatest common factor

The greatest common factor is the largest expression that is a factor of two or more expressions is the greatest common factor (GCF).

Factor Quadratic Trinomials with Leading Coefficient 1

By the end of this section, you will be able to:

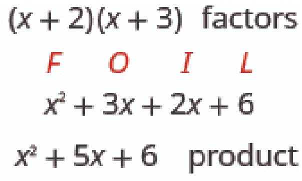
- Factor trinomials of the form $x^2 + bx + c$
- Factor trinomials of the form $x^2 + bxy + cy^2$

Note:
Before you get started, take this readiness quiz.

1. Multiply: $(x+4)(x+5)$.
   If you missed this problem, review [link].
2. Simplify: ⓐ $-9+(-6)$ ⓑ $-9+6$.
   If you missed this problem, review [link].
3. Simplify: ⓐ $-9(6)$ ⓑ $-9(-6)$.
   If you missed this problem, review [link].
4. Simplify: ⓐ $|-5|$ ⓑ $|3|$.
   If you missed this problem, review [link].

Factor Trinomials of the Form $x^2 + bx + c$

You have already learned how to multiply binomials using FOIL. Now you'll need to "undo" this multiplication—to start with the product and end up with the factors. Let's look at an example of multiplying binomials to refresh your memory.



$(x+2)(x+3)$ factors

F O I L

$x^2 + 3x + 2x + 6$

$x^2 + 5x + 6$ product

To factor the trinomial means to start with the product, $x^2 + 5x + 6$, and end with the factors, $(x+2)(x+3)$. You need to think about where each of the terms in the trinomial came from.

The *first term* came from multiplying the first term in each binomial. So to get x^2 in the product, each binomial must start with an *x*.

Equation:

$$\begin{array}{c} x^2+5x+6 \\ (x\quad)(x\quad) \end{array}$$

The *last term* in the trinomial came from multiplying the last term in each binomial. So the last terms must multiply to 6.

What two numbers multiply to 6?

The factors of 6 could be 1 and 6, or 2 and 3. How do you know which pair to use?

Consider the *middle term*. It came from adding the outer and inner terms.

So the numbers that must have a product of 6 will need a sum of 5. We'll test both possibilities and summarize the results in [link]—the table will be very helpful when you work with numbers that can be factored in many different ways.

Factors of 6	**Sum of factors**
$1,6$	$1+6=7$
$2,3$	$2+3=5$

We see that 2 and 3 are the numbers that multiply to 6 and add to 5. So we have the factors of x^2+5x+6. They are $(x+2)(x+3)$.

Equation:

$$\begin{array}{ll} x^2+5x+6 & \text{product} \\ (x+2)(x+3) & \text{factors} \end{array}$$

You should check this by multiplying.

Looking back, we started with x^2+5x+6, which is of the form x^2+bx+c, where $b=5$ and $c=6$. We factored it into two binomials of the form $(x+m)$ and $(x+n)$.

Equation:

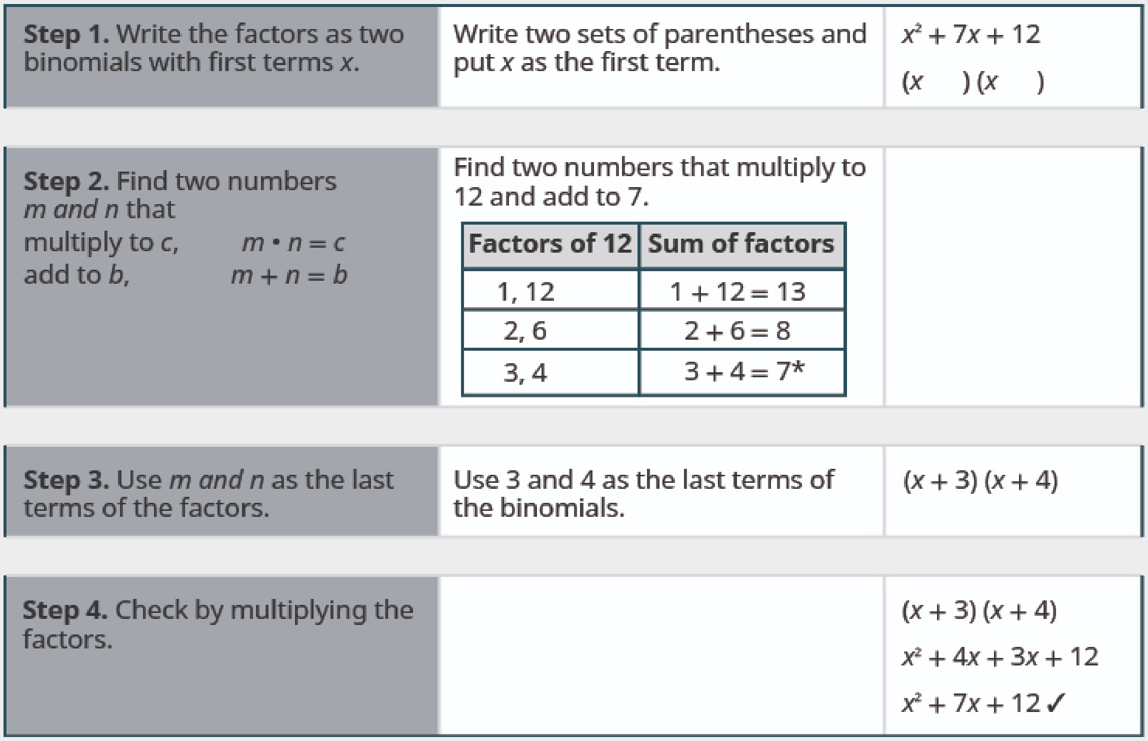
$$\begin{array}{ll} x^2+5x+6 & x^2+bx+c \\ (x+2)(x+3) & (x+m)(x+n) \end{array}$$

To get the correct factors, we found two numbers m and n whose product is c and sum is b.

Example:
How to Factor Trinomials of the Form x^2+bx+c
Exercise:

Problem: Factor: $x^2+7x+12$.

Solution:
Solution



Step 1. Write the factors as two binomials with first terms *x*.	Write two sets of parentheses and put *x* as the first term.	$x^2+7x+12$ $(x\quad)(x\quad)$
Step 2. Find two numbers *m and n* that multiply to *c*, $m \cdot n = c$ add to *b*, $m + n = b$	Find two numbers that multiply to 12 and add to 7.	

Factors of 12	Sum of factors
1, 12	$1+12=13$
2, 6	$2+6=8$
3, 4	$3+4=7$*

Step 3. Use *m and n* as the last terms of the factors.	Use 3 and 4 as the last terms of the binomials.	$(x+3)(x+4)$
Step 4. Check by multiplying the factors.		$(x+3)(x+4)$ $x^2+4x+3x+12$ $x^2+7x+12$ ✓

Note:
Exercise:

Problem: Factor: $x^2 + 6x + 8$.

Solution:

$(x+2)\,(x+4)$

Note:
Exercise:

Problem: Factor: $y^2 + 8y + 15$.

Solution:

$(y+3)\,(y+5)$

Let's summarize the steps we used to find the factors.

Note:
Factor trinomials of the form $x^2 + bx + c$.

Write the factors as two binomials with first terms x: $(x \quad)\,(x \quad)$.
Find two numbers m and n that Multiply to c, $m \cdot n = c$ Add to b, $m + n = b$
Use m and n as the last terms of the factors: $(x+m)(x+n)$.
Check by multiplying the factors.

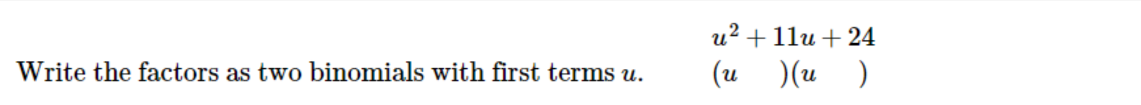
Example:
Exercise:

Problem: Factor: $u^2 + 11u + 24$.

Solution:

Solution

Notice that the variable is u, so the factors will have first terms u.



Write the factors as two binomials with first terms u. $u^2 + 11u + 24$

$(u \quad)(u \quad)$

Find two numbers that: multiply to 24 and add to 11.

Factors of 24	**Sum of factors**
$1, 24$	$1 + 24 = 25$
$2, 12$	$2 + 12 = 14$
$3, 8$	$3 + 8 = 11$*
$4, 6$	$4 + 6 = 10$

Use 3 and 8 as the last terms of the binomials. $(u + 3)\,(u + 8)$

Check.

$$\begin{array}{l} (u+3)\,(u+8) \\ u^2 + 3u + 8u + 24 \\ u^2 + 11u + 24\ \checkmark \end{array}$$

Note:
Exercise:

Problem: Factor: $q^2 + 10q + 24$.

Solution:

$(q+4)(q+6)$

Note:
Exercise:

Problem: Factor: $t^2 + 14t + 24$.

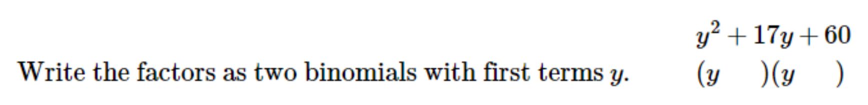
Solution:

$(t+2)(t+12)$

Example:
Exercise:

Problem: Factor: $y^2 + 17y + 60$.

Solution:
Solution



Write the factors as two binomials with first terms y.

$y^2 + 17y + 60$
$(y \quad\ )(y \quad\ )$

Find two numbers that multiply to 60 and add to 17.

Factors of 60	**Sum of factors**
$1, 60$	$1 + 60 = 61$
$2, 30$	$2 + 30 = 32$
$3, 20$	$3 + 20 = 23$

Factors of 60	**Sum of factors**
$4, 15$	$4 + 15 = 19$
$5, 12$	$5 + 12 = 17$*
$6, 10$	$6 + 10 = 16$

Use 5 and 12 as the last terms. $\left(y+5\right)\left(y+12\right)$

Check.

$$(y+5)(y+12)$$
$$(y^2+12y+5y+60)$$
$$(y^2+17y+60)✓$$

Note:
Exercise:

Problem: Factor: $x^2+19x+60$.

Solution:

$\left(x+4\right)\left(x+15\right)$

Note:
Exercise:

Problem: Factor: $v^2+23v+60$.

Solution:

$\left(v+3\right)\left(v+20\right)$

Factor Trinomials of the Form $x^2 + bx + c$ with b Negative, c Positive

In the examples so far, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

Remember: To get a negative sum and a positive product, the numbers must both be negative.

Again, think about FOIL and where each term in the trinomial came from. Just as before,

- the first term, x^2, comes from the product of the two first terms in each binomial factor, x and y;
- the positive last term is the product of the two last terms
- the negative middle term is the sum of the outer and inner terms.

How do you get a *positive product* and a *negative sum*? With two negative numbers.

Example:
Exercise:

Problem: Factor: $t^2 - 11t + 28$.

Solution:
Solution

Again, with the positive last term, 28, and the negative middle term, $-11t$, we need two negative factors. Find two numbers that multiply 28 and add to -11.

$$t^2 - 11t + 28$$

Write the factors as two binomials with first terms t. $\quad (t \quad)(t \quad)$

Find two numbers that: multiply to 28 and add to -11.

Factors of 28	Sum of factors
$-1, -28$	$-1 + (-28) = -29$
$-2, -14$	$-2 + (-14) = -16$
$-4, -7$	$-4 + (-7) = -11$*

Use $-4, -7$ as the last terms of the binomials. $(t-4)(t-7)$

Check.

$$(t-4)(t-7)$$
$$t^2 - 7t - 4t + 28$$
$$t^2 - 11t + 28 \checkmark$$

Note:
Exercise:

Problem: Factor: $u^2 - 9u + 18$.

Solution:

$(u-3)(u-6)$

Note:
Exercise:

Problem: Factor: $y^2 - 16y + 63$.

Solution:

$(y-7)(y-9)$

Factor Trinomials of the Form $x^2 + bx + c$ with *c* Negative

Now, what if the last term in the trinomial is negative? Think about FOIL. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

Remember: To get a negative product, the numbers must have different signs.

Example:
Exercise:

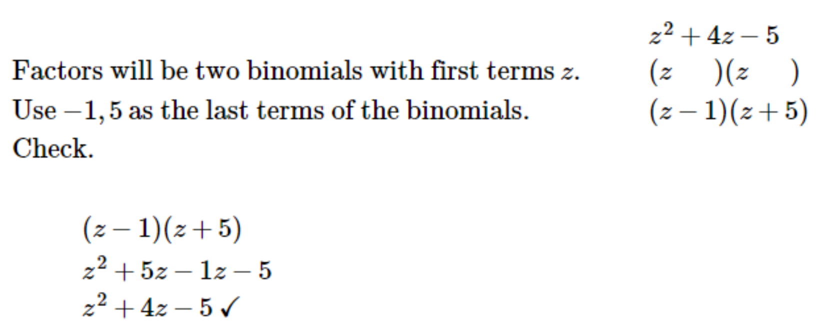
Problem: Factor: $z^2 + 4z - 5$.

Solution:
Solution

To get a negative last term, multiply one positive and one negative. We need factors of -5 that add to positive 4.

Factors of -5	**Sum of factors**
$1, -5$	$1 + (-5) = -4$
$-1, 5$	$-1 + 5 = 4$*

Notice: We listed both $1, -5$ and $-1, 5$ to make sure we got the sign of the middle term correct.



z^2+4z-5

Factors will be two binomials with first terms z. $(z\quad)(z\quad)$

Use $-1,5$ as the last terms of the binomials. $(z-1)(z+5)$

Check.

$$(z-1)(z+5)$$
$$z^2+5z-1z-5$$
$$z^2+4z-5\ ✓$$

Note:
Exercise:

Problem: Factor: $h^2+4h-12$.

Solution:

$(h-2)(h+6)$

Note:
Exercise:

Problem: Factor: k^2+k-20.

Solution:

$(k-4)(k+5)$

Let’s make a minor change to the last trinomial and see what effect it has on the factors.

Example:

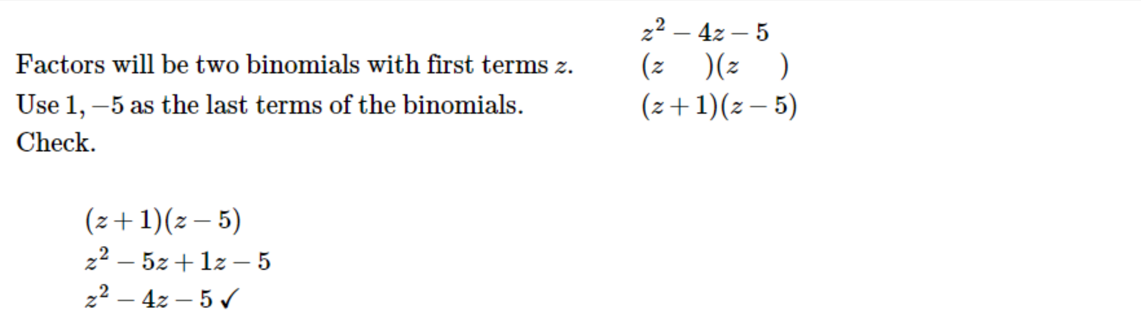
Exercise:

Problem: Factor: $z^2 - 4z - 5$.

Solution:
Solution

This time, we need factors of -5 that add to -4.

Factors of -5	**Sum of factors**
$1, -5$	$1 + (-5) = -4$*
$-1, 5$	$-1 + 5 = 4$



	$z^2 - 4z - 5$
Factors will be two binomials with first terms z.	$(z \quad)(z \quad)$
Use $1, -5$ as the last terms of the binomials.	$(z+1)(z-5)$
Check.	

$$(z+1)(z-5)$$
$$z^2 - 5z + 1z - 5$$
$$z^2 - 4z - 5 \checkmark$$

Notice that the factors of $z^2 - 4z - 5$ are very similar to the factors of $z^2 + 4z - 5$. It is very important to make sure you choose the factor pair that results in the correct sign of the middle term.

Note:
Exercise:

Problem: Factor: $x^2 - 4x - 12$.

Solution:

$(x+2)(x-6)$

Note:
Exercise:

Problem: Factor: y^2-y-20.

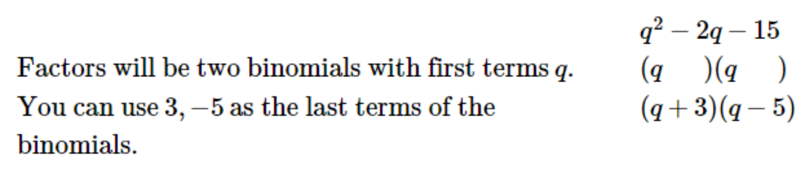
Solution:

$(y+4)(y-5)$

Example:
Exercise:

Problem: Factor: $q^2-2q-15$.

Solution:
Solution



Factors will be two binomials with first terms q.
You can use $3, -5$ as the last terms of the binomials.

$$q^2-2q-15$$
$$(q\quad)(q\quad)$$
$$(q+3)(q-5)$$

Factors of -15	Sum of factors
$1, -15$	$1+(-15)=-14$
$-1, 15$	$-1+15=14$
$3, -5$	$3+(-5)=-2$*

Factors of -15	**Sum of factors**
$-3, 5$	$-3+5=2$

Check.

$$(q+3)(q-5)$$

$$q^2-5q+3q-15$$

$$q^2-2q-15\ ✓$$

Note:
Exercise:

Problem: Factor: $r^2-3r-40$.

Solution:

$(r+5)(r-8)$

Note:
Exercise:

Problem: Factor: $s^2-3s-10$.

Solution:

$(s+2)(s-5)$

Some trinomials are prime. The only way to be certain a trinomial is prime is to list all the possibilities and show that none of them work.

Example:
Exercise:

Problem: Factor: $y^2 - 6y + 15$.

Solution:
Solution

$$y^2 - 6y + 15$$

Factors will be two binomials with first terms y.

$$(y \quad)(y \quad)$$

Factors of 15	**Sum of factors**
$-1, -15$	$-1 + (-15) = -16$
$-3, -5$	$-3 + (-5) = -8$

As shown in the table, none of the factors add to -6; therefore, the expression is prime.

Note:
Exercise:

Problem: Factor: $m^2 + 4m + 18$.

Solution:

prime

Note:
Exercise:

Problem: Factor: $n^2 - 10n + 12$.

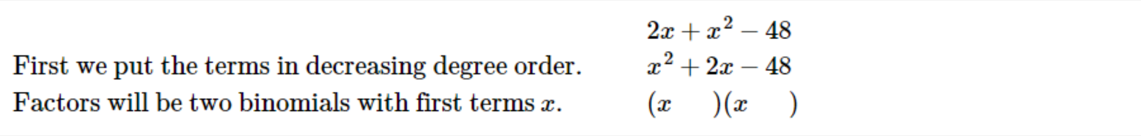
Solution:

prime

Example:
Exercise:

Problem: Factor: $2x + x^2 - 48$.

Solution:
Solution



	$2x + x^2 - 48$
First we put the terms in decreasing degree order.	$x^2 + 2x - 48$
Factors will be two binomials with first terms x.	$(x \quad)(x \quad)$

As shown in the table, you can use $-6, 8$ as the last terms of the binomials.
Equation:

$$(x - 6)(x + 8)$$

Factors of -48	**Sum of factors**
$-1, 48$	$-1 + 48 = 47$

Factors of -48	**Sum of factors**
$-2, 24$ $-3, 16$ $-4, 12$ $-6, 8$	$-2+24=22$ $-3+16=13$ $-4+12=8$ $-6+8=2$

Check.

$$(x-6)(x+8)$$

$$x^2-6q+8q-48$$

$$x^2+2x-48\checkmark$$

Note:
Exercise:

Problem: Factor: $9m+m^2+18$.

Solution:

$(m+3)(m+6)$

Note:
Exercise:

Problem: Factor: $-7n+12+n^2$.

Solution:

$(n-3)(n-4)$

Let’s summarize the method we just developed to factor trinomials of the form $x^2 + bx + c$.

Note:

Factor trinomials.

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

Equation:

$$\begin{array}{c} x^2 + bx + c \\ (x + m)(x + n) \end{array}$$

When *c* is positive, *m* and *n* have the same sign.

Equation:

$$\begin{array}{cc} b \text{ positive} & b \text{ negative} \\ m, n \text{ positive} & m, n \text{ negative} \\ x^2 + 5x + 6 & x^2 - 6x + 8 \\ (x + 2)(x + 3) & (x - 4)(x - 2) \\ \text{same signs} & \text{same signs} \end{array}$$

When *c* is negative, *m* and *n* have opposite signs.

Equation:

$$\begin{array}{cc} x^2 + x - 12 & x^2 - 2x - 15 \\ (x + 4)(x - 3) & (x - 5)(x + 3) \\ \text{opposite signs} & \text{opposite signs} \end{array}$$

Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.

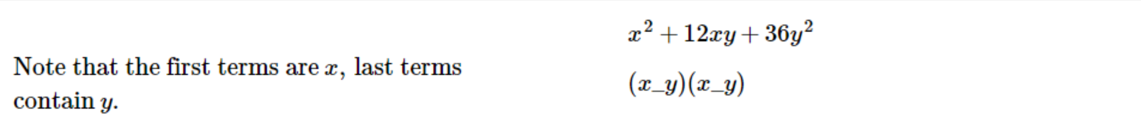
Factor Trinomials of the Form $x^2 + bxy + cy^2$

Sometimes you’ll need to factor trinomials of the form $x^2 + bxy + cy^2$ with two variables, such as $x^2 + 12xy + 36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x \cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain y. To get the coefficients b and c, you use the same process summarized in the previous objective.

Example:
Exercise:

Problem: Factor: $x^2 + 12xy + 36y^2$.

Solution:
Solution



	$x^2 + 12xy + 36y^2$
Note that the first terms are x, last terms contain y.	$(x\_y)(x\_y)$

Find the numbers that multiply to 36 and add to 12.

Factors of 36	**Sum of factors**
$1, 36$	$1 + 36 = 37$
$2, 18$	$2 + 18 = 20$
$3, 12$	$3 + 12 = 15$
$4, 9$	$4 + 9 = 13$
$6, 6$	$6 + 6 = 12$*

Use 6 and 6 as the coefficients of the last terms. $(x + 6y)(x + 6y)$

Check your answer.

$$\begin{gathered}(x + 6y)(x + 6y)\\ x^2 + 6xy + 6xy + 36y^2\\ x^2 + 12xy + 36y^2 \checkmark\end{gathered}$$

Note:
Exercise:

Problem: Factor: $u^2 + 11uv + 28v^2$.

Solution:

$(u + 4v)(u + 7v)$

Note:
Exercise:

Factor: $x^2 + 13xy + 42y^2$.

Problem:

Solution:

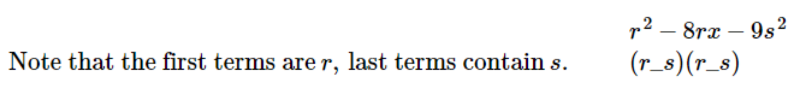
$(x + 6y)(x + 7y)$

Example:
Exercise:

Problem: Factor: $r^2 - 8rx - 9s^2$.

Solution:
Solution

We need r in the first term of each binomial and s in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.



Note that the first terms are r, last terms contain s.

$r^2 - 8rx - 9s^2$

$(r\_s)(r\_s)$

Find the numbers that multiply to -9 and add to -8.

Factors of -9	**Sum of factors**
$1, -9$	$1 + (-9) = -8$*
$-1, 9$	$-1 + 9 = 8$
$3, -3$	$3 + (-3) = 0$

Use $1, -9$ as coefficients of the last terms. $(r + s)(r - 9s)$

Check your answer.

$$(r - 9s)(r + s)$$
$$r^2 + rs - 9rs - 9s^2$$
$$r^2 - 8rs - 9s^2 \checkmark$$

Note:
Exercise:

Problem: Factor: $a^2 - 11ab + 10b^2$.

Solution:

$(a - b)(a - 10b)$

Note:
Exercise:

Problem: Factor: $m^2 - 13mn + 12n^2$.

Solution:

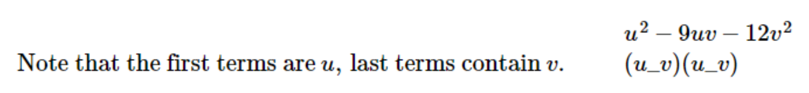
$(m - n)(m - 12n)$

Example:
Exercise:

Problem: Factor: $u^2 - 9uv - 12v^2$.

Solution:
Solution

We need u in the first term of each binomial and v in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.



Note that the first terms are u, last terms contain v.

$u^2 - 9uv - 12v^2$
$(u\_v)(u\_v)$

Find the numbers that multiply to -12 and add to -9.

Factors of -12	**Sum of factors**
$1, -12$	$1 + (-12) = -11$

Factors of -12	**Sum of factors**
$-1, 12$	$-1+12=11$
$2, -6$	$2+(-6)=-4$
$-2, 6$	$-2+6=4$
$3, -4$	$3+(-4)=-1$
$-3, 4$	$-3+4=1$

Note there are no factor pairs that give us -9 as a sum. The trinomial is prime.

Note:
Exercise:

Problem: Factor: $x^2-7xy-10y^2$.

Solution:

prime

Note:
Exercise:

Problem: Factor: $p^2+15pq+20q^2$.

Solution:

prime

Key Concepts

- **Factor trinomials of the form $x^2 + bx + c$**

  Write the factors as two binomials with first terms x: $(x\quad)\,(x\quad)$.
  Find two numbers m and n that Multiply to c, $m \cdot n = c$ Add to b, $m + n = b$
  Use m and n as the last terms of the factors: $(x + m)(x + n)$.
  Check by multiplying the factors.

Practice Makes Perfect

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

Exercise:

Problem: $x^2 + 4x + 3$

Solution:

$(x + 1)\,(x + 3)$

Exercise:

Problem: $y^2 + 8y + 7$

Exercise:

Problem: $m^2 + 12m + 11$

Solution:

$(m + 1)\,(m + 11)$

Exercise:

Problem: $b^2 + 14b + 13$

Exercise:

Problem: $a^2 + 9a + 20$

Solution:

$(a+4)(a+5)$

Exercise:

Problem: $m^2+7m+12$

Exercise:

Problem: $p^2+11p+30$

Solution:

$(p+5)(p+6)$

Exercise:

Problem: $w^2+10x+21$

Exercise:

Problem: $n^2+19n+48$

Solution:

$(n+3)(n+16)$

Exercise:

Problem: $b^2+14b+48$

Exercise:

Problem: $a^2+25a+100$

Solution:

$(a+5)(a+20)$

Exercise:

Problem: $u^2+101u+100$

Exercise:

Problem: $x^2 - 8x + 12$

Solution:

$(x - 2)(x - 6)$

Exercise:

Problem: $q^2 - 13q + 36$

Exercise:

Problem: $y^2 - 18x + 45$

Solution:

$(y - 3)(y - 15)$

Exercise:

Problem: $m^2 - 13m + 30$

Exercise:

Problem: $x^2 - 8x + 7$

Solution:

$(x - 1)(x - 7)$

Exercise:

Problem: $y^2 - 5y + 6$

Exercise:

Problem: $p^2 + 5p - 6$

Solution:

$(p - 1)(p + 6)$

Exercise:

Problem: $n^2 + 6n - 7$

Exercise:

Problem: $y^2 - 6y - 7$

Solution:

$(y + 1)\,(y - 7)$

Exercise:

Problem: $v^2 - 2v - 3$

Exercise:

Problem: $x^2 - x - 12$

Solution:

$(x - 4)(x + 3)$

Exercise:

Problem: $r^2 - 2r - 8$

Exercise:

Problem: $a^2 - 3a - 28$

Solution:

$(a - 7)(a + 4)$

Exercise:

Problem: $b^2 - 13b - 30$

Exercise:

Problem: $w^2 - 5w - 36$

Solution:

$(w-9)(w+4)$

Exercise:

Problem: $t^2-3t-54$

Exercise:

Problem: x^2+x+5

Solution:

prime

Exercise:

Problem: x^2-3x-9

Exercise:

Problem: $8-6x+x^2$

Solution:

$(x-4)(x-2)$

Exercise:

Problem: $7x+x^2+6$

Exercise:

Problem: $x^2-12-11x$

Solution:

$(x-12)(x+1)$

Exercise:

Problem: $-11-10x+x^2$

Factor Trinomials of the Form $x^2 + bxy + cy^2$

In the following exercises, factor each trinomial of the form $x^2 + bxy + cy^2$.

Exercise:

Problem: $p^2 + 3pq + 2q^2$

Solution:

$(p + q)(p + 2q)$

Exercise:

Problem: $m^2 + 6mn + 5n^2$

Exercise:

Problem: $r^2 + 15rs + 36s^2$

Solution:

$(r + 3s)(r + 12s)$

Exercise:

Problem: $u^2 + 10uv + 24v^2$

Exercise:

Problem: $m^2 - 12mn + 20n^2$

Solution:

$(m - 2n)(m - 10n)$

Exercise:

Problem: $p^2 - 16pq + 63q^2$

Exercise:

Problem: $x^2 - 2xy - 80y^2$

Solution:

$(x+8y)(x-10y)$

Exercise:

Problem: $p^2-8pq-65q^2$

Exercise:

Problem: $m^2-64mn-65n^2$

Solution:

$(m+n)(m-65n)$

Exercise:

Problem: $p^2-2pq-35q^2$

Exercise:

Problem: $a^2+5ab-24b^2$

Solution:

$(a+8b)\,(a-3b)$

Exercise:

Problem: $r^2+3rs-28s^2$

Exercise:

Problem: $x^2-3xy-14y^2$

Solution:

prime

Exercise:

Problem: $u^2-8uv-24v^2$

Exercise:

Problem: $m^2 - 5mn + 30n^2$

Solution:

prime

Exercise:

Problem: $c^2 - 7cd + 18d^2$

Mixed Practice

In the following exercises, factor each expression.

Exercise:

Problem: $u^2 - 12u + 36$

Solution:

$(u - 6)(u - 6)$

Exercise:

Problem: $w^2 + 4w - 32$

Exercise:

Problem: $x^2 - 14x - 32$

Solution:

$(x + 2)(x - 16)$

Exercise:

Problem: $y^2 + 41y + 40$

Exercise:

Problem: $r^2 - 20rs + 64s^2$

Solution:

$(r - 4s)(r - 16s)$

Exercise:

Problem: $x^2 - 16xy + 64y^2$

Exercise:

Problem: $k^2 + 34k + 120$

Solution:

$(k + 4)(k + 30)$

Exercise:

Problem: $m^2 + 29m + 120$

Exercise:

Problem: $y^2 + 10y + 15$

Solution:

prime

Exercise:

Problem: $z^2 - 3z + 28$

Exercise:

Problem: $m^2 + mn - 56n^2$

Solution:

$(m + 8n)(m - 7n)$

Exercise:

Problem: $q^2 - 29qr - 96r^2$

Exercise:

Problem: $u^2 - 17uv + 30v^2$

Solution:

$(u - 15v)(u - 2v)$

Exercise:

Problem: $m^2 - 31mn + 30n^2$

Exercise:

Problem: $c^2 - 8cd + 26d^2$

Solution:

prime

Exercise:

Problem: $r^2 + 11rs + 36s^2$

Everyday Math

Exercise:

Problem:

Consecutive integers Deirdre is thinking of two consecutive integers whose product is 56. The trinomial $x^2 + x - 56$ describes how these numbers are related. Factor the trinomial.

Solution:

$(x + 8)(x - 7)$

Exercise:

Problem:

Consecutive integers Deshawn is thinking of two consecutive integers whose product is 182. The trinomial $x^2 + x - 182$ describes how these numbers are related. Factor the trinomial.

Writing Exercises

Exercise:

Problem:

Many trinomials of the form $x^2 + bx + c$ factor into the product of two binomials $(x + m)(x + n)$. Explain how you find the values of m and n.

Solution:

Answers may vary

Exercise:

Problem:

How do you determine whether to use plus or minus signs in the binomial factors of a trinomial of the form $x^2 + bx + c$ where b and c may be positive or negative numbers?

Exercise:

Problem:

Will factored $x^2 - x - 20$ as $(x + 5)(x - 4)$. Bill factored it as $(x + 4)(x - 5)$. Phil factored it as $(x - 5)(x - 4)$. Who is correct? Explain why the other two are wrong.

Solution:

Answers may vary

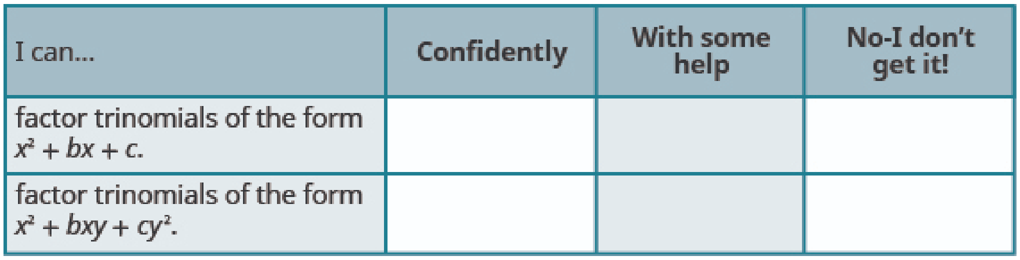
Exercise:

Problem:

Look at [link], where we factored $y^2 + 17y + 60$. We made a table listing all pairs of factors of 60 and their sums. Do you find this kind of table helpful? Why or why not?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
factor trinomials of the form $x^2 + bx + c$.			
factor trinomials of the form $x^2 + bxy + cy^2$.			

ⓑ After reviewing this checklist, what will you do to become confident for all goals?

Factor Quadratic Trinomials with Leading Coefficient Other than 1

By the end of this section, you will be able to:

- Recognize a preliminary strategy to factor polynomials completely
- Factor trinomials of the form $ax^2 + bx + c$ with a GCF
- Factor trinomials using trial and error
- Factor trinomials using the ‘ac’ method

Note:
Before you get started, take this readiness quiz.

1. Find the GCF of $45p^2$ and $30p^6$.
   If you missed this problem, review [link].
2. Multiply $(3y + 4)(2y + 5)$.
   If you missed this problem, review [link].
3. Combine like terms $12x^2 + 3x + 5x + 9$.
   If you missed this problem, review [link].

Recognize a Preliminary Strategy for Factoring

Let’s summarize where we are so far with factoring polynomials. In the first two sections of this chapter, we used three methods of factoring: factoring the GCF, factoring by grouping, and factoring a trinomial by “undoing” FOIL. More methods will follow as you continue in this chapter, as well as later in your studies of algebra.

How will you know when to use each factoring method? As you learn more methods of factoring, how will you know when to apply each method and not get them confused? It will help to organize the factoring methods into a strategy that can guide you to use the correct method.

As you start to factor a polynomial, always ask first, “Is there a greatest common factor?” If there is, factor it first.

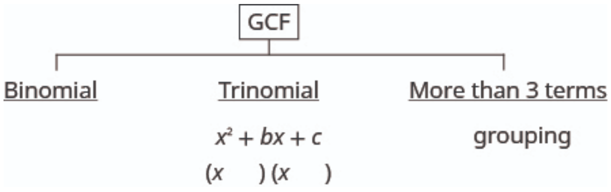
The next thing to consider is the type of polynomial. How many terms does it have? Is it a binomial? A trinomial? Or does it have more than three terms?

If it is a trinomial where the leading coefficient is one, $x^2 + bx + c$, use the “undo FOIL” method.

If it has more than three terms, try the grouping method. This is the only method to use for polynomials of more than three terms.

Some polynomials cannot be factored. They are called “prime.”

Below we summarize the methods we have so far. These are detailed in Choose a strategy to factor polynomials completely.



Note:
Choose a strategy to factor polynomials completely.

Is there a greatest common factor?
- Factor it out.

Is the polynomial a binomial, trinomial, or are there more than three terms?
- If it is a binomial, right now we have no method to factor it.
- If it is a trinomial of the form $x^2 + bx + c$: Undo FOIL $(x \quad)(x \quad)$
- If it has more than three terms: Use the grouping method.

Check by multiplying the factors.

Use the preliminary strategy to completely factor a polynomial. A polynomial is factored completely if, other than monomials, all of its factors are prime.

Example:
Exercise:

Problem: Identify the best method to use to factor each polynomial.

ⓐ $6y^2 - 72$
ⓑ $r^2 - 10r - 24$
ⓒ $p^2 + 5p + pq + 5q$

Solution:
Solution

ⓐ

	$6y^2 - 72$
Is there a greatest common factor?	Yes, 6.
Factor out the 6.	$6\left(y^2 - 12\right)$
Is it a binomial, trinomial, or are there more than 3 terms?	Binomial, we have no method to factor binomials yet.

ⓑ

	$r^2 - 10r - 24$
Is there a greatest common factor?	No, there is no common factor.
Is it a binomial, trinomial, or are there more than three terms?	Trinomial, with leading coefficient 1, so “undo” FOIL.

ⓒ	$p^2 + 5p + pq + 5q$
Is there a greatest common factor?	No, there is no common factor.
Is it a binomial, trinomial, or are there more than three terms?	More than three terms, so factor using grouping.

Note:
Exercise:

Problem: Identify the best method to use to factor each polynomial:

ⓐ $4y^2 + 32$
ⓑ $y^2 + 10y + 21$
ⓒ $yz + 2y + 3z + 6$

Solution:

ⓐ no method ⓑ undo using FOIL ⓒ factor with grouping

Note:
Exercise:

Problem: Identify the best method to use to factor each polynomial:

ⓐ $ab + a + 4b + 4$
ⓑ $3k^2 + 15$
ⓒ $p^2 + 9p + 8$

Solution:

ⓐ factor using grouping ⓑ no method ⓒ undo using FOIL

Factor Trinomials of the form $ax^2 + bx + c$ with a GCF

Now that we have organized what we've covered so far, we are ready to factor trinomials whose leading coefficient is not 1, trinomials of the form $ax^2 + bx + c$.

Remember to always check for a GCF first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods in the last section. Let's do a few examples to see how this works.

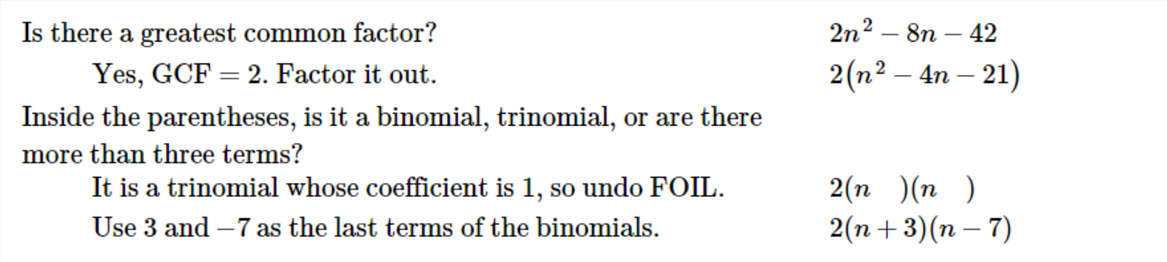
Watch out for the signs in the next two examples.

Example:
Exercise:

Problem: Factor completely: $2n^2 - 8n - 42$.

Solution:
Solution

Use the preliminary strategy.



Is there a greatest common factor?	$2n^2 - 8n - 42$
Yes, GCF $= 2$. Factor it out.	$2(n^2 - 4n - 21)$
Inside the parentheses, is it a binomial, trinomial, or are there more than three terms?	
It is a trinomial whose coefficient is 1, so undo FOIL.	$2(n \quad)(n \quad)$
Use 3 and -7 as the last terms of the binomials.	$2(n+3)(n-7)$

Factors of -21	Sum of factors
$1, -21$	$1 + (-21) = -20$
$3, -7$	$3 + (-7) = -4$*

Check.

$$2(n+3)(n-7)$$

$$2(n^2 - 7n + 3n - 21)$$

$$2(n^2 - 4n - 21)$$

$$2n^2 - 8n - 42 \checkmark$$

Note:
Exercise:

Problem: Factor completely: $4m^2 - 4m - 8$.

Solution:

$4(m+1)(m-2)$

Note:
Exercise:

Problem: Factor completely: $5k^2 - 15k - 50$.

Solution:

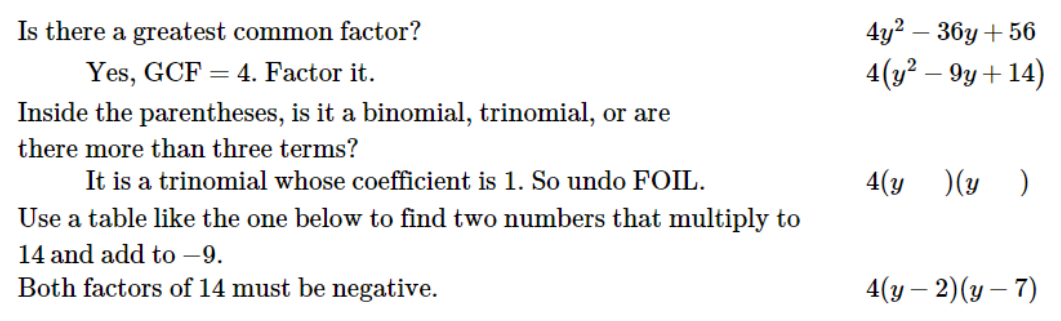
$5\left(k+2\right)\left(k-5\right)$

Example:
Exercise:

Problem: Factor completely: $4y^2 - 36y + 56$.

Solution:
Solution

Use the preliminary strategy.



Is there a greatest common factor?	$4y^2 - 36y + 56$
Yes, GCF $= 4$. Factor it.	$4\left(y^2 - 9y + 14\right)$
Inside the parentheses, is it a binomial, trinomial, or are there more than three terms?	
It is a trinomial whose coefficient is 1. So undo FOIL.	$4(y \quad)(y \quad)$
Use a table like the one below to find two numbers that multiply to 14 and add to -9.	
Both factors of 14 must be negative.	$4(y-2)(y-7)$

Factors of 14	**Sum of factors**
$-1, -14$	$-1 + (-14) = -15$
$-2, -7$	$-2 + (-7) = -9$*

Check.

$4\left(y-2\right)\left(y-7\right)$

$4(y^2 - 7y - 2y + 14)$

$4(y^2 - 9y + 14)$

$4y^2 - 36y + 42$ ✓

Note:
Exercise:

Problem: Factor completely: $3r^2 - 9r + 6$.

Solution:

$3\left(r-1\right)\left(r-2\right)$

Note:
Exercise:

Problem: Factor completely: $2t^2 - 10t + 12$.

Solution:

$2\left(t-2\right)\left(t-3\right)$

In the next example the GCF will include a variable.

Example:
Exercise:

Problem: Factor completely: $4u^3 + 16u^2 - 20u$.

Solution:
Solution

Use the preliminary strategy.

$$\begin{array}{ll}\text{Is there a greatest common factor?} & 4u^3+16u^2-20u \\ \qquad \text{Yes, GCF} = 4u.\ \text{Factor it.} & 4u\left(u^2+4u-5\right) \\ \text{Binomial, trinomial, or more than three terms?} & \\ \qquad \text{It is a trinomial. So “undo FOIL.”} & 4u\left(u\quad\right)\left(u\quad\right) \\ \text{Use a table like the table below to find two numbers that} & 4u\left(u-1\right)\left(u+5\right) \\ \text{multiply to } -5 \text{ and add to } 4. & \end{array}$$

Factors of -5	**Sum of factors**
$-1, 5$	$-1+5=4$*

Factors of -5	**Sum of factors**
$1, -5$	$1 + (-5) = -4$

Check.

$$4u\left(u-1\right)\left(u+5\right)$$

$$4u(u^2+5u-u-5)$$

$$4u(u^2+4u-5)$$

$$4u^3+16u^2-20u \checkmark$$

Note:
Exercise:

Problem: Factor completely: $5x^3+15x^2-20x$.

Solution:

$5x\left(x-1\right)\left(x+4\right)$

Note:
Exercise:

Problem: Factor completely: $6y^3+18y^2-60y$.

Solution:

$6y\left(y-2\right)\left(y+5\right)$

Factor Trinomials using Trial and Error

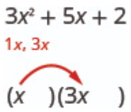
What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let’s factor the trinomial $3x^2+5x+2$.

From our earlier work we expect this will factor into two binomials.
Equation:

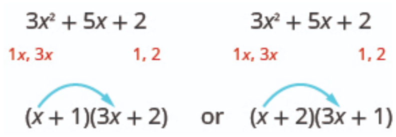
$$\begin{array}{c}3x^2+5x+2\\ (\qquad)(\qquad)\end{array}$$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only factors of $3x^2$ are $1x, 3x$. We can place them in the binomials.

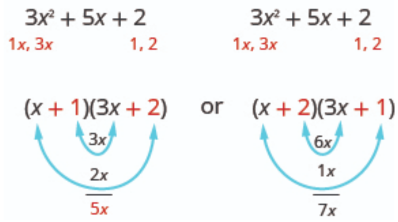


Check. Does $1x \cdot 3x = 3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1 and 2. But we now have two cases to consider as it will make a difference if we write 1, 2, or 2, 1.



Which factors are correct? To decide that, we multiply the inner and outer terms.



Since the middle term of the trinomial is 5*x*, the factors in the first case will work. Let's FOIL to check.
Equation:

$$\begin{gathered}(x+1)\,(3x+2)\\ 3x^2+2x+3x+2\\ 3x^2+5x+2\,\checkmark\end{gathered}$$

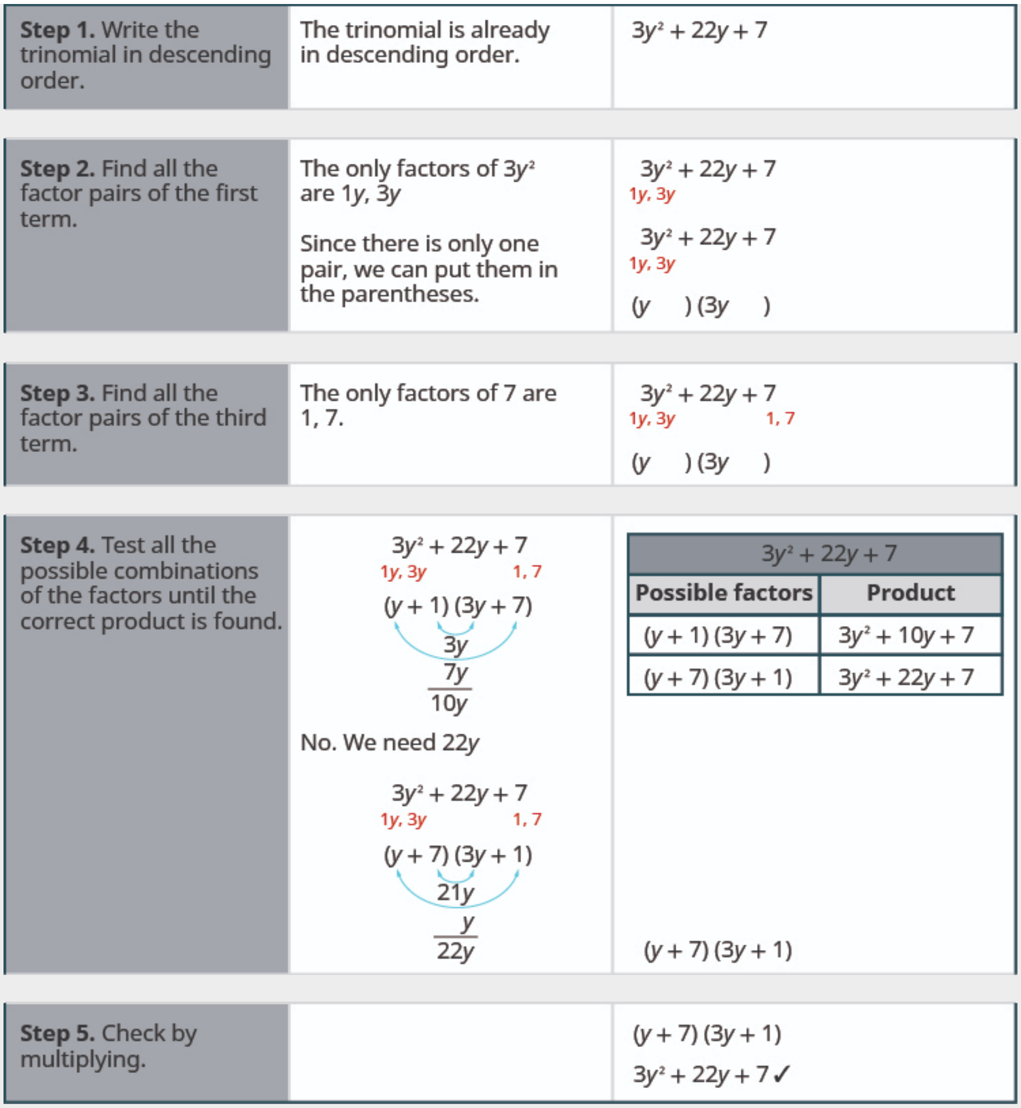
Our result of the factoring is:
Equation:

$$\begin{gathered}3x^2+5x+2\\ (x+1)\,(3x+2)\end{gathered}$$

Example:
How to Factor Trinomials of the Form ax^2+bx+c Using Trial and Error
Exercise:

Problem: Factor completely: $3y^2+22y+7$.

Solution:
Solution



Step 1. Write the trinomial in descending order.	The trinomial is already in descending order.	$3y^2 + 22y + 7$
Step 2. Find all the factor pairs of the first term.	The only factors of $3y^2$ are $1y$, $3y$ Since there is only one pair, we can put them in the parentheses.	$\underset{1y,\ 3y}{3y^2} + 22y + 7$ $\underset{1y,\ 3y}{3y^2} + 22y + 7$ $(y \quad)(3y \quad)$
Step 3. Find all the factor pairs of the third term.	The only factors of 7 are 1, 7.	$\underset{1y,\ 3y}{3y^2} + 22y + \underset{1,\ 7}{7}$ $(y \quad)(3y \quad)$
Step 4. Test all the possible combinations of the factors until the correct product is found.	$\underset{1y,\ 3y}{3y^2} + 22y + \underset{1,\ 7}{7}$ $(y + 1)(3y + 7)$ $3y$ $\underline{7y}$ $10y$ No. We need $22y$ $\underset{1y,\ 3y}{3y^2} + 22y + \underset{1,\ 7}{7}$ $(y + 7)(3y + 1)$ $21y$ $\underline{y}$ $22y$	$(y + 7)(3y + 1)$
Step 5. Check by multiplying.		$(y + 7)(3y + 1)$ $3y^2 + 22y + 7$ ✓

$3y^2 + 22y + 7$	
Possible factors	**Product**
$(y + 1)(3y + 7)$	$3y^2 + 10y + 7$
$(y + 7)(3y + 1)$	$3y^2 + 22y + 7$

Note:
Exercise:

Problem: Factor completely: $2a^2 + 5a + 3$.

Solution:

$(a + 1)(2a + 3)$

Note:
Exercise:

Problem: Factor completely: $4b^2 + 5b + 1$.

Solution:

$(b+1)\,(4b+1)$

Note:
Factor trinomials of the form ax^2+bx+c using trial and error.

Write the trinomial in descending order of degrees.
Find all the factor pairs of the first term.
Find all the factor pairs of the third term.
Test all the possible combinations of the factors until the correct product is found.
Check by multiplying.

When the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

Example:
Exercise:

Problem: Factor completely: $6b^2-13b+5$.

Solution:
Solution

The trinomial is already in descending order.	$6b^2-13b+5$
Find the factors of the first term.	$6b^2-13b+5$ $1b\cdot 6b$ $2b\cdot 3b$
Find the factors of the last term. Consider the signs. Since the last term, 5 is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.	$6b^2-13b+5$ $1b\cdot 6b$ $-1,-5$ $2b\cdot 3b$

Consider all the combinations of factors.

$6b^2-13b+5$

$6b^2 - 13b + 5$

Possible factors	**Product**

Possible factors	**Product**
$(b-1)(6b-5)$	$6b^2 - 11b + 5$
$(b-5)(6b-1)$	$6b^2 - 31b + 5$
$(2b-1)(3b-5)$	$6b^2 - 13b + 5$ *
$(2b-5)(3b-1)$	$6b^2 - 17b + 5$

The correct factors are those whose product is the original trinomial. $(2b-1)(3b-5)$

Check by multiplying.

$$\begin{array}{l} (2b-1)(3b-5) \\ 6b^2 - 10b - 3b + 5 \\ 6b^2 - 13b + 5 \checkmark \end{array}$$

Note:
Exercise:

Problem: Factor completely: $8x^2 - 13x + 3$.

Solution:

$(2x-3)(4x-1)$

Note:
Exercise:

Problem: Factor completely: $10y^2 - 37y + 7$.

Solution:

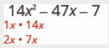
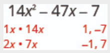
$(2y-7)(5y-1)$

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

Example:
Exercise:

Problem: Factor completely: $14x^2 - 47x - 7$.

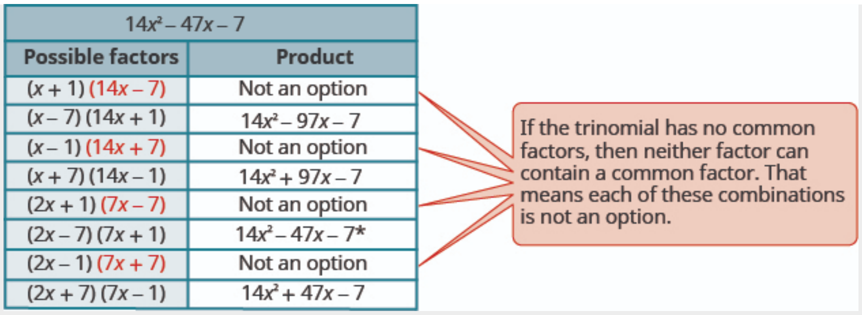
Solution:
Solution

The trinomial is already in descending order.	$14x^2 - 47x - 7$
Find the factors of the first term.	 $14x^2 - 47x - 7$ $1x \cdot 14x$ $2x \cdot 7x$
Find the factors of the last term. Consider the signs. Since it is negative, one factor must be positive and one negative.	 $14x^2 - 47x - 7$ $1x \cdot 14x$ $1, -7$ $2x \cdot 7x$ $-1, 7$

Consider all the combinations of factors. We use each pair of the factors of $14x^2$ with each pair of factors of -7.

Factors of $14x^2$	Pair with	Factors of -7
x, $14x$		1, -7 -7, 1 (reverse order)
x, $14x$		-1, 7 7, -1 (reverse order)
$2x, 7x$		1, -7 -7, 1 (reverse order)
$2x, 7x$		-1, 7 7, -1 (reverse order)

These pairings lead to the following eight combinations.



$14x^2 - 47x - 7$	
Possible factors	**Product**
$(x + 1)(14x - 7)$	Not an option
$(x - 7)(14x + 1)$	$14x^2 - 97x - 7$
$(x - 1)(14x + 7)$	Not an option
$(x + 7)(14x - 1)$	$14x^2 + 97x - 7$
$(2x + 1)(7x - 7)$	Not an option
$(2x - 7)(7x + 1)$	$14x^2 - 47x - 7$*
$(2x - 1)(7x + 7)$	Not an option
$(2x + 7)(7x - 1)$	$14x^2 + 47x - 7$

If the trinomial has no common factors, then neither factor can contain a common factor. That means each of these combinations is not an option.

The correct factors are those whose product is the original trinomial. $(2x - 7)(7x + 1)$

Check by multiplying.

$$(2x - 7)(7x + 1)$$

$$14x^2 + 2x - 49x - 7$$

$$14x^2 - 47x - 7 \checkmark$$

Note:
Exercise:

Problem: Factor completely: $8a^2 - 3a - 5$.

Solution:

$(a - 1)(8a + 5)$

Note:
Exercise:

Problem: Factor completely: $6b^2 - b - 15$.

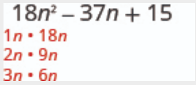
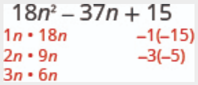
Solution:

$(2b + 3)(3b - 5)$

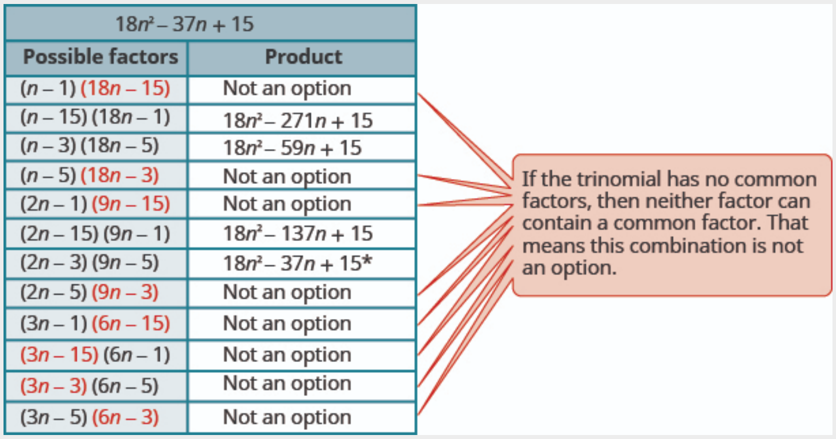
Example:
Exercise:

Problem: Factor completely: $18n^2 - 37n + 15$.

Solution:
Solution

The trinomial is already in descending order.	$18n^2 - 37n + 15$
Find the factors of the first term.	 $18n^2 - 37n + 15$ $1n \cdot 18n$ $2n \cdot 9n$ $3n \cdot 6n$
Find the factors of the last term. Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative facotrs.	 $18n^2 - 37n + 15$ $1n \cdot 18n \quad -1(-15)$ $2n \cdot 9n \quad -3(-5)$ $3n \cdot 6n$

Consider all the combinations of factors.



$18n^2 - 37n + 15$	
Possible factors	**Product**
$(n-1)(18n-15)$	Not an option
$(n-15)(18n-1)$	$18n^2 - 271n + 15$
$(n-3)(18n-5)$	$18n^2 - 59n + 15$
$(n-5)(18n-3)$	Not an option
$(2n-1)(9n-15)$	Not an option
$(2n-15)(9n-1)$	$18n^2 - 137n + 15$
$(2n-3)(9n-5)$	$18n^2 - 37n + 15$*
$(2n-5)(9n-3)$	Not an option
$(3n-1)(6n-15)$	Not an option
$(3n-15)(6n-1)$	Not an option
$(3n-3)(6n-5)$	Not an option
$(3n-5)(6n-3)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial.

$(2n-3)(9n-5)$

Check by multiplying.

$$\begin{gathered}(2n-3)(9n-5)\\ 18n^2 - 10n - 27n + 15\\ 18n^2 - 37n + 15 \checkmark\end{gathered}$$

Note:
Exercise:

Problem: Factor completely: $18x^2 - 3x - 10$.

Solution:

$(3x+2)(6x-5)$

Note:
Exercise:

Problem: Factor completely: $30y^2 - 53y - 21$.

Solution:

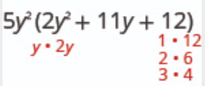
$(3y + 1)(10y - 21)$

Don’t forget to look for a GCF first.

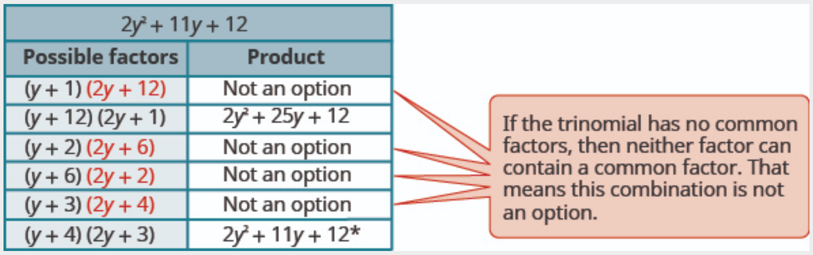
Example:
Exercise:

Problem: Factor completely: $10y^4 + 55y^3 + 60y^2$.

Solution:
Solution

	$10y^4 + 55y^3 + 60y^2$
Notice the greatest common factor, and factor it first.	$5y^2\left(2y^2 + 11y + 12\right)$
Factor the trinomial.	 $5y^2(2y^2 + 11y + 12)$ $y \cdot 2y$ $1 \cdot 12$ $2 \cdot 6$ $3 \cdot 4$

Consider all the combinations.



$2y^2 + 11y + 12$	
Possible factors	**Product**
$(y + 1)(2y + 12)$	Not an option
$(y + 12)(2y + 1)$	$2y^2 + 25y + 12$
$(y + 2)(2y + 6)$	Not an option
$(y + 6)(2y + 2)$	Not an option
$(y + 3)(2y + 4)$	Not an option
$(y + 4)(2y + 3)$	$2y^2 + 11y + 12$*

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. Remember to include the factor $5y^2$.
Check by multiplying.

$5y^2\left(y+4\right)\left(2y+3\right)$

$$5y^2\left(y+4\right)\left(2y+3\right)$$
$$5y^2\left(2y^2+8y+3y+12\right)$$
$$10y^4+55y^3+60y^2 \checkmark$$

Note:
Exercise:

Problem: Factor completely: $15n^3-85n^2+100n$.

Solution:

$5n\left(n-4\right)\left(3n-5\right)$

Note:
Exercise:

Problem: Factor completely: $56q^3+320q^2-96q$.

Solution:

$8q\left(q+6\right)\left(7q-2\right)$

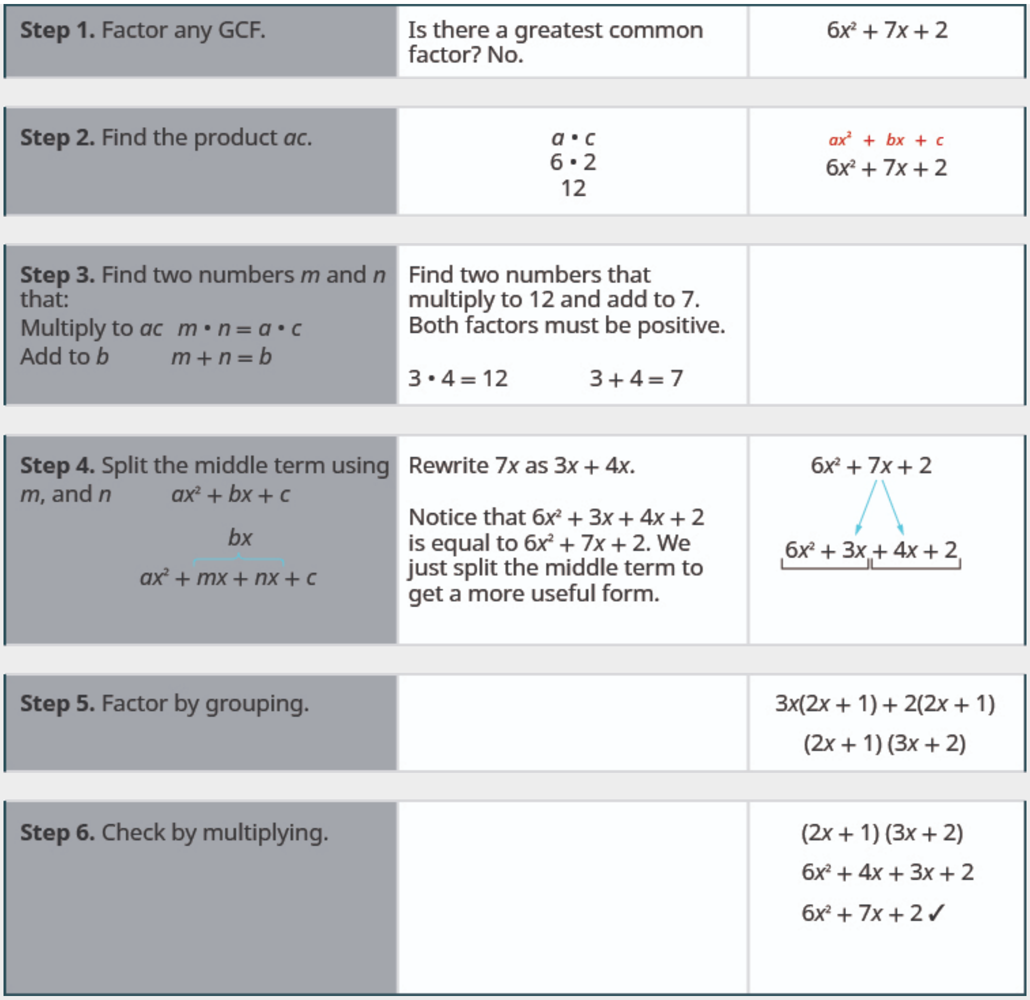
Factor Trinomials using the “ac” Method

Another way to factor trinomials of the form ax^2+bx+c is the “ac” method. (The “ac” method is sometimes called the grouping method.) The “ac” method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

Example:
How to Factor Trinomials Using the “ac” Method
Exercise:

Problem: Factor: $6x^2+7x+2$.

Solution:
Solution



Step 1. Factor any GCF.	Is there a greatest common factor? No.	$6x^2 + 7x + 2$
Step 2. Find the product *ac*.	$a \cdot c$ $6 \cdot 2$ 12	$ax^2 + bx + c$ $6x^2 + 7x + 2$
Step 3. Find two numbers *m* and *n* that: Multiply to *ac* $m \cdot n = a \cdot c$ Add to *b* $m + n = b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12$ $3 + 4 = 7$	
Step 4. Split the middle term using *m*, and *n* $ax^2 + bx + c$ $ax^2 + \overbrace{mx + nx}^{bx} + c$	Rewrite 7*x* as 3*x* + 4*x*. Notice that $6x^2 + 3x + 4x + 2$ is equal to $6x^2 + 7x + 2$. We just split the middle term to get a more useful form.	$6x^2 + 7x + 2$ $\underbrace{6x^2 + 3x} + \underbrace{4x + 2}$
Step 5. Factor by grouping.		$3x(2x + 1) + 2(2x + 1)$ $(2x + 1)(3x + 2)$
Step 6. Check by multiplying.		$(2x + 1)(3x + 2)$ $6x^2 + 4x + 3x + 2$ $6x^2 + 7x + 2$ ✓

Note:
Exercise:

Problem: Factor: $6x^2 + 13x + 2$.

Solution:

$(x + 2)(6x + 1)$

Note:
Exercise:

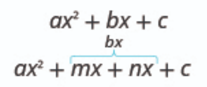
Problem: Factor: $4y^2 + 8y + 3$.

Solution:

$(2y+1)\,(2y+3)$

Note:
Factor trinomials using the "AC" method.

Factor any GCF.
Find the product ac.
Find two numbers m and n that: Multiply to ac $\quad m \cdot n = a \cdot c$
Add to b $\quad m + n = b$
Split the middle term using m and n:



$$ax^2 + bx + c$$
$$ax^2 + \overbrace{mx + nx}^{bx} + c$$

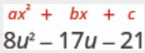
Factor by grouping.
Check by multiplying the factors.

When the third term of the trinomial is negative, the factors of the third term will have opposite signs.

Example:
Exercise:

Problem: Factor: $8u^2 - 17u - 21$.

Solution:
Solution

Is there a greatest common factor? No.		 $\overset{ax^2}{8u^2} \overset{+\ bx}{-\ 17u} \overset{+\ c}{-\ 21}$
Find $a \cdot c$.	$a \cdot c$	
	$8(-21)$	
	-168	

Find two numbers that multiply to -168 and add to -17. The larger factor must be negative.

Factors of -168	**Sum of factors**
$1, -168$	$1 + (-168) = -167$
$2, -84$	$2 + (-84) = -82$
$3, -56$	$3 + (-56) = -53$
$4, -42$	$4 + (-42) = -38$
$6, -28$	$6 + (-28) = -22$
$7, -24$	$7 + (-24) = -17^*$
$8, -21$	$8 + (-21) = -13$

Split the middle term using $7u$ and $-24u$.

$$\begin{array}{c} 8u^2 - 17u - 21 \\ \swarrow\searrow \\ \underbrace{8u^2 + 7u}\,\underbrace{-24u - 21} \end{array}$$

Factor by grouping.

$$u(8u+7) - 3(8u+7)$$
$$(8u+7)(u-3)$$

Check by multiplying.

$$(8u+7)(u-3)$$
$$8u^2 - 24u + 7u - 21$$
$$8u^2 - 17u - 21 \checkmark$$

Note:
Exercise:

Problem: Factor: $20h^2 + 13h - 15$.

Solution:

$(4n+5)(5n-3)$

Note:
Exercise:

Problem: Factor: $6g^2 + 19g - 20$.

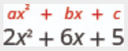
Solution:

$(g+4)(6g-5)$

Example:
Exercise:

Problem: Factor: $2x^2+6x+5$.

Solution:
Solution

Is there a greatest common factor? No.	 $\overset{ax^2}{2x^2}+\overset{bx}{6x}+\overset{c}{5}$
Find $a \cdot c$.	$a \cdot c$
	$2(5)$
	10

Find two numbers that multiply to 10 and add to 6.

Factors of 10	**Sum of factors**
$1, 10$	$1+10=11$
$2, 5$	$2+5=7$

There are no factors that multiply to 10 and add to 6. The polynomial is prime.

Note:
Exercise:

Problem: Factor: $10t^2+19t-15$.

Solution:

$(2t+5)(5t-3)$

Note:
Exercise:

Problem: Factor: $3u^2 + 8u + 5$.

Solution:

$(u+1)(3u+5)$

Don’t forget to look for a common factor!

Example:
Exercise:

Problem: Factor: $10y^2 - 55y + 70$.

Solution:
Solution

Is there a greatest common factor? Yes. The GCF is 5.	$10y^2 - 55y + 70$
Factor it. Be careful to keep the factor of 5 all the way through the solution!	$5(2y^2 - 11y + 14)$
The trinomial inside the parentheses has a leading coefficient that is not 1.	$ax^2 + bx + c$ $5(2y^2 - 11y + 14)$
Factor the trinomial.	$5(y-2)(2y-7)$
Check by mulitplying all three factors.	
$5\left(2y^2 - 2y - 4y + 14\right)$	
$5\left(2y^2 - 11y + 14\right)$	
$10y^2 - 55y + 70$✓	

Note:
Exercise:

Problem: Factor: $16x^2 - 32x + 12$.

Solution:

$4\left(2x-3\right)\left(2x-1\right)$

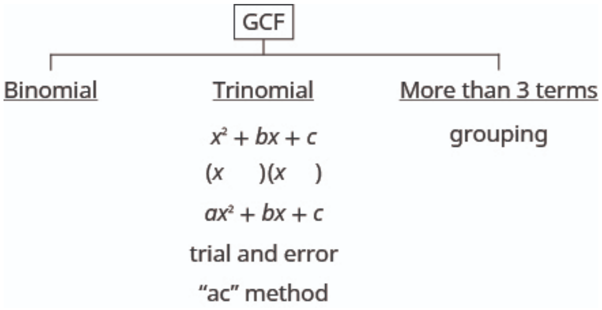
Note:
Exercise:

Problem: Factor: $18w^2 - 39w + 18$.

Solution:

$3\left(3w-2\right)\left(2w-3\right)$

We can now update the Preliminary Factoring Strategy, as shown in [link] and detailed in Choose a strategy to factor polynomials completely (updated), to include trinomials of the form $ax^2 + bx + c$. Remember, some polynomials are prime and so they cannot be factored.



Note:
Choose a strategy to factor polynomials completely (updated).

Is there a greatest common factor?

- Factor it.

Is the polynomial a binomial, trinomial, or are there more than three terms?

- If it is a binomial, right now we have no method to factor it.
- If it is a trinomial of the form $x^2 + bx + c$
Undo FOIL $\left(x \quad\right)\left(x \quad\right)$.
- If it is a trinomial of the form $ax^2 + bx + c$
Use Trial and Error or the "ac" method.

- If it has more than three terms
  Use the grouping method.

Check by multiplying the factors.

Note:
Access these online resources for additional instruction and practice with factoring trinomials of the form $ax^2 + bx + c$.

- Factoring Trinomials, a is not 1

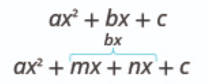
Key Concepts

- **Factor Trinomials of the Form $ax^2 + bx + c$ using Trial and Error:** See [link].

  Write the trinomial in descending order of degrees.
  Find all the factor pairs of the first term.
  Find all the factor pairs of the third term.
  Test all the possible combinations of the factors until the correct product is found.
  Check by multiplying.

- **Factor Trinomials of the Form $ax^2 + bx + c$ Using the "ac" Method:** See [link].

  Factor any GCF.
  Find the product ac.
  Find two numbers m and n that: Multiply to ac $\quad m \cdot n = a \cdot c$
  Add to b $\quad m + n = b$
  Split the middle term using m and n:

  Factor by grouping.
  Check by multiplying the factors.

- **Choose a strategy to factor polynomials completely (updated):**

  Is there a greatest common factor? Factor it.

Is the polynomial a binomial, trinomial, or are there more than three terms?	If it is a binomial, right now we have no method to factor it.	If it is a trinomial of the form $x^2 + bx + c$	Undo FOIL $(x \quad)(x \quad)$.	If it is a trinomial of the form $ax^2 + bx + c$	Use Trial and Error or the "ac" method.	If it has more than three terms	Use the grouping method.

  Check by multiplying the factors.

Practice Makes Perfect

Recognize a Preliminary Strategy to Factor Polynomials Completely

In the following exercises, identify the best method to use to factor each polynomial.

Exercise:

Problem:

ⓐ $10q^2 + 50$
ⓑ $a^2 - 5a - 14$
ⓒ $uv + 2u + 3v + 6$

Solution:

ⓐ factor the GCF, binomial ⓑ Undo FOIL ⓒ factor by grouping

Exercise:

Problem:

ⓐ $n^2 + 10n + 24$
ⓑ $8u^2 + 16$
ⓒ $pq + 5p + 2q + 10$

Exercise:

Problem:

ⓐ $x^2 + 4x - 21$
ⓑ $ab + 10b + 4a + 40$
ⓒ $6c^2 + 24$

Solution:

ⓐ undo FOIL ⓑ factor by grouping ⓒ factor the GCF, binomial

Exercise:

Problem:

ⓐ $20x^2 + 100$
ⓑ $uv + 6u + 4v + 24$
ⓒ $y^2 - 8y + 15$

Factor Trinomials of the form $ax^2 + bx + c$ with a GCF

In the following exercises, factor completely.

Exercise:

Problem: $5x^2 + 35x + 30$

Solution:

$5(x + 1)(x + 6)$

Exercise:

Problem: $12s^2 + 24s + 12$

Exercise:

Problem: $2z^2 - 2z - 24$

Solution:

$2(z-4)(z+3)$

Exercise:

Problem: $3u^2 - 12u - 36$

Exercise:

Problem: $7v^2 - 63v + 56$

Solution:

$7(v-1)(v-8)$

Exercise:

Problem: $5w^2 - 30w + 45$

Exercise:

Problem: $p^3 - 8p^2 - 20p$

Solution:

$p(p-10)(p+2)$

Exercise:

Problem: $q^3 - 5q^2 - 24q$

Exercise:

Problem: $3m^3 - 21m^2 + 30m$

Solution:

$3m(m-5)(m-2)$

Exercise:

Problem: $11n^3 - 55n^2 + 44n$

Exercise:

Problem: $5x^4 + 10x^3 - 75x^2$

Solution:

$5x^2(x-3)(x+5)$

Exercise:

Problem: $6y^4+12y^3-48y^2$

Factor Trinomials Using Trial and Error

In the following exercises, factor.

Exercise:

Problem: $2t^2+7t+5$

Solution:

$(2t+5)(t+1)$

Exercise:

Problem: $5y^2+16y+11$

Exercise:

Problem: $11x^2+34x+3$

Solution:

$(11x+1)(x+3)$

Exercise:

Problem: $7b^2+50b+7$

Exercise:

Problem: $4w^2-5w+1$

Solution:

$(4w-1)\,(w-1)$

Exercise:

Problem: $5x^2-17x+6$

Exercise:

Problem: $6p^2-19p+10$

Solution:

$(3p-2)\,(2p-5)$

Exercise:

Problem: $21m^2-29m+10$

Exercise:

Problem: $4q^2 - 7q - 2$

Solution:

$(4q+1)(q-2)$

Exercise:

Problem: $10y^2 - 53y - 11$

Exercise:

Problem: $4p^2 + 17p - 15$

Solution:

$(4p-3)(p+5)$

Exercise:

Problem: $6u^2 + 5u - 14$

Exercise:

Problem: $16x^2 - 32x + 16$

Solution:

$16(x-1)(x-1)$

Exercise:

Problem: $81a^2 + 153a - 18$

Exercise:

Problem: $30q^3 + 140q^2 + 80q$

Solution:

$10q(3q+2)(q+4)$

Exercise:

Problem: $5y^3 + 30y^2 - 35y$

Factor Trinomials using the ‘ac’ Method

In the following exercises, factor.

Exercise:

Problem: $5n^2 + 21n + 4$

Solution:

$(5n+1)(n+4)$

Exercise:

Problem: $8w^2+25w+3$

Exercise:

Problem: $9z^2+15z+4$

Solution:

$(3z+1)(3z+4)$

Exercise:

Problem: $3m^2+26m+48$

Exercise:

Problem: $4k^2-16k+15$

Solution:

$(2k-3)(2k-5)$

Exercise:

Problem: $4q^2-9q+5$

Exercise:

Problem: $5s^2-9s+4$

Solution:

$(5s-4)(s-1)$

Exercise:

Problem: $4r^2-20r+25$

Exercise:

Problem: $6y^2+y-15$

Solution:

$(3y+5)(2y-3)$

Exercise:

Problem: $6p^2+p-22$

Exercise:

Problem: $2n^2-27n-45$

Solution:

$(2n+3)(n-15)$

Exercise:

Problem: $12z^2-41z-11$

Exercise:

Problem: $3x^2+5x+4$

Solution:

prime

Exercise:

Problem: $4y^2+15y+6$

Exercise:

Problem: $60y^2+290y-50$

Solution:

$10(6y-1)(y+5)$

Exercise:

Problem: $6u^2-46u-16$

Exercise:

Problem: $48z^3-102z^2-45z$

Solution:

$3z(8z+3)(2z-5)$

Exercise:

Problem: $90n^3+42n^2-216n$

Exercise:

Problem: $16s^2+40s+24$

Solution:

$8\left(2s+3\right)\left(s+1\right)$

Exercise:

Problem: $24p^2+160p+96$

Exercise:

Problem: $48y^2+12y-36$

Solution:

$12\left(4y-3\right)\left(y+1\right)$

Exercise:

Problem: $30x^2+105x-60$

Mixed Practice

In the following exercises, factor.

Exercise:

Problem: $12y^2-29y+14$

Solution:

$\left(4y-7\right)\left(3y-2\right)$

Exercise:

Problem: $12x^2+36y-24z$

Exercise:

Problem: a^2-a-20

Solution:

$\left(a-5\right)\left(a+4\right)$

Exercise:

Problem: m^2-m-12

Exercise:

Problem: $6n^2+5n-4$

Solution:

$\left(2n-1\right)\left(3n+4\right)$

Exercise:

Problem: $12y^2-37y+21$

Exercise:

Problem: $2p^2+4p+3$

Solution:

prime

Exercise:

Problem: $3q^2 + 6q + 2$

Exercise:

Problem: $13z^2 + 39z - 26$

Solution:

$13\left(z^2 + 3z - 2\right)$

Exercise:

Problem: $5r^2 + 25r + 30$

Exercise:

Problem: $x^2 + 3x - 28$

Solution:

$(x + 7)(x - 4)$

Exercise:

Problem: $6u^2 + 7u - 5$

Exercise:

Problem: $3p^2 + 21p$

Solution:

$3p(p + 7)$

Exercise:

Problem: $7x^2 - 21x$

Exercise:

Problem: $6r^2 + 30r + 36$

Solution:

$6(r + 2)(r + 3)$

Exercise:

Problem: $18m^2 + 15m + 3$

Exercise:

Problem: $24n^2 + 20n + 4$

Solution:

$4\left(2n+1\right)\left(3n+1\right)$

Exercise:

Problem: $4a^2+5a+2$

Exercise:

Problem: $x^2+2x-24$

Solution:

$\left(x+6\right)\left(x-4\right)$

Exercise:

Problem: $2b^2-7b+4$

Everyday Math

Exercise:

Problem:

Height of a toy rocket The height of a toy rocket launched with an initial speed of 80 feet per second from the balcony of an apartment building is related to the number of seconds, t, since it is launched by the trinomial $-16t^2+80t+96$. Completely factor the trinomial.

Solution:

$-16\left(t-6\right)\left(t+1\right)$

Exercise:

Problem:

Height of a beach ball The height of a beach ball tossed up with an initial speed of 12 feet per second from a height of 4 feet is related to the number of seconds, t, since it is tossed by the trinomial $-16t^2+12t+4$. Completely factor the trinomial.

Writing Exercises

Exercise:

Problem:

List, in order, all the steps you take when using the “ac” method to factor a trinomial of the form ax^2+bx+c.

Solution:

Answers may vary.

Exercise:

Problem: How is the “ac” method similar to the “undo FOIL” method? How is it different?

Exercise:

Problem:

What are the questions, in order, that you ask yourself as you start to factor a polynomial? What do you need to do as a result of the answer to each question?

Solution:

Answers may vary.

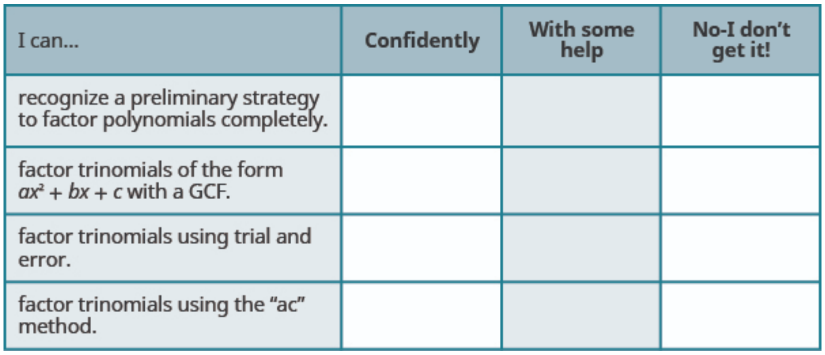
Exercise:

Problem:

On your paper draw the chart that summarizes the factoring strategy. Try to do it without looking at the book. When you are done, look back at the book to finish it or verify it.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don’t get it!
recognize a preliminary strategy to factor polynomials completely.			
factor trinomials of the form $ax^2 + bx + c$ with a GCF.			
factor trinomials using trial and error.			
factor trinomials using the “ac” method.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

prime polynomials

Polynomials that cannot be factored are prime polynomials.

Factor Special Products

By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes
- Choose method to factor a polynomial completely

Note:

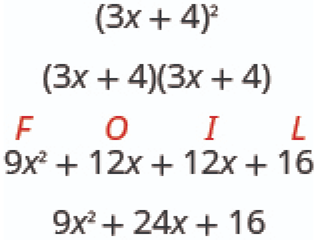
Before you get started, take this readiness quiz.

1. Simplify: $(12x)^2$.
   If you missed this problem, review [link].
2. Multiply: $(m+4)^2$.
   If you missed this problem, review [link].
3. Multiply: $(p-9)^2$.
   If you missed this problem, review [link].
4. Multiply: $(k+3)(k-3)$.
   If you missed this problem, review [link].

The strategy for factoring we developed in the last section will guide you as you factor most binomials, trinomials, and polynomials with more than three terms. We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. You can square a binomial by using FOIL, but using the Binomial Squares pattern you saw in a previous chapter saves you a step. Let’s review the Binomial Squares pattern by squaring a binomial using FOIL.



$$(3x+4)^2$$

$$(3x+4)(3x+4)$$

F O I L

$$9x^2+12x+12x+16$$

$$9x^2+24x+16$$

The first term is the square of the first term of the binomial and the last term is the square of the last. The middle term is twice the product of the two terms of the binomial.

Equation:

$$\begin{gathered}(3x)^2+2\left(3x\cdot 4\right)+4^2\\ 9x^2+24x+16\end{gathered}$$

The trinomial $9x^2+24x+16$ is called a perfect square trinomial. It is the square of the binomial $3x+4$.

We’ll repeat the Binomial Squares Pattern here to use as a reference in factoring.

Note:

Binomial Squares Pattern

If a and b are real numbers,

Equation:

$$(a+b)^2=a^2+2ab+b^2 \qquad (a-b)^2=a^2-2ab+b^2$$

When you square a binomial, the product is a perfect square trinomial. In this chapter, you are learning to factor—now, you will start with a perfect square trinomial and factor it into its prime factors.

You could factor this trinomial using the methods described in the last section, since it is of the form ax^2+bx+c. But if you recognize that the first and last terms are squares and the trinomial fits the **perfect square trinomials pattern**, you will save yourself a lot of work.

Here is the pattern—the reverse of the binomial squares pattern.

Note:
Perfect Square Trinomials Pattern
If a and b are real numbers,
Equation:

$$a^2+2ab+b^2=(a+b)^2 \qquad a^2-2ab+b^2=(a-b)^2$$

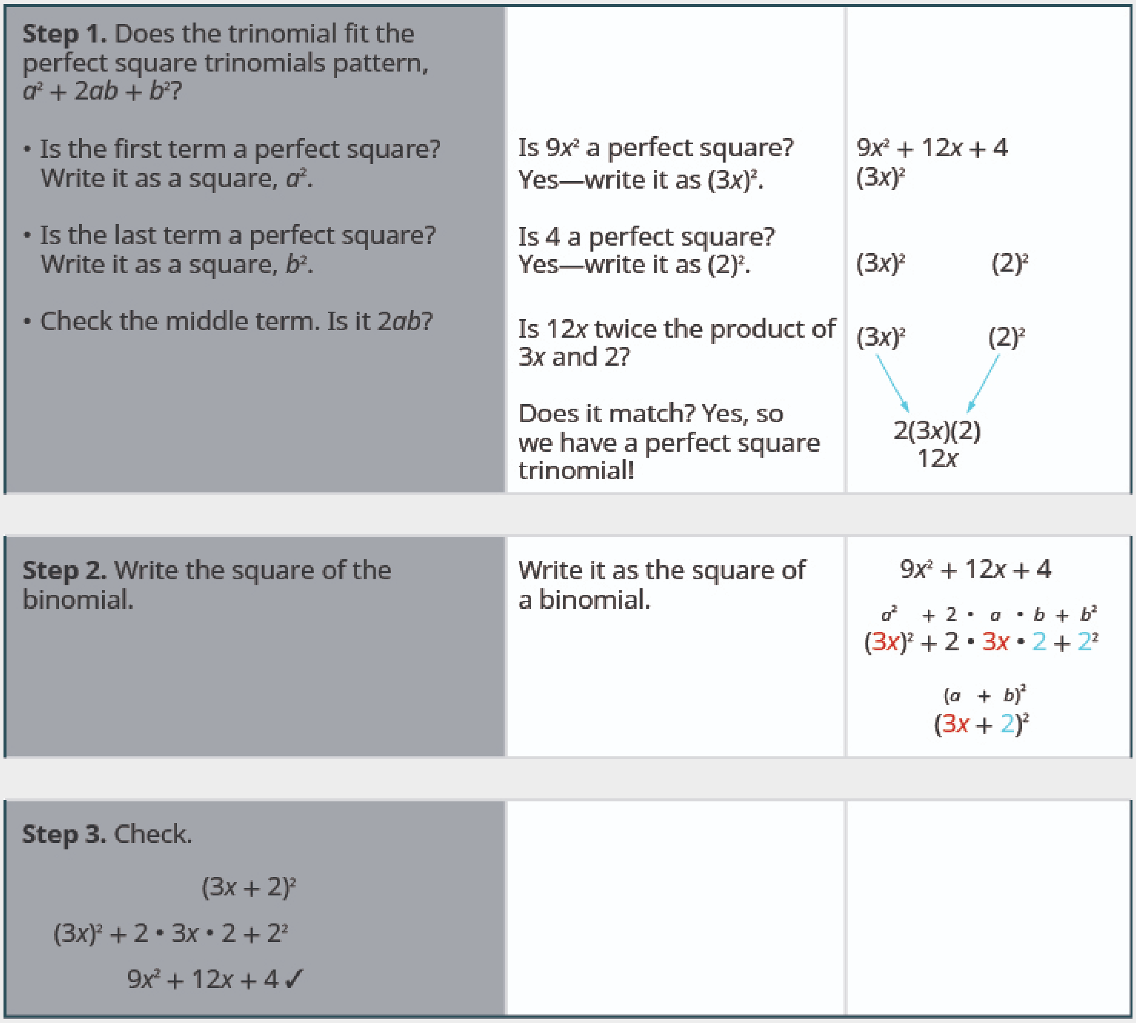
To make use of this pattern, you have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next check that the last term is a perfect square, b^2. Then check the middle term —is it twice the product, 2 ab? If everything checks, you can easily write the factors.

Example:
How to Factor Perfect Square Trinomials
Exercise:

Problem: Factor: $9x^2+12x+4$.

Solution:

Solution



Step 1. Does the trinomial fit the perfect square trinomials pattern, $a^2 + 2ab + b^2$?		
- Is the first term a perfect square? Write it as a square, a^2.	Is $9x^2$ a perfect square? Yes—write it as $(3x)^2$.	$9x^2 + 12x + 4$ $(3x)^2$
- Is the last term a perfect square? Write it as a square, b^2.	Is 4 a perfect square? Yes—write it as $(2)^2$.	$(3x)^2 \quad (2)^2$
- Check the middle term. Is it $2ab$?	Is $12x$ twice the product of $3x$ and 2? Does it match? Yes, so we have a perfect square trinomial!	$(3x)^2 \quad (2)^2$ $2(3x)(2)$ $12x$
Step 2. Write the square of the binomial.	Write it as the square of a binomial.	$9x^2 + 12x + 4$ $a^2 + 2 \cdot a \cdot b + b^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $(a + b)^2$ $(3x + 2)^2$
Step 3. Check. $(3x + 2)^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $9x^2 + 12x + 4$ ✓		

Note:
Exercise:

Problem: Factor: $4x^2 + 12x + 9$.

Solution:

$(2x + 3)^2$

Note:
Exercise:

Problem: Factor: $9y^2 + 24y + 16$.

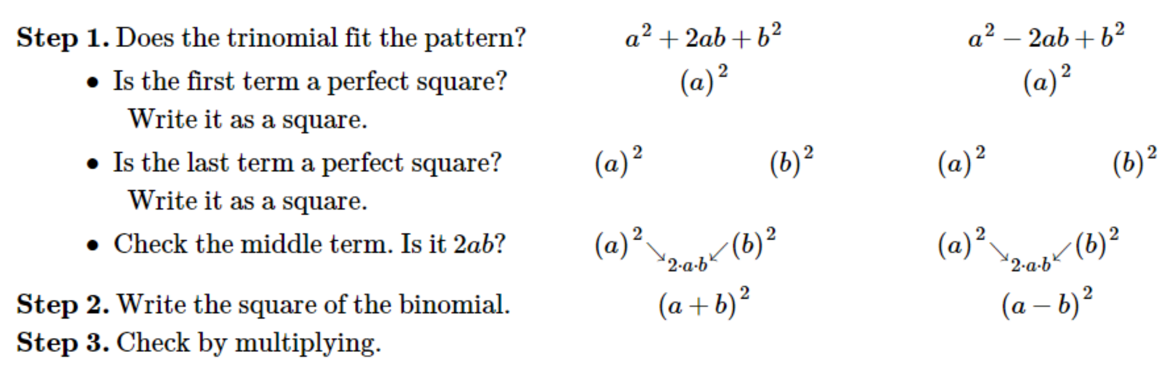
Solution:

$(3y + 4)^2$

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2 - 2ab + b^2$, which factors to $(a - b)^2$.

The steps are summarized here.

Note:
Factor perfect square trinomials.



Step 1. Does the trinomial fit the pattern?	$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
• Is the first term a perfect square? Write it as a square.	$(a)^2$	$(a)^2$
• Is the last term a perfect square? Write it as a square.	$(a)^2 \qquad (b)^2$	$(a)^2 \qquad (b)^2$
• Check the middle term. Is it $2ab$?	$(a)^2 \searrow_{2\cdot a\cdot b} \swarrow (b)^2$	$(a)^2 \searrow_{2\cdot a\cdot b} \swarrow (b)^2$
Step 2. Write the square of the binomial.	$(a + b)^2$	$(a - b)^2$
Step 3. Check by multiplying.		

We’ll work one now where the middle term is negative.

Example:
Exercise:

Problem: Factor: $81y^2 - 72y + 16$.

Solution:
Solution

The first and last terms are squares. See if the middle term fits the pattern of a perfect square trinomial. The middle term is negative, so the binomial square would be $(a-b)^2$.

	$81y^2 - 72y + 16$
Are the first and last terms perfect squares?	$(9y)^2 \qquad (4)^2$
Check the middle term.	$(9y)^2 \qquad (4)^2$ $2(9y)(4)$ $72y$
Does is match $(a-b)^2$? Yes.	$\overset{a^2}{(9y)^2} \overset{-2}{-2} \cdot \overset{a}{9y} \cdot \overset{b}{4} \overset{+}{+} \overset{b^2}{4^2}$
Write the square of a binomial.	$(9y-4)^2$

Check by mulitplying.	
$(9y-4)^2$	
$(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$	
$81y^2 - 72y + 16$✓	

Note:
Exercise:

Problem: Factor: $64y^2 - 80y + 25$.

Solution:

$(8y-5)^2$

Note:
Exercise:

Problem: Factor: $16z^2 - 72z + 81$.

Solution:

$(4z-9)^2$

The next example will be a perfect square trinomial with two variables.

Example:
Exercise:

Problem: Factor: $36x^2+84xy+49y^2$.

Solution:
Solution

	$36x^2+84xy+49y^2$
Test each term to verify the pattern.	$\overset{a^2}{(6x)^2}\overset{+\,2}{+2}\cdot\overset{a}{6x}\cdot\overset{b}{7y}\overset{+}{+}\overset{b^2}{(7y)^2}$
Factor.	$(6x+7y)^2$
Check by mulitplying.	
$(6x+7y)^2$	
$(6x)^2+2\cdot 6x\cdot 7y+(7y)^2$	
$36x^2+84xy+49y^2$✓	

Note:
Exercise:

Problem: Factor: $49x^2 + 84xy + 36y^2$.

Solution:

$(7x + 6y)^2$

Note:
Exercise:

Problem: Factor: $64m^2 + 112mn + 49n^2$.

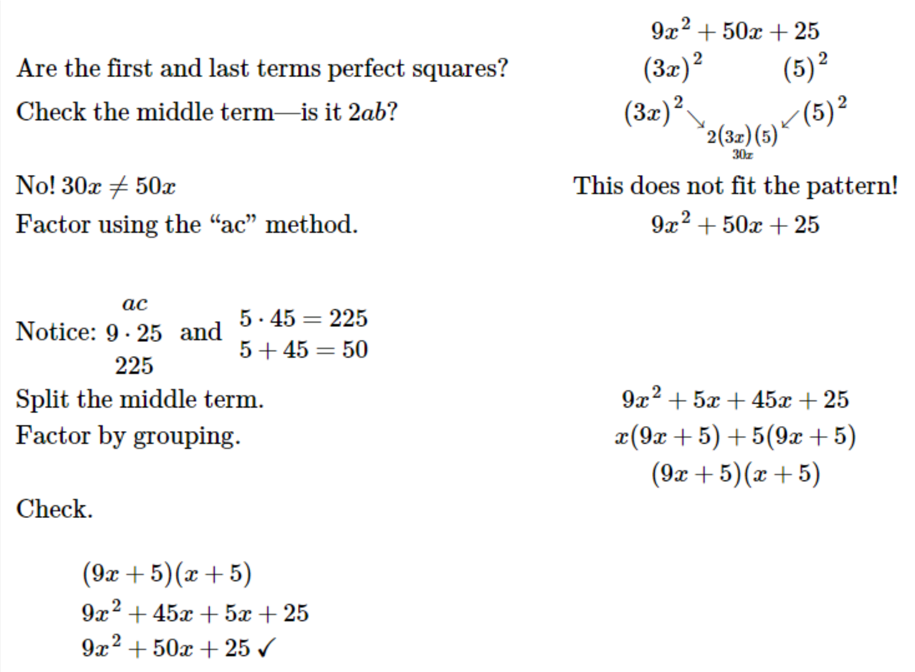
Solution:

$(8m + 7n)^2$

Example:
Exercise:

Problem: Factor: $9x^2 + 50x + 25$.

Solution:
Solution



	$9x^2+50x+25$
Are the first and last terms perfect squares?	$(3x)^2 \qquad (5)^2$
Check the middle term—is it $2ab$?	$(3x)^2 \searrow \underbrace{2(3x)(5)}_{30x} \swarrow (5)^2$
No! $30x \neq 50x$	This does not fit the pattern!
Factor using the “ac” method.	$9x^2+50x+25$
Notice: $\overset{ac}{\underset{225}{9\cdot 25}}$ and $\begin{array}{c} 5\cdot 45=225 \\ 5+45=50 \end{array}$	
Split the middle term.	$9x^2+5x+45x+25$
Factor by grouping.	$x(9x+5)+5(9x+5)$ $(9x+5)(x+5)$
Check.	

$$(9x+5)(x+5)$$
$$9x^2+45x+5x+25$$
$$9x^2+50x+25\ ✓$$

Note:
Exercise:

Problem: Factor: $16r^2+30rs+9s^2$.

Solution:

$(8r+3s)\,(2r+3s)$

Note:
Exercise:

Problem: Factor: $9u^2+87u+100$.

Solution:

$(3u+4)\,(3u+25)$

Remember the very first step in our Strategy for Factoring Polynomials? It was to ask “is there a greatest common factor?” and, if there was, you factor the GCF before going any further. Perfect square trinomials may have a GCF in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, you will recognize a perfect square trinomial.

Example:
Exercise:

Problem: Factor: $36x^2y-48xy+16y$.

Solution:
Solution

	$36x^2y-48xy+16y$
Is there a GCF? Yes, 4*y*, so factor it out.	$4y\left(9x^2-12x+4\right)$
Is this a perfect square trinomial?	
Verify the pattern.	$4y[\overset{a^2}{(3x)^2}\overset{-\,2}{-\,2}\cdot\overset{a}{3x}\cdot\overset{b}{2}\overset{+\,b^2}{+\,2^2}]$

Factor.	$4y(3x-2)^2$
Remember: Keep the factor 4*y* in the final product.	
Check.	
$4y(3x-2)^2$	
$4y\left[(3x)^2 - 2 \cdot 3x \cdot 2 + 2^2\right]$	
$4y(9x)^2 - 12x + 4$	
$36x^2y - 48xy + 16y$✓	

Note:
Exercise:

Problem: Factor: $8x^2y - 24xy + 18y$.

Solution:

$2y(2x-3)^2$

Note:
Exercise:

Problem: Factor: $27p^2q + 90pq + 75q$.

Solution:

$3q(3p+5)^2$

Factor Differences of Squares

The other special product you saw in the previous was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Here’s an example:
Equation:

$$\begin{gathered}(3x-4)\left(3x+4\right)\\9x^2-16\end{gathered}$$

Remember, when you multiply conjugate binomials, the middle terms of the product add to 0. All you have left is a binomial, the difference of squares.

Multiplying conjugates is the only way to get a binomial from the product of two binomials.

Note:
Product of Conjugates Pattern
If a and b are real numbers
Equation:

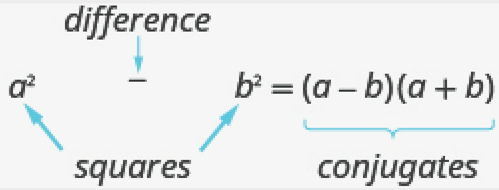
$$(a-b)\left(a+b\right)=a^2-b^2$$

The product is called a difference of squares.

To factor, we will use the product pattern “in reverse” to factor the difference of squares. A **difference of squares** factors to a product of conjugates.

Note:
Difference of Squares Pattern
If a and b are real numbers,

$$a^2 - b^2 = (a-b)(a+b)$$

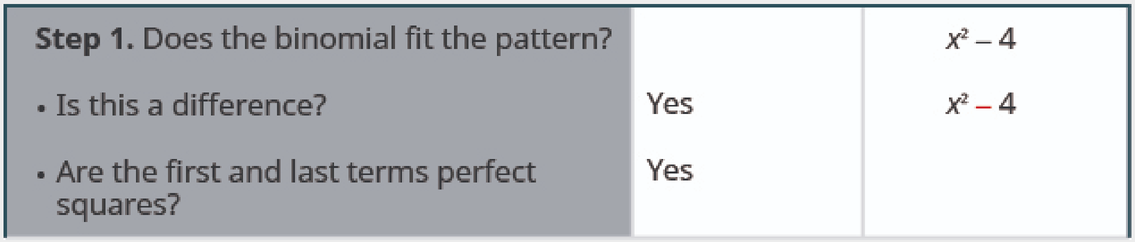


Remember, “difference” refers to subtraction. So, to use this pattern you must make sure you have a binomial in which two squares are being subtracted.

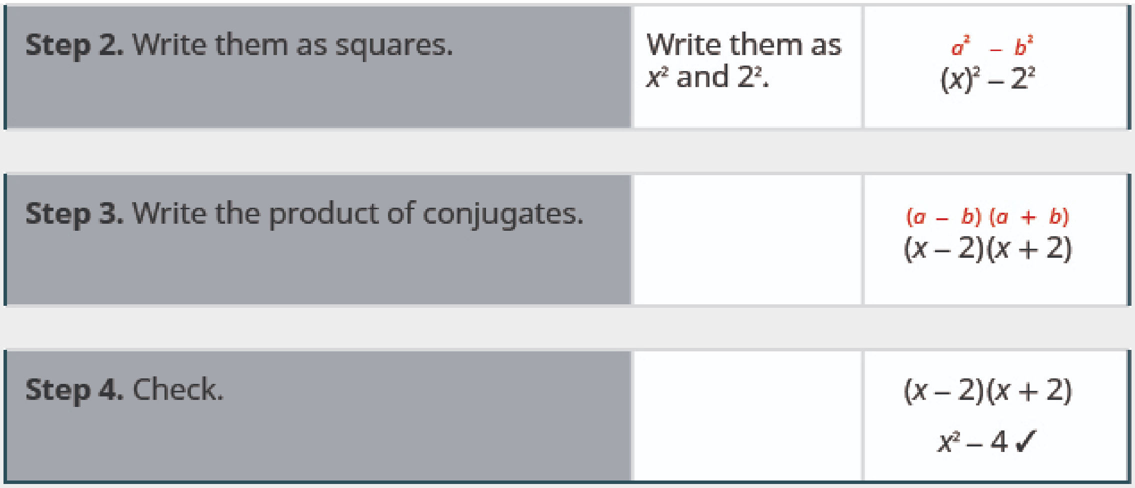
Example:
How to Factor Differences of Squares
Exercise:

Problem: Factor: $x^2 - 4$.

Solution:
Solution



Step 1. Does the binomial fit the pattern?		$x^2 - 4$
• Is this a difference?	Yes	$x^2 - 4$
• Are the first and last terms perfect squares?	Yes	



Step 2. Write them as squares.	Write them as x^2 and 2^2.	$a^2 - b^2$ $(x)^2 - 2^2$
Step 3. Write the product of conjugates.		$(a - b)(a + b)$ $(x - 2)(x + 2)$
Step 4. Check.		$(x - 2)(x + 2)$ $x^2 - 4$ ✓

Note:
Exercise:

Problem: Factor: $h^2 - 81$.

Solution:

$(h - 9)\,(h + 9)$

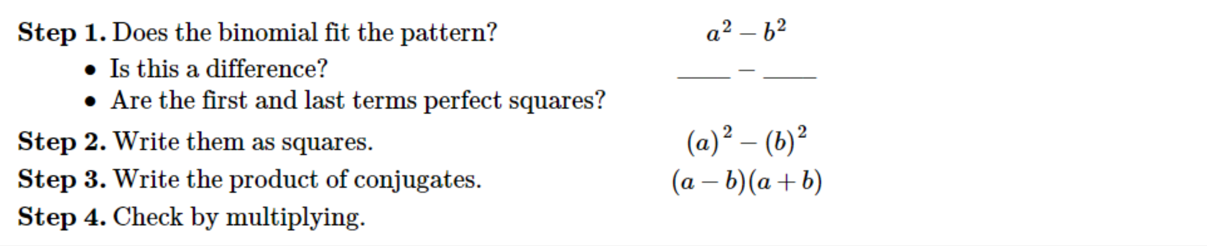
Note:
Exercise:

Problem: Factor: $k^2 - 121$.

Solution:

$(k - 11)\,(k + 11)$

Note:
Factor differences of squares.



Step 1. Does the binomial fit the pattern? $a^2 - b^2$
- Is this a difference? $\_\_\_\_ - \_\_\_\_$
- Are the first and last terms perfect squares?

Step 2. Write them as squares. $(a)^2 - (b)^2$

Step 3. Write the product of conjugates. $(a-b)(a+b)$

Step 4. Check by multiplying.

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression $a^2 + b^2$ is prime!

Don’t forget that 1 is a perfect square. We’ll need to use that fact in the next example.

Example:
Exercise:

Problem: Factor: $64y^2 - 1$.

Solution:
Solution

	$64y^2 - 1$
Is this a difference? Yes.	$64y^2 - 1$
Are the first and last terms perfect squares?	
Yes - write them as squares.	$\overset{a^2}{(8y)^2} \overset{-}{-} \overset{b^2}{1^2}$
Factor as the product of conjugates.	$\overset{(a\;-\;b)}{(8y-1)}\overset{(a\;+\;b)}{(8y+1)}$
Check by multiplying.	
$(8y-1)(8y+1)$	
$64y^2 - 1$✓	

Note:
Exercise:

Problem: Factor: $m^2 - 1$.

Solution:

$(m-1)\,(m+1)$

Note:
Exercise:

Problem: Factor: $81y^2 - 1$.

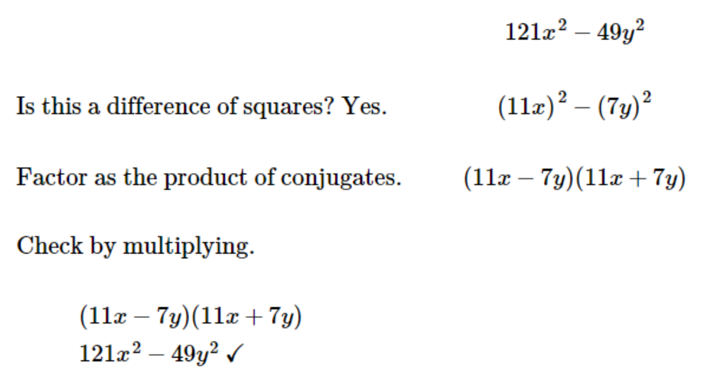
Solution:

$(9y - 1)(9y + 1)$

Example:
Exercise:

Problem: Factor: $121x^2 - 49y^2$.

Solution:
Solution



$121x^2 - 49y^2$

Is this a difference of squares? Yes. $(11x)^2 - (7y)^2$

Factor as the product of conjugates. $(11x - 7y)(11x + 7y)$

Check by multiplying.

$(11x - 7y)(11x + 7y)$
$121x^2 - 49y^2$ ✓

Note:
Exercise:

Problem: Factor: $196m^2 - 25n^2$.

Solution:

$(14m - 5n)(14m + 5n)$

Note:
Exercise:

Problem: Factor: $144p^2 - 9q^2$.

Solution:

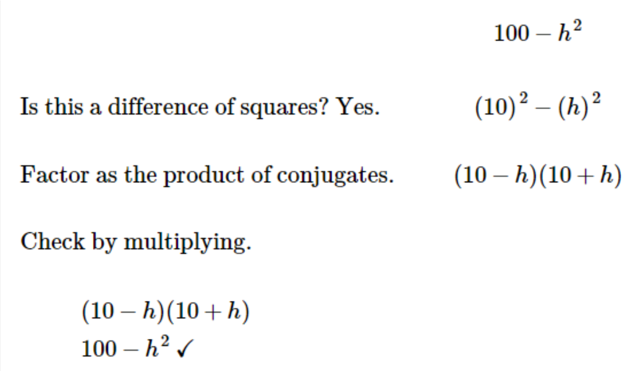
$(12p - 3q)(12p + 3q)$

The binomial in the next example may look “backwards,” but it’s still the difference of squares.

Example:
Exercise:

Problem: Factor: $100 - h^2$.

Solution:
Solution



$100-h^2$

Is this a difference of squares? Yes. $(10)^2-(h)^2$

Factor as the product of conjugates. $(10-h)(10+h)$

Check by multiplying.

$(10-h)(10+h)$
$100-h^2$ ✓

Be careful not to rewrite the original expression as h^2-100.

Factor h^2-100 on your own and then notice how the result differs from $(10-h)(10+h)$.

Note:
Exercise:

Problem: Factor: $144-x^2$.

Solution:

$(12-x)(12+x)$

Note:
Exercise:

Problem: Factor: $169-p^2$.

Solution:

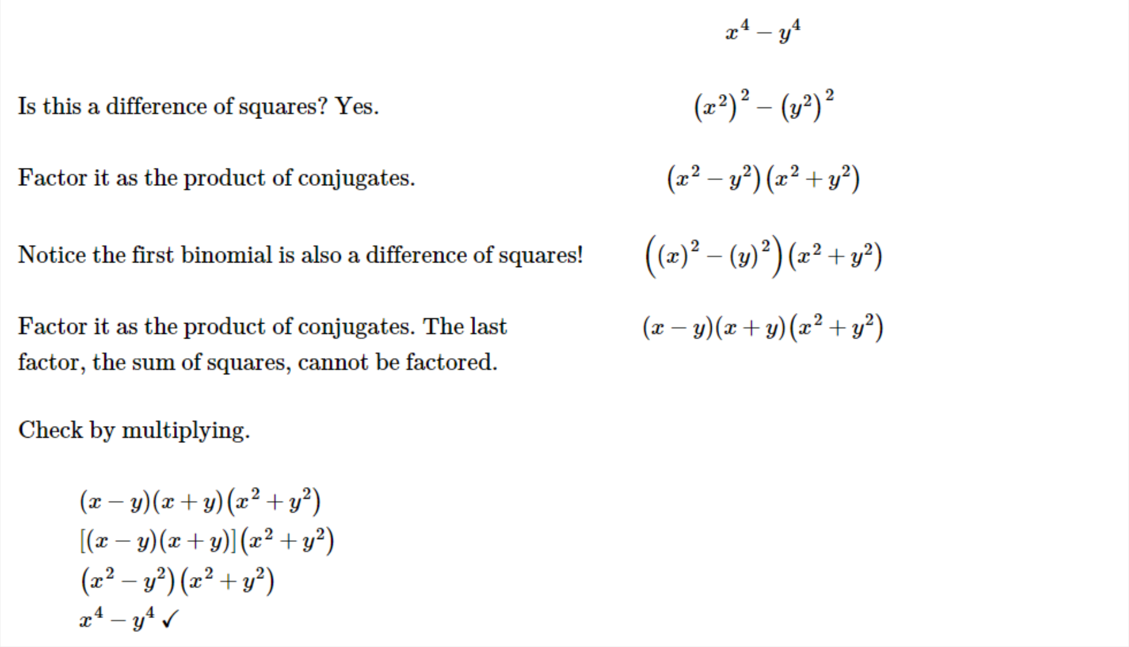
$(13-p)(13+p)$

To completely factor the binomial in the next example, we'll factor a difference of squares twice!

Example:
Exercise:

Problem: Factor: $x^4 - y^4$.

Solution:
Solution



	$x^4 - y^4$
Is this a difference of squares? Yes.	$(x^2)^2 - (y^2)^2$
Factor it as the product of conjugates.	$(x^2 - y^2)(x^2 + y^2)$
Notice the first binomial is also a difference of squares!	$\left((x)^2 - (y)^2\right)(x^2 + y^2)$
Factor it as the product of conjugates. The last factor, the sum of squares, cannot be factored.	$(x - y)(x + y)(x^2 + y^2)$

Check by multiplying.

$$(x - y)(x + y)(x^2 + y^2)$$
$$[(x - y)(x + y)](x^2 + y^2)$$
$$(x^2 - y^2)(x^2 + y^2)$$
$$x^4 - y^4 \checkmark$$

Note:
Exercise:

Problem: Factor: $a^4 - b^4$.

Solution:

$\left(a^2+b^2\right)(a+b)(a-b)$

Note:
Exercise:

Problem: Factor: x^4-16.

Solution:

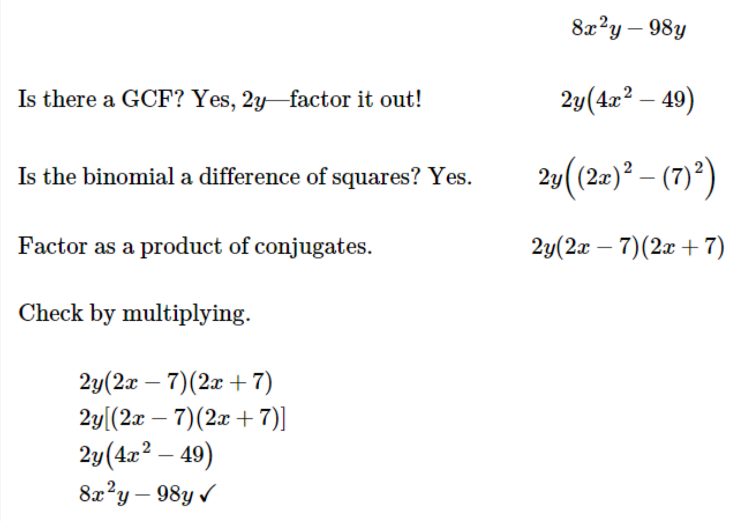
$\left(x^2+4\right)(x+2)(x-2)$

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may “disguise” the difference of squares and you won’t recognize the perfect squares until you factor the GCF.

Example:
Exercise:

Problem: Factor: $8x^2y-18y$.

Solution:
Solution



	$8x^2y - 98y$
Is there a GCF? Yes, $2y$—factor it out!	$2y\left(4x^2 - 49\right)$
Is the binomial a difference of squares? Yes.	$2y\left((2x)^2 - (7)^2\right)$
Factor as a product of conjugates.	$2y(2x - 7)(2x + 7)$

Check by multiplying.

$$\begin{array}{l} 2y(2x-7)(2x+7) \\ 2y[(2x-7)(2x+7)] \\ 2y\left(4x^2-49\right) \\ 8x^2y-98y\ \checkmark \end{array}$$

Note:
Exercise:

Problem: Factor: $7xy^2 - 175x$.

Solution:

$7x\,(y-5)\,(y+5)$

Note:
Exercise:

Problem: Factor: $45a^2b - 80b$.

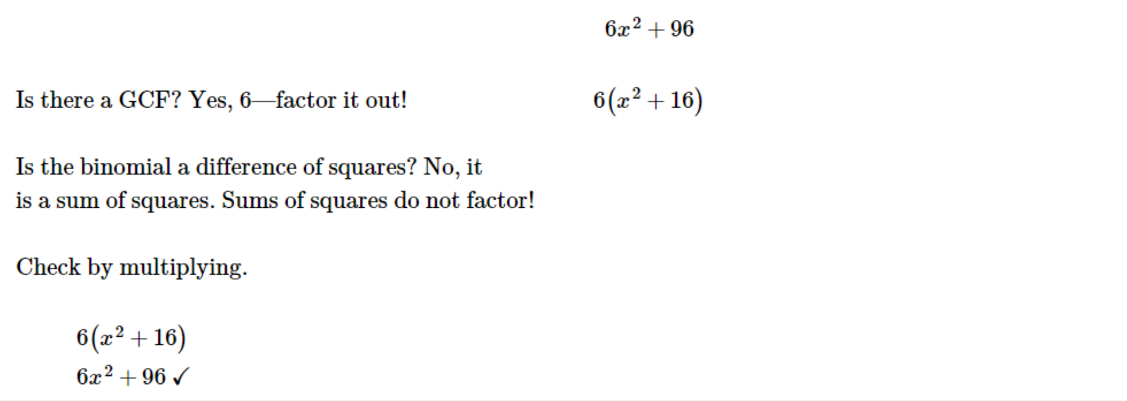
Solution:

$5b\,(3a-4)\,(3a+4)$

Example:
Exercise:

Problem: Factor: $6x^2 + 96$.

Solution:
Solution



$6x^2 + 96$

Is there a GCF? Yes, 6—factor it out! $6(x^2 + 16)$

Is the binomial a difference of squares? No, it is a sum of squares. Sums of squares do not factor!

Check by multiplying.

$6(x^2 + 16)$
$6x^2 + 96$ ✓

Note:
Exercise:

Problem: Factor: $8a^2 + 200$.

Solution:

$8\left(a^2 + 25\right)$

Note:
Exercise:

Problem: Factor: $36y^2 + 81$.

Solution:

$9\left(4y^2+9\right)$

Factor Sums and Differences of Cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes. We will write these formulas first and then check them by multiplication.

Equation:

$$a^3+b^3=(a+b)\left(a^2-ab+b^2\right)$$
$$a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

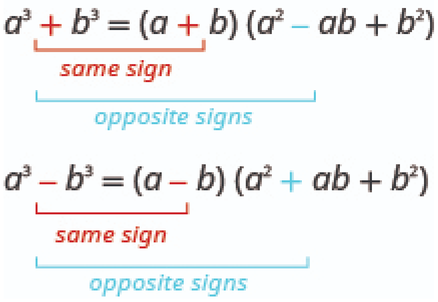
We'll check the first pattern and leave the second to you.

	$(a+b)\left(a^2-ab+b^2\right)$
Distribute.	$a\left(a^2-ab+b^2\right)+b\left(a^2-ab+b^2\right)$
Multiply.	$a^3-a^2b+ab^2+a^2b-ab^2+b^3$
Combine like terms.	a^3+b^3

Note:
Sum and Difference of Cubes Pattern
Equation:

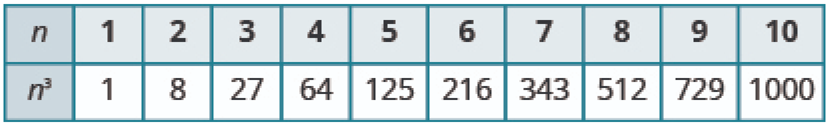
$$\begin{aligned} a^3 + b^3 &= (a+b)\left(a^2 - ab + b^2\right) \\ a^3 - b^3 &= (a-b)\left(a^2 + ab + b^2\right) \end{aligned}$$

The two patterns look very similar, don’t they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.


$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
same sign
opposite signs
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
same sign
opposite signs

The trinomial factor in the sum and difference of cubes pattern cannot be factored.

It can be very helpful if you learn to recognize the cubes of the integers from 1 to 10, just like you have learned to recognize squares. We have listed the cubes of the integers from 1 to 10 in [link].



n	1	2	3	4	5	6	7	8	9	10
n^3	1	8	27	64	125	216	343	512	729	1000

Example:

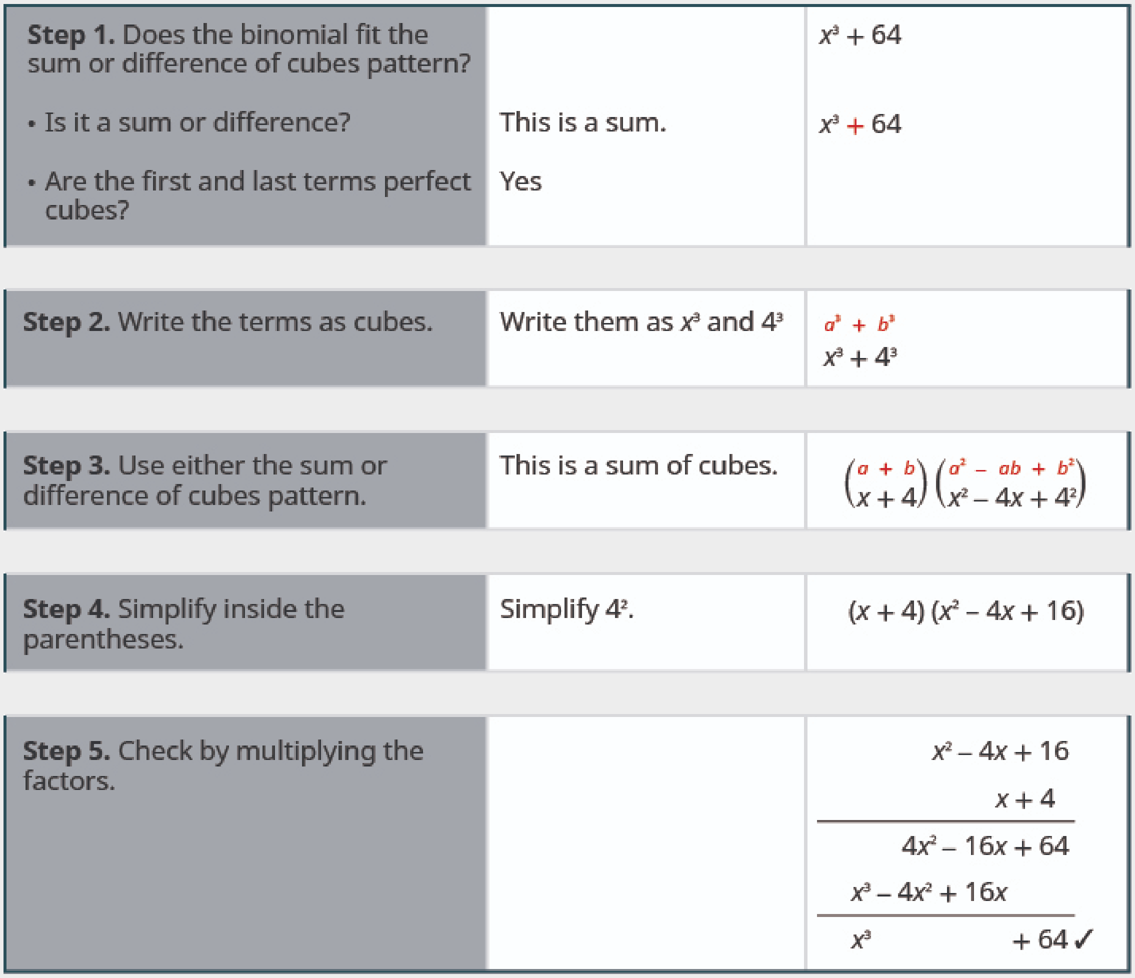
How to Factor the Sum or Difference of Cubes

Exercise:

Problem: Factor: $x^3 + 64$.

Solution:

Solution



Step 1. Does the binomial fit the sum or difference of cubes pattern? • Is it a sum or difference? • Are the first and last terms perfect cubes?	 This is a sum. Yes	x^3+64 x^3+64
Step 2. Write the terms as cubes.	Write them as x^3 and 4^3	$\overset{a^3\ +\ b^3}{x^3+4^3}$
Step 3. Use either the sum or difference of cubes pattern.	This is a sum of cubes.	$\left(\overset{a\ +\ b}{x+4}\right)\left(\overset{a^2\ -\ ab\ +\ b^2}{x^2-4x+4^2}\right)$
Step 4. Simplify inside the parentheses.	Simplify 4^2.	$(x+4)(x^2-4x+16)$
Step 5. Check by multiplying the factors.		$x^2-4x+16$ $x+4$ $4x^2-16x+64$ x^3-4x^2+16x $x^3 \qquad +64$ ✓

Note:

Exercise:

Problem: Factor: $x^3 + 27$.

Solution:

$(x+3)\left(x^2 - 3x + 9\right)$

Note:
Exercise:

Problem: Factor: $y^3 + 8$.

Solution:

$(y+2)\left(y^2 - 2y + 4\right)$

Note:
Factor the sum or difference of cubes.
To factor the sum or difference of cubes:

Does the binomial fit the sum or difference of cubes pattern?

- Is it a sum or difference?
- Are the first and last terms perfect cubes?

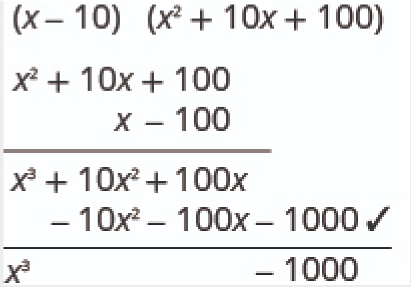
Write them as cubes.
Use either the sum or difference of cubes pattern.
Simplify inside the parentheses
Check by multiplying the factors.

Example:
Exercise:

Problem: Factor: $x^3 - 1000$.

Solution:
Solution

	$x^3 - 1000$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{x^3} \overset{-}{-} \overset{b^3}{10^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{x} \overset{-}{-} \overset{b}{10}\right) \left(\overset{a^2}{x^2} \overset{+}{+} \overset{ab}{10 \cdot x} \overset{+}{+} \overset{b^2}{10^2}\right)$
Simplify.	$\left(\overset{a}{x} \overset{-}{-} \overset{b}{10}\right) \left(\overset{a^2}{x^2} \overset{+}{+} \overset{ab}{10x} \overset{+}{+} \overset{b^2}{100}\right)$
Check by multiplying.	



$$(x-10)\quad(x^2+10x+100)$$

$$\begin{array}{r}
x^2+10x+100 \\
x-100 \\
\hline
x^3+10x^2+100x \qquad\qquad \\
-10x^2-100x-1000\ ✓ \\
\hline
x^3 \qquad\qquad\qquad\qquad -1000
\end{array}$$

Note:
Exercise:

Problem: Factor: u^3-125.

Solution:

$(u-5)\left(u^2+5u+25\right)$

Note:
Exercise:

Problem: Factor: v^3-343.

Solution:

$(v-7)\left(v^2+7v+49\right)$

Be careful to use the correct signs in the factors of the sum and difference of cubes.

Example:
Exercise:

Problem: Factor: $512 - 125p^3$.

Solution:
Solution

	$512 - 125p^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{8^3} \overset{-}{-} \overset{b^3}{(5p)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{8} \overset{-}{-} \overset{b}{5p}\right)\left(\overset{a^2}{8^2} \overset{+}{+} \overset{ab}{8 \cdot 5p} \overset{+}{+} \overset{b^2}{(5p)^2}\right)$
Simplify.	$\left(\overset{a}{8} \overset{-}{-} \overset{b}{5p}\right)\left(\overset{a^2}{64} \overset{+}{+} \overset{ab}{40p} \overset{+}{+} \overset{b^2}{25p^2}\right)$
Check by multiplying.	We'll leave the check to you.

Note:
Exercise:

Problem: Factor: $64 - 27x^3$.

Solution:

$(4 - 3x)\left(16 + 12x + 9x^2\right)$

Note:
Exercise:

Problem: Factor: $27 - 8y^3$.

Solution:

$(3 - 2y)\left(9 + 6y + 4y^2\right)$

Example:
Exercise:

Problem: Factor: $27u^3 - 125v^3$.

Solution:
Solution

	$27u^3 - 125v^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{(3u)^3} \overset{-}{-} \overset{b^3}{(5v)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{3u} \overset{-}{-} \overset{b}{5v}\right)\left(\overset{a^2}{(3u)^2} \overset{+}{+} \overset{ab}{3u \cdot 5v} \overset{+}{+} \overset{b^2}{(5v)^2}\right)$
Simplify.	$\left(\overset{a}{3u} \overset{-}{-} \overset{b}{5v}\right)\left(\overset{a^2}{9u^2} \overset{+}{+} \overset{ab}{15uv} \overset{+}{+} \overset{b^2}{25v^2}\right)$
Check by multiplying.	We'll leave the check to you.

Note:
Exercise:

Problem: Factor: $8x^3 - 27y^3$.

Solution:

$(2x - 3y)\left(4x^2 + 6xy + 9y^2\right)$

Note:
Exercise:

Problem: Factor: $1000m^3 - 125n^3$.

Solution:

$(10m - 5n)\left(100m^2 + 50mn + 25n^2\right)$

In the next example, we first factor out the GCF. Then we can recognize the sum of cubes.

Example:
Exercise:

Problem: Factor: $5m^3 + 40n^3$.

Solution:
Solution

	$5m^3 + 40n^3$
Factor the common factor.	$5(m^3 + 8n^3)$
This binomial is a sum. The first and last terms are perfect cubes.	

Write the terms as cubes.	$5\left(\overset{a^3}{m^3} \overset{+}{+} \overset{b^3}{(2n)^3}\right)$
Use the sum of cubes pattern.	$5\left(\overset{a}{m} \overset{+}{+} \overset{b}{2n}\right)\left(\overset{a^2}{m^2} \overset{-}{-} \overset{ab}{m \cdot 2n} \overset{+}{+} \overset{b^2}{(2n)^2}\right)$
Simplify.	$5\left(\overset{a}{m} \overset{+}{+} \overset{b}{2n}\right)\left(\overset{a^2}{m^2} \overset{-}{-} \overset{ab}{2mn} \overset{+}{+} \overset{b^2}{4n^2}\right)$

Check. To check, you may find it easier to multiply the sum of cubes factors first, then multiply that product by 5. We’ll leave the multiplication for you.

$$5\left(m+2n\right)\left(m^2-2mn+4n^2\right)$$

Note:
Exercise:

Problem: Factor: $500p^3+4q^3$.

Solution:

$$4(5p+q)\left(25p^2-5pq+q^2\right)$$

Note:
Exercise:

Problem: Factor: $432c^3+686d^3$.

Solution:

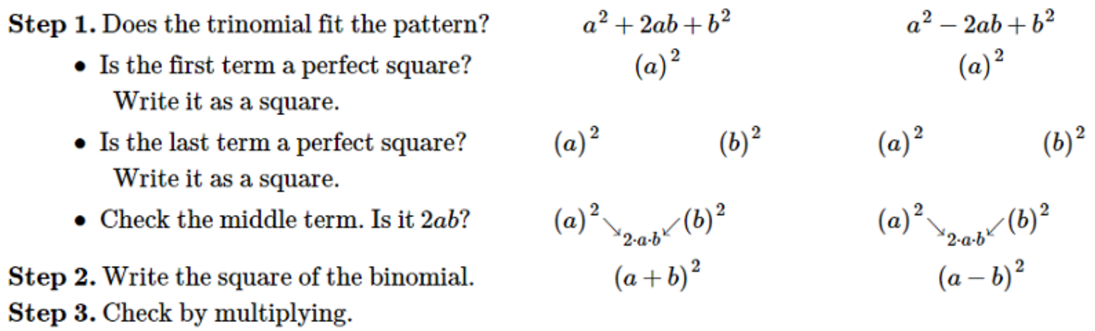
$$2\left(6c+7d\right)\left(36c^2-42cd+49d^2\right)$$

Note:
Access these online resources for additional instruction and practice with factoring special products.

- Sum of Difference of Cubes
- Difference of Cubes Factoring

Key Concepts

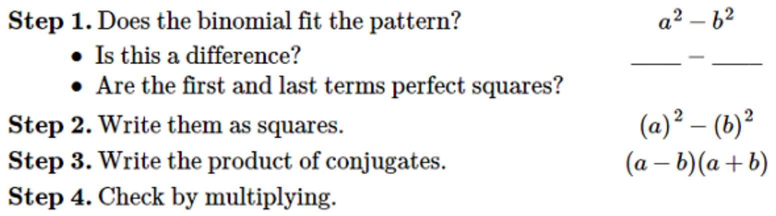
- **Factor perfect square trinomials** See [link].



Step 1. Does the trinomial fit the pattern? $a^2+2ab+b^2$ $a^2-2ab+b^2$
- Is the first term a perfect square? Write it as a square. $(a)^2$ $(a)^2$
- Is the last term a perfect square? Write it as a square. $(a)^2 \quad (b)^2$ $(a)^2 \quad (b)^2$
- Check the middle term. Is it $2ab$? $(a)^2 \searrow_{2\cdot a\cdot b} \swarrow (b)^2$ $(a)^2 \searrow_{2\cdot a\cdot b} \swarrow (b)^2$

Step 2. Write the square of the binomial. $(a+b)^2$ $(a-b)^2$
Step 3. Check by multiplying.

- **Factor differences of squares** See [link].



Step 1. Does the binomial fit the pattern? a^2-b^2
- Is this a difference? ____ − ____
- Are the first and last terms perfect squares?

Step 2. Write them as squares. $(a)^2-(b)^2$
Step 3. Write the product of conjugates. $(a-b)(a+b)$
Step 4. Check by multiplying.

- **Factor sum and difference of cubes** To factor the sum or difference of cubes: See [link].

  Does the binomial fit the sum or difference of cubes pattern? Is it a sum or difference? Are the first and last terms perfect cubes?
  Write them as cubes.
  Use either the sum or difference of cubes pattern.
  Simplify inside the parentheses
  Check by multiplying the factors.

Practice Makes Perfect

Factor Perfect Square Trinomials

In the following exercises, factor.
Exercise:

Problem: $16y^2 + 24y + 9$

Solution:

$(4y + 3)^2$

Exercise:

Problem: $25v^2 + 20v + 4$

Exercise:

Problem: $36s^2 + 84s + 49$

Solution:

$(6s + 7)^2$

Exercise:

Problem: $49s^2 + 154s + 121$

Exercise:

Problem: $100x^2 - 20x + 1$

Solution:

$(10x - 1)^2$

Exercise:

Problem: $64z^2 - 16z + 1$

Exercise:

Problem: $25n^2 - 120n + 144$

Solution:

$(5n - 12)^2$

Exercise:

Problem: $4p^2 - 52p + 169$

Exercise:

Problem: $49x^2 - 28xy + 4y^2$

Solution:

$(7x - 2y)^2$

Exercise:

Problem: $25r^2 - 60rs + 36s^2$

Exercise:

Problem: $25n^2 + 25n + 4$

Solution:

$\left(5n + 4\right)\left(5n + 1\right)$

Exercise:

Problem: $100y^2 - 52y + 1$

Exercise:

Problem: $64m^2 - 34m + 1$

Solution:

$\left(32m - 1\right)\left(2m - 1\right)$

Exercise:

Problem: $100x^2 - 25x + 1$

Exercise:

Problem: $10k^2 + 80k + 160$

Solution:

$10(k + 4)^2$

Exercise:

Problem: $64x^2 - 96x + 36$

Exercise:

Problem: $75u^3 - 30u^2v + 3uv^2$

Solution:

$3u(5u - v)^2$

Exercise:

Problem: $90p^3 + 300p^2q + 250pq^2$

Factor Differences of Squares

In the following exercises, factor.

Exercise:

Problem: $x^2 - 16$

Solution:

$(x - 4)\,(x + 4)$

Exercise:

Problem: $n^2 - 9$

Exercise:

Problem: $25v^2 - 1$

Solution:

$(5v - 1)\,(5v + 1)$

Exercise:

Problem: $169q^2 - 1$

Exercise:

Problem: $121x^2 - 144y^2$

Solution:

$(11x - 12y)\,(11x + 12y)$

Exercise:

Problem: $49x^2 - 81y^2$

Exercise:

Problem: $169c^2 - 36d^2$

Solution:

$(13c - 6d)\,(13c + 6d)$

Exercise:

Problem: $36p^2 - 49q^2$

Exercise:

Problem: $4 - 49x^2$

Solution:

$(2 - 7x)(2 + 7x)$

Exercise:

Problem: $121 - 25s^2$

Exercise:

Problem: $16z^4 - 1$

Solution:

$(2z-1)(2z+1)(4z^2+1)$

Exercise:

Problem: $m^4 - n^4$

Exercise:

Problem: $5q^2 - 45$

Solution:

$5(q-3)(q+3)$

Exercise:

Problem: $98r^3 - 72r$

Exercise:

Problem: $24p^2 + 54$

Solution:

$6\left(4p^2+9\right)$

Exercise:

Problem: $20b^2 + 140$

Factor Sums and Differences of Cubes

In the following exercises, factor.

Exercise:

Problem: $x^3 + 125$

Solution:

$(x + 5)(x^2 - 5x + 25)$

Exercise:

Problem: $n^3 + 512$

Exercise:

Problem: $z^3 - 27$

Solution:

$(z - 3)(z^2 + 3z + 9)$

Exercise:

Problem: $v^3 - 216$

Exercise:

Problem: $8 - 343t^3$

Solution:

$(2 - 7t)(4 + 14t + 49t^2)$

Exercise:

Problem: $125 - 27w^3$

Exercise:

Problem: $8y^3 - 125z^3$

Solution:

$(2y - 5z)(4y^2 + 10yz + 25z^2)$

Exercise:

Problem: $27x^3 - 64y^3$

Exercise:

Problem: $7k^3 + 56$

Solution:

$7(k + 2)(k^2 - 2k + 4)$

Exercise:

Problem: $6x^3 - 48y^3$

Exercise:

Problem: $2 - 16y^3$

Solution:

$2(1 - 2y)(1 + 2y + 4y^2)$

Exercise:

Problem: $-2x^3 - 16y^3$

Mixed Practice

In the following exercises, factor.

Exercise:

Problem: $64a^2 - 25$

Solution:

$\left(8a - 5\right)\left(8a + 5\right)$

Exercise:

Problem: $121x^2 - 144$

Exercise:

Problem: $27q^2 - 3$

Solution:

$3\left(3q - 1\right)\left(3q + 1\right)$

Exercise:

Problem: $4p^2 - 100$

Exercise:

Problem: $16x^2 - 72x + 81$

Solution:

$\left(4x - 9\right)^2$

Exercise:

Problem: $36y^2 + 12y + 1$

Exercise:

Problem: $8p^2 + 2$

Solution:

$2\left(4p^2 + 1\right)$

Exercise:

Problem: $81x^2 + 169$

Exercise:

Problem: $125 - 8y^3$

Solution:

$(5 - 2y)(25 + 10y + 4y^2)$

Exercise:

Problem: $27u^3 + 1000$

Exercise:

Problem: $45n^2 + 60n + 20$

Solution:

$5(3n + 2)^2$

Exercise:

Problem: $48q^3 - 24q^2 + 3q$

Everyday Math

Exercise:

Problem:

Landscaping Sue and Alan are planning to put a 15 foot square swimming pool in their backyard. They will surround the pool with a tiled deck, the same width on all sides. If the width of the deck is w, the total area of the pool and deck is given by the trinomial $4w^2 + 60w + 225$. Factor the trinomial.

Solution:

$(2w + 15)^2$

Exercise:

Problem:

Home repair The height a twelve foot ladder can reach up the side of a building if the ladder's base is b feet from the building is the square root of the binomial $144 - b^2$. Factor the binomial.

Writing Exercises

Exercise:

Problem:

Why was it important to practice using the binomial squares pattern in the chapter on multiplying polynomials?

Solution:

Answers may vary.

Exercise:

Problem: How do you recognize the binomial squares pattern?

Exercise:

Problem:

Explain why $n^2 + 25 \neq (n + 5)^2$. Use algebra, words, or pictures.

Solution:

Answers may vary.

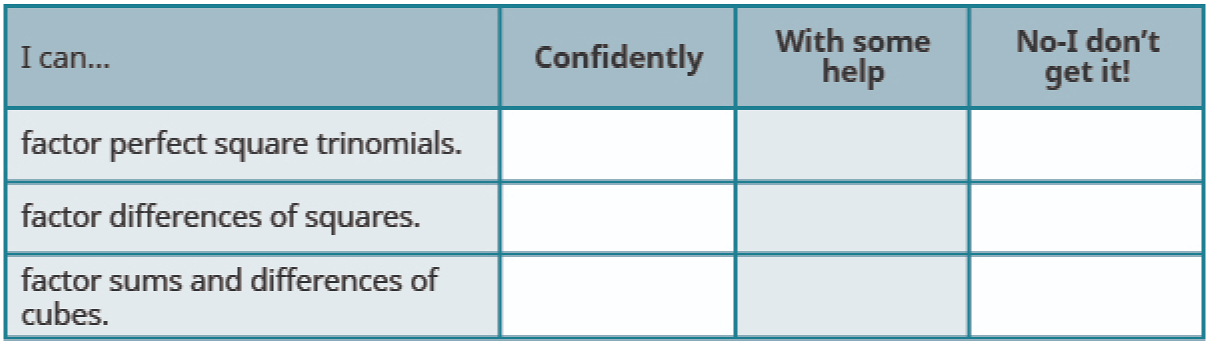
Exercise:

Problem:

Maribel factored $y^2 - 30y + 81$ as $(y - 9)^2$. Was she right or wrong? How do you know?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
factor perfect square trinomials.			
factor differences of squares.			
factor sums and differences of cubes.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

perfect square trinomials pattern

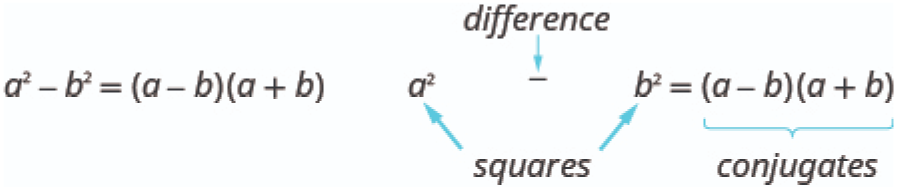
If a and b are real numbers,

Equation:

$$a^2 + 2ab + b^2 = (a+b)^2 \qquad a^2 - 2ab + b^2 = (a-b)^2$$

difference of squares pattern

If a and b are real numbers,



$$a^2 - b^2 = (a-b)(a+b) \qquad a^2 - b^2 = (a-b)(a+b)$$

sum and difference of cubes pattern

Equation:

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

General Strategy for Factoring Polynomials

By the end of this section, you will be able to:

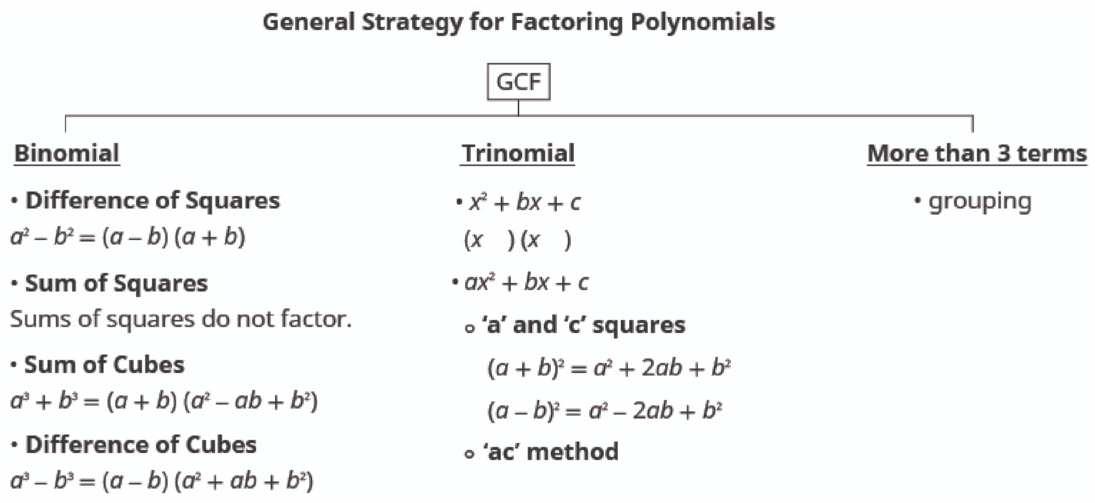
- Recognize and use the appropriate method to factor a polynomial completely

Note:
Before you get started, take this readiness quiz.

1. Factor $y^2 - 2y - 24$.
   If you missed this problem, review [link].
2. Factor $3t^2 + 17t + 10$.
   If you missed this problem, review [link].
3. Factor $36p^2 - 60p + 25$.
   If you missed this problem, review [link].
4. Factor $5x^2 - 80$.
   If you missed this problem, review [link].

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

You have now become acquainted with all the methods of factoring that you will need in this course. (In your next algebra course, more methods will be added to your repertoire.) The figure below summarizes all the factoring methods we have covered. [link] outlines a strategy you should use when factoring polynomials.


General Strategy for Factoring Polynomials
GCF
Binomial
• Difference of Squares
$a^2 - b^2 = (a - b)(a + b)$
• Sum of Squares
Sums of squares do not factor.
• Sum of Cubes
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
• Difference of Cubes
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
Trinomial
• $x^2 + bx + c$
$(x \quad)(x \quad)$
• $ax^2 + bx + c$
○ 'a' and 'c' squares
$(a + b)^2 = a^2 + 2ab + b^2$
$(a - b)^2 = a^2 - 2ab + b^2$
○ 'ac' method
More than 3 terms
• grouping

Note:
Factor polynomials.

Is there a greatest common factor?

- Factor it out.

Is the polynomial a binomial, trinomial, or are there more than three terms?

- If it is a binomial:
  Is it a sum?
    - Of squares? Sums of squares do not factor.
    - Of cubes? Use the sum of cubes pattern.

  Is it a difference?
    - Of squares? Factor as the product of conjugates.
    - Of cubes? Use the difference of cubes pattern.
- If it is a trinomial:
  Is it of the form $x^2 + bx + c$? Undo FOIL.
  Is it of the form $ax^2 + bx + c$?
    - If a and c are squares, check if it fits the trinomial square pattern.
    - Use the trial and error or “ac” method.
- If it has more than three terms:
  Use the grouping method.

Check.
- Is it factored completely?
- Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are prime!

Example:
Exercise:

Problem: Factor completely: $4x^5 + 12x^4$.

Solution:
Solution

Is there a GCF?	Yes, $4x^4$.	$4x^5+12x^4$
	Factor out the GCF.	$4x^4\left(x+3\right)$
In the parentheses, is it a binomial, a trinomial, or are there more than three terms?	Binomial.	
Is it a sum?		Yes.
Of squares? Of cubes?		No.
Check.		
Is the expression factored completely?		Yes.
Multiply.		

$$\begin{array}{l} 4x^4\left(x+3\right) \\ 4x^4\cdot x+4x^4\cdot 3 \\ 4x^5+12x^4\ \checkmark \end{array}$$

Note:
Exercise:

Problem: Factor completely: $3a^4+18a^3$.

Solution:

$3a^3\left(a+6\right)$

Note:
Exercise:

Problem: Factor completely: $45b^6+27b^5$.

Solution:

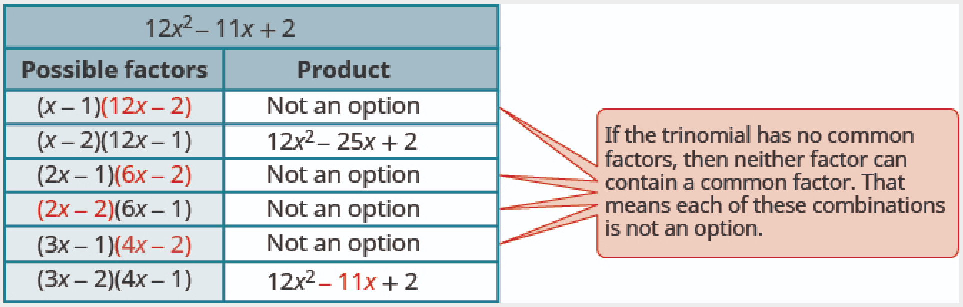
$9b^5\left(5b+3\right)$

Example:
Exercise:

Problem: Factor completely: $12x^2-11x+2$.

Solution:
Solution

		$12x^2 - 11x + 2$
Is there a GCF?	No.	
Is it a binomial, trinomial, or are there more than three terms?	Trinomial.	
Are *a* and *c* perfect squares?	No, $a = 12$, not a perfect square.	
Use trial and error or the “ac” method. We will use trial and error here.		$12x^2 - 11x + 2$ $1x, 12x$ $-1, -2$ $2x, 6x$ $3x, 4x$



$12x^2 - 11x + 2$	
Possible factors	**Product**
$(x - 1)(12x - 2)$	Not an option
$(x - 2)(12x - 1)$	$12x^2 - 25x + 2$
$(2x - 1)(6x - 2)$	Not an option
$(2x - 2)(6x - 1)$	Not an option
$(3x - 1)(4x - 2)$	Not an option
$(3x - 2)(4x - 1)$	$12x^2 - 11x + 2$

If the trinomial has no common factors, then neither factor can contain a common factor. That means each of these combinations is not an option.

Check.

$$(3x - 2)(4x - 1)$$

$$12x^2 - 3x - 8x + 2$$

$$12x^2 - 11x + 2\ \checkmark$$

Note:
Exercise:

Problem: Factor completely: $10a^2 - 17a + 6$.

Solution:

$(5a - 6)(2a - 1)$

Note:
Exercise:

Problem: Factor completely: $8x^2 - 18x + 9$.

Solution:

$\left(2x - 3\right)\left(4x - 3\right)$

Example:
Exercise:

Problem: Factor completely: $g^3 + 25g$.

Solution:
Solution

Is there a GCF?	Yes, g.	$g^3 + 25g$
Factor out the GCF.		$g\left(g^2 + 25\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum ? Of squares?	Yes.	Sums of squares are prime.

Check.

Is the expression factored completely?	Yes.

Multiply.

$g\left(g^2 + 25\right)$

$g^3 + 25g$ ✓

Note:
Exercise:

Problem: Factor completely: $x^3 + 36x$.

Solution:

$x\left(x^2 + 36\right)$

Note:
Exercise:

Problem: Factor completely: $27y^2 + 48$.

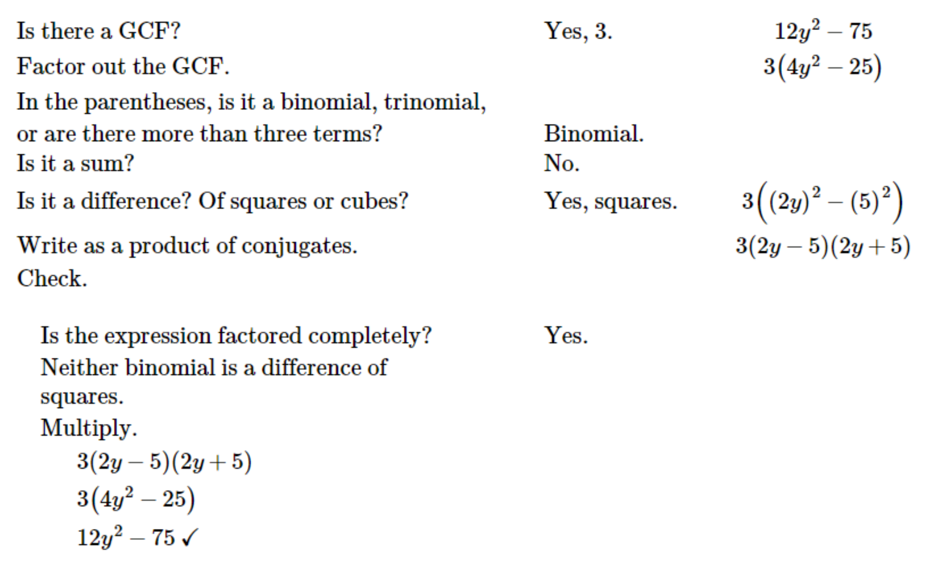
Solution:

$3\left(9y^2+16\right)$

Example:
Exercise:

Problem: Factor completely: $12y^2-75$.

Solution:
Solution



Is there a GCF?	Yes, 3.	$12y^2-75$
Factor out the GCF.		$3\left(4y^2-25\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum?	No.	
Is it a difference? Of squares or cubes?	Yes, squares.	$3\left((2y)^2-(5)^2\right)$
Write as a product of conjugates.		$3(2y-5)(2y+5)$
Check.		
Is the expression factored completely? Neither binomial is a difference of squares.	Yes.	
Multiply. $3(2y-5)(2y+5)$ $3\left(4y^2-25\right)$ $12y^2-75$ ✓		

Note:
Exercise:

Problem: Factor completely: $16x^3-36x$.

Solution:

$4x\left(2x-3\right)\left(2x+3\right)$

Note:
Exercise:

Problem: Factor completely: $27y^2-48$.

Solution:

$3\left(3y-4\right)\left(3y+4\right)$

Example:
Exercise:

Problem: Factor completely: $4a^2-12ab+9b^2$.

Solution:
Solution

Is there a GCF?	No.	$4a^2 - 12ab + 9b^2$
Is it a binomial, trinomial, or are there more terms?		
Trinomial with $a \neq 1$. But the first term is a perfect square.		
Is the last term a perfect square?	Yes.	$(2a)^2 - 12ab + (3b)^2$
Does it fit the pattern, $a^2-2ab+b^2$?	Yes.	$(2a)^2 - 12ab + (3b)^2$ $-2(2a)(3b)$ $12ab$
Write it as a square.		$(2a - 3b)^2$
Check your answer.		
Is the expression factored completely?		
Yes.		
The binomial is not a difference of squares.		
Multiply.		
$\left(2a-3b\right)^2$		
$\left(2a\right)^2-2\cdot 2a\cdot 3b+\left(3b\right)^2$		

$4a^2 - 12ab + 9b^2$✓		

Note:
Exercise:

Problem: Factor completely: $4x^2 + 20xy + 25y^2$.

Solution:

$(2x + 5y)^2$

Note:
Exercise:

Problem: Factor completely: $9m^2 + 42mn + 49n^2$.

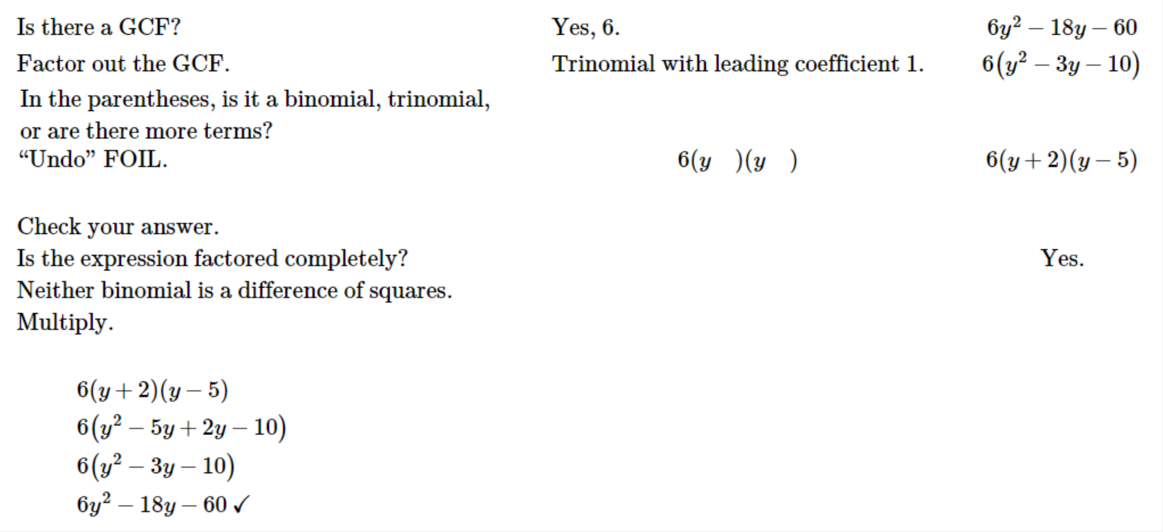
Solution:

$(3m + 7n)^2$

Example:
Exercise:

Problem: Factor completely: $6y^2 - 18y - 60$.

Solution:
Solution



Is there a GCF?	Yes, 6.	$6y^2-18y-60$
Factor out the GCF.	Trinomial with leading coefficient 1.	$6\left(y^2-3y-10\right)$
In the parentheses, is it a binomial, trinomial, or are there more terms?		
“Undo” FOIL.	$6(y\quad)(y\quad)$	$6(y+2)(y-5)$
Check your answer.		
Is the expression factored completely?		Yes.
Neither binomial is a difference of squares.		
Multiply.		

$$6(y+2)(y-5)$$
$$6\left(y^2-5y+2y-10\right)$$
$$6\left(y^2-3y-10\right)$$
$$6y^2-18y-60\,\checkmark$$

Note:
Exercise:

Problem: Factor completely: $8y^2+16y-24$.

Solution:

$8\,(y-1)\,(y+3)$

Note:
Exercise:

Factor completely: $5u^2-15u-270$.
Problem:

Solution:

$5\,(u-9)\,(u+6)$

Example:
Exercise:

Problem: Factor completely: $24x^3+81$.

Solution:
Solution

Is there a GCF?	Yes, 3.	$24x^3+81$
Factor it out.		$3\left(8x^3+27\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum or difference?	Sum.	
Of squares or cubes?	Sum of cubes.	$3\left(\overset{a^3}{(2x)^3}\overset{+}{+}\overset{b^3}{(3)^3}\right)$
Write it using the sum of cubes pattern.		$3\left(\overset{a}{2x}\overset{+}{+}\overset{b}{3}\right)\left(\overset{a^2}{(2x)^2}\overset{-}{-}\overset{ab}{2x\cdot 3}\overset{+}{+}\overset{b^2}{3^2}\right)$
Is the expression factored completely?	Yes.	$3\left(2x+3\right)\left(4x^2-6x+9\right)$
Check by multiplying.		We leave the check to you.

Note:
Exercise:

Problem: Factor completely: $250m^3+432$.

Solution:

$2\left(5m+6\right)\left(25m^2-30m+36\right)$

Note:
Exercise:

Problem: Factor completely: $81q^3+192$.

Solution:

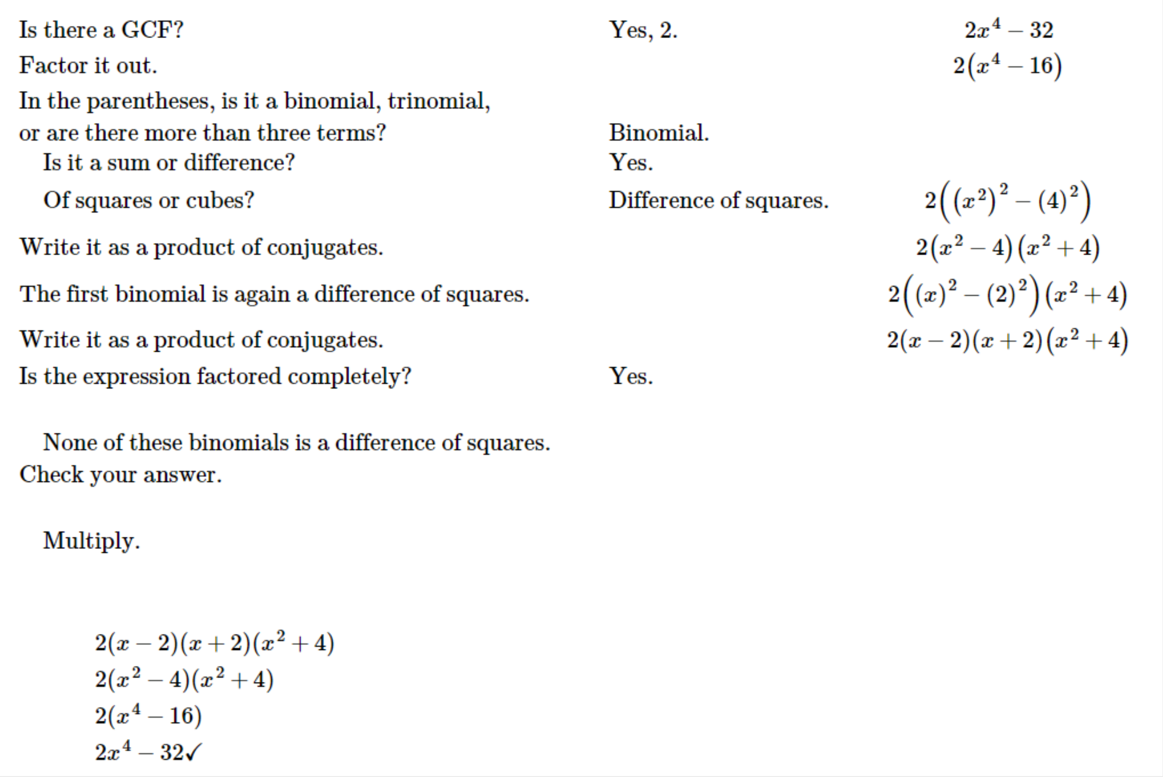
$3\left(3q+4\right)\left(9q^2-12q+16\right)$

Example:

Exercise:

Problem: Factor completely: $2x^4 - 32$.

Solution:
Solution



Is there a GCF?	Yes, 2.	$2x^4 - 32$
Factor it out.		$2(x^4 - 16)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum or difference?	Yes.	
Of squares or cubes?	Difference of squares.	$2\left((x^2)^2 - (4)^2\right)$
Write it as a product of conjugates.		$2(x^2 - 4)(x^2 + 4)$
The first binomial is again a difference of squares.		$2\left((x)^2 - (2)^2\right)(x^2 + 4)$
Write it as a product of conjugates.		$2(x - 2)(x + 2)(x^2 + 4)$
Is the expression factored completely?	Yes.	

None of these binomials is a difference of squares.
Check your answer.

Multiply.

$$\begin{array}{l} 2(x-2)(x+2)(x^2+4) \\ 2(x^2-4)(x^2+4) \\ 2(x^4-16) \\ 2x^4-32\checkmark \end{array}$$

Note:
Exercise:

Problem: Factor completely: $4a^4 - 64$.

Solution:

$4\left(a^2+4\right)(a-2)(a+2)$

Note:
Exercise:

Problem: Factor completely: $7y^4 - 7$.

Solution:

$7\left(y^2+1\right)(y-1)(y+1)$

Example:
Exercise:

Problem: Factor completely: $3x^2+6bx-3ax-6ab$.

Solution:
Solution

Is there a GCF?	Yes, 3.	$3x^2+6bx-3ax-6ab$
Factor out the GCF.		$3\left(x^2+2bx-ax-2ab\right)$
In the parentheses, is it a binomial, trinomial, or are there more terms?	More than 3 terms.	
Use grouping.		$3\left[x(x+2b)-a(x+2b)\right]$ $3(x+2b)(x-a)$
Check your answer.		

Is the expression factored completely? Yes.
Multiply.

$3(x+2b)(x-a)$
$3\left(x^2-ax+2bx-2ab\right)$
$3x^2-3ax+6bx-6ab$ ✓

Note:
Exercise:

Problem: Factor completely: $6x^2-12xc+6bx-12bc$.

Solution:

$6(x+b)(x-2c)$

Note:
Exercise:

Problem: Factor completely: $16x^2+24xy-4x-6y$.

Solution:

$2(4x-1)(2x+3y)$

Example:
Exercise:

Problem: Factor completely: $10x^2-34x-24$.

Solution:
Solution

Is there a GCF?	Yes, 2.	$10x^2-34x-24$
Factor out the GCF.		$2\left(5x^2-17x-12\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Trinomial with $a \neq 1$.	
Use trial and error or the "ac" method.		$2(5x^2-17x-12)$ $2\left(5x+3\right)\left(x-4\right)$
Check your answer. Is the expression factored completely? Yes.		

Multiply.

$$2\left(5x+3\right)\left(x-4\right)$$
$$2\left(5x^2-20x+3x-12\right)$$
$$2\left(5x^2-17x-12\right)$$
$$10x^2-34x-24 \checkmark$$

Note:
Exercise:

Problem: Factor completely: $4p^2-16p+12$.

Solution:

$4\left(p-1\right)\left(p-3\right)$

Note:

Exercise:

Problem: Factor completely: $6q^2 - 9q - 6$.

Solution:

$3\left(q - 2\right)\left(2q + 1\right)$

Key Concepts

- **General Strategy for Factoring Polynomials See [link].**
- **How to Factor Polynomials**

  Is there a greatest common factor? Factor it out.
  Is the polynomial a binomial, trinomial, or are there more than three terms?

  - If it is a binomial:
    Is it a sum?
    - Of squares? Sums of squares do not factor.
    - Of cubes? Use the sum of cubes pattern.

    Is it a difference?
    - Of squares? Factor as the product of conjugates.
    - Of cubes? Use the difference of cubes pattern.
  - If it is a trinomial:
    Is it of the form $x^2 + bx + c$? Undo FOIL.
    Is it of the form $ax^2 + bx + c$?
    - If 'a' and 'c' are squares, check if it fits the trinomial square pattern.
    - Use the trial and error or 'ac' method.
  - If it has more than three terms:
    Use the grouping method.

  Check. Is it factored completely? Do the factors multiply back to the original polynomial?

Practice Makes Perfect

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.
Exercise:

Problem: $10x^4 + 35x^3$

Solution:

$5x^3(2x+7)$

Exercise:

Problem: $18p^6+24p^3$

Exercise:

Problem: $y^2+10y-39$

Solution:

$(y-3)(y+13)$

Exercise:

Problem: $b^2-17b+60$

Exercise:

Problem: $2n^2+13n-7$

Solution:

$(2n-1)(n+7)$

Exercise:

Problem: $8x^2-9x-3$

Exercise:

Problem: a^5+9a^3

Solution:

$a^3(a^2+9)$

Exercise:

Problem: $75m^3+12m$

Exercise:

Problem: $121r^2-s^2$

Solution:

$(11r-s)(11r+s)$

Exercise:

Problem: $49b^2 - 36a^2$

Exercise:

Problem: $8m^2 - 32$

Solution:

$8(m-2)(m+2)$

Exercise:

Problem: $36q^2 - 100$

Exercise:

Problem: $25w^2 - 60w + 36$

Solution:

$(5w-6)^2$

Exercise:

Problem: $49b^2 - 112b + 64$

Exercise:

Problem: $m^2 + 14mn + 49n^2$

Solution:

$(m+7n)^2$

Exercise:

Problem: $64x^2 + 16xy + y^2$

Exercise:

Problem: $7b^2 + 7b - 42$

Solution:

$7(b+3)(b-2)$

Exercise:

Problem: $3n^2 + 30n + 72$

Exercise:

Problem: $3x^3 - 81$

Solution:

$3\left(x-3\right)\left({x}^{2}+3x+9\right)$

Exercise:

Problem: $5{t}^{3}-40$

Exercise:

Problem: ${k}^{4}-16$

Solution:

$(k-2)(k+2)({k}^{2}+4)$

Exercise:

Problem: ${m}^{4}-81$

Exercise:

Problem: $15pq-15p+12q-12$

Solution:

$3(5p+4)(q-1)$

Exercise:

Problem: $12ab-6a+10b-5$

Exercise:

Problem: $4{x}^{2}+40x+84$

Solution:

$4\left(x+3\right)\left(x+7\right)$

Exercise:

Problem: $5{q}^{2}-15q-90$

Exercise:

Problem: ${u}^{5}+{u}^{2}$

Solution:

${u}^{2}\left(u+1\right)\left({u}^{2}-u+1\right)$

Exercise:

Problem: $5n^3 + 320$

Exercise:

Problem: $4c^2 + 20cd + 81d^2$

Solution:

prime

Exercise:

Problem: $25x^2 + 35xy + 49y^2$

Exercise:

Problem: $10m^4 - 6250$

Solution:

$10\left(m-5\right)\left(m+5\right)\left(m^2+25\right)$

Exercise:

Problem: $3v^4 - 768$

Everyday Math

Exercise:

Problem:

Watermelon drop A springtime tradition at the University of California San Diego is the Watermelon Drop, where a watermelon is dropped from the seventh story of Urey Hall.

ⓐ The binomial $-16t^2 + 80$ gives the height of the watermelon t seconds after it is dropped. Factor the greatest common factor from this binomial.
ⓑ If the watermelon is thrown down with initial velocity 8 feet per second, its height after t seconds is given by the trinomial $-16t^2 - 8t + 80$. Completely factor this trinomial.

Solution:

ⓐ $-16\left(t^2-5\right)$ ⓑ $-8\left(2t+5\right)\left(t-2\right)$

Exercise:

Problem:

Pumpkin drop A fall tradition at the University of California San Diego is the Pumpkin Drop, where a pumpkin is dropped from the eleventh story of Tioga Hall.

ⓐ The binomial $-16t^2+128$ gives the height of the pumpkin t seconds after it is dropped. Factor the greatest common factor from this binomial.
ⓑ If the pumpkin is thrown down with initial velocity 32 feet per second, its height after t seconds is given by the trinomial $-16t^2-32t+128$. Completely factor this trinomial.

Writing Exercises

Exercise:

Problem:

The difference of squares y^4-625 can be factored as $\left(y^2-25\right)\left(y^2+25\right)$. But it is not *completely* factored. What more must be done to completely factor it?

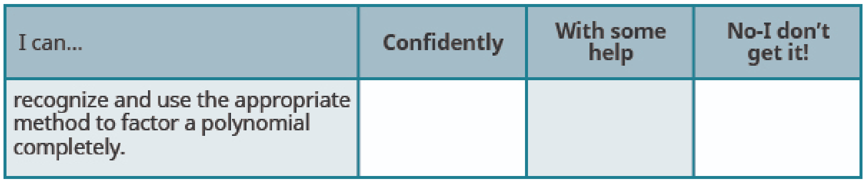
Exercise:

Problem:

Of all the factoring methods covered in this chapter (GCF, grouping, undo FOIL, ‘ac’ method, special products) which is the easiest for you? Which is the hardest? Explain your answers.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don’t get it!
recognize and use the appropriate method to factor a polynomial completely.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Quadratic Equations
By the end of this section, you will be able to:

- Solve quadratic equations by using the Zero Product Property
- Solve quadratic equations by factoring
- Solve applications modeled by quadratic equations

Note:
Before you get started, take this readiness quiz.

1. Solve: $5y - 3 = 0$.
   If you missed this problem, review [link].
2. Solve: $10a = 0$.
   If you missed this problem, review [link].
3. Combine like terms: $12x^2 - 6x + 4x$.
   If you missed this problem, review [link].
4. Factor $n^3 - 9n^2 - 22n$ completely.
   If you missed this problem, review [link].

We have already solved linear equations, equations of the form $ax + by = c$. In linear equations, the variables have no exponents. Quadratic equations are equations in which the variable is squared. Listed below are some examples of quadratic equations:
Equation:

$$x^2 + 5x + 6 = 0 \qquad 3y^2 + 4y = 10 \qquad 64u^2 - 81 = 0 \qquad n\left(n + 1\right) = 42$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get $n^2 + n$.

The general form of a quadratic equation is $ax^2 + bx + c = 0$, with $a \neq 0$.

Note:
Quadratic Equation
An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation.
Equation:

$$a, b, \text{ and } c \text{ are real numbers and } a \neq 0$$

To solve quadratic equations we need methods different than the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Solve Quadratic Equations Using the Zero Product Property

We will first solve some quadratic equations by using the Zero Product Property. The **Zero Product Property** says that if the product of two quantities is zero, it must be that at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

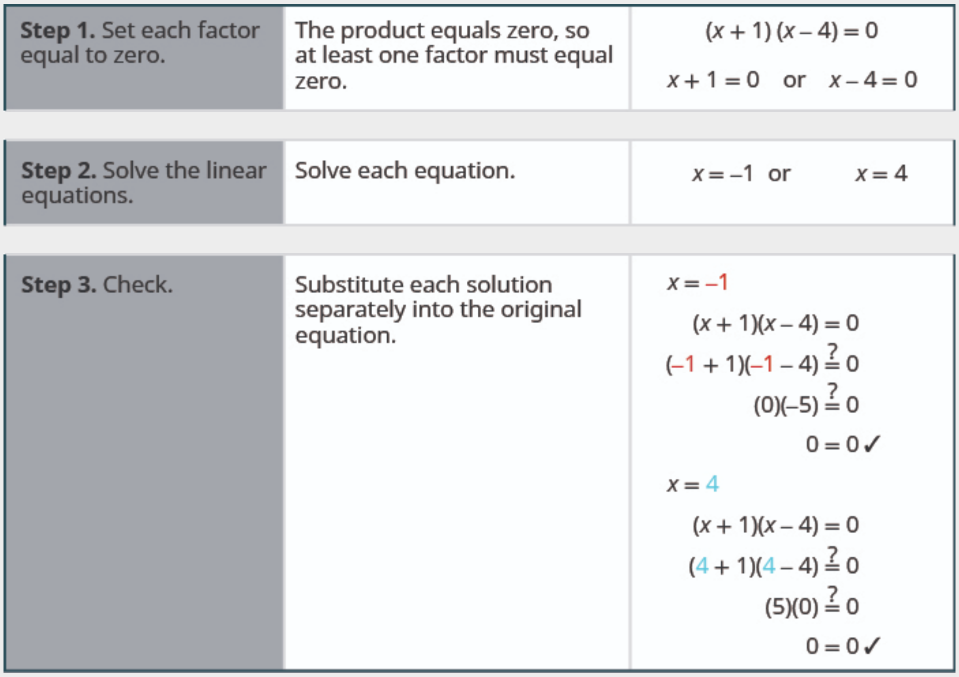
Note:
Zero Product Property
If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.

We will now use the Zero Product Property, to solve a quadratic equation.

Example:
How to Use the Zero Product Property to Solve a Quadratic Equation
Exercise:

Problem: Solve: $\left(x+1\right)\left(x-4\right)=0$.

Solution:
Solution



Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(x+1)(x-4)=0$ $x+1=0 \quad \text{or} \quad x-4=0$
Step 2. Solve the linear equations.	Solve each equation.	$x=-1 \quad \text{or} \quad x=4$
Step 3. Check.	Substitute each solution separately into the original equation.	$x=-1$ $(x+1)(x-4)=0$ $(-1+1)(-1-4)\overset{?}{=}0$ $(0)(-5)\overset{?}{=}0$ $0=0\checkmark$ $x=4$ $(x+1)(x-4)=0$ $(4+1)(4-4)\overset{?}{=}0$ $(5)(0)\overset{?}{=}0$ $0=0\checkmark$

Therefore, the solutions to this equation are $x=-1$ and $x=4$.

Note:
Exercise:

Problem: Solve: $\left(x-3\right)\left(x+5\right)=0$.

Solution:

$x = 3, x = -5$

Note:
Exercise:

Problem: Solve: $(y - 6)(y + 9) = 0$.

Solution:

$y = 6, y = -9$

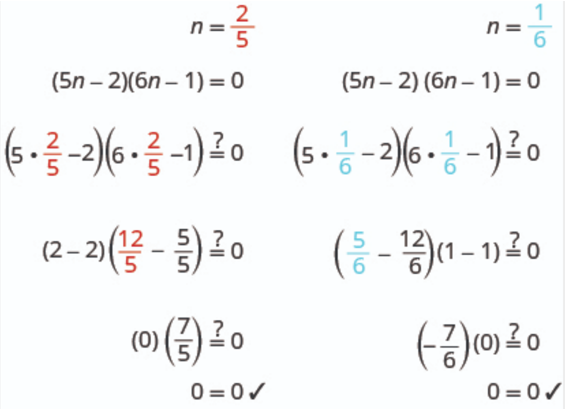
We usually will do a little more work than we did in this last example to solve the linear equations that result from using the Zero Product Property.

Example:
Exercise:

Problem: Solve: $(5n - 2)(6n - 1) = 0$.

Solution:
Solution

	$(5n-2)(6n-1)=0$	
Use the Zero Product Property to set each factor to 0.	$5n-2=0$	$6n-1=0$
Solve the equations.	$n = \frac{2}{5}$	$n = \frac{1}{6}$
Check your answers.		



$$n = \frac{2}{5} \qquad n = \frac{1}{6}$$

$$(5n-2)(6n-1)=0 \qquad (5n-2)(6n-1)=0$$

$$\left(5\cdot\frac{2}{5}-2\right)\left(6\cdot\frac{2}{5}-1\right)\stackrel{?}{=}0 \qquad \left(5\cdot\frac{1}{6}-2\right)\left(6\cdot\frac{1}{6}-1\right)\stackrel{?}{=}0$$

$$(2-2)\left(\frac{12}{5}-\frac{5}{5}\right)\stackrel{?}{=}0 \qquad \left(\frac{5}{6}-\frac{12}{6}\right)(1-1)\stackrel{?}{=}0$$

$$(0)\left(\frac{7}{5}\right)\stackrel{?}{=}0 \qquad \left(-\frac{7}{6}\right)(0)\stackrel{?}{=}0$$

$$0=0\checkmark \qquad 0=0\checkmark$$

This image shows the steps for solving (5 n – 2)(6 n – 1) = 0. First, use the zero factor property to set each factor equal to 0, 5 n – 2 = 0 or 6 n – 1 = 0. Then, solve the equations, n = 2/5 or n = 1/6. Finally, check the answers by substituting the two solutions back into the original equation.

Note:
Exercise:

Problem: Solve: $(3m-2)(2m+1)=0$.

Solution:

$m=\frac{2}{3}, m=-\frac{1}{2}$

Note:
Exercise:

Problem: Solve: $(4p+3)(4p-3)=0$.

Solution:

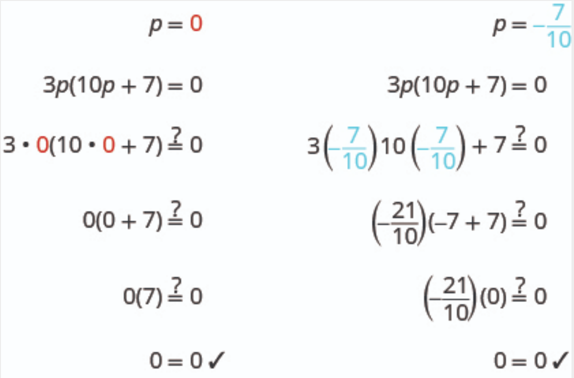
$p=-\frac{3}{4}, p=\frac{3}{4}$

Notice when we checked the solutions that each of them made just one factor equal to zero. But the product was zero for both solutions.

Example:
Exercise:

Problem: Solve: $3p(10p+7)=0$.

Solution:
Solution

	$3p\left(10p+7\right)=0$	
Use the Zero Product Property to set each factor to 0.	$3p=0$	$10p+7=0$
Solve the equations.	$p=0$	$10p=-7$
		$p=-\frac{7}{10}$
Check your answers.		
 $p=0$ $3p(10p+7)=0$ $3\cdot 0(10\cdot 0+7)\overset{?}{=}0$ $0(0+7)\overset{?}{=}0$ $0(7)\overset{?}{=}0$ $0=0$ ✓ $p=-\frac{7}{10}$ $3p(10p+7)=0$ $3\left(-\frac{7}{10}\right)10\left(-\frac{7}{10}\right)+7\overset{?}{=}0$ $\left(-\frac{21}{10}\right)(-7+7)\overset{?}{=}0$ $\left(-\frac{21}{10}\right)(0)\overset{?}{=}0$ $0=0$ ✓		

This image shows the steps for solving 3 p (10 p + 7) = 0. The first step is using the zero product property to set each factor equal to 0, 3p = 0 or 10 p + 7 = 0. The next step is solving both equations, p = 0 or p = negative 7/10. Finally, check the solutions by substituting the answers into the original equation.

Note:
Exercise:

Problem: Solve: $2u\left(5u-1\right)=0.$

Solution:

$u=0,u=\frac{1}{5}$

Note:
Exercise:

Problem: Solve: $w\left(2w+3\right)=0$.

Solution:

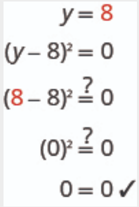
$w=0,w=-\frac{3}{2}$

It may appear that there is only one factor in the next example. Remember, however, that ${\left(y-8\right)}^{2}$ means $\left(y-8\right)\left(y-8\right)$.

Example:
Exercise:

Problem: Solve: ${\left(y-8\right)}^{2}=0$.

Solution:
Solution

	${\left(y-8\right)}^{2}=0$	
Rewrite the left side as a product.	$(y-8)(y-8)=0$	
Use the Zero Product Property and set each factor to 0.	$y-8=0$	$y-8=0$
Solve the equations.	$y=8$	$y=8$
When a solution repeats, we call it a double root.		
Check your answer.		
 $y=8$ $(y-8)^2=0$ $(8-8)^2\overset{?}{=}0$ $(0)^2\overset{?}{=}0$ $0=0\,✓$		

This image shows the steps for solving the equation (y – 8) squared = 0. The first step is to write the left hand side as a product, (y – 8)(y – 8) = 0. The next step is using the zero product property and set each factor equal to 0, y – 8 = 0 and y – 8 -= 0. Solve both equations, y = 8 and y = 8. When the solution repeats, it is a double root. Finally, check the solution by substituting back into the original equation.

Note:
Exercise:

Problem: Solve: $(x+1)^2=0.$

Solution:

$x=-1$

Note:
Exercise:

Problem: Solve: $(v-2)^2=0.$

Solution:

$v=2$

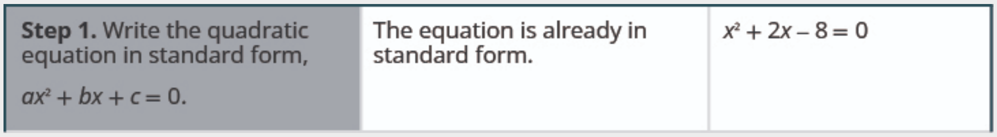
Solve Quadratic Equations by Factoring

Each of the equations we have solved in this section so far had one side in factored form. In order to use the Zero Product Property, the quadratic equation must be factored, with zero on one side. So we be sure to start with the quadratic equation in standard form, $ax^2+bx+c=0$. Then we factor the expression on the left.

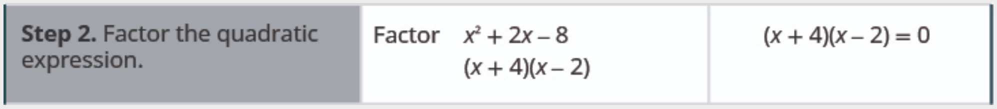
Example:
How to Solve a Quadratic Equation by Factoring
Exercise:

Problem: Solve: $x^2+2x-8=0.$

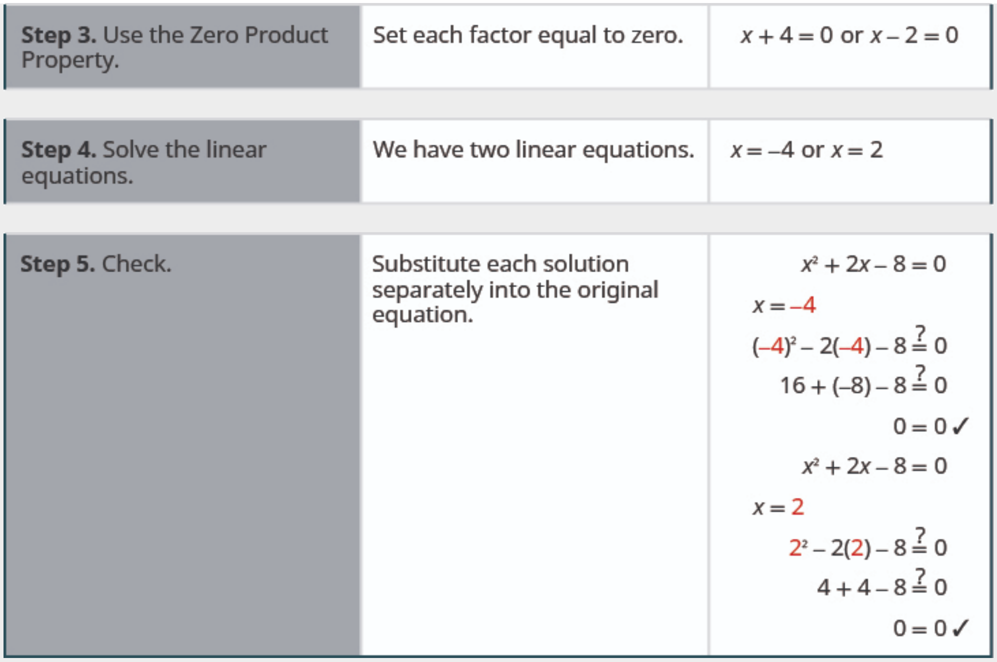
Solution:
Solution



Step 1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.	The equation is already in standard form.	$x^2+2x-8=0$



Step 2. Factor the quadratic expression.	Factor x^2+2x-8 $(x+4)(x-2)$	$(x+4)(x-2)=0$



Step 3. Use the Zero Product Property.	Set each factor equal to zero.	$x+4=0$ or $x-2=0$
Step 4. Solve the linear equations.	We have two linear equations.	$x=-4$ or $x=2$
Step 5. Check.	Substitute each solution separately into the original equation.	$x^2+2x-8=0$ $x=-4$ $(-4)^2-2(-4)-8\overset{?}{=}0$ $16+(-8)-8\overset{?}{=}0$ $0=0$ ✓ $x^2+2x-8=0$ $x=2$ $2^2-2(2)-8\overset{?}{=}0$ $4+4-8\overset{?}{=}0$ $0=0$ ✓

Note:
Exercise:

Problem: Solve: $x^2-x-12=0$.

Solution:

$x=4, x=-3$

Note:
Exercise:

Problem: Solve: $b^2+9b+14=0$.

Solution:

$b=-2, b=-7$

Note:
Solve a quadratic equation by factoring.

Write the quadratic equation in standard form, $ax^2+bx+c=0$.
Factor the quadratic expression.
Use the Zero Product Property.
Solve the linear equations.

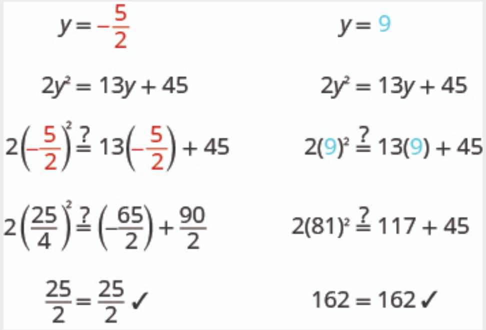
Check.

Before we factor, we must make sure the quadratic equation is in standard form.

Example:
Exercise:

Problem: Solve: $2y^2 = 13y + 45$.

Solution:
Solution

	$2y^2 = 13y + 45$	
Write the quadratic equation in standard form.	$2y^2 - 13y - 45 = 0$	
Factor the quadratic expression.	$(2y + 5)(y - 9) = 0$	
Use the Zero Product Property to set each factor to 0.	$2y + 5 = 0$	$y - 9 = 0$
Solve each equation.	$y = -\frac{5}{2}$	$y = 9$
Check your answers.		
 $y = -\frac{5}{2}$ $\quad$ $y = 9$ $2y^2 = 13y + 45$ $\quad$ $2y^2 = 13y + 45$ $2\left(-\frac{5}{2}\right)^2 \overset{?}{=} 13\left(-\frac{5}{2}\right) + 45$ $\quad$ $2(9)^2 \overset{?}{=} 13(9) + 45$ $2\left(\frac{25}{4}\right)^2 \overset{?}{=} \left(-\frac{65}{2}\right) + \frac{90}{2}$ $\quad$ $2(81)^2 \overset{?}{=} 117 + 45$ $\frac{25}{2} = \frac{25}{2}$ ✓ $\quad$ $162 = 162$ ✓		

This image shows the steps for solving the equation 2 y squared = 13 y + 45. The first step is writing the equation in standard quadratic form, 2 y squared – 13 y – 45 = 0. The second step is to factor the quadratic expression, (2 y + 5)(y – 9) = 0. The third step is to use the zero product property to set each factor equal to 0, 2 y + 5 = 0 or y – 9 = 0. Solve each equation, y = −5/2 or y = 9. Finally, check the answers by substituting them back into the original equation.

Note:
Exercise:

Problem: Solve: $3c^2 = 10c - 8$.

Solution:

$c = 2, c = \frac{4}{3}$

Note:
Exercise:

Problem: Solve: $2d^2 - 5d = 3$.

Solution:

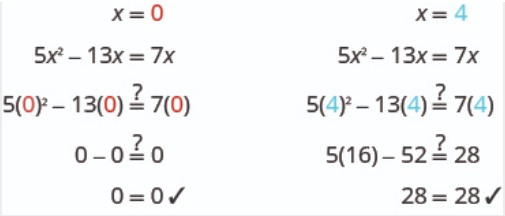
$d = 3, d = -\frac{1}{2}$

Example:
Exercise:

Problem: Solve: $5x^2 - 13x = 7x$.

Solution:
Solution

	$5x^2 - 13x = 7x$	
Write the quadratic equation in standard form.	$5x^2 - 20x = 0$	
Factor the left side of the equation.	$5x(x - 4) = 0$	
Use the Zero Product Property to set each factor to 0.	$5x = 0$	$x - 4 = 0$
Solve each equation.	$x = 0$	$x = 4$
Check your answers.		



$x = 0$	$x = 4$
$5x^2 - 13x = 7x$	$5x^2 - 13x = 7x$
$5(0)^2 - 13(0) \overset{?}{=} 7(0)$	$5(4)^2 - 13(4) \overset{?}{=} 7(4)$
$0 - 0 \overset{?}{=} 0$	$5(16) - 52 \overset{?}{=} 28$
$0 = 0$ ✓	$28 = 28$ ✓

This image shows the steps for solving the equation 5 x squared – 13 x = 7 x. The first step is writing the equation in standard quadratic form, 5 x squared – 20 x = 0. The second step is to factor the quadratic expression, 5 x(x – 4)= 0. The third step is to use the zero product property to set each factor equal to 0, 5 x = 0 or x − 4 = 0. Solve each equation, x = 0 or x = 4. Finally, check the answers by substituting them back into the original equation.

Note:
Exercise:

Problem: Solve: $6a^2 + 9a = 3a$.

Solution:

$a = 0, a = -1$

Note:
Exercise:

Problem: Solve: $45b^2 - 2b = -17b$.

Solution:

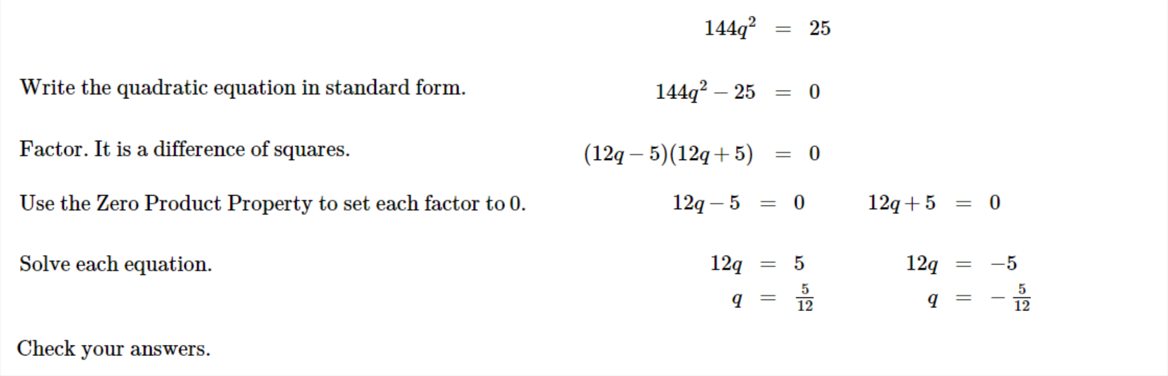
$b = 0, b = -\frac{1}{3}$

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

Example:
Exercise:

Problem: Solve: $144q^2 = 25$.

Solution:
Solution



	$144q^2 = 25$
Write the quadratic equation in standard form.	$144q^2 - 25 = 0$
Factor. It is a difference of squares.	$(12q-5)(12q+5) = 0$
Use the Zero Product Property to set each factor to 0.	$12q - 5 = 0 \qquad 12q + 5 = 0$
Solve each equation.	$12q = 5 \qquad 12q = -5$ $q = \frac{5}{12} \qquad q = -\frac{5}{12}$
Check your answers.	

Note:
Exercise:

Problem: Solve: $25p^2 = 49$.

Solution:

$p = \frac{7}{5}$, p $= -\frac{7}{5}$

Note:
Exercise:

Problem: Solve: $36x^2 = 121$.

Solution:

$x = \frac{11}{6}, x = -\frac{11}{6}$

The left side in the next example is factored, but the right side is not zero. In order to use the Zero Product Property, one side of the equation must be zero. We’ll multiply the factors and then write the equation in standard form.

Example:
Exercise:

Problem: Solve: $(3x-8)(x-1) = 3x$.

Solution:
Solution

	$(3x-8)(x-1) = 3x$	
Multiply the binomials.	$3x^2-11x+8 = 3x$	
Write the quadratic equation in standard form.	$3x^2-14x+8 = 0$	
Factor the trinomial.	$(3x-2)(x-4) = 0$	
Use the Zero Product Property to set each factor to 0.	$3x-2 = 0$	$x-4 = 0$
Solve each equation.	$3x = 2$	$x = 4$
	$x = \frac{2}{3}$	
Check your answers.	The check is left to you!	

Note:
Exercise:

Problem: Solve: $(2m+1)(m+3) = 12m$.

Solution:

$m = 1, m = \frac{3}{2}$

Note:
Exercise:

Problem: Solve: $(k+1)(k-1) = 8$.

Solution:

$k = 3, k = -3$

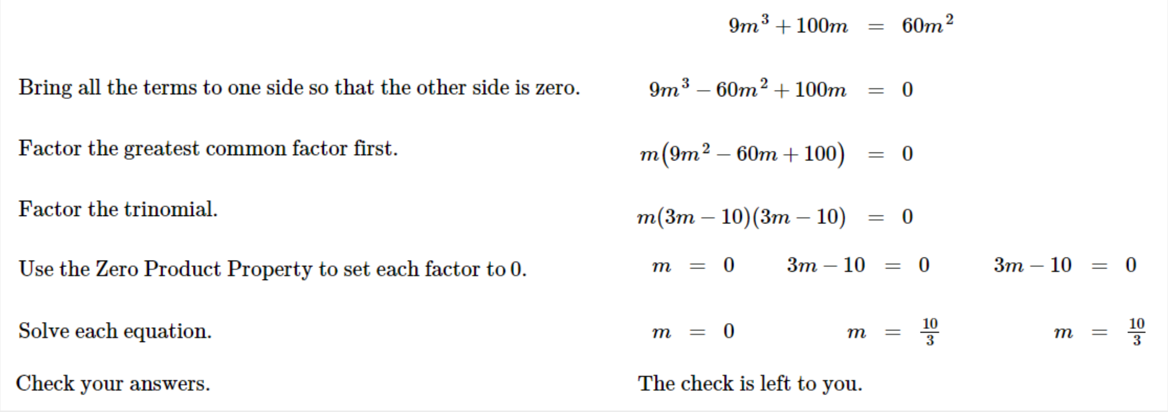
The Zero Product Property also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree more than two by using the Zero Product Property, just like we solved quadratic equations.

Example:
Exercise:

Problem: Solve: $9m^3+100m = 60m^2$.

Solution:

Solution



	$9m^3 + 100m = 60m^2$
Bring all the terms to one side so that the other side is zero.	$9m^3 - 60m^2 + 100m = 0$
Factor the greatest common factor first.	$m(9m^2 - 60m + 100) = 0$
Factor the trinomial.	$m(3m - 10)(3m - 10) = 0$
Use the Zero Product Property to set each factor to 0.	$m = 0 \quad 3m - 10 = 0 \quad 3m - 10 = 0$
Solve each equation.	$m = 0 \quad m = \frac{10}{3} \quad m = \frac{10}{3}$
Check your answers.	The check is left to you.

Note:
Exercise:

Problem: Solve: $8x^3 = 24x^2 - 18x$.

Solution:

$x = 0, x = \frac{3}{2}$

Note:
Exercise:

Problem: Solve: $16y^2 = 32y^3 + 2y$.

Solution:

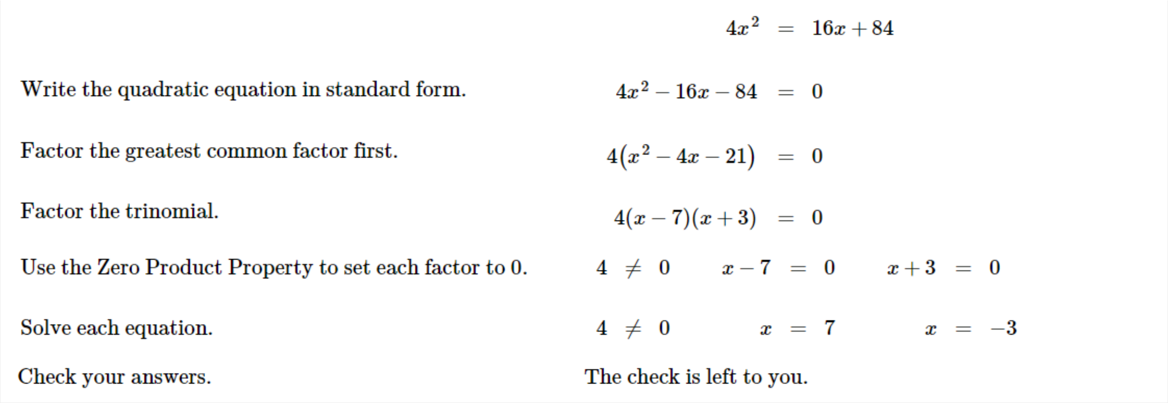
$y = 0, y = \frac{1}{4}$

When we factor the quadratic equation in the next example we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

Example:
Exercise:

Problem: Solve: $4x^2 = 16x + 84$.

Solution:
Solution



	$4x^2 = 16x + 84$		
Write the quadratic equation in standard form.	$4x^2 - 16x - 84 = 0$		
Factor the greatest common factor first.	$4(x^2 - 4x - 21) = 0$		
Factor the trinomial.	$4(x-7)(x+3) = 0$		
Use the Zero Product Property to set each factor to 0.	$4 \neq 0$	$x - 7 = 0$	$x + 3 = 0$
Solve each equation.	$4 \neq 0$	$x = 7$	$x = -3$
Check your answers.	The check is left to you.		

Note:
Exercise:

Problem: Solve: $18a^2 - 30 = -33a$.

Solution:

$a = -\frac{5}{2}, a = \frac{2}{3}$

Note:
Exercise:

Problem: Solve: $123b = -6 - 60b^2$.

Solution:

$b = -2, b = -\frac{1}{20}$

Solve Applications Modeled by Quadratic Equations

The problem solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to quadratic equations. We will copy the problem solving strategy here so we can use it for reference.

Note:
Use a problem-solving strategy to solve word problems.

Readthe problem. Make sure all the words and ideas are understood.
Identifywhat we are looking for.
Namewhat we are looking for. Choose a variable to represent that quantity.
Translateinto an equation. It may be helpful to restate the problem in one sentence with all the important

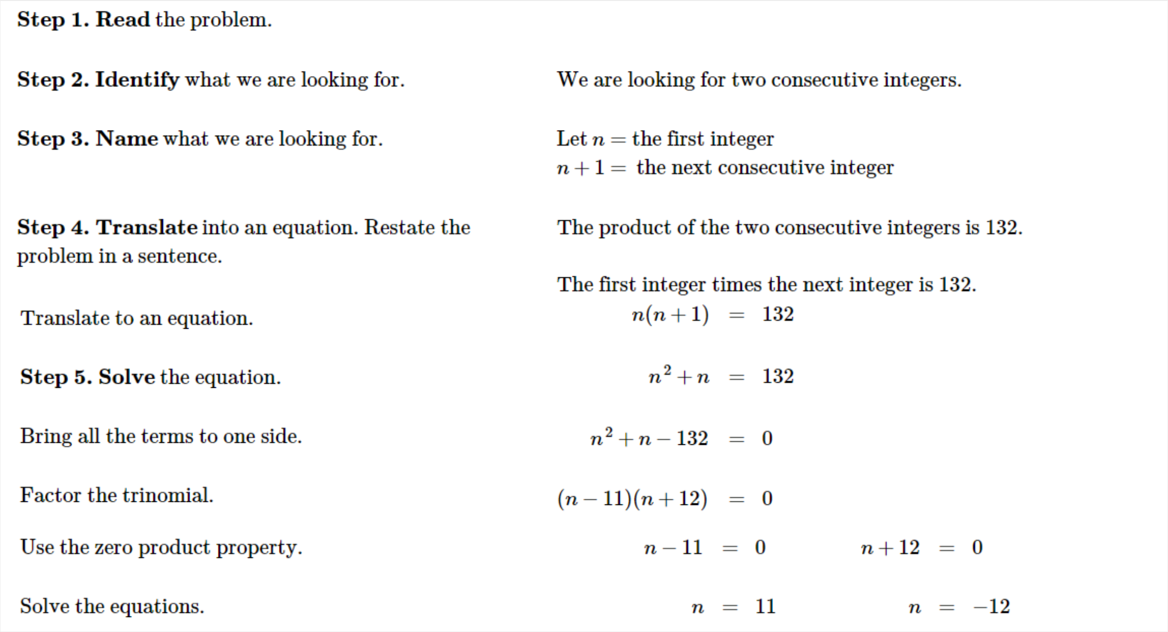
information. Then, translate the English sentence into an algebra equation.
Solvethe equation using good algebra techniques.
Checkthe answer in the problem and make sure it makes sense.
Answerthe question with a complete sentence.

We will start with a number problem to get practice translating words into a quadratic equation.

Example:
Exercise:

Problem: The product of two consecutive integers is 132. Find the integers.

Solution:
Solution



Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two consecutive integers.
Step 3. Name what we are looking for.	Let $n =$ the first integer $n+1 =$ the next consecutive integer
Step 4. Translate into an equation. Restate the problem in a sentence.	The product of the two consecutive integers is 132. The first integer times the next integer is 132.
Translate to an equation.	$n(n+1) = 132$
Step 5. Solve the equation.	$n^2+n = 132$
Bring all the terms to one side.	$n^2+n-132 = 0$
Factor the trinomial.	$(n-11)(n+12) = 0$
Use the zero product property.	$n-11 = 0 \qquad n+12 = 0$
Solve the equations.	$n = 11 \qquad n = -12$

There are two values for n that are solutions to this problem. So there are two sets of consecutive integers that will work.

If the first integer is $n = 11$	If the first integer is $n = -12$
then the next integer is $n+1$	then the next integer is $n+1$
$11+1$	$-12+1$
12	-11

Step 6. Check the answer.

The consecutive integers are $11, 12$ and $-11, -12$. The product $11 \cdot 12 = 132$ and the product $-11\,(-12) = 132$. Both pairs of consecutive integers are solutions.

Step 7. Answer the question. The consecutive integers are $11, 12$ and $-11, -12$.

Note:
Exercise:

Problem: The product of two consecutive integers is 240. Find the integers.

Solution:

$-15, -16$ and $15, 16$

Note:
Exercise:

Problem: The product of two consecutive integers is 420. Find the integers.

Solution:

$-21, -20$ and $20, 21$

Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give 132.

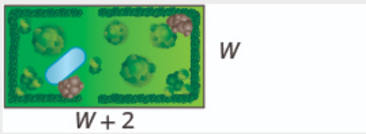
In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

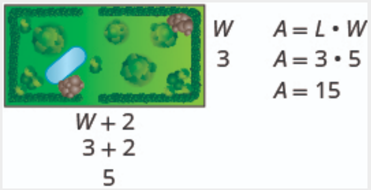
Example:
Exercise:

Problem:

A rectangular garden has an area 15 square feet. The length of the garden is two feet more than the width. Find the length and width of the garden.

Solution:
Solution

Step 1. Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation.	

Step 2. Identify what you are looking for.	We are looking for the length and width.	
Step 3. Name what you are looking for. The length is two feet more than width.	Let W = the width of the garden. $W + 2$ = the length of the garden	
Step 4. Translate into an equation. Restate the important information in a sentence.	The area of the rectangular garden is 15 square feet.	
Use the formula for the area of a rectangle.	$A = L \cdot W$	
Substitute in the variables.	$15 = (W + 2)W$	
Step 5. Solve the equation. Distribute first.	$15 = W^2 + 2W$	
Get zero on one side.	$0 = W^2 + 2W - 15$	
Factor the trinomial.	$0 = (W + 5)(W - 3)$	
Use the Zero Product Property.	$0 = W + 5$	$0 = W - 3$
Solve each equation.	$-5 = W$	$3 = W$
Since W is the width of the garden, it does not make sense for it to be negative. We eliminate that value for W.	$\cancel{-5 = W}$ $W = 3$	$3 = W$ Width is 3 feet.
Find the value of the length.	$W + 2 = \text{length}$	
	$3 + 2$	
	5	Length is 5 feet.
Step 6. Check the answer. Does the answer make sense?		
 W 3 A = L • W A = 3 • 5 A = 15 W + 2 3 + 2 5		
	Yes, this makes sense.	
Step 7. Answer the question.	The width of the garden is 3 feet and the length is 5 feet.	

This figure shows the steps for solving the rectangular garden question. The garden is a rectangle and is labeled w for the width and w + 2 for the length. Next, the formula for area, A = L times W, is used to create an equation. The equation is 15 = (w + 2)w. The equation is simplified and put in standard quadratic form, 0 = w squared + 2 w – 15. Then, the quadratic expression is factored, 0 = (w + 5)(w −3). The equation is solved with the zero product property, negative 5 = w and 3 = w. The negative answer does not makes sense. The solution then is, w = 3. Finally, the width is 3 feet and the length is 3 + 2 = 5 feet. The garden is 3 feet by 5 feet.

Note:
Exercise:

Problem:

A rectangular sign has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

Solution:

5 feet and 6 feet

Note:
Exercise:

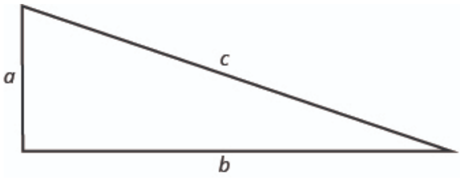
Problem:

A rectangular patio has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

Solution:

12 feet and 15 feet

In an earlier chapter, we used the Pythagorean Theorem $\left(a^2+b^2=c^2\right)$. It gave the relation between the legs and the hypotenuse of a right triangle.

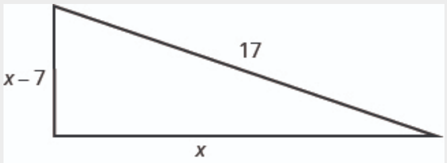


We will use this formula to in the next example.

Example:
Exercise:

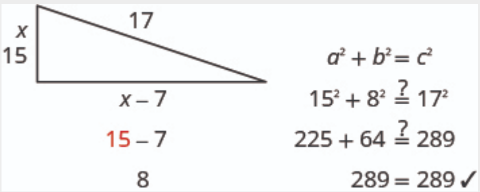
Problem:

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle, as shown below. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the deck.



Solution:
Solution

Step 1. Read the problem.			
Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the deck.		
Step 3. Name what you are looking for. One side is 7 less than the other.	Let *x* = length of a side of the deck *x* − 7 = length of other side		
Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a^2+b^2=c^2$		
Substitute in the variables.	$x^2+(x-7)^2=17^2$		
Step 5. Solve the equation.	$x^2+x^2-14x+49=289$		
Simplify.	$2x^2-14x+49=289$		
It is a quadratic equation, so get zero on one side.	$2x^2-14x-240=0$		
Factor the greatest common factor.	$2(x^2-7x-120)=0$		
Factor the trinomial.	$2(x-15)(x+8)=0$		
Use the Zero Product Property.	$2\neq 0$	$x-15=0$	$x+8=0$
Solve.	$2\neq 0$	$x=15$	$x=-8$
Since *x* is a side of the triangle, $x=-8$ does not make sense.	$2\neq 0$	$x=15$	$\cancel{x=-8}$
Find the length of the other side.			

If the length of one side is		$x = 15$	
then the length of the other side is		$x - 7$	
		$15 - 7$	
	8 is the length of the other side.		
Step 6. Check the answer. Do these numbers make sense?			
 17 x 15 x – 7 15 – 7 8 $a^2 + b^2 = c^2$ $15^2 + 8^2 \overset{?}{=} 17^2$ $225 + 64 \overset{?}{=} 289$ $289 = 289$ ✓			
Step 7. Answer the question.	The sides of the deck are 8, 15, and 17 feet.		

This image shows the steps for solving the above deck problem. The first step is reading the problem. The second step is identify what you are looking for, which is the lengths of the sides of the deck. The third step is naming what you are looking for, which is x = length of one side of the deck and x – 7 the length of the other side. The fourth step is translating into an equation. This problem uses the Pythagorean theorem, x squared + (x – 7) squared = 17 squared. The fifth step is to simplify the equation and solve. The equation simplifies to 2 x squared – 14 x + 49 = 289. Then, the equation is put into standard quadratic form, 2 x squared – 14 x – 240 = 0. The equation is then factored into 2 (x – 15)(x + 8) = 0. Each factor is set equal to 0 and solved, x = 15 or x = −8. Since x = negative 8 does not make sense, the solution is x = 15. The sixth step is to see if the answer makes sense. The last step is to answer the question, the sides of the deck are 8, 15, and 17 feet.

Note:

Exercise:

Problem:

A boat’s sail is a right triangle. The length of one side of the sail is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the sail.

Solution:

5 feet and 12 feet

Note:

Exercise:

Problem:

A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of one of the other legs. Find the lengths of the hypotenuse and the other leg.

Solution:

24 feet and 25 feet

Key Concepts

- **Zero Product Property** If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both. See [link].
- **Solve a quadratic equation by factoring** To solve a quadratic equation by factoring: See [link].

  Write the quadratic equation in standard form,$ax^2 + bx + c = 0$.
  Factor the quadratic expression.
  Use the Zero Product Property.
  Solve the linear equations.
  Check.

- **Use a problem solving strategy to solve word problems** See [link].

  Readthe problem. Make sure all the words and ideas are understood.
  Identifywhat we are looking for.
  Namewhat we are looking for. Choose a variable to represent that quantity.
  Translateinto an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  Solvethe equation using good algebra techniques.
  Checkthe answer in the problem and make sure it makes sense.
  Answerthe question with a complete sentence.

Practice Makes Perfect

Use the Zero Product Property

In the following exercises, solve.
Exercise:

Problem: $(x - 3)(x + 7) = 0$

Solution:

$x = 3, x = -7$

Exercise:

Problem: $(y - 11)(y + 1) = 0$

Exercise:

Problem: $(3a-10)(2a-7)=0$

Solution:

$a=10/3, a=7/2$

Exercise:

Problem: $(5b+1)(6b+1)=0$

Exercise:

Problem: $6m(12m-5)=0$

Solution:

$m=0, m=5/12$

Exercise:

Problem: $2x(6x-3)=0$

Exercise:

Problem: $(y-3)^2=0$

Solution:

$y=3$

Exercise:

Problem: $(b+10)^2=0$

Exercise:

Problem: $(2x-1)^2=0$

Solution:

$x=1/2$

Exercise:

Problem: $(3y+5)^2=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

Exercise:

Problem: $x^2+7x+12=0$

Solution:

$x=-3, x=-4$

Exercise:

Problem: $y^2-8y+15=0$

Exercise:

Problem: $5a^2-26a=24$

Solution:

$a=-4/5, a=6$

Exercise:

Problem: $4b^2+7b=-3$

Exercise:

Problem: $4m^2=17m-15$

Solution:

$m=5/4, m=3$

Exercise:

Problem: $n^2=5-6n\ n^2=5n-6$

Exercise:

Problem: $7a^2+14a=7a$

Solution:

$a=-1, a=0$

Exercise:

Problem: $12b^2-15b=-9b$

Exercise:

Problem: $49m^2=144$

Solution:

$m=12/7, m=-12/7$

Exercise:

Problem: $625=x^2$

Exercise:

Problem: $(y-3)(y+2)=4y$

Solution:

$y = -1, y = 6$

Exercise:

Problem: $(p - 5)(p + 3) = -7$

Exercise:

Problem: $(2x + 1)(x - 3) = -4x$

Solution:

$x = 3/2, x = -1$

Exercise:

Problem: $(x + 6)(x - 3) = -8$

Exercise:

Problem: $16p^3 = 24p^2 - 9p$

Solution:

$p = 0, p = \frac{3}{4}$

Exercise:

Problem: $m^3 - 2m^2 = -m$

Exercise:

Problem: $20x^2 - 60x = -45$

Solution:

$x = \frac{3}{2}$

Exercise:

Problem: $3y^2 - 18y = -27$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

Exercise:

Problem: The product of two consecutive integers is 56. Find the integers.

Solution:

7 and 8; -8 and -7

Exercise:

Problem: The product of two consecutive integers is 42. Find the integers.

Exercise:

Problem:

The area of a rectangular carpet is 28 square feet. The length is three feet more than the width. Find the length and the width of the carpet.

Solution:

$4 \text{ feet and } 7 \text{ feet}$

Exercise:

Problem:

A rectangular retaining wall has area 15 square feet. The height of the wall is two feet less than its length. Find the height and the length of the wall.

Exercise:

Problem:

A pennant is shaped like a right triangle, with hypotenuse 10 feet. The length of one side of the pennant is two feet longer than the length of the other side. Find the length of the two sides of the pennant.

Solution:

$6 \text{ feet and } 8 \text{ feet}$

Exercise:

Problem:

A reflecting pool is shaped like a right triangle, with one leg along the wall of a building. The hypotenuse is 9 feet longer than the side along the building. The third side is 7 feet longer than the side along the building. Find the lengths of all three sides of the reflecting pool.

Mixed Practice

In the following exercises, solve.

Exercise:

Problem: $\left(x+8\right)\left(x-3\right)=0$

Solution:

$x=-8, x=3$

Exercise:

Problem: $\left(3y-5\right)\left(y+7\right)=0$

Exercise:

Problem: $p^2+12p+11=0$

Solution:

$p = -1, p = -11$

Exercise:

Problem: $q^2 - 12q - 13 = 0$

Exercise:

Problem: $m^2 = 6m + 16$

Solution:

$m = -2, m = 8$

Exercise:

Problem: $4n^2 + 19n = 5$

Exercise:

Problem: $a^3 - a^2 - 42a = 0$

Solution:

$a = 0, a = -6, a = 7$

Exercise:

Problem: $4b^2 - 60b + 224 = 0$

Exercise:

Problem: The product of two consecutive integers is 110. Find the integers.

Solution:

$10 \text{ and } 11; -11 \text{ and } -10$

Exercise:

Problem:

The length of one leg of a right triangle is three more than the other leg. If the hypotenuse is 15, find the lengths of the two legs.

Everyday Math

Exercise:

Problem:

Area of a patio If each side of a square patio is increased by 4 feet, the area of the patio would be 196 square feet. Solve the equation $(s + 4)^2 = 196$ for s to find the length of a side of the patio.

Solution:

10 feet

Exercise:

Problem:

Watermelon drop A watermelon is dropped from the tenth story of a building. Solve the equation $-16t^2 + 144 = 0$ for t to find the number of seconds it takes the watermelon to reach the ground.

Writing Exercises

Exercise:

Problem:

Explain how you solve a quadratic equation. How many answers do you expect to get for a quadratic equation?

Solution:

Answers may vary.

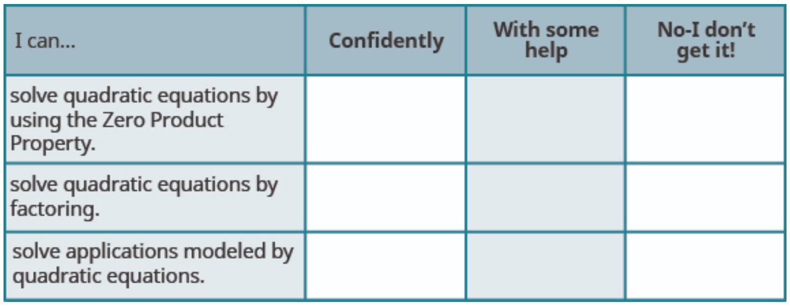
Exercise:

Problem:

Give an example of a quadratic equation that has a GCF and none of the solutions to the equation is zero.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations by using the Zero Product Property.			
solve quadratic equations by factoring.			
solve applications modeled by quadratic equations.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Greatest Common Factor and Factor by Grouping

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

Exercise:

Problem: $42, 60$

Solution:

6

Exercise:

Problem: $450, 420$

Exercise:

Problem: $90, 150, 105$

Solution:

15

Exercise:

Problem: $60, 294, 630$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

Exercise:

Problem: $24x - 42$

Solution:

$6(4x - 7)$

Exercise:

Problem: $35y + 84$

Exercise:

Problem: $15m^4 + 6m^2n$

Solution:

$3m^2(5m^2 + 2n)$

Exercise:

Problem: $24pt^4 + 16t^7$

Factor by Grouping

In the following exercises, factor by grouping.

Exercise:

Problem: $ax - ay + bx - by$

Solution:

$(a+b)\,(x-y)$

Exercise:

Problem: $x^2y - xy^2 + 2x - 2y$

Exercise:

Problem: $x^2 + 7x - 3x - 21$

Solution:

$(x-3)\,(x+7)$

Exercise:

Problem: $4x^2 - 16x + 3x - 12$

Exercise:

Problem: $m^3 + m^2 + m + 1$

Solution:

$(m^2+1)(m+1)$

Exercise:

Problem: $5x - 5y - y + x$

Factor Trinomials of the form $x^2 + bx + c$

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

Exercise:

Problem: $u^2 + 17u + 72$

Solution:

$(u+8)\,(u+9)$

Exercise:

Problem: $a^2 + 14a + 33$

Exercise:

Problem: $k^2 - 16k + 60$

Solution:

$(k-6)\,(k-10)$

Exercise:

Problem: $r^2 - 11r + 28$

Exercise:

Problem: $y^2 + 6y - 7$

Solution:

$(y + 7)(y - 1)$

Exercise:

Problem: $m^2 + 3m - 54$

Exercise:

Problem: $s^2 - 2s - 8$

Solution:

$(s - 4)(s + 2)$

Exercise:

Problem: $x^2 - 3x - 10$

Factor Trinomials of the Form $x^2 + bxy + cy^2$

In the following examples, factor each trinomial of the form $x^2 + bxy + cy^2$.

Exercise:

Problem: $x^2 + 12xy + 35y^2$

Solution:

$(x + 5y)(x + 7y)$

Exercise:

Problem: $u^2 + 14uv + 48v^2$

Exercise:

Problem: $a^2 + 4ab - 21b^2$

Solution:

$(a + 7b)(a - 3b)$

Exercise:

Problem: $p^2 - 5pq - 36q^2$

Factoring Trinomials of the form $ax^2 + bx + c$

Recognize a Preliminary Strategy to Factor Polynomials Completely

In the following exercises, identify the best method to use to factor each polynomial.
Exercise:

Problem: $y^2 - 17y + 42$

Solution:

Undo FOIL

Exercise:

Problem: $12r^2 + 32r + 5$

Exercise:

Problem: $8a^3 + 72a$

Solution:

Factor the GCF

Exercise:

Problem: $4m - mn - 3n + 12$

Factor Trinomials of the Form $ax^2 + bx + c$ with a GCF

In the following exercises, factor completely.
Exercise:

Problem: $6x^2 + 42x + 60$

Solution:

$6\left(x + 2\right)\left(x + 5\right)$

Exercise:

Problem: $8a^2 + 32a + 24$

Exercise:

Problem: $3n^4 - 12n^3 - 96n^2$

Solution:

$3n^2\left(n - 8\right)\left(n + 4\right)$

Exercise:

Problem: $5y^4 + 25y^2 - 70y$

Factor Trinomials Using the “ac” Method

In the following exercises, factor.

Exercise:

Problem: $2x^2 + 9x + 4$

Solution:

$(x + 4)\,(2x + 1)$

Exercise:

Problem: $3y^2 + 17y + 10$

Exercise:

Problem: $18a^2 - 9a + 1$

Solution:

$(3a - 1)\,(6a - 1)$

Exercise:

Problem: $8u^2 - 14u + 3$

Exercise:

Problem: $15p^2 + 2p - 8$

Solution:

$(5p + 4)\,(3p - 2)$

Exercise:

Problem: $15x^2 + 6x - 2$

Exercise:

Problem: $40s^2 - s - 6$

Solution:

$(5s - 2)\,(8s + 3)$

Exercise:

Problem: $20n^2 - 7n - 3$

Factor Trinomials with a GCF Using the “ac” Method

In the following exercises, factor.

Exercise:

Problem: $3x^2 + 3x - 36$

Solution:

$3\left(x + 4\right)\left(x - 3\right)$

Exercise:

Problem: $4x^2 + 4x - 8$

Exercise:

Problem: $60y^2 - 85y - 25$

Solution:

$5\left(4y + 1\right)\left(3y - 5\right)$

Exercise:

Problem: $18a^2 - 57a - 21$

Factoring Special Products

Factor Perfect Square Trinomials

In the following exercises, factor.

Exercise:

Problem: $25x^2 + 30x + 9$

Solution:

$\left(5x + 3\right)^2$

Exercise:

Problem: $16y^2 + 72y + 81$

Exercise:

Problem: $36a^2 - 84ab + 49b^2$

Solution:

$\left(6a - 7b\right)^2$

Exercise:

Problem: $64r^2 - 176rs + 121s^2$

Exercise:

Problem: $40x^2 + 360x + 810$

Solution:

$10(2x + 9)^2$

Exercise:

Problem: $75u^2 + 180u + 108$

Exercise:

Problem: $2y^3 - 16y^2 + 32y$

Solution:

$2y(y - 4)^2$

Exercise:

Problem: $5k^3 - 70k^2 + 245k$

Factor Differences of Squares

In the following exercises, factor.

Exercise:

Problem: $81r^2 - 25$

Solution:

$(9r - 5)(9r + 5)$

Exercise:

Problem: $49a^2 - 144$

Exercise:

Problem: $169m^2 - n^2$

Solution:

$(13m + n)(13m - n)$

Exercise:

Problem: $64x^2 - y^2$

Exercise:

Problem: $25p^2 - 1$

Solution:

$(5p-1)(5p+1)$

Exercise:

Problem: $1-16s^2$

Exercise:

Problem: $9-121y^2$

Solution:

$(3+11y)(3-11y)$

Exercise:

Problem: $100k^2-81$

Exercise:

Problem: $20x^2-125$

Solution:

$5(2x-5)(2x+5)$

Exercise:

Problem: $18y^2-98$

Exercise:

Problem: $49u^3-9u$

Solution:

$u(7u+3)(7u-3)$

Exercise:

Problem: $169n^3-n$

Factor Sums and Differences of Cubes

In the following exercises, factor.

Exercise:

Problem: a^3-125

Solution:

$(a-5)\left(a^2+5a+25\right)$

Exercise:

Problem: b^3-216

Exercise:

Problem: $2m^3 + 54$

Solution:

$2\left(m+3\right)\left(m^2-3m+9\right)$

Exercise:

Problem: $81x^3 + 3$

General Strategy for Factoring Polynomials

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

Exercise:

Problem: $24x^3 + 44x^2$

Solution:

$4x^2\left(6x+11\right)$

Exercise:

Problem: $24a^4 - 9a^3$

Exercise:

Problem: $16n^2 - 56mn + 49m^2$

Solution:

$\left(4n-7m\right)^2$

Exercise:

Problem: $6a^2 - 25a - 9$

Exercise:

Problem: $5r^2 + 22r - 48$

Solution:

$\left(r+6\right)\left(5r-8\right)$

Exercise:

Problem: $5u^4 - 45u^2$

Exercise:

Problem: $n^4 - 81$

Solution:

$\left(n^2 + 9\right)(n + 3)(n - 3)$

Exercise:

Problem: $64j^2 + 225$

Exercise:

Problem: $5x^2 + 5x - 60$

Solution:

$5(x - 3)(x + 4)$

Exercise:

Problem: $b^3 - 64$

Exercise:

Problem: $m^3 + 125$

Solution:

$(m + 5)(m^2 - 5m + 25)$

Exercise:

Problem: $2b^2 - 2bc + 5cb - 5c^2$

Quadratic Equations

Use the Zero Product Property

In the following exercises, solve.

Exercise:

Problem: $(a - 3)(a + 7) = 0$

Solution:

$a = 3\ a = -7$

Exercise:

Problem: $(b - 3)(b + 10) = 0$

Exercise:

Problem: $3m(2m-5)(m+6)=0$

Solution:

$m=0$, $m=-6$, $m=\frac{5}{2}$

Exercise:

Problem: $7n(3n+8)(n-5)=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

Exercise:

Problem: $x^2+9x+20=0$

Solution:

$x=-4, x=-5$

Exercise:

Problem: $y^2-y-72=0$

Exercise:

Problem: $2p^2-11p=40$

Solution:

$p=-\frac{5}{2}, p=8$

Exercise:

Problem: $q^3+3q^2+2q=0$

Exercise:

Problem: $144m^2-25=0$

Solution:

$m=\frac{5}{12}, m=-\frac{5}{12}$

Exercise:

Problem: $4n^2=36$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

Exercise:

Problem: The product of two consecutive numbers is 462. Find the numbers.

Solution:

$-21, -22$ $21, 22$

Exercise:

Problem:

The area of a rectangular shaped patio 400 square feet. The length of the patio is 9 feet more than its width. Find the length and width.

Practice Test

In the following exercises, find the Greatest Common Factor in each expression.

Exercise:

Problem: $14y - 42$

Solution:

$7\left(y - 6\right)$

Exercise:

Problem: $-6x^2 - 30x$

Exercise:

Problem: $80a^2 + 120a^3$

Solution:

$40a^2(2 + 3a)$

Exercise:

Problem: $5m\left(m - 1\right) + 3\left(m - 1\right)$

In the following exercises, factor completely.

Exercise:

Problem: $x^2 + 13x + 36$

Solution:

$(x + 7)(x + 6)$

Exercise:

Problem: $p^2 + pq - 12q^2$

Exercise:

Problem: $3a^3 - 6a^2 - 72a$

Solution:

$3a(a^2 - 2a - 14)$

Exercise:

Problem: $s^2 - 25s + 84$

Exercise:

Problem: $5n^2 + 30n + 25$

Solution:

$5\,(n+1)\,(n+5)$

Exercise:

Problem: $64y^2 - 49$

Exercise:

Problem: $xy - 8y + 7x - 56$

Solution:

$(x-8)\,(y+7)$

Exercise:

Problem: $40r^2 + 810$

Exercise:

Problem: $9s^2 - 12s + 4$

Solution:

$(3s-2)^2$

Exercise:

Problem: $n^2 + 12n + 36$

Exercise:

Problem: $100 - a^2$

Solution:

$(10-a)\,(10+a)$

Exercise:

Problem: $6x^2 - 11x - 10$

Exercise:

Problem: $3x^2 - 75y^2$

Solution:

$3\left(x + 5y\right)\left(x - 5y\right)$

Exercise:

Problem: $c^3 - 1000d^3$

Exercise:

Problem: $ab - 3b - 2a + 6$

Solution:

$\left(a - 3\right)\left(b - 2\right)$

Exercise:

Problem: $6u^2 + 3u - 18$

Exercise:

Problem: $8m^2 + 22m + 5$

Solution:

$\left(4m + 1\right)\left(2m + 5\right)$

In the following exercises, solve.

Exercise:

Problem: $x^2 + 9x + 20 = 0$

Exercise:

Problem: $y^2 = y + 132$

Solution:

$\text{y} = -11, \text{y} = 12$

Exercise:

Problem: $5a^2 + 26a = 24$

Exercise:

Problem: $9b^2 - 9 = 0$

Solution:

$b=1, b=-1$

Exercise:

Problem: $16-m^2=0$

Exercise:

Problem: $4n^2+19n+21=0$

Solution:

$n=-\frac{7}{4}, n=-3$

Exercise:

Problem: $(x-3)(x+2)=6$

Exercise:

Problem: The product of two consecutive integers is 156. Find the integers.

Solution:

12 and 13; -13 and -12

Exercise:

Problem:

The area of a rectangular place mat is 168 square inches. Its length is two inches longer than the width. Find the length and width of the placemat.

Glossary

quadratic equations

are equations in which the variable is squared.

Zero Product Property

The Zero Product Property states that, if the product of two quantities is zero, at least one of the quantities is zero.

Introduction class="introduction"

Rowing a boat downstream can be very relaxing, but it takes much more effort to row the boat upstream.

Like rowing a boat, riding a bicycle is a situation in which going in one direction, downhill, is easy, but going in the opposite direction, uphill, can be more work. The trip to reach a destination may be quick, but the return trip whether upstream or uphill will take longer.

Rational equations are used to model situations like these. In this chapter, we will work with rational expressions, solve rational equations, and use them to solve problems in a variety of applications.

Simplify Rational Expressions

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Evaluate rational expressions
- Simplify rational expressions
- Simplify rational expressions with opposite factors

Note:

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Simplify: $\frac{90y}{15y^2}$.
   If you missed this problem, review [link].
2. Factor: $6x^2 - 7x + 2$.
   If you missed this problem, review [link].
3. Factor: $n^3 + 8$.
   If you missed this problem, review [link].

In Chapter 1, we reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers, and the denominator is not zero.

In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call these rational expressions.

Note:

Rational Expression

A **rational expression** is an expression of the form $\frac{p(x)}{q(x)}$, where p and q are polynomials and $q \neq 0$.

Remember, division by 0 is undefined.

Here are some examples of rational expressions:
Equation:

$$-\frac{13}{42} \qquad \frac{7y}{8z} \qquad \frac{5x+2}{x^2-7} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{13}{42}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will perform same operations with rational expressions that we do with fractions. We will simplify, add, subtract, multiply, divide, and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

When we work with a numerical fraction, it is easy to avoid dividing by zero, because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

Note:
Determine the Values for Which a Rational Expression is Undefined.

Set the denominator equal to zero.
Solve the equation in the set of reals, if possible.

Example:
Exercise:

Problem:

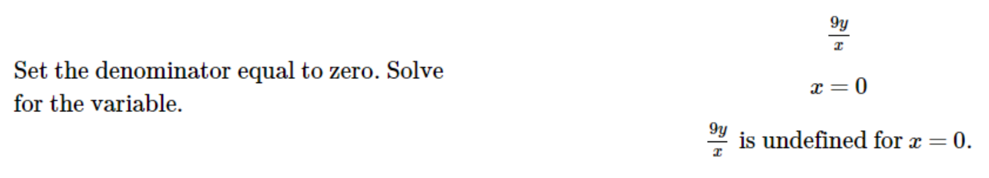
Determine the values for which the rational expression is undefined:

ⓐ $\frac{9y}{x}$ ⓑ $\frac{4b-3}{2b+5}$ ⓒ $\frac{x+4}{x^2+5x+6}$

Solution:
Solution

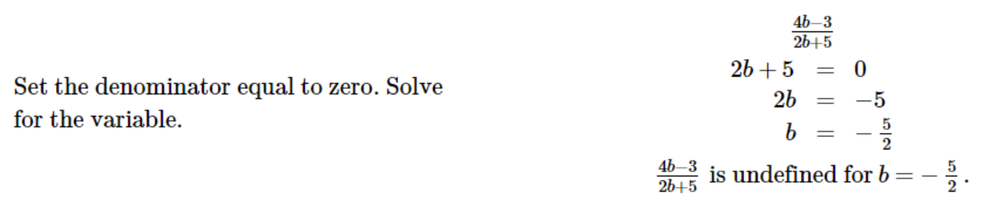
The expression will be undefined when the denominator is zero.

ⓐ



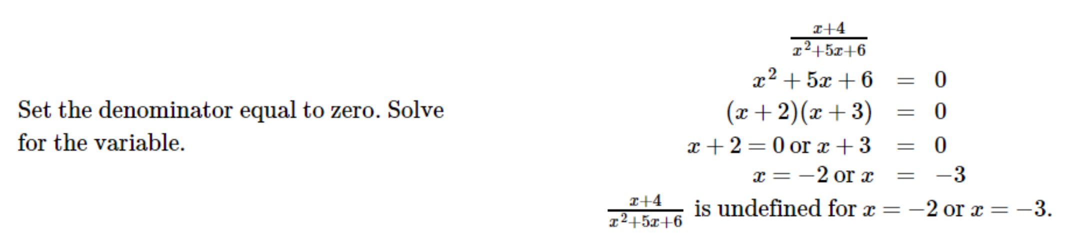
Set the denominator equal to zero. Solve for the variable.	$\frac{9y}{x}$ $x = 0$ $\frac{9y}{x}$ is undefined for $x = 0$.

ⓑ



Set the denominator equal to zero. Solve for the variable.	$\frac{4b-3}{2b+5}$ $\begin{array}{rcl} 2b+5 & = & 0 \\ 2b & = & -5 \\ b & = & -\frac{5}{2} \end{array}$ $\frac{4b-3}{2b+5}$ is undefined for $b = -\frac{5}{2}$.

ⓒ



Set the denominator equal to zero. Solve for the variable.	$\frac{x+4}{x^2+5x+6}$ $\begin{array}{rcl} x^2+5x+6 & = & 0 \\ (x+2)(x+3) & = & 0 \\ x+2=0 \text{ or } x+3 & = & 0 \\ x=-2 \text{ or } x & = & -3 \end{array}$ $\frac{x+4}{x^2+5x+6}$ is undefined for $x = -2$ or $x = -3$.

Saying that the rational expression $\frac{x+4}{x^2+5x+6}$ is undefined for $x=-2$ or $x=-3$ is similar to writing the phrase “void where prohibited” in contest rules.

Note:
Exercise:

Problem:

Determine the values for which the rational expression is undefined:

ⓐ $\frac{3y}{x}$ ⓑ $\frac{8n-5}{3n+1}$ ⓒ $\frac{a+10}{a^2+4a+3}$

Solution:

ⓐ $x=0$ ⓑ $n=-\frac{1}{3}$ ⓒ $a=-1, a=-3$

Note:
Exercise:

Problem:

Determine the values for which the rational expression is undefined:

ⓐ $\frac{4p}{5q}$ ⓑ $\frac{y-1}{3y+2}$ ⓒ $\frac{m-5}{m^2+m-6}$

Solution:

ⓐ $q=0$ ⓑ $y=-\frac{2}{3}$ ⓒ $m=2, m=-3$

Evaluate Rational Expressions

To evaluate a rational expression, we substitute values of the variables into the expression and simplify, just as we have for many other expressions in this book.

Example:
Exercise:

Problem: Evaluate $\frac{2x+3}{3x-5}$ for each value:

ⓐ $x=0$ ⓑ $x=2$ ⓒ $x=-3$

Solution:
Solution

ⓐ

	$\frac{2x+3}{3x-5}$
Substitute 0 for *x*.	$\frac{2(0)+3}{3(0)-5}$
Simplify.	$-\frac{3}{5}$

ⓑ

	$\frac{2x+3}{3x-5}$

Substitute 2 for *x*.	$\frac{2(2)+3}{3(2)-5}$
Simplify.	$\frac{4+3}{6-5}$
	$\frac{7}{1}$
	7

ⓒ

	$\frac{2x+3}{3x-5}$
Substitute –3 for *x*.	$\frac{2(-3)+3}{3(-3)-5}$
Simplify.	$\frac{-6+3}{-9-5}$
	$\frac{-3}{-14}$

	$\frac{3}{14}$

Note:
Exercise:

Problem: Evaluate $\frac{y+1}{2y-3}$ for each value:

ⓐ $y=1$ ⓑ $y=-3$ ⓒ $y=0$

Solution:

ⓐ -2 ⓑ $\frac{2}{9}$ ⓒ $-\frac{1}{3}$

Note:
Exercise:

Problem: Evaluate $\frac{5x-1}{2x+1}$ for each value:

ⓐ $x=1$ ⓑ $x=-1$ ⓒ $x=0$

Solution:

ⓐ $\frac{4}{3}$ ⓑ 6 ⓒ -1

Example:
Exercise:

Problem: Evaluate $\frac{x^2+8x+7}{x^2-4}$ for each value:

ⓐ $x=0$ ⓑ $x=2$ ⓒ $x=-1$

Solution:
Solution

ⓐ

	$\frac{x^2+8x+7}{x^2-4}$
Substitute 0 for *x*.	$\frac{(0)^2+8(0)+7}{(0)^2-4}$
Simplify.	$\frac{7}{-4}$
	$-\frac{7}{4}$

ⓑ

	$\frac{x^2+8x+7}{x^2-4}$
Substitute 2 for *x*.	$\frac{(2)^2+8(2)+7}{(2)^2-4}$
Simplify.	$\frac{4+16+7}{4-4}$
	$\frac{27}{0}$
This rational expression is undefined for *x* = 2.	

Ⓒ

	$\frac{x^2+8x+7}{x^2-4}$
Substitute –1 for *x*.	$\frac{(-1)^2+8(-1)+7}{(-1)^2-4}$
Simplify.	$\frac{1-8+7}{1-4}$
	$\frac{-7+7}{-3}$

	$\frac{0}{-3}$
	0

Note:
Exercise:

Problem: Evaluate $\frac{x^2+1}{x^2-3x+2}$ for each value:

ⓐ $x=0$ ⓑ $x=-1$ ⓒ $x=3$

Solution:

ⓐ $\frac{1}{2}$ ⓑ $\frac{1}{3}$ ⓒ 2

Note:
Exercise:

Problem: Evaluate $\frac{x^2+x-6}{x^2-9}$ for each value:

ⓐ $x=0$ ⓑ $x=-2$ ⓒ $x=1$

Solution:

ⓐ $\frac{2}{3}$ ⓑ $\frac{4}{5}$ ⓒ $\frac{1}{2}$

Remember that a fraction is simplified when it has no common factors, other than 1, in its numerator and denominator. When we evaluate a rational expression, we make sure to simplify the resulting fraction.

Example:
Exercise:

Problem: Evaluate $\frac{a^2+2ab+b^2}{3ab^3}$ for each value:

ⓐ $a=1, b=2$ ⓑ $a=-2, b=-1$ ⓒ $a=\frac{1}{3}, b=0$

Solution:
Solution

ⓐ

	$\frac{a^2+2ab+b^2}{3ab^2}$ when $a=1, b=2.$
Substitute 1 for *a* and 2 for *b*.	$\frac{(1)^2+2(1)(2)+(2)^2}{3(1)(2)^2}$
Simplify.	$\frac{1+4+4}{3(4)}$
	$\frac{9}{12}$
	$\frac{3}{4}$

ⓑ

	$\frac{a^2+2ab+b^2}{3ab^2}$ when $a=-2, b=-1.$
Substitute −2 for *a* and −1 for *b*.	$\frac{(-2)^2+2(-2)(-1)+(-1)^2}{3(-2)(-1)^2}$
Simplify.	$\frac{4+4+1}{-6}$
	$-\frac{9}{6}$
	$-\frac{3}{2}$

ⓒ

	$\frac{a^2+2ab+b^2}{3ab^2}$ when $a=\frac{1}{3}, b=0.$
Substitute $\frac{1}{3}$ for *a* and 0 for *b*.	$\frac{\left(\frac{1}{3}\right)^2+2\left(\frac{1}{3}\right)(0)+(0)^2}{3\left(\frac{1}{3}\right)(0)^2}$
Simplify.	$\frac{\frac{1}{9}+0+0}{0}$
	$\frac{\frac{1}{9}}{0}$
	The expression is undefined.

Note:
Exercise:

Problem: Evaluate $\frac{2a^3b}{a^2+2ab+b^2}$ for each value:

ⓐ $a=-1, b=2$ ⓑ $a=0, b=-1$ ⓒ $a=1, b=\frac{1}{2}$

Solution:

ⓐ -4 ⓑ 0 ⓒ $\frac{4}{9}$

Note:

Exercise:

Problem: Evaluate $\frac{a^2-b^2}{8ab^3}$ for each value:

ⓐ $a=1, b=-1$ ⓑ $a=\frac{1}{2}, b=-1$ ⓒ $a=-2, b=1$

Solution:

ⓐ 0 ⓑ $\frac{3}{16}$ ⓒ $-\frac{3}{16}$

Simplify Rational Expressions

Just like a fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator, a rational expression is *simplified* if it has no common factors, other than 1, in its numerator and denominator.

Note:
Simplified Rational Expression
A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example:

- $\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.
- $\frac{2x}{3x}$ is not simplified because *x* is a common factor of 2*x* and 3*x*.

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

Note:
Equivalent Fractions Property
If *a*, *b*, and *c* are numbers where $b \neq 0, c \neq 0,$ then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0, c \neq 0$ clearly stated. Every time we write a rational expression, we should make a similar statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

Let’s start by reviewing how we simplify numerical fractions.

Example:
Exercise:

Problem: Simplify: $-\frac{36}{63}$.

Solution:
Solution

	$-\frac{36}{63}$
Rewrite the numerator and denominator showing the common factors.	$-\frac{4 \cdot 9}{7 \cdot 9}$
Simplify using the Equivalent Fractions Property.	$-\frac{4}{7}$

Notice that the fraction $-\frac{4}{7}$ is simplified because there are no more common factors.

Note:
Exercise:

Problem: Simplify: $-\frac{45}{81}$.

Solution:

$-\frac{5}{9}$.

Note:
Exercise:

Problem: Simplify: $-\frac{42}{54}$.

Solution:

$-\frac{7}{9}$

Throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$ and $y \neq 0$.

Example:
Exercise:

Problem: Simplify: $\frac{3xy}{18x^2y^2}$.

Solution:
Solution

	$\frac{3xy}{18x^2y^2}$
Rewrite the numerator and denominator showing the common factors.	$\frac{1 \cdot 3xy}{6xy \cdot 3xy}$
Simplify using the Equivalent Fractions Property.	$\frac{1}{6xy}$

Did you notice that these are the same steps we took when we divided monomials in Polynomials?

Note:
Exercise:

Problem: Simplify: $\frac{4x^2y}{12xy^2}$.

Solution:

$\frac{x}{3y}$

Note:
Exercise:

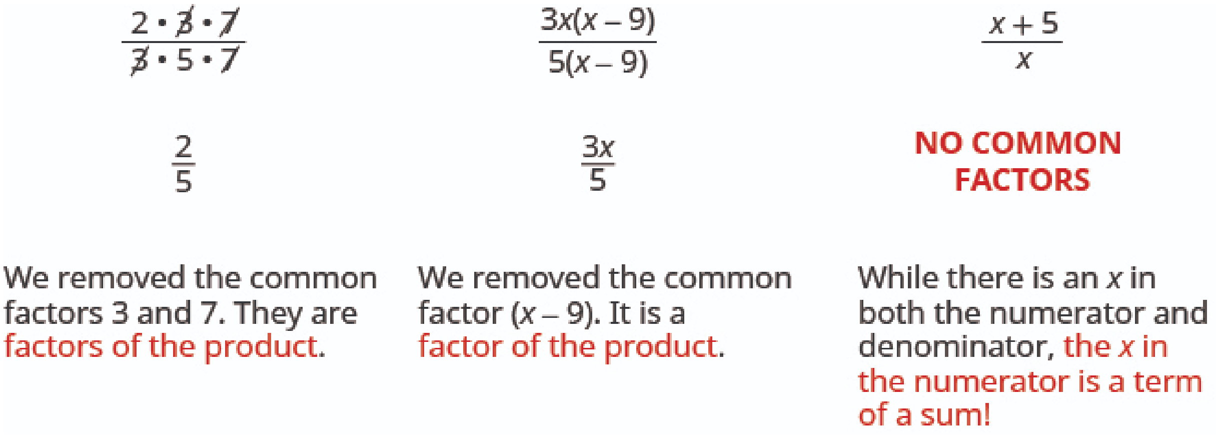
Problem: Simplify: $\frac{16x^2y}{2xy^2}$.

Solution:

$\frac{8x}{y}$

To simplify rational expressions we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.



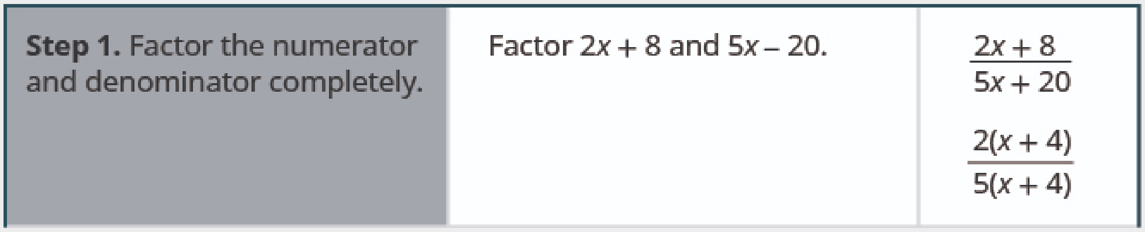
$\frac{2\cdot\cancel{3}\cdot\cancel{7}}{\cancel{3}\cdot 5\cdot\cancel{7}}$	$\frac{3x(x-9)}{5(x-9)}$	$\frac{x+5}{x}$
$\frac{2}{5}$	$\frac{3x}{5}$	**NO COMMON FACTORS**
We removed the common factors 3 and 7. They are factors of the product.	We removed the common factor $(x-9)$. It is a factor of the product.	While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Note that removing the x’s from $\frac{x+5}{x}$ would be like cancelling the 2’s in the fraction $\frac{2+5}{2}$!

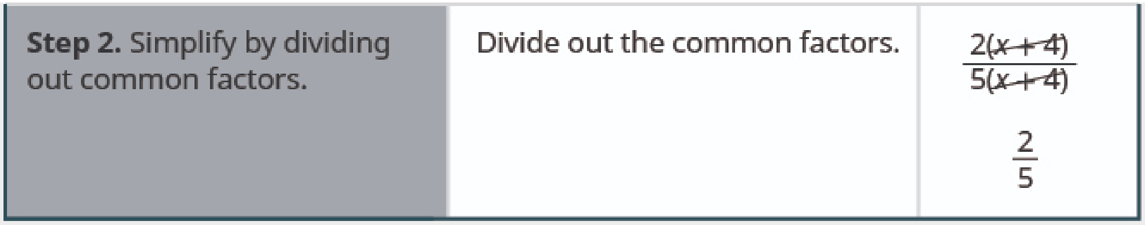
Example:
How to Simplify Rational Binomials
Exercise:

Problem: Simplify: $\frac{2x+8}{5x+20}$.

Solution:
Solution



Step 1. Factor the numerator and denominator completely.	Factor $2x+8$ and $5x-20$.	$\frac{2x+8}{5x+20}$ $\frac{2(x+4)}{5(x+4)}$



Step 2. Simplify by dividing out common factors.	Divide out the common factors.	$\frac{2\cancel{(x+4)}}{5\cancel{(x+4)}}$ $\frac{2}{5}$

Note:
Exercise:

Problem: Simplify: $\frac{3x-6}{2x-4}$.

Solution:

$\frac{3}{2}$

Note:
Exercise:

Problem: Simplify: $\frac{7y+35}{5y+25}$.

Solution:

$\frac{7}{5}$

We now summarize the steps you should follow to simplify rational expressions.

Note:
Simplify a Rational Expression.

Factor the numerator and denominator completely.
Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way it is easy to check that we have removed all the common factors!

We'll use the methods we covered in Factoring to factor the polynomials in the numerators and denominators in the following examples.

Example:
Exercise:

Problem: Simplify: $\frac{x^2+5x+6}{x^2+8x+12}$.

Solution:
Solution

$$\frac{x^2+5x+6}{x^2+8x+12}$$

Factor the numerator and denominator. $\frac{(x+2)(x+3)}{(x+2)(x+6)}$

Remove the common factor $x+2$ from the numerator and the denominator. $\frac{\cancel{(x+2)}\,(x+3)}{\cancel{(x+2)}\,(x+6)}$

$$\frac{x+3}{x+6}$$

Can you tell which values of *x* must be excluded in this example?

Note:
Exercise:

Problem: Simplify: $\frac{x^2-x-2}{x^2-3x+2}$.

Solution:

$\frac{x+1}{x-1}$

Note:
Exercise:

Problem: Simplify: $\frac{x^2-3x-10}{x^2+x-2}$.

Solution:

$\frac{x-5}{x-1}$

Example:
Exercise:

Problem: Simplify: $\frac{y^2+y-42}{y^2-36}$.

Solution:
Solution

$$\frac{y^2+y-42}{y^2-36}$$

Factor the numerator and denominator.

$$\frac{(y+7)(y-6)}{(y+6)(y-6)}$$

Remove the common factor $y-6$ from the numerator and the denominator.

$$\frac{(y+7)\,\cancel{(y-6)}}{(y+6)\,\cancel{(y-6)}}$$

$$\frac{y+7}{y+6}$$

Note:

Exercise:

Problem: Simplify: $\frac{x^2+x-6}{x^2-4}$.

Solution:

$\frac{x+3}{x+2}$

Note:
Exercise:

Problem: Simplify: $\frac{x^2+8x+7}{x^2-49}$.

Solution:

$\frac{x+1}{x-7}$

Example:
Exercise:

Problem: Simplify: $\frac{p^3-2p^2+2p-4}{p^2-7p+10}$.

Solution:
Solution

$$\frac{p^3-2p^2+2p-4}{p^2-7p+10}$$

Factor the numerator and denominator, using grouping to factor the numerator.

$$\frac{p^2(p-2)+2(p-2)}{(p-5)(p-2)}$$

$$\frac{\left(p^2+2\right)(p-2)}{(p-5)(p-2)}$$

Remove the common factor of $p-2$ from the numerator and the denominator.

$$\frac{\left(p^2+2\right)\cancel{(p-2)}}{(p-5)\cancel{(p-2)}}$$

$$\frac{p^2+2}{p-5}$$

Note:
Exercise:

Problem: Simplify: $\frac{y^3-3y^2+y-3}{y^2-y-6}$.

Solution:

$\frac{y^2+1}{y+2}$

Note:
Exercise:

Problem: Simplify: $\frac{p^3-p^2+2p-2}{p^2+4p-5}$.

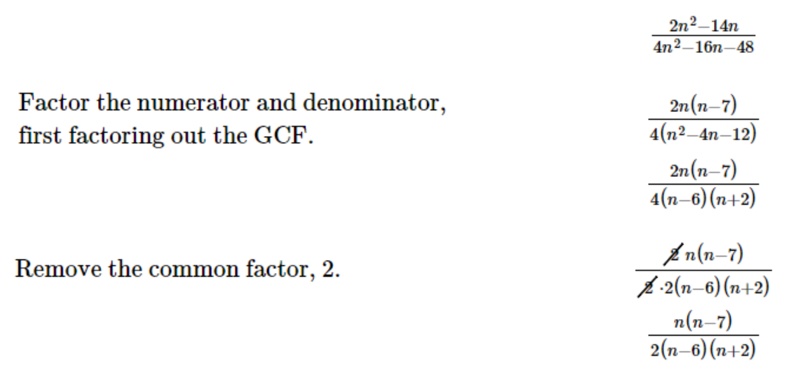
Solution:

$\frac{p^2+2}{p+5}$

Example:
Exercise:

Problem: Simplify: $\frac{2n^2-14n}{4n^2-16n-48}$.

Solution:
Solution



	$\frac{2n^2-14n}{4n^2-16n-48}$
Factor the numerator and denominator, first factoring out the GCF.	$\frac{2n(n-7)}{4(n^2-4n-12)}$
	$\frac{2n(n-7)}{4(n-6)(n+2)}$
Remove the common factor, 2.	$\frac{\cancel{2}n(n-7)}{\cancel{2}\cdot 2(n-6)(n+2)}$
	$\frac{n(n-7)}{2(n-6)(n+2)}$

Note:
Exercise:

Problem: Simplify: $\frac{2n^2-10n}{4n^2-16n-20}$.

Solution:

$\frac{n}{2(n+1)}$

Note:
Exercise:

Problem: Simplify: $\frac{4x^2-16x}{8x^2-16x-64}$.

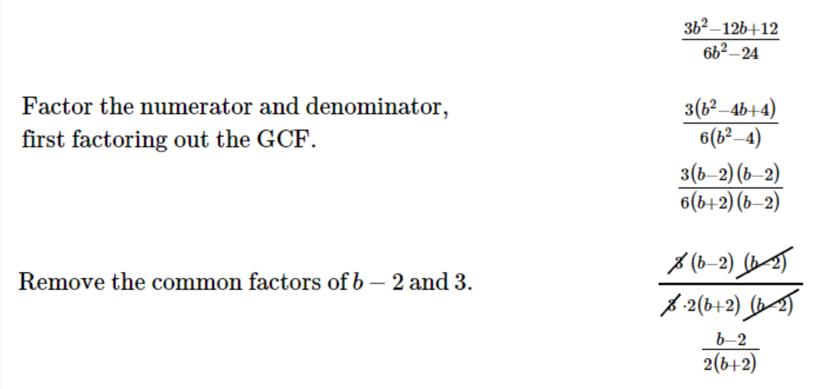
Solution:

$$\frac{x}{2(x+2)}$$

Example:
Exercise:

Problem: Simplify: $\frac{3b^2-12b+12}{6b^2-24}$.

Solution:
Solution



	$\frac{3b^2-12b+12}{6b^2-24}$
Factor the numerator and denominator, first factoring out the GCF.	$\frac{3(b^2-4b+4)}{6(b^2-4)}$ $\frac{3(b-2)(b-2)}{6(b+2)(b-2)}$
Remove the common factors of $b-2$ and 3.	$\frac{\cancel{3}(b-2)\cancel{(b-2)}}{\cancel{3}\cdot 2(b+2)\cancel{(b-2)}}$ $\frac{b-2}{2(b+2)}$

Note:
Exercise:

Problem: Simplify: $\frac{2x^2-12x+18}{3x^2-27}$.

Solution:

$$\frac{2(x-3)}{3(x+3)}$$

Note:

Exercise:

Problem: Simplify: $\frac{5y^2-30y+25}{2y^2-50}$.

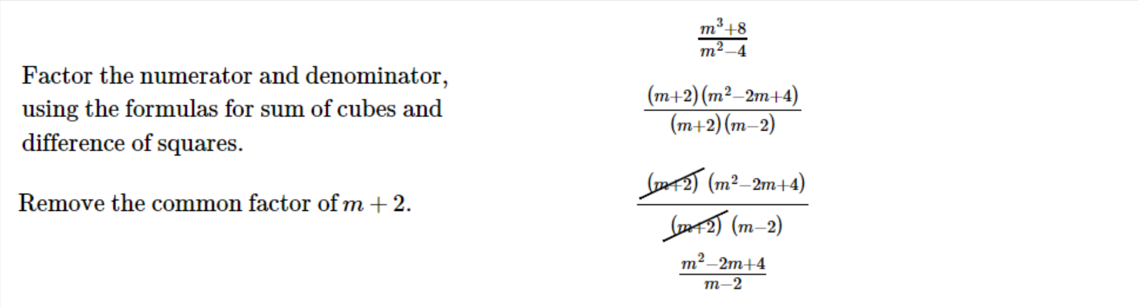
Solution:

$\frac{5(x-1)}{2(x+5)}$

Example:
Exercise:

Problem: Simplify: $\frac{m^3+8}{m^2-4}$.

Solution:
Solution



$$\frac{m^3+8}{m^2-4}$$

Factor the numerator and denominator, using the formulas for sum of cubes and difference of squares.

$$\frac{(m+2)(m^2-2m+4)}{(m+2)(m-2)}$$

Remove the common factor of $m+2$.

$$\frac{\cancel{(m+2)}\,(m^2-2m+4)}{\cancel{(m+2)}\,(m-2)}$$

$$\frac{m^2-2m+4}{m-2}$$

Note:
Exercise:

Problem: Simplify: $\frac{p^3-64}{p^2-16}$.

Solution:

$\frac{p^2+4p+16}{p+4}$

Note:
Exercise:

Problem: Simplify: $\frac{x^3+8}{x^2-4}$.

Solution:

$\frac{x^2-2x+4}{x-2}$

Simplify Rational Expressions with Opposite Factors

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. Let's start with a numerical fraction, say $\frac{7}{-7}$. We know this fraction simplifies to -1. We also recognize that the numerator and denominator are opposites.

In Foundations, we introduced opposite notation: the opposite of a is $-a$. We remember, too, that $-a = -1 \cdot a$.

We simplify the fraction $\frac{a}{-a}$, whose numerator and denominator are opposites, in this way:

	$\frac{a}{-a}$
We could rewrite this.	$\frac{1 \cdot a}{-1 \cdot a}$
Remove the common factors.	$\frac{1}{-1}$
Simplify.	-1

So, in the same way, we can simplify the fraction $\frac{x-3}{-(x-3)}$:

We could rewrite this. $\frac{1\cdot(x-3)}{-1\cdot(x-3)}$

Remove the common factors. $\frac{1}{-1}$

Simplify. -1

But the opposite of $x-3$ could be written differently:

$-(x-3)$

Distribute. $-x+3$

Rewrite. $3-x$

This means the fraction $\frac{x-3}{3-x}$ simplifies to -1.

In general, we could write the opposite of $a-b$ as $b-a$. So the rational expression $\frac{a-b}{b-a}$ simplifies to -1.

Note:
Opposites in a Rational Expression
The opposite of $a-b$ is $b-a$.
Equation:

$$\frac{a-b}{b-a}=-1 \qquad a\neq b$$

An expression and its opposite divide to -1.

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators.

Example:
Exercise:

Problem: Simplify: $\frac{x-8}{8-x}$.

Solution:
Solution

	$\frac{x-8}{8-x}$
Recognize that $x-8$ and $8-x$ are opposites.	-1

Note:
Exercise:

Problem: Simplify: $\frac{y-2}{2-y}$.

Solution:

-1

Note:
Exercise:

Problem: Simplify: $\frac{n-9}{9-n}$.

Solution:

-1

Remember, the first step in simplifying a rational expression is to factor the numerator and denominator completely.

Example:
Exercise:

Problem: Simplify: $\frac{14-2x}{x^2-49}$.

Solution:
Solution

	$\frac{14-2x}{x^2-49}$
Factor the numerator and denominator.	$\frac{2(7-x)}{(x+7)(x-7)}$
Recognize that $7-x$ and $x-7$ are opposites.	$\frac{2\cancel{(7-x)}}{(x+7)\cancel{(x-7)}}(-1)$
Simplify.	$-\frac{2}{x+7}$

Note:
Exercise:

Problem: Simplify: $\frac{10-2y}{y^2-25}$.

Solution:

$-\frac{2}{y+5}$.

Note:
Exercise:

Problem: Simplify: $\frac{3y-27}{81-y^2}$.

Solution:

$-\frac{3}{9+y}$

Example:
Exercise:

Problem: Simplify: $\frac{x^2-4x-32}{64-x^2}$.

Solution:
Solution

	$\frac{x^2-4x-32}{64-x^2}$
Factor the numerator and denominator.	$\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	

	$(-1)\frac{\cancel{(x-8)}(x+4)}{\cancel{(8-x)}(8+x)}$
Simplify.	$-\frac{x+4}{x+8}$

Note:
Exercise:

Problem: Simplify: $\frac{x^2-4x-5}{25-x^2}$.

Solution:

$-\frac{x+1}{x+5}$

Note:
Exercise:

Problem: Simplify: $\frac{x^2+x-2}{1-x^2}$.

Solution:

$-\frac{x+2}{x+1}$

Key Concepts

- **Determine the Values for Which a Rational Expression is Undefined**

  Set the denominator equal to zero.

Solve the equation, if possible.

- **Simplified Rational Expression**
  - A rational expression is considered simplified if there are no common factors in its numerator and denominator.
- **Simplify a Rational Expression**

  Factor the numerator and denominator completely.
  Simplify by dividing out common factors.
- **Opposites in a Rational Expression**
  - The opposite of $a-b$ is $b-a$.
    $\frac{a-b}{b-a}=-1 \qquad a\neq 0, b\neq 0, \mathrm{a}\neq \mathrm{b}$

Practice Makes Perfect

In the following exercises, determine the values for which the rational expression is undefined.

Exercise:

Problem: ⓐ $\frac{2x}{z}$
ⓑ $\frac{4p-1}{6p-5}$
ⓒ $\frac{n-3}{n^2+2n-8}$

Solution:

ⓐ $z=0$ ⓑ $p=\frac{5}{6}$
ⓒ $n=-4, n=2$

Exercise:

Problem: ⓐ $\frac{10m}{11n}$
ⓑ $\frac{6y+13}{4y-9}$
ⓒ $\frac{b-8}{b^2-36}$

Exercise:

Problem: ⓐ $\frac{4x^2y}{3y}$
ⓑ $\frac{3x-2}{2x+1}$
ⓒ $\frac{u-1}{u^2-3u-28}$

Solution:

ⓐ $y=0$ ⓑ $x=-\frac{1}{2}$
ⓒ $u=-4, u=7$

Exercise:

Problem: ⓐ $\frac{5pq^2}{9q}$
ⓑ $\frac{7a-4}{3a+5}$
ⓒ $\frac{1}{x^2-4}$

Evaluate Rational Expressions

In the following exercises, evaluate the rational expression for the given values.

Exercise:

Problem: $\frac{2x}{x-1}$

ⓐ $x=0$
ⓑ $x=2$
ⓒ $x=-1$

Solution:

ⓐ 0 ⓑ 4 ⓒ 1

Exercise:

Problem: $\frac{4y-1}{5y-3}$

ⓐ $y=0$
ⓑ $y=2$
ⓒ $y=-1$

Exercise:

Problem: $\frac{2p+3}{p^2+1}$

ⓐ $p=0$
ⓑ $p=1$
ⓒ $p=-2$

Solution:

ⓐ 3 ⓑ $\frac{5}{2}$ ⓒ $-\frac{1}{5}$

Exercise:

Problem: $\frac{x+3}{2-3x}$

ⓐ $x=0$
ⓑ $x=1$
ⓒ $x=-2$

Exercise:

Problem: $\frac{y^2+5y+6}{y^2-1}$

ⓐ $y=0$
ⓑ $y=2$
ⓒ $y=-2$

Solution:

ⓐ -6 ⓑ $\frac{20}{3}$ ⓒ 0

Exercise:

Problem: $\frac{z^2+3z-10}{z^2-1}$

ⓐ $z=0$
ⓑ $z=2$
ⓒ $z=-2$

Exercise:

Problem: $\frac{a^2-4}{a^2+5a+4}$

ⓐ $a=0$
ⓑ $a=1$
ⓒ $a=-2$

Solution:

ⓐ -1 ⓑ $-\frac{3}{10}$ ⓒ 0

Exercise:

Problem: $\frac{b^2+2}{b^2-3b-4}$

ⓐ $b=0$
ⓑ $b=2$
ⓒ $b=-2$

Exercise:

Problem: $\frac{x^2+3xy+2y^2}{2x^3y}$

ⓐ $x=1, y=-1$
ⓑ $x=2, y=1$
ⓒ $x=-1, y=-2$

Solution:

ⓐ 0 ⓑ $\frac{3}{4}$ ⓒ $\frac{15}{4}$

Exercise:

Problem: $\frac{c^2+cd-2d^2}{cd^3}$

ⓐ $c=2, d=-1$
ⓑ $c=1, d=-1$
ⓒ $c=-1, d=2$

Exercise:

Problem: $\frac{m^2-4n^2}{5mn^3}$

ⓐ $m=2, n=1$
ⓑ $m=-1, n=-1$
ⓒ $m=3, n=2$

Solution:

ⓐ 0 ⓑ $-\frac{3}{5}$ ⓒ $-\frac{7}{120}$

Exercise:

Problem: $\frac{2s^2t}{s^2-9t^2}$

ⓐ $s=4, t=1$
ⓑ $s=-1, t=-1$
ⓒ $s=0, t=2$

Simplify Rational Expressions

In the following exercises, simplify.

Exercise:

Problem: $-\frac{4}{52}$

Solution:

$-\frac{1}{13}$

Exercise:

Problem: $-\frac{44}{55}$

Exercise:

Problem: $\frac{56}{63}$

Solution:

$\frac{8}{9}$

Exercise:

Problem: $\frac{65}{104}$

Exercise:

Problem: $\frac{6ab^2}{12a^2b}$

Solution:

$\frac{b}{2a}$

Exercise:

Problem: $\frac{15xy}{3x^3y^3}$

Exercise:

Problem: $\frac{8m^3n}{12mn^2}$

Solution:

$\frac{2m^2}{3n}$

Exercise:

Problem: $\frac{36v^3w^2}{27vw^3}$

Exercise:

Problem: $\frac{3a+6}{4a+8}$

Solution:

$\frac{3}{4}$

Exercise:

Problem: $\frac{5b+5}{6b+6}$

Exercise:

Problem: $\frac{3c-9}{5c-15}$

Solution:

$\frac{3}{5}$

Exercise:

Problem: $\frac{4d+8}{9d+18}$

Exercise:

Problem: $\frac{7m+63}{5m+45}$

Solution:

$\frac{7}{5}$

Exercise:

Problem: $\frac{8n-96}{3n-36}$

Exercise:

Problem: $\frac{12p-240}{5p-100}$

Solution:

$\frac{12}{5}$

Exercise:

Problem: $\frac{6q+210}{5q+175}$

Exercise:

Problem: $\frac{a^2-a-12}{a^2-8a+16}$

Solution:

$\frac{a+3}{a-4}$

Exercise:

Problem: $\frac{x^2+4x-5}{x^2-2x+1}$

Exercise:

Problem: $\frac{y^2+3y-4}{y^2-6y+5}$

Solution:

$\frac{y+4}{y-5}$

Exercise:

Problem: $\frac{v^2+8v+15}{v^2-v-12}$

Exercise:

Problem: $\frac{x^2-25}{x^2+2x-15}$

Solution:

$\frac{x-5}{x-3}$

Exercise:

Problem: $\frac{a^2-4}{a^2+6a-16}$

Exercise:

Problem: $\frac{y^2-2y-3}{y^2-9}$

Solution:

$\frac{y+1}{y+3}$

Exercise:

Problem: $\frac{b^2+9b+18}{b^2-36}$

Exercise:

Problem: $\frac{y^3+y^2+y+1}{y^2+2y+1}$

Solution:

$\frac{y^2+1}{y+1}$

Exercise:

Problem: $\frac{p^3+3p^2+4p+12}{p^2+p-6}$

Exercise:

Problem: $\frac{x^3-2x^2-25x+50}{x^2-25}$

Solution:

$x-2$

Exercise:

Problem: $\frac{q^3+3q^2-4q-12}{q^2-4}$

Exercise:

Problem: $\frac{3a^2+15a}{6a^2+6a-36}$

Solution:

$\frac{a(a+5)}{2(a+3)(a-2)}$

Exercise:

Problem: $\frac{8b^2-32b}{2b^2-6b-80}$

Exercise:

Problem: $\frac{-5c^2-10c}{-10c^2+30c+100}$

Solution:

$\frac{c}{2(c-5)}$

Exercise:

Problem: $\frac{4d^2-24d}{2d^2-4d-48}$

Exercise:

Problem: $\frac{3m^2+30m+75}{4m^2-100}$

Solution:

$\frac{3(m+5)}{4(m-5)}$

Exercise:

Problem: $\frac{5n^2+30n+45}{2n^2-18}$

Exercise:

Problem: $\frac{5r^2+30r-35}{r^2-49}$

Solution:

$\frac{5(r-1)}{r+7}$

Exercise:

Problem: $\frac{3s^2+30s+24}{3s^2-48}$

Exercise:

Problem: $\frac{t^3-27}{t^2-9}$

Solution:

$\frac{t^2+3t+9}{t+3}$

Exercise:

Problem: $\frac{v^3-1}{v^2-1}$

Exercise:

Problem: $\frac{w^3+216}{w^2-36}$

Solution:

$\frac{w^2-6w+36}{w-6}$

Exercise:

Problem: $\frac{v^3+125}{v^2-25}$

Simplify Rational Expressions with Opposite Factors

In the following exercises, simplify each rational expression.

Exercise:

Problem: $\frac{a-5}{5-a}$

Solution:

-1

Exercise:

Problem: $\frac{b-12}{12-b}$

Exercise:

Problem: $\frac{11-c}{c-11}$

Solution:

-1

Exercise:

Problem: $\frac{5-d}{d-5}$

Exercise:

Problem: $\frac{12-2x}{x^2-36}$

Solution:

$-\frac{2}{x+6}$

Exercise:

Problem: $\frac{20-5y}{y^2-16}$

Exercise:

Problem: $\frac{4v-32}{64-v^2}$

Solution:

$-\frac{4}{8+v}$

Exercise:

Problem: $\frac{7w-21}{9-w^2}$

Exercise:

Problem: $\frac{y^2-11y+24}{9-y^2}$

Solution:

$-\frac{y-8}{3+y}$

Exercise:

Problem: $\frac{z^2-9z+20}{16-z^2}$

Exercise:

Problem: $\frac{a^2-5z-36}{81-a^2}$

Solution:

$-\frac{a+4}{9+a}$

Exercise:

Problem: $\frac{b^2+b-42}{36-b^2}$

Everyday Math

Exercise:

Problem:

Tax Rates For the tax year 2015, the amount of tax owed by a single person earning between \$37,450 and \$90,750, can be found by evaluating the formula $0.25x-4206.25$, where *x* is income. The average tax rate for this income can be found by evaluating the formula $\frac{0.25x-4206.25}{x}$. What would be the average tax rate for a single person earning \$50,000?

Solution:

16.5%

Exercise:

Problem:

Work The length of time it takes for two people for perform the same task if they work together can be found by evaluating the formula $\frac{xy}{x+y}$. If Tom can paint the den in $x=45$ minutes and his brother Bobby can paint it in $y=60$ minutes, how many minutes will it take them if they work together?

Writing Exercises

Exercise:

Problem:

Explain how you find the values of *x* for which the rational expression $\frac{x^2-x-20}{x^2-4}$ is undefined.

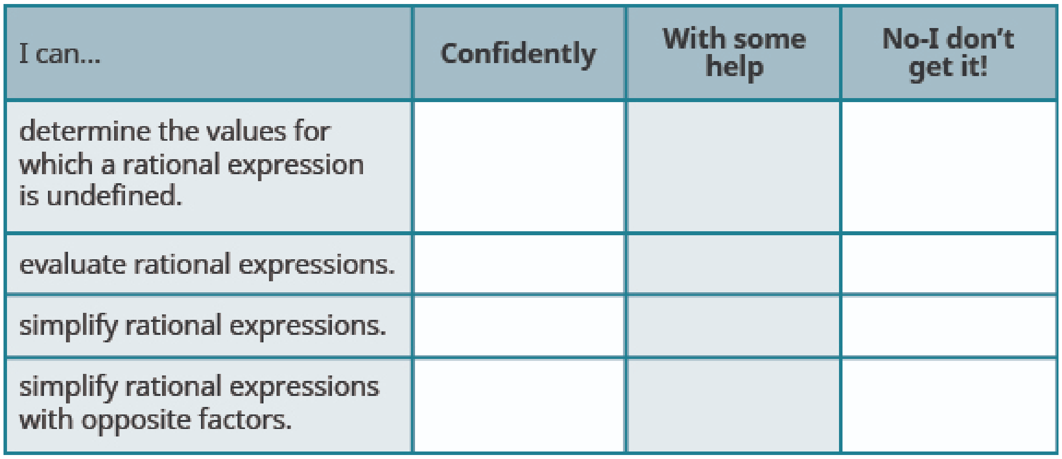
Exercise:

Problem:

Explain all the steps you take to simplify the rational expression $\frac{p^2+4p-21}{9-p^2}$.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
determine the values for which a rational expression is undefined.			
evaluate rational expressions.			
simplify rational expressions.			
simplify rational expressions with opposite factors.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

rational expression

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Multiply and Divide Rational Expressions
By the end of this section, you will be able to:

- Multiply rational expressions
- Divide rational expressions

Note:
Before you get started, take this readiness quiz.
If you miss a problem, go back to the section listed and review the material.

1. Multiply: $\frac{14}{15} \cdot \frac{6}{35}$.
   If you missed this problem, review [link].
2. Divide: $\frac{14}{15} \div \frac{6}{35}$.
   If you missed this problem, review [link].
3. Factor completely: $2x^2 - 98$.
   If you missed this problem, review [link].
4. Factor completely: $10n^3 + 10$.
   If you missed this problem, review [link].
5. Factor completely: $10p^2 - 25pq - 15q^2$.
   If you missed this problem, review [link].

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

Note:
Multiplication of Rational Expressions
If p, q, r, s are polynomials where $q \neq 0$ and $s \neq 0$, then
Equation:

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

We’ll do the first example with numerical fractions to remind us of how we multiplied fractions without variables.

Example:
Exercise:

Problem: Multiply: $\frac{10}{28} \cdot \frac{8}{15}$.

Solution:
Solution

	$\frac{10}{28} \cdot \frac{8}{15}$
Multiply the numerators and denominators.	$\frac{10 \cdot 8}{28 \cdot 15}$
Look for common factors, and then remove them.	$\frac{2 \cdot \cancel{5} \cdot 2 \cdot \cancel{4}}{7 \cdot \cancel{4} \cdot 3 \cdot \cancel{5}}$
Simplify.	$\frac{4}{21}$

Note:
Exercise:

Problem: Mulitply: $\frac{6}{10} \cdot \frac{15}{12}$.

Solution:

$\frac{3}{4}$

Note:
Exercise:

Problem: Mulitply: $\frac{20}{15} \cdot \frac{6}{8}$.

Solution:

1

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$ and $y \neq 0$.

Example:
Exercise:

Problem: Mulitply: $\frac{2x}{3y^2} \cdot \frac{6xy^3}{x^2y}$.

Solution:
Solution

	$\frac{2x}{3y^2} \cdot \frac{6xy^3}{x^2y}$
Multiply.	$\frac{2x \cdot 6xy^3}{3y^2 \cdot x^2y}$
Factor the numerator and denominator completely, and then	

remove common factors.	$\frac{2\cdot\cancel{x}\cdot 2\cdot\cancel{3}\cdot\cancel{x}\cdot\cancel{y}\cdot\cancel{y}\cdot\cancel{y}}{\cancel{3}\cdot\cancel{y}\cdot\cancel{y}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{y}}$
Simplify.	4

Note:
Exercise:

Problem: Mulitply: $\frac{3pq}{q^2}\cdot\frac{5p^2q}{6pq}$.

Solution:

$\frac{5p^2}{2q}$

Note:
Exercise:

Problem: Mulitply: $\frac{6x^3y}{7x^2}\cdot\frac{2xy^3}{x^2y}$.

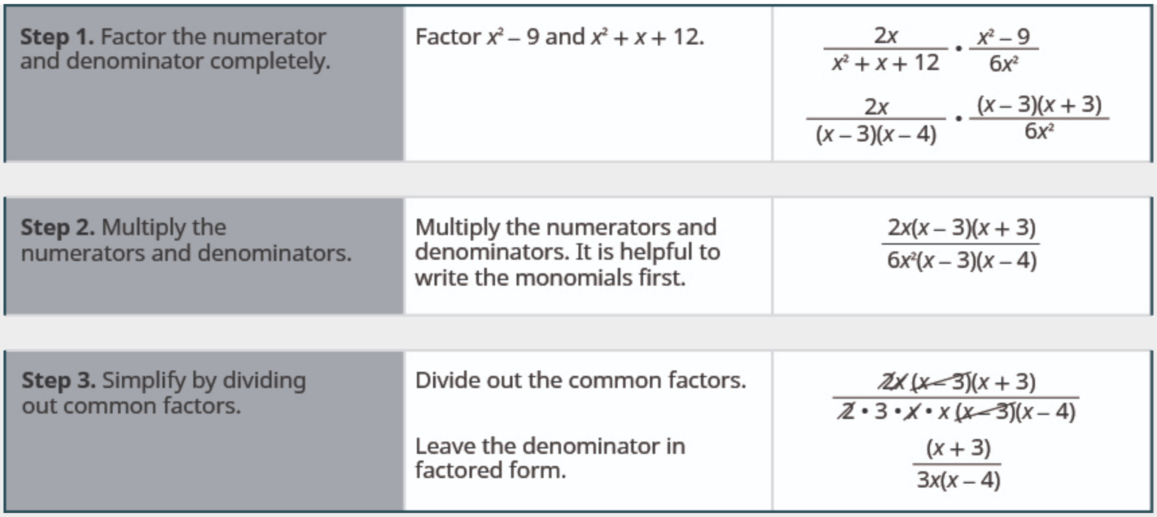
Solution:

$\frac{12y^3}{7}$

Example:
How to Multiply Rational Expressions
Exercise:

Problem: Mulitply: $\frac{2x}{x^2+x+12}\cdot\frac{x^2-9}{6x^2}$.

Solution:
Solution



Step 1. Factor the numerator and denominator completely.	Factor $x^2 - 9$ and $x^2 + x + 12$.	$\frac{2x}{x^2+x+12} \cdot \frac{x^2-9}{6x^2}$ $\frac{2x}{(x-3)(x-4)} \cdot \frac{(x-3)(x+3)}{6x^2}$
Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$\frac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3. Simplify by dividing out common factors.	Divide out the common factors. Leave the denominator in factored form.	$\frac{\cancel{2}\cancel{x}\cancel{(x-3)}(x+3)}{\cancel{2} \cdot 3 \cdot \cancel{x} \cdot x\cancel{(x-3)}(x-4)}$ $\frac{(x+3)}{3x(x-4)}$

Note:
Exercise:

Problem: Mulitply: $\frac{5x}{x^2+5x+6} \cdot \frac{x^2-4}{10x}$.

Solution:

$\frac{x-2}{2(x+3)}$

Note:
Exercise:

Problem: Mulitply: $\frac{9x^2}{x^2+11x+30} \cdot \frac{x^2-36}{3x^2}$.

Solution:

$\frac{3(x-6)}{x+5}$

Note:
Multiply a rational expression.

Factor each numerator and denominator completely.
Multiply the numerators and denominators.
Simplify by dividing out common factors.

Example:
Exercise:

Problem: Multiply: $\frac{n^2-7n}{n^2+2n+1}\cdot\frac{n+1}{2n}$.

Solution:
Solution

	$\frac{n^2-7n}{n^2+2n+1}\cdot\frac{n+1}{2n}$
Factor each numerator and denominator.	$\frac{n(n-7)}{(n+1)(n+1)}\cdot\frac{n+1}{2n}$
Multiply the numerators and the denominators.	$\frac{n(n-7)(n+1)}{(n+1)(n+1)2n}$
Remove common factors.	$\frac{\cancel{n}\,(n-7)\,\cancel{(n+1)}}{(n+1)\,\cancel{(n+1)}\,2\,\cancel{n}}$
Simplify.	$\frac{n-7}{2(n+1)}$

Note:
Exercise:

Problem: Multiply: $\frac{x^2-25}{x^2-3x-10}\cdot\frac{x+2}{x}$.

Solution:

$\frac{x+5}{x}$

Note:
Exercise:

Problem: Multiply: $\frac{x^2-4x}{x^2+5x+6} \cdot \frac{x+2}{x}$.

Solution:

$\frac{x-4}{x+3}$

Example:
Exercise:

Problem: Multiply: $\frac{16-4x}{2x-12} \cdot \frac{x^2-5x-6}{x^2-16}$.

Solution:
Solution

	$\frac{16-4x}{2x-12} \cdot \frac{x^2-5x-6}{x^2-16}$
Factor each numerator and denominator.	$\frac{4(4-x)}{2(x-6)} \cdot \frac{(x-6)(x+1)}{(x-4)(x+4)}$
Multiply the numerators and the denominators.	$\frac{4(4-x)(x-6)(x+1)}{2(x-6)(x-4)(x+4)}$
Remove common factors.	$(-1)\frac{\cancel{2}\cdot 2\,\cancel{(4-x)}\,\cancel{(x-6)}\,(x+1)}{\cancel{2}\,\cancel{(x-6)}\,\cancel{(x-4)}\,(x+4)}$
Simplify.	$-\frac{2(x+1)}{(x+4)}$

Note:
Exercise:

Problem: Multiply: $\frac{12x-6x^2}{x^2+8x} \cdot \frac{x^2+11x+24}{x^2-4}$.

Solution:

$-\frac{6(x+3)}{x+2}$

Note:
Exercise:

Problem: Multiply: $\frac{9v-3v^2}{9v+36} \cdot \frac{v^2+7v+12}{v^2-9}$.

Solution:

$-\frac{v}{3}$

Example:
Exercise:

Problem: Multiply: $\frac{2x-6}{x^2-8x+15} \cdot \frac{x^2-25}{2x+10}$.

Solution:
Solution

	$\frac{2x-6}{x^2-8x+15} \cdot \frac{x^2-25}{2x+10}$
Factor each numerator and denominator.	$\frac{2(x-3)}{(x-3)(x-5)} \cdot \frac{(x-5)(x+5)}{2(x+5)}$
Multiply the numerators and denominators.	$\frac{2(x-3)(x-5)(x+5)}{2(x-3)(x-5)(x+5)}$
Remove common factors.	$\frac{\cancel{2}\cancel{(x-3)}\cancel{(x-5)}\cancel{(x+5)}}{\cancel{2}\cancel{(x-3)}\cancel{(x-5)}\cancel{(x+5)}}$
Simplify.	1

Note:
Exercise:

Problem: Multiply: $\frac{3a-21}{a^2-9a+14} \cdot \frac{a^2-4}{3a+6}$.

Solution:

1

Note:
Exercise:

Problem: Multiply: $\frac{b^2-b}{b^2+9b-10} \cdot \frac{b^2-100}{b^2-10b}$.

Solution:

1

Divide Rational Expressions

To divide rational expressions we multiply the first fraction by the reciprocal of the second, just like we did for numerical fractions.

Remember, the **reciprocal** of $\frac{a}{b}$ is $\frac{b}{a}$. To find the reciprocal we simply put the numerator in the denominator and the denominator in the numerator. We “flip” the fraction.

Note:
Division of Rational Expressions
If p, q, r, s are polynomials where $q \neq 0, r \neq 0, s \neq 0$, then
Equation:

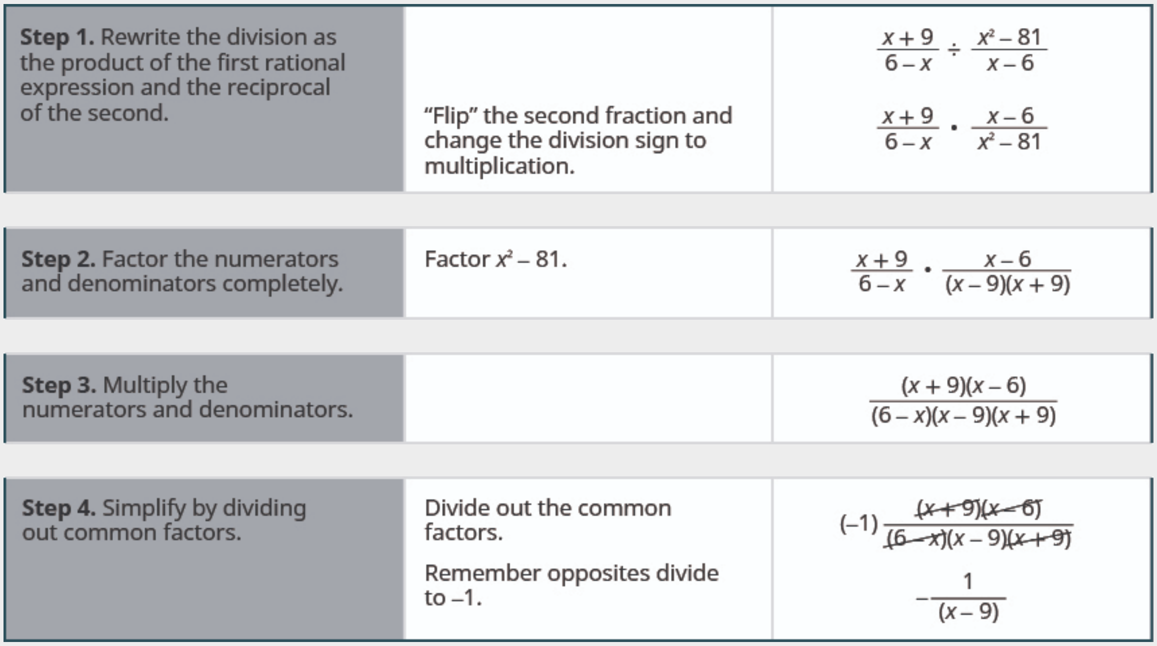
$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

To divide rational expressions multiply the first fraction by the reciprocal of the second.

Example:
How to Divide Rational Expressions
Exercise:

Problem: Divide: $\frac{x+9}{6-x} \div \frac{x^2-81}{x-6}$.

Solution:
Solution



Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	“Flip” the second fraction and change the division sign to multiplication.	$\frac{x+9}{6-x} \div \frac{x^2-81}{x-6}$ $\frac{x+9}{6-x} \cdot \frac{x-6}{x^2-81}$
Step 2. Factor the numerators and denominators completely.	Factor x^2-81.	$\frac{x+9}{6-x} \cdot \frac{x-6}{(x-9)(x+9)}$
Step 3. Multiply the numerators and denominators.		$\frac{(x+9)(x-6)}{(6-x)(x-9)(x+9)}$
Step 4. Simplify by dividing out common factors.	Divide out the common factors. Remember opposites divide to –1.	$(-1)\,\frac{\cancel{(x+9)}\cancel{(x-6)}}{\cancel{(6-x)}(x-9)\cancel{(x+9)}}$ $-\frac{1}{(x-9)}$

Note:
Exercise:

Problem: Divide: $\frac{c+3}{5-c} \div \frac{c^2-9}{c-5}$.

Solution:

$-\frac{1}{c-3}$

Note:
Exercise:

Problem: Divide: $\frac{2-d}{d-4} \div \frac{4-d^2}{4-d}$.

Solution:

$-\frac{1}{2+d}$

Note:
Divide rational expressions.

Rewrite the division as the product of the first rational expression and the reciprocal of the second.
Factor the numerators and denominators completely.
Multiply the numerators and denominators together.
Simplify by dividing out common factors.

Example:
Exercise:

Problem: Divide: $\frac{3n^2}{n^2-4n} \div \frac{9n^2-45n}{n^2-7n+10}$.

Solution:
Solution

	$\frac{3n^2}{n^2-4n} \div \frac{9n^2-45n}{n^2-7n+10}$
Rewrite the division as the product of the first rational expression and the reciprocal of the second.	$\frac{3n^2}{n^2-4n} \cdot \frac{n^2-7n+10}{9n^2-45n}$
Factor the numerators and denominators and then multiply.	

	$\frac{3\cdot n\cdot n\cdot(n-5)(n-2)}{n(n-4)\cdot 3\cdot 3\cdot n\cdot(n-5)}$
Simplify by dividing out common factors.	$\frac{\cancel{3}\cdot\cancel{n}\cdot\cancel{n(n-5)}(n-2)}{\cancel{n}(n-4)\cancel{3}\cdot 3\cdot\cancel{n(n-5)}}$
	$\frac{n-2}{3(n-4)}$

Note:
Exercise:

Problem: Divide: $\frac{2m^2}{m^2-8m} \div \frac{8m^2+24m}{m^2+m-6}$.

Solution:

$\frac{(m-2)}{4(m-8)}$

Note:
Exercise:

Problem: Divide: $\frac{15n^2}{3n^2+33n} \div \frac{5n-5}{n^2+9n-22}$.

Solution:

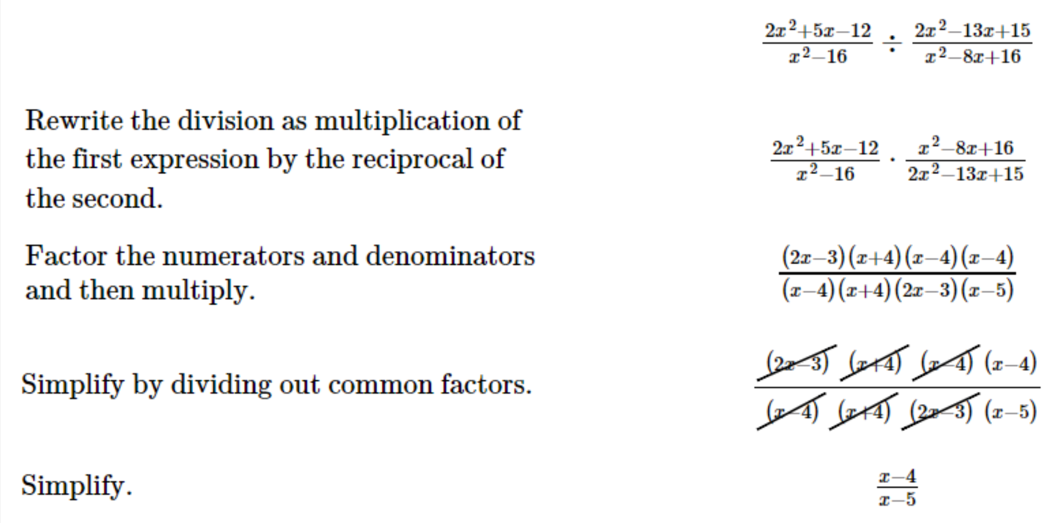
$\frac{n(n-2)}{n-1}$

Remember, first rewrite the division as multiplication of the first expression by the reciprocal of the second. Then factor everything and look for common factors.

Example:
Exercise:

Problem: Divide: $\frac{2x^2+5x-12}{x^2-16} \div \frac{2x^2-13x+15}{x^2-8x+16}$.

Solution:
Solution



	$\frac{2x^2+5x-12}{x^2-16} \div \frac{2x^2-13x+15}{x^2-8x+16}$
Rewrite the division as multiplication of the first expression by the reciprocal of the second.	$\frac{2x^2+5x-12}{x^2-16} \cdot \frac{x^2-8x+16}{2x^2-13x+15}$
Factor the numerators and denominators and then multiply.	$\frac{(2x-3)(x+4)(x-4)(x-4)}{(x-4)(x+4)(2x-3)(x-5)}$
Simplify by dividing out common factors.	$\frac{\cancel{(2x-3)}\ \cancel{(x+4)}\ \cancel{(x-4)}\ (x-4)}{\cancel{(x-4)}\ \cancel{(x+4)}\ \cancel{(2x-3)}\ (x-5)}$
Simplify.	$\frac{x-4}{x-5}$

Note:
Exercise:

Problem: Divide: $\frac{3a^2-8a-3}{a^2-25} \div \frac{3a^2-14a-5}{a^2+10a+25}$.

Solution:

$\frac{(a-3)(a+5)}{(a-5)(a-5)}$

Note:
Exercise:

Problem: Divide: $\frac{4b^2+7b-2}{1-b^2} \div \frac{4b^2+15b-4}{b^2-2b+1}$.

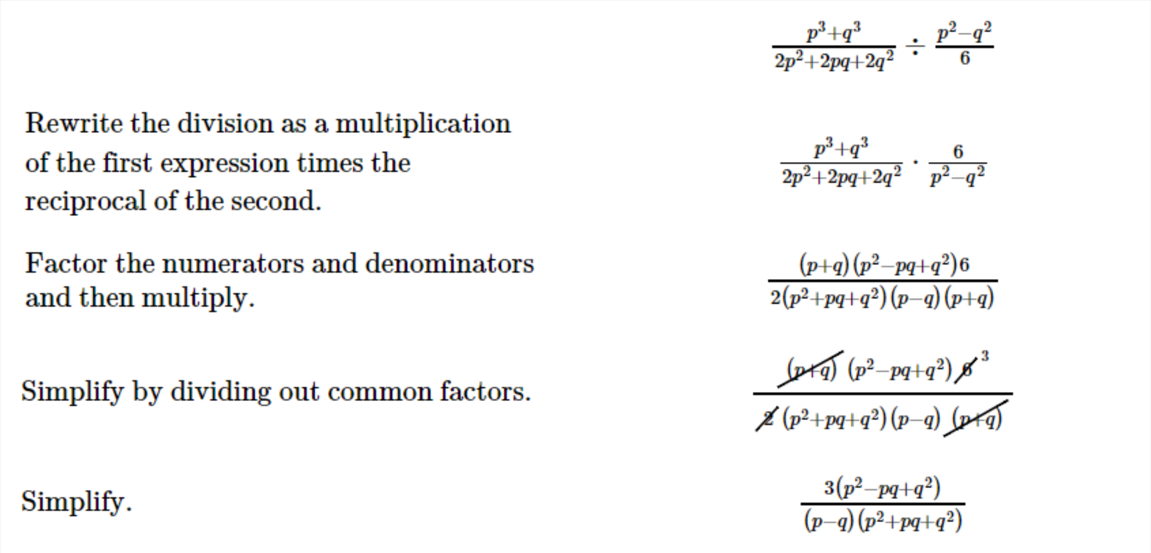
Solution:

$-\frac{(b+2)(b-1)}{(1+b)(b+4)}$

Example:
Exercise:

Problem: Divide: $\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$.

Solution:
Solution



	$\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$
Rewrite the division as a multiplication of the first expression times the reciprocal of the second.	$\frac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \frac{6}{p^2-q^2}$
Factor the numerators and denominators and then multiply.	$\frac{(p+q)(p^2-pq+q^2)6}{2(p^2+pq+q^2)(p-q)(p+q)}$
Simplify by dividing out common factors.	$\frac{\cancel{(p+q)}\,(p^2-pq+q^2)\,\cancel{6}^{\,3}}{\cancel{2}\,(p^2+pq+q^2)(p-q)\,\cancel{(p+q)}}$
Simplify.	$\frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

Note:
Exercise:

Problem: Divide: $\frac{x^3-8}{3x^2-6x+12} \div \frac{x^2-4}{6}$.

Solution:

$\frac{2(x^2+2x+4)}{(x+2)(x^2-2x+4)}$

Note:
Exercise:

Problem: Divide: $\frac{2z^2}{z^2-1} \div \frac{z^3-z^2+z}{z^3-1}$.

Solution:

$\frac{2z(z^2+z+1)}{(z+1)(z^2-z+1)}$

Before doing the next example, let's look at how we divide a fraction by a whole number. When we divide $\frac{3}{5} \div 4$, we first write 4 as a fraction so that we can find its reciprocal.

Equation:

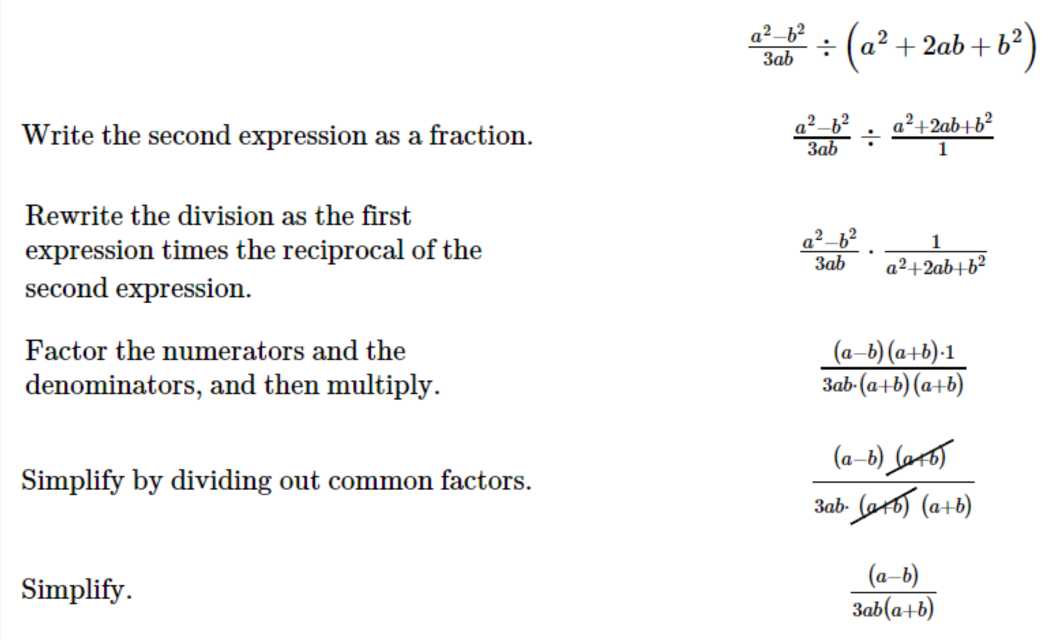
$$\begin{array}{c} \frac{3}{5} \div 4 \\ \frac{3}{5} \div \frac{4}{1} \\ \frac{3}{5} \cdot \frac{1}{4} \end{array}$$

We do the same thing when we divide rational expressions.

Example:
Exercise:

Problem: Divide: $\frac{a^2-b^2}{3ab} \div \left(a^2 + 2ab + b^2\right)$.

Solution:
Solution



	$\frac{a^2-b^2}{3ab} \div \left(a^2+2ab+b^2\right)$
Write the second expression as a fraction.	$\frac{a^2-b^2}{3ab} \div \frac{a^2+2ab+b^2}{1}$
Rewrite the division as the first expression times the reciprocal of the second expression.	$\frac{a^2-b^2}{3ab} \cdot \frac{1}{a^2+2ab+b^2}$
Factor the numerators and the denominators, and then multiply.	$\frac{(a-b)(a+b)\cdot 1}{3ab\cdot(a+b)(a+b)}$
Simplify by dividing out common factors.	$\frac{(a-b)\,\cancel{(a+b)}}{3ab\cdot\,\cancel{(a+b)}\,(a+b)}$
Simplify.	$\frac{(a-b)}{3ab(a+b)}$

Note:
Exercise:

Problem: Divide: $\frac{2x^2-14x-16}{4} \div (x^2+2x+1)$.

Solution:

$\frac{x-8}{2(x+1)}$

Note:
Exercise:

Problem: Divide: $\frac{y^2-6y+8}{y^2-4y} \div (3y^2-12y)$.

Solution:

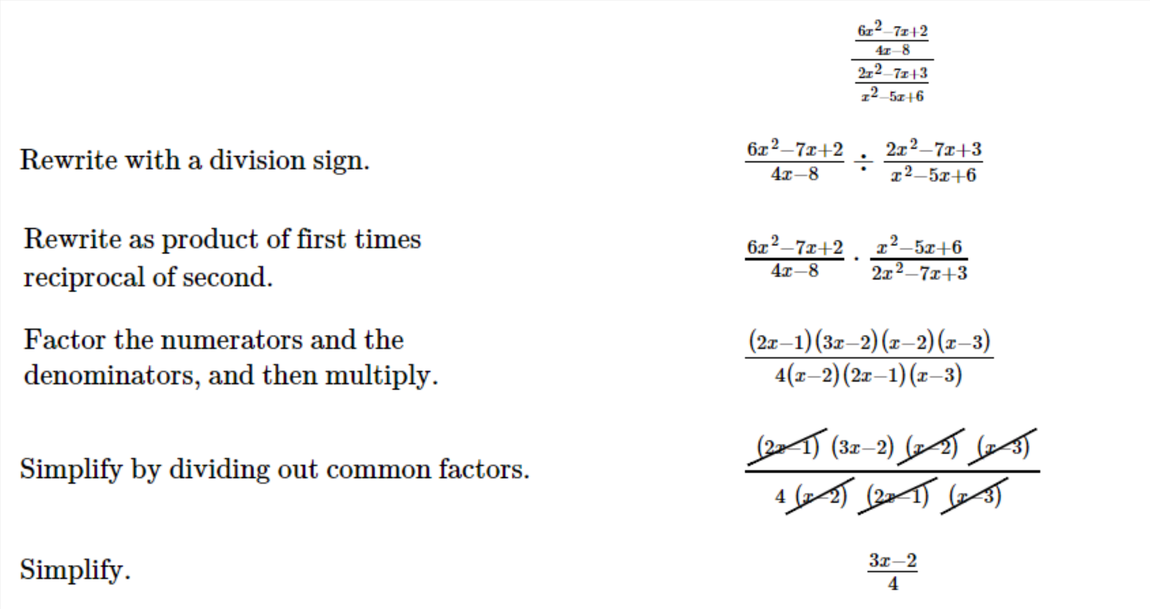
$\frac{y-2}{3y(y-4)}$

Remember a fraction bar means division. A complex fraction is another way of writing division of two fractions.

Example:
Exercise:

Problem: Divide: $\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$.

Solution:
Solution



	$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$
Rewrite with a division sign.	$\frac{6x^2-7x+2}{4x-8} \div \frac{2x^2-7x+3}{x^2-5x+6}$
Rewrite as product of first times reciprocal of second.	$\frac{6x^2-7x+2}{4x-8} \cdot \frac{x^2-5x+6}{2x^2-7x+3}$
Factor the numerators and the denominators, and then multiply.	$\frac{(2x-1)(3x-2)(x-2)(x-3)}{4(x-2)(2x-1)(x-3)}$
Simplify by dividing out common factors.	$\frac{\cancel{(2x-1)}\ (3x-2)\ \cancel{(x-2)}\ \cancel{(x-3)}}{4\ \cancel{(x-2)}\ \cancel{(2x-1)}\ \cancel{(x-3)}}$
Simplify.	$\frac{3x-2}{4}$

Note:
Exercise:

Problem: Divide: $\frac{\frac{3x^2+7x+2}{4x+24}}{\frac{3x^2-14x-5}{x^2+x-30}}$.

Solution:

$\frac{x+2}{4}$

Note:
Exercise:

Problem: Divide: $\frac{\frac{y^2-36}{2y^2+11y-6}}{\frac{2y^2-2y-60}{8y-4}}$.

Solution:

$\frac{2}{y+5}$

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then we factor and multiply.

Example:
Exercise:

Problem: Divide: $\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$.

Solution:
Solution

	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$
Rewrite the division as multiplication by the reciprocal.	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \cdot \frac{8x+16}{2x+12}$
Factor the numerators and the denominators, and then multiply.	$\frac{3 \cdot 8(x-2)(x+3)(x-1)(x+2)}{4 \cdot 2(x-1)(x+2)(x-5)(x+6)}$
Simplify by dividing out common factors.	

	$\frac{3\cdot\cancel{8}(x-2)(x+3)\cancel{(x-1)}\cancel{(x+2)}}{\cancel{4}\cdot\cancel{2}\cancel{(x-1)}\cancel{(x+2)}(x-5)(x+6)}$
Simplify.	$\frac{3(x-2)(x+3)}{(x-5)(x+6)}$

Note:
Exercise:

Problem: Divide: $\frac{4m+4}{3m-15}\cdot\frac{m^2-3m-10}{m^2-4m-32}\div\frac{12m-36}{6m-48}$.

Solution:

$\frac{2(m+1)(m+2)}{3(m+4)(m-3)}$

Note:
Exercise:

Problem: Divide: $\frac{2n^2+10n}{n-1}\div\frac{n^2+10n+24}{n^2+8n-9}\cdot\frac{n+4}{8n^2+12n}$.

Solution:

$\frac{(n+5)(n+9)}{2(n+6)(2n+3)}$

Key Concepts

- **Multiplication of Rational Expressions**
  - If p, q, r, s are polynomials where $q \neq 0$, $s \neq 0$, then $\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs}$.
  - To multiply rational expressions, multiply the numerators and multiply the denominators
- **Multiply a Rational Expression**

Factor each numerator and denominator completely.
Multiply the numerators and denominators.
Simplify by dividing out common factors.

- **Division of Rational Expressions**
  - If p, q, r, s are polynomials where $q \neq 0, r \neq 0, s \neq 0$, then $\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$.
  - To divide rational expressions multiply the first fraction by the reciprocal of the second.
- **Divide Rational Expressions**

  Rewrite the division as the product of the first rational expression and the reciprocal of the second.
  Factor the numerators and denominators completely.
  Multiply the numerators and denominators together.
  Simplify by dividing out common factors.

Practice Makes Perfect

Multiply Rational Expressions

In the following exercises, multiply.

Exercise:

Problem: $\frac{12}{16} \cdot \frac{4}{10}$

Solution:

$\frac{3}{10}$

Exercise:

Problem: $\frac{32}{5} \cdot \frac{16}{24}$

Exercise:

Problem: $\frac{18}{10} \cdot \frac{4}{30}$

Solution:

$\frac{6}{25}$

Exercise:

Problem: $\frac{21}{36} \cdot \frac{45}{24}$

Exercise:

Problem: $\frac{5x^2y^4}{12xy^3} \cdot \frac{6x^2}{20y^2}$

Solution:

$\frac{x^3}{8y}$

Exercise:

Problem: $\frac{8w^3y}{9y^2} \cdot \frac{3y}{4w^4}$

Exercise:

Problem: $\frac{12a^3b}{b^2} \cdot \frac{2ab^2}{9b^3}$

Solution:

$\frac{8a^4}{3b^2}$

Exercise:

Problem: $\frac{4mn^2}{5n^3} \cdot \frac{mn^3}{8m^2n^2}$

Exercise:

Problem: $\frac{5p^2}{p^2-5p-36} \cdot \frac{p^2-16}{10p}$

Solution:

$\frac{p(p-4)}{2(p-9)}$

Exercise:

Problem: $\frac{3q^2}{q^2+q-6} \cdot \frac{q^2-9}{9q}$

Exercise:

Problem: $\frac{4r}{r^2-3r-10} \cdot \frac{r^2-25}{8r^2}$

Solution:

$\frac{r+5}{2r(r+2)}$

Exercise:

Problem: $\frac{s}{s^2-9s+14} \cdot \frac{s^2-49}{7s^2}$

Exercise:

Problem: $\frac{x^2-7x}{x^2+6x+9} \cdot \frac{x+3}{4x}$

Solution:

$\frac{x-7}{4(x+3)}$

Exercise:

Problem: $\frac{2y^2-10y}{y^2+10y+25} \cdot \frac{y+5}{6y}$

Exercise:

Problem: $\frac{z^2+3z}{z^2-3z-4} \cdot \frac{z-4}{z^2}$

Solution:

$\frac{z+3}{z(z+1)}$

Exercise:

Problem: $\frac{2a^2+8a}{a^2-9a+20} \cdot \frac{a-5}{a^2}$

Exercise:

Problem: $\frac{28-4b}{3b-3} \cdot \frac{b^2+8b-9}{b^2-49}$

Solution:

$-\frac{4(b+9)}{3(b+7)}$

Exercise:

Problem: $\frac{18c-2c^2}{6c+30} \cdot \frac{c^2+7c+10}{c^2-81}$

Exercise:

Problem: $\frac{35d-7d^2}{d^2+7d} \cdot \frac{d^2+12d+35}{d^2-25}$

Solution:

-7

Exercise:

Problem: $\frac{72m-12m^2}{8m+32} \cdot \frac{m^2+10m+24}{m^2-36}$

Exercise:

Problem: $\frac{4n+20}{n^2+n-20} \cdot \frac{n^2-16}{4n+16}$

Solution:

1

Exercise:

Problem: $\frac{6p^2-6p}{p^2+7p-18} \cdot \frac{p^2-81}{3p^2-27p}$

Exercise:

Problem: $\frac{q^2-2q}{q^2+6q-16} \cdot \frac{q^2-64}{q^2-8q}$

Solution:

1

Exercise:

Problem: $\frac{2r^2-2r}{r^2+4r-5} \cdot \frac{r^2-25}{2r^2-10r}$

Divide Rational Expressions

In the following exercises, divide.

Exercise:

Problem: $\frac{3-t}{t-6} \div \frac{t^2-9}{t-5}$

Solution:

$\frac{5-t}{t^2-3t-18}$

Exercise:

Problem: $\frac{v-5}{11-v} \div \frac{v^2-25}{v-11}$

Exercise:

Problem: $\frac{10+w}{w-8} \div \frac{100-w^2}{8-w}$

Solution:

$-\frac{1}{10-w}$

Exercise:

Problem: $\frac{7+x}{x-6} \div \frac{49-x}{x+6}^{2}$

Exercise:

Problem: $\frac{27y^2}{3y-21} \div \frac{3y^2+18}{y^2+13y+42}$

Solution:

$\frac{3y^2(y+6)(y+7)}{(y-7)(y^2+6)}$

Exercise:

Problem: $\frac{24z^2}{2z-8} \div \frac{4z-28}{z^2-11z+28}$

Exercise:

Problem: $\frac{16a^2}{4a+36} \div \frac{4a^2-24a}{a^2+4a-45}$

Solution:

$\frac{a(a-5)}{a-6}$

Exercise:

Problem: $\frac{24b^2}{2b-4} \div \frac{12b^2+36b}{b^2-11b+18}$

Exercise:

Problem: $\frac{5c^2+9c+2}{c^2-25} \div \frac{3c^2-14c-5}{c^2+10c+25}$

Solution:

$\frac{(c+2)(c+2)}{(c-2)(c-3)}$

Exercise:

Problem: $\frac{2d^2+d-3}{d^2-16} \div \frac{2d^2-9d-18}{d^2-8d+16}$

Exercise:

Problem: $\frac{6m^2-7m-10}{9-m^2} \div \frac{6m^2+29m+20}{m^2-6m+9}$

Solution:

$-\frac{(m-2)(m-3)}{(3+m)(m+4)}$

Exercise:

Problem: $\frac{2n^2-3n-14}{25-n^2} \div \frac{2n^2-13n+21}{n^2-10n+25}$

Exercise:

Problem: $\frac{3s^2}{s^2-16} \div \frac{s^3-4s^2+16s}{s^3-64}$

Solution:

$\frac{3s}{s+4}$

Exercise:

Problem: $\frac{r^2-9}{15} \div \frac{r^3-27}{5r^2+15r+45}$

Exercise:

Problem: $\frac{p^3+q^3}{3p^2+3pq+3q^2} \div \frac{p^2-q^2}{12}$

Solution:

$\frac{4(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

Exercise:

Problem: $\frac{v^3-8w^3}{2v^2+4vw+8w^2} \div \frac{v^2-4w^2}{4}$

Exercise:

Problem: $\frac{t^2-9}{2t} \div (t^2-6t+9)$

Solution:

$\frac{t+3}{2t(t-3)}$

Exercise:

Problem: $\frac{x^2+3x-10}{4x} \div (2x^2+20x+50)$

Exercise:

Problem: $\frac{2y^2-10yz-48z^2}{2y-1} \div (4y^2-32yz)$

Solution:

$\frac{y+3z}{2y(2y-1)}$

Exercise:

Problem: $\frac{2m^2-98n^2}{2m+6} \div (m^2-7mn)$

Exercise:

Problem: $\frac{\frac{2a^2-a-21}{5a+20}}{\frac{a^2+7a+12}{a^2+8a+16}}$

Solution:

$\frac{2a-7}{5}$

Exercise:

Problem: $\frac{\frac{3b^2+2b-8}{12b+18}}{\frac{3b^2+2b-8}{2b^2-7b-15}}$

Exercise:

Problem: $\frac{\frac{12c^2-12}{2c^2-3c+1}}{\frac{4c+4}{6c^2-13c+5}}$

Solution:

$3\left(3c-5\right)$

Exercise:

Problem: $\frac{\frac{4d^2+7d-2}{35d+10}}{\frac{d^2-4}{7d^2-12d-4}}$

Exercise:

Problem: $\frac{10m^2+80m}{3m-9}\cdot\frac{m^2+4m-21}{m^2-9m+20}$
$\div\frac{5m^2+10m}{2m-10}$

Solution:

$\frac{4(m+8)(m+7)}{3(m-4)(m+2)}$

Exercise:

Problem: $\frac{4n^2+32n}{3n+2}\cdot\frac{3n^2-n-2}{n^2+n-30}$
$\div\frac{108n^2-24n}{n+6}$

Exercise:

Problem: $\frac{12p^2+3p}{p+3}\div\frac{p^2+2p-63}{p^2-p-12}$
$\cdot\frac{p-7}{9p^3-9p^2}$

Solution:

$\frac{(4p+1)(p-4)}{3p(p+9)(p-1)}$

Exercise:

Problem: $\frac{6q+3}{9q^2-9q}\div\frac{q^2+14q+33}{q^2+4q-5}$
$\cdot\frac{4q^2+12q}{12q+6}$

Everyday Math

Exercise:

Problem:

Probability The director of large company is interviewing applicants for two identical jobs. If $w =$ the number of women applicants and $m =$ the number of men applicants, then the probability that two women are selected for the jobs is $\frac{w}{w+m} \cdot \frac{w-1}{w+m-1}$.

ⓐ Simplify the probability by multiplying the two rational expressions.
ⓑ Find the probability that two women are selected when $w = 5$ and $m = 10$.

Solution:

ⓐ $\frac{w(w-1)}{(w+m)(w+m-1)}$
ⓑ $\frac{2}{21}$

Exercise:

Problem:

Area of a triangle The area of a triangle with base b and height h is $\frac{bh}{2}$. If the triangle is stretched to make a new triangle with base and height three times as much as in the original triangle, the area is $\frac{9bh}{2}$. Calculate how the area of the new triangle compares to the area of the original triangle by dividing $\frac{9bh}{2}$ by $\frac{bh}{2}$.

Writing Exercises

Exercise:

Problem:

ⓐ Multiply $\frac{7}{4} \cdot \frac{9}{10}$ and explain all your steps.
ⓑ Multiply $\frac{n}{n-3} \cdot \frac{9}{n+3}$ and explain all your steps.
ⓒ Evaluate your answer to part (b) when $n = 7$. Did you get the same answer you got in part (a)? Why or why not?

Solution:

Answers will vary.

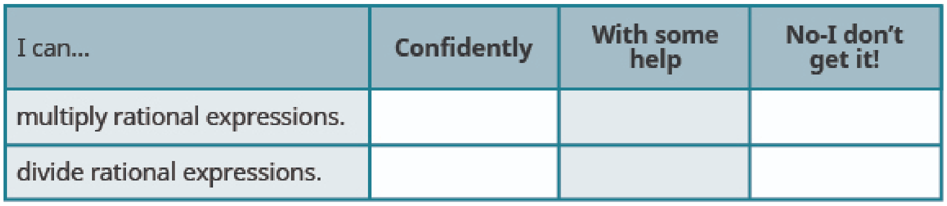
Exercise:

Problem:

ⓐ Divide $\frac{24}{5} \div 6$ and explain all your steps.
ⓑ Divide $\frac{x^2-1}{x} \div (x+1)$ and explain all your steps.
ⓒ Evaluate your answer to part (b) when $x = 5$. Did you get the same answer you got in part (a)? Why or why not?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
multiply rational expressions.			
divide rational expressions.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Add and Subtract Rational Expressions with a Common Denominator
By the end of this section, you will be able to:

- Add rational expressions with a common denominator
- Subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites

Note:
Before you get started, take this readiness quiz.
If you miss a problem, go back to the section listed and review the material.

1. Add: $\frac{y}{3}+\frac{9}{3}$.
   If you missed this problem, review [link].
2. Subtract: $\frac{10}{x}-\frac{2}{x}$.
   If you missed this problem, review [link].
3. Factor completely: $8n^5-20n^3$.
   If you missed this problem, review [link].
4. Factor completely: $45a^3-5ab^2$.
   If you missed this problem, review [link].

Add Rational Expressions with a Common Denominator

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

Note:
Rational Expression Addition

If $p, q,$ and r are polynomials where $r \neq 0$, then
Equation:

$$\frac{p}{r} + \frac{q}{r} = \frac{p+q}{r}$$

To add rational expressions with a common denominator, add the numerators and place the sum over the common denominator.

We will add two numerical fractions first, to remind us of how this is done.

Example:
Exercise:

Problem: Add: $\frac{5}{18} + \frac{7}{18}$.

Solution:
Solution

	$\frac{5}{18} + \frac{7}{18}$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{5+7}{18}$
Add in the numerator.	$\frac{12}{18}$
Factor the numerator and denominator to show the common factors.	$\frac{6\cdot 2}{6\cdot 3}$
Remove common factors.	$\frac{\cancel{6}\cdot 2}{\cancel{6}\cdot 3}$
Simplify.	$\frac{2}{3}$

Note:
Exercise:

Problem: Add: $\frac{7}{16} + \frac{5}{16}$.

Solution:

$\frac{3}{4}$

Note:
Exercise:

Problem: Add: $\frac{3}{10} + \frac{1}{10}$.

Solution:

$\frac{2}{5}$

Remember, we do not allow values that would make the denominator zero. What value of y should be excluded in the next example?

Example:
Exercise:

Problem: Add: $\frac{3y}{4y-3} + \frac{7}{4y-3}$.

Solution:
Solution

$\frac{3y}{4y-3}+\frac{7}{4y-3}$

The fractions have a common denominator, so add the numerators and place the sum over the common denominator.

$\frac{3y+7}{4y-3}$

The numerator and denominator cannot be factored. The fraction is simplified.

Note:
Exercise:

Problem: Add: $\frac{5x}{2x+3}+\frac{2}{2x+3}$.

Solution:

$\frac{5x+2}{2x+3}$.

Note:
Exercise:

Problem: Add: $\frac{x}{x-2}+\frac{1}{x-2}$.

Solution:

$\frac{x+1}{x-2}$

Example:
Exercise:

Problem: Add: $\frac{7x+12}{x+3}+\frac{x^2}{x+3}$.

Solution:

Solution

	$\frac{7x+12}{x+3}+\frac{x^2}{x+3}$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{7x+12+x^2}{x+3}$
Write the degrees in descending order.	$\frac{x^2+7x+12}{x+3}$
Factor the numerator.	$\frac{(x+3)(x+4)}{x+3}$
Simplify by removing common factors.	$\frac{\cancel{(x+3)}\,(x+4)}{\cancel{x+3}}$
Simplify.	$x+4$

Note:
Exercise:

Problem: Add: $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

Solution:

$x+2$

Note:
Exercise:

Problem: Add: $\frac{x^2+8x}{x+5}+\frac{15}{x+5}$.

Solution:

$x+3$

Subtract Rational Expressions with a Common Denominator

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator.

Note:
Rational Expression Subtraction
If $p, q,$ and r are polynomials where $r \neq 0$, then
Equation:

$$\frac{p}{r} - \frac{q}{r} = \frac{p - q}{r}$$

To subtract rational expressions, subtract the numerators and place the difference over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.

Example:
Exercise:

Problem: Subtract: $\frac{n^2}{n-10} - \frac{100}{n-10}$.

Solution:
Solution

	$\frac{n^2}{n-10} - \frac{100}{n-10}$
The fractions have a common denominator, so subtract the numerators and place the difference over the common denominator.	$\frac{n^2-100}{n-10}$
Factor the numerator.	$\frac{(n-10)(n+10)}{n-10}$
Simplify by removing common factors.	$\frac{\cancel{(n-10)}\,(n+10)}{\cancel{n-10}}$
Simplify.	$n+10$

Note:
Exercise:

Problem: Subtract: $\frac{x^2}{x+3} - \frac{9}{x+3}$.

Solution:

$x-3$

Note:
Exercise:

Problem: Subtract: $\frac{4x^2}{2x-5} - \frac{25}{2x-5}$.

Solution:

$2x+5$

Be careful of the signs when you subtract a binomial!

Example:
Exercise:

Problem: Subtract: $\frac{y^2}{y-6} - \frac{2y+24}{y-6}$.

Solution:
Solution

	$\frac{y^2}{y-6} - \frac{2y+24}{y-6}$
The fractions have a common denominator, so subtract the numerators and place the difference over the common denominator.	$\frac{y^2-(2y+24)}{y-6}$
Distribute the sign in the numerator.	$\frac{y^2-2y-24}{y-6}$
Factor the numerator.	$\frac{(y-6)(y+4)}{y-6}$
Remove common factors.	$\frac{\cancel{(y-6)}\,(y+4)}{\cancel{y-6}}$
Simplify.	$y+4$

Note:
Exercise:

Problem: Subtract: $\frac{n^2}{n-4} - \frac{n+12}{n-4}$.

Solution:

$n+3$

Note:

Exercise:

Problem: Subtract: $\frac{y^2}{y-1} - \frac{9y-8}{y-1}$.

Solution:

$y - 8$

Example:
Exercise:

Problem: Subtract: $\frac{5x^2-7x+3}{x^2-3x+18} - \frac{4x^2+x-9}{x^2-3x+18}$.

Solution:
Solution

	$\frac{5x^2-7x+3}{x^2-3x+18} - \frac{4x^2+x-9}{x^2-3x+18}$
Subtract the numerators and place the difference over the common denominator.	$\frac{5x^2-7x+3-\left(4x^2+x-9\right)}{x^2-3x+18}$
Distribute the sign in the numerator.	$\frac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$
Combine like terms.	$\frac{x^2-8x+12}{x^2-3x-18}$
Factor the numerator and the denominator.	$\frac{(x-2)(x-6)}{(x+3)(x-6)}$
Simplify by removing common factors.	$\frac{(x-2)\cancel{(x-6)}}{(x+3)\cancel{(x-6)}}$
Simplify.	$\frac{(x-2)}{(x+3)}$

Note:

Exercise:

Problem: Subtract: $\frac{4x^2-11x+8}{x^2-3x+2}-\frac{3x^2+x-3}{x^2-3x+2}$.

Solution:

$\frac{x-11}{x-2}$

Note:
Exercise:

Problem: Subtract: $\frac{6x^2-x+20}{x^2-81}-\frac{5x^2+11x-7}{x^2-81}$.

Solution:

$\frac{x-3}{x+9}$

Add and Subtract Rational Expressions whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$.

Let’s see how this works.

	$\frac{7}{d}+\frac{5}{-d}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{7}{d}+\frac{(-1)5}{(-1)(-d)}$

The denominators are the same.	$\frac{7}{d}+\frac{-5}{d}$
Simplify.	$\frac{2}{d}$

Example:
Exercise:

Problem: Add: $\frac{4u-1}{3u-1}+\frac{u}{1-3u}$.

Solution:
Solution

	$\frac{4u-1}{3u-1}+\frac{u}{1-3u}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{4u-1}{3u-1}+\frac{(-1)u}{(-1)(1-3u)}$
Simplify the second fraction.	$\frac{4u-1}{3u-1}+\frac{-u}{3u-1}$
The denominators are the same. Add the numerators.	$\frac{4u-1-u}{3u-1}$
Simplify.	$\frac{3u-1}{3u-1}$

Simplify.	1

Note:
Exercise:

Problem: Add: $\frac{8x-15}{2x-5}+\frac{2x}{5-2x}$.

Solution:

3

Note:
Exercise:

Problem: Add: $\frac{6y^2+7y-10}{4y-7}+\frac{2y^2+2y+11}{7-4y}$.

Solution:

$y+3$

Example:
Exercise:

Problem: Subtract: $\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$.

Solution:
Solution

	$\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2-6m}{m^2-1}-\frac{-1(3m+2)}{-1(1-m^2)}$
Simplify the second fraction.	$\frac{m^2-6m}{m^2-1}-\frac{-3m-2}{m^2-1}$
The denominators are the same. Subtract the numerators.	$\frac{m^2-6m-(-3m-2)}{m^2-1}$
Distribute.	$\frac{m^2-6m+3m+2}{m^2-1}$
Combine like terms.	$\frac{m^2-3m+2}{m^2-1}$
Factor the numerator and denominator.	$\frac{(m-1)(m-2)}{(m-1)(m+1)}$
Simplify by removing common factors.	$\frac{\cancel{(m-1)}(m-2)}{\cancel{(m-1)}(m+1)}$
Simplify.	$\frac{m-2}{m+1}$

Note:
Exercise:

Problem: Subtract: $\frac{y^2-5y}{y^2-4}-\frac{6y-6}{4-y^2}$.

Solution:

$\frac{y+3}{y+2}$

Note:
Exercise:

Problem: Subtract: $\frac{2n^2+8n-1}{n^2-1} - \frac{n^2-7n-1}{1-n^2}$.

Solution:

$\frac{3n-2}{n-1}$

Key Concepts

- **Rational Expression Addition**
    - If $p, q,$ and r are polynomials where $r \neq 0$, then
      Equation:

      $$\frac{p}{r} + \frac{q}{r} = \frac{p+q}{r}$$
    - To add rational expressions with a common denominator, add the numerators and place the sum over the common denominator.
- **Rational Expression Subtraction**
    - If $p, q,$ and r are polynomials where $r \neq 0$, then
      Equation:

      $$\frac{p}{r} - \frac{q}{r} = \frac{p-q}{r}$$
    - To subtract rational expressions, subtract the numerators and place the difference over the common denominator.

Practice Makes Perfect

Add Rational Expressions with a Common Denominator

In the following exercises, add.

Exercise:

Problem: $\frac{2}{15} + \frac{7}{15}$

Solution:

$\frac{3}{5}$

Exercise:

Problem: $\frac{4}{21} + \frac{3}{21}$

Exercise:

Problem: $\frac{7}{24} + \frac{11}{24}$

Solution:

$\frac{3}{4}$

Exercise:

Problem: $\frac{7}{36} + \frac{13}{36}$

Exercise:

Problem: $\frac{3a}{a-b} + \frac{1}{a-b}$

Solution:

$\frac{3a+1}{a+b}$

Exercise:

Problem: $\frac{3c}{4c-5} + \frac{5}{4c-5}$

Exercise:

Problem: $\frac{d}{d+8}+\frac{5}{d+8}$

Solution:

$\frac{d+5}{d+8}$

Exercise:

Problem: $\frac{7m}{2m+n}+\frac{4}{2m+n}$

Exercise:

Problem: $\frac{p^2+10p}{p+2}+\frac{16}{p+2}$

Solution:

$p+8$

Exercise:

Problem: $\frac{q^2+12q}{q+3}+\frac{27}{q+3}$

Exercise:

Problem: $\frac{2r^2}{2r-1}+\frac{15r-8}{2r-1}$

Solution:

$r+8$

Exercise:

Problem: $\frac{3s^2}{3s-2}+\frac{13s-10}{3s-2}$

Exercise:

Problem: $\frac{8t^2}{t+4}+\frac{32t}{t+4}$

Solution:

$8t$

Exercise:

Problem: $\frac{6v^2}{v+5} + \frac{30v}{v+5}$

Exercise:

Problem: $\frac{2w^2}{w^2-16} + \frac{8w}{w^2-16}$

Solution:

$\frac{2w}{w-4}$

Exercise:

Problem: $\frac{7x^2}{x^2-9} + \frac{21x}{x^2-9}$

Subtract Rational Expressions with a Common Denominator

In the following exercises, subtract.

Exercise:

Problem: $\frac{y^2}{y+8} - \frac{64}{y+8}$

Solution:

$y - 8$

Exercise:

Problem: $\frac{z^2}{z+2} - \frac{4}{z+2}$

Exercise:

Problem: $\frac{9a^2}{3a-7} - \frac{49}{3a-7}$

Solution:

$3a + 7$

Exercise:

Problem: $\frac{25b^2}{5b-6} - \frac{36}{5b-6}$

Exercise:

Problem: $\frac{c^2}{c-8} - \frac{6c+16}{c-8}$

Solution:

$c + 2$

Exercise:

Problem: $\frac{d^2}{d-9} - \frac{6d+27}{d-9}$

Exercise:

Problem: $\frac{3m^2}{6m-30} - \frac{21m-30}{6m-30}$

Solution:

$\frac{m-2}{3}$

Exercise:

Problem: $\frac{2n^2}{4n-32} - \frac{30n-16}{4n-32}$

Exercise:

Problem: $\frac{6p^2+3p+4}{p^2+4p-5} - \frac{5p^2+p+7}{p^2+4p-5}$

Solution:

$\frac{p+3}{p+5}$

Exercise:

Problem: $\frac{5q^2+3q-9}{q^2+6q+8} - \frac{4q^2+9q+7}{q^2+6q+8}$

Exercise:

Problem: $\frac{5r^2+7r-33}{r^2-49} - \frac{4r^2-5r-30}{r^2-49}$

Solution:

$\frac{r+9}{r+7}$

Exercise:

Problem: $\frac{7t^2-t-4}{t^2-25} - \frac{6t^2+2t-1}{t^2-25}$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add.

Exercise:

Problem: $\frac{10v}{2v-1} + \frac{2v+4}{1-2v}$

Solution:

4

Exercise:

Problem: $\frac{20w}{5w-2} + \frac{5w+6}{2-5w}$

Exercise:

Problem: $\frac{10x^2+16x-7}{8x-3} + \frac{2x^2+3x-1}{3-8x}$

Solution:

$x+2$

Exercise:

Problem: $\frac{6y^2+2y-11}{3y-7} + \frac{3y^2-3y+17}{7-3y}$

In the following exercises, subtract.

Exercise:

Problem: $\frac{z^2+6z}{z^2-25} - \frac{3z+20}{25-z^2}$

Solution:

$\frac{z+4}{z-5}$

Exercise:

Problem: $\frac{a^2+3a}{a^2-9} - \frac{3a-27}{9-a^2}$

Exercise:

Problem: $\frac{2b^2+30b-13}{b^2-49} - \frac{2b^2-5b-8}{49-b^2}$

Solution:

$\frac{4b-3}{b-7}$

Exercise:

Problem: $\frac{c^2+5c-10}{c^2-16} - \frac{c^2-8c-10}{16-c^2}$

Everyday Math

Exercise:

Problem:

Sarah ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. If r represents Sarah's speed when she ran, then her running time is modeled by the expression $\frac{8}{r}$ and her biking time is modeled by the expression $\frac{24}{r+4}$. Add the rational expressions $\frac{8}{r} + \frac{24}{r+4}$ to get an expression for the total amount of time Sarah ran and biked.

Solution:

$\frac{32r+32}{r(r+4)}$

Exercise:

Problem:

If Pete can paint a wall in p hours, then in one hour he can paint $\frac{1}{p}$ of the wall. It would take Penelope 3 hours longer than Pete to paint the wall, so in one hour she can paint $\frac{1}{p+3}$ of the wall. Add the rational expressions $\frac{1}{p}+\frac{1}{p+3}$ to get an expression for the part of the wall Pete and Penelope would paint in one hour if they worked together.

Writing Exercises

Exercise:

Problem: Donald thinks that $\frac{3}{x}+\frac{4}{x}$ is $\frac{7}{2x}$. Is Donald correct? Explain.

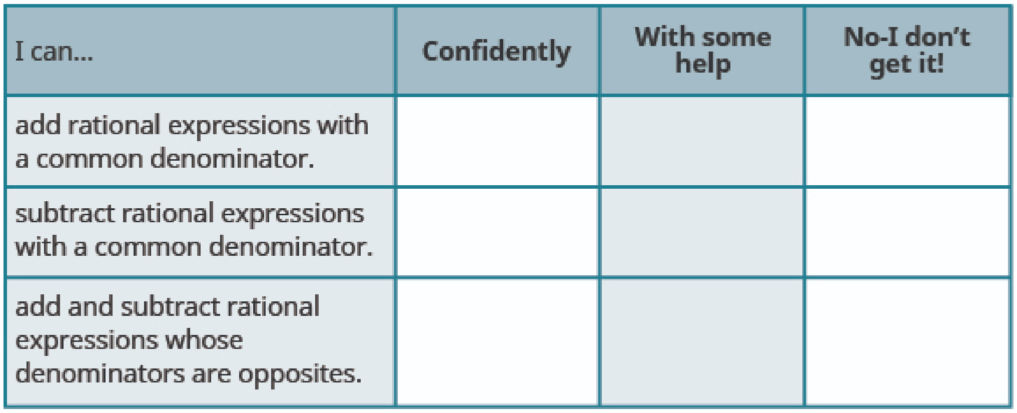
Exercise:

Problem:

Explain how you find the Least Common Denominator of x^2+5x+4 and x^2-16.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
add rational expressions with a common denominator.			
subtract rational expressions with a common denominator.			
add and subtract rational expressions whose denominators are opposites.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Add and Subtract Rational Expressions with Unlike Denominators
By the end of this section, you will be able to:

- Find the least common denominator of rational expressions
- Find equivalent rational expressions
- Add rational expressions with different denominators
- Subtract rational expressions with different denominators

Note:
Before you get started, take this readiness quiz.
If you miss a problem, go back to the section listed and review the material.

1. Add: $\frac{7}{10}+\frac{8}{15}$.
   If you missed this problem, review [link].
2. Subtract: $6\left(2x+1\right)-4\left(x-5\right)$.
   If you missed this problem, review [link].
3. Find the Greatest Common Factor of $9x^2y^3$ and $12xy^5$.
   If you missed this problem, review [link].
4. Factor completely $-48n-12$.
   If you missed this problem, review [link].

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let’s look at the example $\frac{7}{12}+\frac{5}{18}$ from Foundations. Since the denominators are not the same, the first step was to find the least common denominator (LCD). Remember, the LCD is the least common multiple of the denominators. It is the smallest number we can use as a common denominator.

To find the LCD of 12 and 18, we factored each number into primes, lining up any common primes in columns. Then we “brought down” one prime from each column. Finally, we multiplied the factors to find the LCD.
Equation:

$$\begin{aligned} 12 &= 2 \cdot 2 \cdot 3 \\ 18 &= 2 \cdot \quad \cdot 3 \cdot 3 \\ \text{LCD} &= 2 \cdot 2 \cdot 3 \cdot 3 \\ \text{LCD} &= 36 \end{aligned}$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

Note:
Find the least common denominator of rational expressions.

Factor each expression completely.
List the factors of each expression. Match factors vertically when possible.
Bring down the columns.
Multiply the factors.

Remember, we always exclude values that would make the denominator zero. What values of x should we exclude in this next example?

Example:
Exercise:

Problem: Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.

Solution:
Solution

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.

Factor each expression completely, lining up common factors.
Bring down the columns.

$$\begin{array}{r} x^2-2x-3=(x+1)(x-2) \qquad\quad \\ x^2+4x+3=(x+1) \qquad (x+3) \\ \hline \text{LCD}=(x+1)(x-2)(x+3) \end{array}$$

Multiply the factors.

The LCD is $(x+1)(x-3)(x+3)$.

Note:
Exercise:

Problem: Find the LCD for $\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$.

Solution:

$(x-4)(x+4)(x+3)$

Note:
Exercise:

Problem: Find the LCD for $\frac{x}{x^2+8x+15}, \frac{5}{x^2+9x+18}$.

Solution:

$(x+3)(x+6)(x+5)$

Find Equivalent Rational Expressions

When we add numerical fractions, once we find the LCD, we rewrite each fraction as an equivalent fraction with the LCD.

$$\frac{7}{12}+\frac{5}{18}$$

$$\frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2}$$

$$\frac{21}{36}+\frac{10}{36}$$

$$\begin{array}{l} 12 = 2\cdot 2\cdot 3 \\ 18 = 2\cdot \, 3\cdot 3 \\ \hline \text{LCD} = 2\cdot 2\cdot 3\cdot 3 \\ \text{LCD} = 36 \end{array}$$

We will do the same thing for rational expressions.

Example:
Exercise:

Problem:

Rewrite as equivalent rational expressions with denominator $(x+1)(x-3)(x+3)$: $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.

Solution:
Solution

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each denominator.	$\frac{8}{(x+1)(x-3)}, \frac{3x}{(x+1)(x+3)}$
Find the LCD. $x^2-2x-3=(x+1)(x-3)\bigcirc$ $x^2+4x+3=(x+1)\bigcirc(x+3)$ $\text{LCD} \quad =(x+1)(x-3)(x+3)$	
Multiply each denominator by the 'missing' factor and multiply each numerator by the same factor.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}, \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}, \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$

Note:
Exercise:

Problem:

Rewrite as equivalent rational expressions with denominator $(x+3)(x-4)(x+4)$: $\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$.

Solution:

$\frac{2x+8}{(x-4)(x+3)(x+4)}, \frac{x+3}{(x-4)(x+3)(x+4)}$

Note:
Exercise:

Problem:

Rewrite as equivalent rational expressions with denominator $(x+3)(x+5)(x+6)$: $\frac{x}{x^2+8x+15}$, $\frac{5}{x^2+9x+18}$.

Solution:

$\frac{x^2+6x}{(x+3)(x+5)(x+6)}$, $\frac{5(x+3)}{(x+3)(x+5)(x+6)}$

Add Rational Expressions with Different Denominators

Now we have all the steps we need to add rational expressions with different denominators. As we have done previously, we will do one example of adding numerical fractions first.

Example:
Exercise:

Problem: Add: $\frac{7}{12}+\frac{5}{18}$.

Solution:
Solution

	$\frac{7}{12}+\frac{5}{18}$
Find the LCD of 12 and 18. $12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$ $\text{LCD} = 2 \cdot 2 \cdot 3 \cdot 3$ $\text{LCD} = 36$	

Rewrite each fraction as an equivalent fraction with the LCD.	$\frac{7 \cdot 3}{12 \cdot 3} + \frac{5 \cdot 2}{18 \cdot 2}$
Add the fractions.	$\frac{21}{36} + \frac{10}{36}$
The fraction cannot be simplified.	$\frac{31}{36}$

Note:
Exercise:

Problem: Add: $\frac{11}{30} + \frac{7}{12}$.

Solution:

$\frac{57}{60}$

Note:
Exercise:

Problem: Add: $\frac{3}{8} + \frac{9}{20}$.

Solution:

$\frac{33}{40}$

Now we will add rational expressions whose denominators are monomials.

Example:
Exercise:

Problem: Add: $\frac{5}{12x^2y}+\frac{4}{21xy^2}$.

Solution:
Solution

		$\frac{5}{12x^2y}+\frac{4}{21xy^2}$
Find the LCD of $12x^2y$ and $21xy^2$. $\begin{array}{lll} 12x^2y & = 2\cdot2\cdot3\cdotx\cdot x\cdot y \\ 21xy^2 & = 3\cdot7\cdot x\cdoty\cdot y \\ \hline \text{LCD} & = 2\cdot2\cdot3\cdot7\cdot x\cdot x\cdot y\cdot y \\ \text{LCD} & = 84x^2y^2 \end{array}$		
		$\frac{5}{12x^2y}+\frac{4}{21xy^2}$
Rewrite each rational expression as an equivalent fraction with the LCD.		$\frac{5\cdot7y}{12x^2y\cdot7y}+\frac{4\cdot4x}{21xy^2\cdot4x}$
Simplify.		$\frac{35y}{84x^2y^2}+\frac{16x}{84x^2y^2}$
Add the rational expressions.		$\frac{16x+35y}{84x^2y^2}$
There are no factors common to the numerator and denominator. The fraction cannot be simplified.		

Note:
Exercise:

Problem: Add: $\frac{2}{15a^2b} + \frac{5}{6ab^2}$.

Solution:

$\frac{4b+25a}{30a^2b^2}$

Note:
Exercise:

Problem: Add: $\frac{5}{16c} + \frac{3}{8cd^2}$.

Solution:

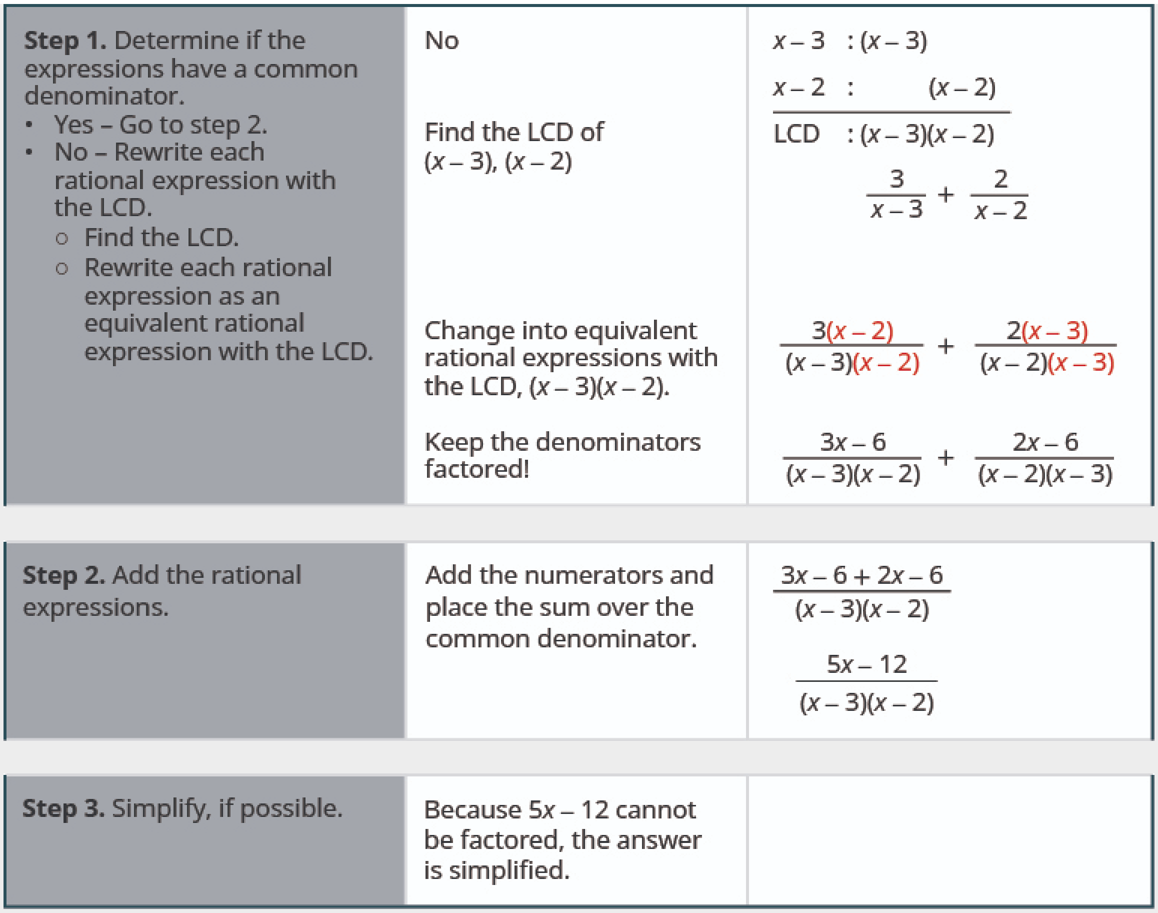
$\frac{5d^2+6}{16cd^2}$

Now we are ready to tackle polynomial denominators.

Example:
How to Add Rational Expressions with Different Denominators
Exercise:

Problem: Add: $\frac{3}{x-3} + \frac{2}{x-2}$.

Solution:
Solution



Step 1. Determine if the expressions have a common denominator. • Yes – Go to step 2. • No – Rewrite each rational expression with the LCD. ○ Find the LCD. ○ Rewrite each rational expression as an equivalent rational expression with the LCD.	No	$x-3 \ : (x-3)$ $x-2 \ : \qquad (x-2)$
	Find the LCD of $(x-3)$, $(x-2)$	LCD $: (x-3)(x-2)$ $\frac{3}{x-3}+\frac{2}{x-2}$
	Change into equivalent rational expressions with the LCD, $(x-3)(x-2)$.	$\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}$
	Keep the denominators factored!	$\frac{3x-6}{(x-3)(x-2)}+\frac{2x-6}{(x-2)(x-3)}$
Step 2. Add the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$ $\frac{5x-12}{(x-3)(x-2)}$
Step 3. Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	

Note:
Exercise:

Problem: Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

Solution:

$\frac{7x-4}{(x+3)(x-2)}$

Note:
Exercise:

Problem: Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

Solution:

$$\frac{7m+25}{(m+3)(m+4)}$$

The steps to use to add rational expressions are summarized in the following procedure box.

Note:
Add rational expressions

Determine if the expressions have a common denominator.	**Yes**– go to step 2.	**No**– Rewrite each rational expression with the LCD.	Find the LCD.	Rewrite each rational expression as an equivalent rational expression with the LCD.

Add the rational expressions.
Simplify, if possible.

Example:
Exercise:

Problem: Add: $\frac{2a}{2ab+b^2}+\frac{3a}{4a^2-b^2}$.

Solution:
Solution

	$\frac{2a}{2ab+b^2}+\frac{3a}{4a^2-b^2}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD.	

$\begin{array}{ll} 2ab+b^2 & = b(2a+b) \\ 4a^2-b^2 & = (2a+b)(2a-b) \\ \hline \text{LCD} & = b(2a+b)(2a-b) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2a(2a-b)}{b(2a+b)(2a-b)}+\frac{3a\cdot b}{(2a+b)(2a-b)\cdot b}$
Simplify the numerators.	$\frac{4a^2-2ab}{b(2a+b)(2a-b)}+\frac{3ab}{b(2a+b)(2a-b)}$
Add the rational expressions.	$\frac{4a^2-2ab+3ab}{b(2a+b)(2a-b)}$
Simplify the numerator.	$\frac{4a^2+ab}{b(2a+b)(2a-b)}$
Factor the numerator.	$\frac{a(4a+b)}{b(2a+b)(2a-b)}$
There are no factors common to the numerator and denominator. The fraction cannot be simplified.	

Note:
Exercise:

Problem: Add: $\frac{5x}{xy-y^2}+\frac{2x}{x^2-y^2}$.

Solution:

$\frac{x(5x+7y)}{y(x-y)(x+y)}$

Note:
Exercise:

Problem: Add: $\frac{7}{2m+6} + \frac{4}{m^2+4m+3}$.

Solution:

$\frac{7m+15}{2(m+3)(m+1)}$

Avoid the temptation to simplify too soon! In the example above, we must leave the first rational expression as $\frac{2a(2a-b)}{b(2a+b)(2a-b)}$ to be able to add it to $\frac{3a\cdot b}{(2a+b)(2a-b)\cdot b}$. Simplify only after you have combined the numerators.

Example:
Exercise:

Problem: Add: $\frac{8}{x^2-2x-3} + \frac{3x}{x^2+4x+3}$.

Solution:
Solution

	$\frac{8}{x^2-2x-3} + \frac{3x}{x^2+4x+3}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{array}{rl} x^2-2x-3 &= (x+1)(x-3) \\ x^2+4x+3 &= (x+1)(x+3) \\ \hline \text{LCD} &= (x+1)(x-3)(x+3) \end{array}$	
Rewrite each rational expression as an equivalent fraction with the LCD.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)} + \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$

Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)} + \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$
Add the rational expressions.	$\frac{8x+24+3x^2+9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$\frac{3x^2-x^2+24}{(x+1)(x-3)(x+3)}$
The numerator is prime, so there are no common factors.	

Note:
Exercise:

Problem: Add: $\frac{1}{m^2-m-2} + \frac{5m}{m^2+3m+2}$.

Solution:

$\frac{5m^2-9m+2}{(m-2)(m+1)(m+2)}$

Note:
Exercise:

Problem: Add: $\frac{2n}{n^2-3n-10} + \frac{6}{n^2+5n+6}$.

Solution:

$\frac{2(n^2+6n-15)}{(n+2)(n-5)(n+3)}$

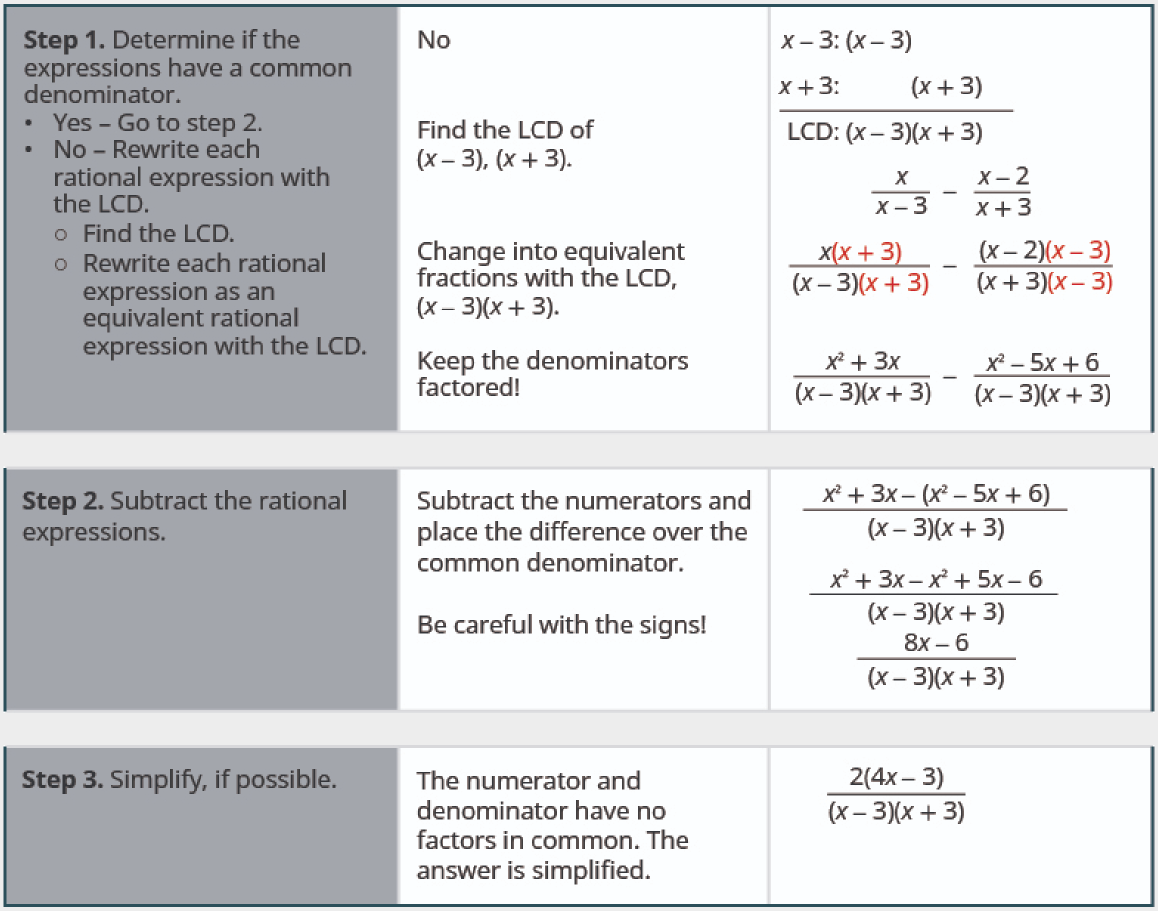
Subtract Rational Expressions with Different Denominators

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

Example:
How to Subtract Rational Expressions with Different Denominators
Exercise:

Problem: Subtract: $\frac{x}{x-3}-\frac{x-2}{x+3}$.

Solution:
Solution



Step 1. Determine if the expressions have a common denominator. - Yes – Go to step 2. - No – Rewrite each rational expression with the LCD. ○ Find the LCD. ○ Rewrite each rational expression as an equivalent rational expression with the LCD.	No	$x-3: (x-3)$ $x+3: \quad (x+3)$
	Find the LCD of $(x-3)$, $(x+3)$.	LCD: $(x-3)(x+3)$
		$\frac{x}{x-3}-\frac{x-2}{x+3}$
	Change into equivalent fractions with the LCD, $(x-3)(x+3)$.	$\frac{x(x+3)}{(x-3)(x+3)}-\frac{(x-2)(x-3)}{(x+3)(x-3)}$
	Keep the denominators factored!	$\frac{x^2+3x}{(x-3)(x+3)}-\frac{x^2-5x+6}{(x-3)(x+3)}$
Step 2. Subtract the rational expressions.	Subtract the numerators and place the difference over the common denominator.	$\frac{x^2+3x-(x^2-5x+6)}{(x-3)(x+3)}$
	Be careful with the signs!	$\frac{x^2+3x-x^2+5x-6}{(x-3)(x+3)}$ $\frac{8x-6}{(x-3)(x+3)}$
Step 3. Simplify, if possible.	The numerator and denominator have no factors in common. The answer is simplified.	$\frac{2(4x-3)}{(x-3)(x+3)}$

Note:
Exercise:

Problem: Subtract: $\frac{y}{y+4} - \frac{y-2}{y-5}$.

Solution:

$\frac{-7y+8}{(y+4)(y-5)}$

Note:
Exercise:

Problem: Subtract: $\frac{z+3}{z+2} - \frac{z}{z+3}$.

Solution:

$\frac{4z+9}{(z+2)(z+3)}$

The steps to take to subtract rational expressions are listed below.

Note:
Subtract rational expressions

Determine if they have a common denominator.	**Yes**– go to step 2.	**No**– Rewrite each rational expression with the LCD.	Find the LCD.	Rewrite each rational expression as an equivalent rational expression with the LCD.

Subtract the rational expressions.
Simplify, if possible.

Example:
Exercise:

Problem: Subtract: $\frac{8y}{y^2-16} - \frac{4}{y-4}$.

Solution:

Solution

	$\frac{8y}{y^2-16}-\frac{4}{y-4}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $y^2-16=(y-4)(y+4)$ $y-4\ \ =y-4$ $\text{LCD}\ \ =(y-4)(y+4)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{(y-4)(y+4)}-\frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{8y}{(y-4)(y+4)}-\frac{4y+16}{(y-4)(y+4)}$
Subtract the rational expressions.	$\frac{8y-4y-16}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{4y-16}{(y-4)(y+4)}$
Factor the numerator to look for common factors.	$\frac{4(y-4)}{(y-4)(y+4)}$
Remove common factors.	$\frac{4\cancel{(y-4)}}{\cancel{(y-4)}(y+4)}$
Simplify.	$\frac{4}{(y+4)}$

Note:
Exercise:

Problem: Subtract: $\frac{2x}{x^2-4} - \frac{1}{x+2}$.

Solution:

$\frac{1}{x-2}$

Note:
Exercise:

Problem: Subtract: $\frac{3}{z+3} - \frac{6z}{z^2-9}$.

Solution:

$\frac{-3}{z-3}$

There are lots of negative signs in the next example. Be extra careful!

Example:
Exercise:

Problem: Subtract: $\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$.

Solution:
Solution

	$\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$
Factor the denominator.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{n+3}{2-n}$
Since $n-2$ and $2-n$ are opposites, we will mutliply the second rational expression by $\frac{-1}{-1}$.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{(-1)(n+3)}{(-1)(2-n)}$
Simplify.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)}{(n-2)}$
Do the expressions have a common denominator? No.	
Find the LCD. $n^2+n-6=(n-2)(n+3)$ $n-2=(n-2)$ $\text{LCD}=(n-2)(n+3)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)(n+3)}{(n-2)(n+3)}$
Simplify the numerators.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{n^2+6n+9}{(n-2)(n+3)}$
Simplify the rational expressions.	$\frac{-3n-9+n^2+6n+9}{(n-2)(n+3)}$
Simplify the numerator.	$\frac{n^2+3n}{(n-2)(n+3)}$
Factor the numerator to look for common factors.	$\frac{n\cancel{(n+3)}}{(n-2)\cancel{(n+3)}}$
Simplify.	$\frac{n}{(n-2)}$

Note:
Exercise:

Problem: Subtract: $\frac{3x-1}{x^2-5x-6} - \frac{2}{6-x}$.

Solution:

$\frac{5x+1}{(x-6)(x+1)}$

Note:
Exercise:

Problem: Subtract: $\frac{-2y-2}{y^2+2y-8} - \frac{y-1}{2-y}$.

Solution:

$\frac{y+3}{y+4}$

When one expression is not in fraction form, we can write it as a fraction with denominator 1.

Example:
Exercise:

Problem: Subtract: $\frac{5c+4}{c-2} - 3$.

Solution:
Solution

	$\frac{5c+4}{c-2} - 3$

Write 3 as $\frac{3}{1}$ to have 2 rational expressions.	$\frac{5c+4}{c-2} - \frac{3}{1}$
Do the rational expressions have a common denominator? No.	
Find the LCD of $c - 2$ and 1. LCD = $c - 2$.	
Rewrite $\frac{3}{1}$ as an equivalent rational expression with the LCD.	$\frac{5c+4}{c-2} - \frac{3(c-2)}{1(c-2)}$
Simplify.	$\frac{5c+4}{c-2} - \frac{3c-6}{c-2}$
Subtract the rational expressions.	$\frac{5c+4-(3c-6)}{c-2}$
Simplify.	$\frac{2c+10}{c-2}$
Factor to check for common factors.	$\frac{2(c+5)}{c-2}$
There are no common factors; the rational expression is simplified.	

Note:
Exercise:

Problem: Subtract: $\frac{2x+1}{x-7} - 3$.

Solution:

$\frac{-x+22}{x-7}$

Note:
Exercise:

Problem: Subtract: $\frac{4y+3}{2y-1} - 5$.

Solution:

$\frac{-2(3y-4)}{2y-1}$

Note:
Add or subtract rational expressions

Determine if the expressions have a common denominator.	**Yes**– go to step 2.	**No**– Rewrite each rational expression with the LCD.	Find the LCD.	Rewrite each rational expression as an equivalent rational expression with the LCD.

Add or subtract the rational expressions.
Simplify, if possible.

We follow the same steps as before to find the LCD when we have more than two rational expressions. In the next example we will start by factoring all three denominators to find their LCD.

Example:
Exercise:

Problem: Simplify: $\frac{2u}{u-1} + \frac{1}{u} - \frac{2u-1}{u^2-u}$.

Solution:
Solution

	$\frac{2u}{u-1} + \frac{1}{u} - \frac{2u-1}{u^2-u}$

Do the rational expressions have a common denominator? No.	
Find the LCD. $\begin{array}{rl} u-1 = & u-1 \\ u = & u \\ u^2-u = & u(u-1) \\ \hline \text{LCD} = & u(u-1) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2u\cdot u}{(u-1)u}+\frac{1\cdot(u-1)}{u\cdot(u-1)}-\frac{2u-1}{u(u-1)}$
	$\frac{2u^2}{(u-1)u}+\frac{u-1}{u\cdot(u-1)}-\frac{2u-1}{u(u-1)}$
Write as one rational expression.	$\frac{2u^2+u-1-2u+1}{u(u-1)}$
Simplify.	$\frac{2u^2-u}{u(u-1)}$
Factor the numerator, and remove common factors.	$\frac{\cancel{u}(2u-1)}{\cancel{u}(u-1)}$
Simplify.	$\frac{2u-1}{u-1}$

Note:
Exercise:

Problem: Simplify: $\frac{v}{v+1}+\frac{3}{v-1}-\frac{6}{v^2-1}$.

Solution:

$\frac{v+3}{v+1}$

Note:
Exercise:

Problem: Simplify: $\frac{3w}{w+2}+\frac{2}{w+7}-\frac{17w+4}{w^2+9w+14}$.

Solution:

$\frac{3w}{w+7}$

Key Concepts

- **Find the Least Common Denominator of Rational Expressions**

  Factor each expression completely.
  List the factors of each expression. Match factors vertically when possible.
  Bring down the columns.
  Multiply the factors.

- **Add or Subtract Rational Expressions**

Determine if the expressions have a common denominator.	**Yes**– go to step 2.	**No**– Rewrite each rational expression with the LCD.	▪ Find the LCD. ▪ Rewrite each rational expression as an equivalent rational expression with the LCD.

  Add or subtract the rational expressions.
  Simplify, if possible.

Practice Makes Perfect

In the following exercises, find the LCD.
Exercise:

Problem: $\frac{5}{x^2-2x-8}$, $\frac{2x}{x^2-x-12}$

Solution:

$$(x-4)(x+2)(x+3)$$

Exercise:

Problem: $\frac{8}{y^2+12y+35}$, $\frac{3y}{y^2+y-42}$

Exercise:

Problem: $\frac{9}{z^2+2z-8}$, $\frac{4z}{z^2-4}$

Solution:

$$(z-2)(z+4)(z+2)$$

Exercise:

Problem: $\frac{6}{a^2+14a+45}$, $\frac{5a}{a^2-81}$

Exercise:

Problem: $\frac{4}{b^2+6b+9}$, $\frac{2b}{b^2-2b-15}$

Solution:

$$(b+3)(b+3)(b-5)$$

Exercise:

Problem: $\frac{5}{c^2-4c+4}$, $\frac{3c}{c^2-10c+16}$

Exercise:

Problem: $\frac{2}{3d^2+14d-5}$, $\frac{5d}{3d^2-19d+6}$

Solution:

$$(3d-1)(d+5)(d-6)$$

Exercise:

Problem: $\frac{3}{5m^2-3m-2}$, $\frac{6m}{5m^2+17m+6}$

In the following exercises, write as equivalent rational expressions with the given LCD.

Exercise:

$$\frac{5}{x^2-2x-8}, \frac{2x}{x^2-x-12}$$

Problem: LCD $(x-4)(x+2)(x+3)$

Solution:

$$\frac{5x+15}{(x-4)(x+2)(x+3)}, \frac{2x^2+4x}{(x-4)(x+2)(x+3)}$$

Exercise:

$$\frac{8}{y^2+12y+35}, \frac{3y}{y^2+y-42}$$

Problem: LCD $(y+7)(y+5)(y-6)$

Exercise:

$$\frac{9}{z^2+2z-8}, \frac{4z}{z^2-4}$$

Problem: LCD $(z-2)(z+4)(z+2)$

Solution:

$$\frac{9z+18}{(z-2)(z+4)(z+2)}, \frac{4z^2+16z}{(z-2)(z+4)(z+2)}$$

Exercise:

$$\frac{6}{a^2+14a+45}, \frac{5a}{a^2-81}$$

Problem: LCD $(a+9)(a+5)(a-9)$

Exercise:

$$\frac{4}{b^2+6b+9}, \frac{2b}{b^2-2b-15}$$

Problem: LCD $(b+3)(b+3)(b-5)$

Solution:

$$\frac{4b-20}{(b+3)(b+3)(b-5)}, \frac{2b^2+6b}{(b+3)(b+3)(b-5)}$$

Exercise:

$$\frac{5}{c^2-4c+4},\ \frac{3c}{c^2-10c+10}$$

Problem: LCD $(c-2)(c-2)(c-8)$

Exercise:

$$\frac{2}{3d^2+14d-5},\ \frac{5d}{3d^2-19d+6}$$

Problem: LCD $(3d-1)(d+5)(d-6)$

Solution:

$$\frac{2d-12}{(3d-1)(d+5)(d-6)},\ \frac{5d^2+25d}{(3d-1)(d+5)(d-6)}$$

Exercise:

$$\frac{3}{5m^2-3m-2},\ \frac{6m}{5m^2+17m+6}$$

Problem: LCD $(5m+2)(m-1)(m+3)$

In the following exercises, add.

Exercise:

Problem: $\frac{5}{24}+\frac{11}{36}$

Solution:

$$\frac{37}{72}$$

Exercise:

Problem: $\frac{7}{30}+\frac{13}{45}$

Exercise:

Problem: $\frac{9}{20}+\frac{11}{30}$

Solution:

$$\frac{49}{60}$$

Exercise:

Problem: $\frac{8}{27} + \frac{7}{18}$

Exercise:

Problem: $\frac{7}{10x^2y} + \frac{4}{15xy^2}$

Solution:

$\frac{21y+8x}{30x^2y^2}$

Exercise:

Problem: $\frac{1}{12a^3b^2} + \frac{5}{9a^2b^3}$

Exercise:

Problem: $\frac{1}{2m} + \frac{7}{8m^2n}$

Solution:

$\frac{4mn+7}{8m^2n}$

Exercise:

Problem: $\frac{5}{6p^2q} + \frac{1}{4p}$

Exercise:

Problem: $\frac{3}{r+4} + \frac{2}{r-5}$

Solution:

$\frac{5r-7}{(r+4)(r-5)}$

Exercise:

Problem: $\frac{4}{s-7} + \frac{5}{s+3}$

Exercise:

Problem: $\frac{8}{t+5} + \frac{6}{t-5}$

Solution:

$\frac{14t-10}{(t+5)(t-5)}$

Exercise:

Problem: $\frac{7}{v+5} + \frac{9}{v-5}$

Exercise:

Problem: $\frac{5}{3w-2} + \frac{2}{w+1}$

Solution:

$\frac{11w+1}{(3w-2)(w+1)}$

Exercise:

Problem: $\frac{4}{2x+5} + \frac{2}{x-1}$

Exercise:

Problem: $\frac{2y}{y+3} + \frac{3}{y-1}$

Solution:

$\frac{2y^2+y+9}{(y+3)(y-1)}$

Exercise:

Problem: $\frac{3z}{z-2} + \frac{1}{z+5}$

Exercise:

Problem: $\frac{5b}{a^2b-2a^2} + \frac{2b}{b^2-4}$

Solution:

$\frac{b(5b+10+2a^2)}{a^2(b-2)(b+2)}$

Exercise:

Problem: $\frac{4}{cd+3c} + \frac{1}{d^2-9}$

Exercise:

Problem: $\frac{2m}{3m-3}+\frac{5m}{m^2+3m-4}$

Solution:

$\frac{2m^2+23m}{3(m-1)(m+4)}$

Exercise:

Problem: $\frac{3}{4n+4}+\frac{6}{n^2-n-2}$

Exercise:

Problem: $\frac{3}{n^2+3n-18}+\frac{4n}{n^2+8n+12}$

Solution:

$\frac{4n^2-9n+6}{(n-3)(n+6)(n+2)}$

Exercise:

Problem: $\frac{6}{q^2-3q-10}+\frac{5q}{q^2-8q+15}$

Exercise:

Problem: $\frac{3r}{r^2+7r+6}+\frac{9}{r^2+4r+3}$

Solution:

$\frac{3(r^2+6r+18)}{(r+1)(r+6)(r+3)}$

Exercise:

Problem: $\frac{2s}{s^2+2s-8}+\frac{4}{s^2+3s-10}$

In the following exercises, subtract.

Exercise:

Problem: $\frac{t}{t-6}-\frac{t-2}{t+6}$

Solution:

$\frac{2(7t-6)}{(t-6)(t+6)}$

Exercise:

Problem: $\frac{v}{v-3} - \frac{v-6}{v+1}$

Exercise:

Problem: $\frac{w+2}{w+4} - \frac{w}{w-2}$

Solution:

$\frac{-4(1+w)}{(w+4)(w-2)}$

Exercise:

Problem: $\frac{x-3}{x+6} - \frac{x}{x+3}$

Exercise:

Problem: $\frac{y-4}{y+1} - \frac{1}{y+7}$

Solution:

$\frac{y^2+2y-29}{(y+1)(y+7)}$

Exercise:

Problem: $\frac{z+8}{z-3} - \frac{z}{z-2}$

Exercise:

Problem: $\frac{5a}{a+3} - \frac{a+2}{a+6}$

Solution:

$\frac{4a^2+25a-6}{(a+3)(a+6)}$

Exercise:

Problem: $\frac{3b}{b-2} - \frac{b-6}{b-8}$

Exercise:

Problem: $\frac{6c}{c^2-25}-\frac{3}{c+5}$

Solution:

$\frac{3}{c-5}$

Exercise:

Problem: $\frac{4d}{d^2-81}-\frac{2}{d+9}$

Exercise:

Problem: $\frac{6}{m+6}-\frac{12m}{m^2-36}$

Solution:

$\frac{-6}{m-6}$

Exercise:

Problem: $\frac{4}{n+4}-\frac{8n}{n^2-16}$

Exercise:

Problem: $\frac{-9p-17}{p^2-4p-21}-\frac{p+1}{7-p}$

Solution:

$\frac{p+2}{p+3}$

Exercise:

Problem: $\frac{7q+8}{q^2-2q-24}-\frac{q+2}{4-q}$

Exercise:

Problem: $\frac{-2r-16}{r^2+6r-16}-\frac{5}{2-r}$

Solution:

$\frac{3}{r-2}$

Exercise:

Problem: $\frac{2t-30}{t^2+6t-27} - \frac{2}{3-t}$

Exercise:

Problem: $\frac{5v-2}{v+3} - 4$

Solution:

$\frac{v-14}{v+3}$

Exercise:

Problem: $\frac{6w+5}{w-1} + 2$

Exercise:

Problem: $\frac{2x+7}{10x-1} + 3$

Solution:

$\frac{4(8x+1)}{10x-1}$

Exercise:

Problem: $\frac{8y-4}{5y+2} - 6$

In the following exercises, add and subtract.

Exercise:

Problem: $\frac{5a}{a-2} + \frac{9}{a} - \frac{2a+18}{a^2-2a}$

Solution:

$\frac{5a^2+7a-36}{a(a-2)}$

Exercise:

Problem: $\frac{2b}{b-5} + \frac{3}{2b} - \frac{2b-15}{2b^2-10b}$

Exercise:

Problem: $\frac{c}{c+2} + \frac{5}{c-2} - \frac{11c}{c^2-4}$

Solution:

$\frac{c^2-8c+10}{(c+2)(c-2)}$

Exercise:

Problem: $\frac{6d}{d-5}+\frac{1}{d+4}-\frac{7d-5}{d^2-d-20}$

In the following exercises, simplify.

Exercise:

Problem: $\frac{6a}{3ab+b^2}+\frac{3a}{9a^2-b^2}$

Solution:

$\frac{3a(6a-b)}{b(3a+b)(3a-b)}$

Exercise:

Problem: $\frac{2c}{2c+10}+\frac{7c}{c^2+9c+20}$

Exercise:

Problem: $\frac{6d}{d^2-64}-\frac{3}{d-8}$

Solution:

$\frac{3}{d+8}$

Exercise:

Problem: $\frac{5}{n+7}-\frac{10n}{n^2-49}$

Exercise:

Problem: $\frac{4m}{m^2+6m-7}+\frac{2}{m^2+10m+21}$

Solution:

$\frac{4m^2+14m-2}{(m+7)(m-1)(m+3)}$

Exercise:

Problem: $\frac{3p}{p^2+4p-12} + \frac{1}{p^2+p-30}$

Exercise:

Problem: $\frac{-5n-5}{n^2+n-6} - \frac{n+1}{2-n}$

Solution:

$\frac{n+1}{n+3}$

Exercise:

Problem: $\frac{-4b-24}{b^2+b-30} + \frac{b+7}{5-b}$

Exercise:

Problem: $\frac{7}{15p} + \frac{5}{18pq}$

Solution:

$\frac{42q+25}{90pq}$

Exercise:

Problem: $\frac{3}{20a^2} + \frac{11}{12ab^2}$

Exercise:

Problem: $\frac{4}{x-2} + \frac{3}{x+5}$

Solution:

$\frac{7(x+2)}{(x-2)(x+5)}$

Exercise:

Problem: $\frac{6}{m+4} + \frac{9}{m-8}$

Exercise:

Problem: $\frac{2q+7}{q+4} - 2$

Solution:

$-\frac{1}{y+4}$

Exercise:

Problem: $\frac{3y-1}{y+4} - 2$

Exercise:

Problem: $\frac{z+2}{z-5} - \frac{z}{z+1}$

Solution:

$\frac{8z+2}{(z-5)(z+1)}$

Exercise:

Problem: $\frac{t}{t-5} - \frac{t-1}{t+5}$

Exercise:

Problem: $\frac{3d}{d+2} + \frac{4}{d} - \frac{d+8}{d^2+2d}$

Solution:

$\frac{3(d+1)}{d+2}$

Exercise:

Problem: $\frac{2q}{q+5} + \frac{3}{q-3} - \frac{13q+15}{q^2+2q-15}$

Everyday Math

Exercise:

Problem:

Decorating cupcakes Victoria can decorate an order of cupcakes for a wedding in t hours, so in 1 hour she can decorate $\frac{1}{t}$ of the cupcakes. It would take her sister 3 hours longer to decorate the same order of cupcakes, so in 1 hour she can decorate $\frac{1}{t+3}$ of the cupcakes.

ⓐ Find the fraction of the decorating job that Victoria and her sister, working together, would complete in one hour by adding the rational expressions $\frac{1}{t} + \frac{1}{t+3}$.

ⓑ Evaluate your answer to part (a) when $t = 5$.

Solution:

$\frac{2t+3}{t(t+3)}$

$\frac{13}{40}$

Exercise:

Problem:

Kayaking When Trina kayaks upriver, it takes her $\frac{5}{3-c}$ hours to go 5 miles, where c is the speed of the river current. It takes her $\frac{5}{3+c}$ hours to kayak 5 miles down the river.

ⓐ Find an expression for the number of hours it would take Trina to kayak 5 miles up the river and then return by adding $\frac{5}{3-c} + \frac{5}{3+c}$.
ⓑ Evaluate your answer to part (a) when $c = 1$ to find the number of hours it would take Trina if the speed of the river current is 1 mile per hour.

Writing Exercises

Exercise:

Problem: Felipe thinks $\frac{1}{x} + \frac{1}{y}$ is $\frac{2}{x+y}$.

ⓐ Choose numerical values for x and y and evaluate $\frac{1}{x} + \frac{1}{y}$.
ⓑ Evaluate $\frac{2}{x+y}$ for the same values of x and y you used in part (a).
ⓒ Explain why Felipe is wrong.
ⓓ Find the correct expression for $\frac{1}{x} + \frac{1}{y}$.

Solution:

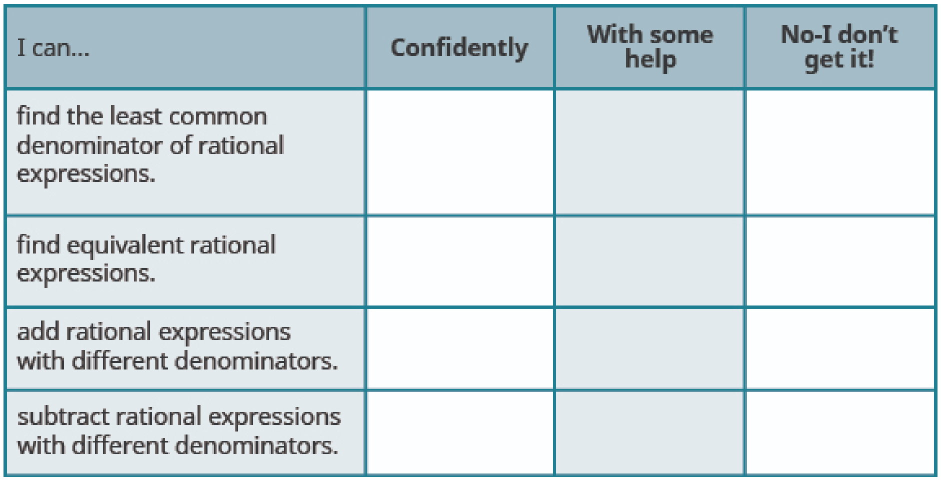
Answers may vary.

Exercise:

Problem: Simplify the expression $\frac{4}{n^2+6n+9} - \frac{1}{n^2-9}$ and explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
find the least common denominator of rational expressions.			
find equivalent rational expressions.			
add rational expressions with different denominators.			
subtract rational expressions with different denominators.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Simplify Complex Rational Expressions

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Note:
Before you get started, take this readiness quiz.
If you miss a problem, go back to the section listed and review the material.

1. Simplify: $\frac{\frac{3}{5}}{\frac{9}{10}}$.
   If you missed this problem, review [link].
2. Simplify: $\frac{1-\frac{1}{3}}{4^2+4\cdot 5}$.
   If you missed this problem, review [link].

Complex fractions are fractions in which the numerator or denominator contains a fraction. In Chapter 1 we simplified complex fractions like these:

Equation:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{xy}{6}}$$

In this section we will simplify *complex rational expressions*, which are rational expressions with rational expressions in the numerator or denominator.

Note:

Complex Rational Expression

A **complex rational expression** is a rational expression in which the numerator or denominator contains a rational expression.

Here are a few complex rational expressions:

Equation:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}} \qquad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} \qquad \frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

Simplify a Complex Rational Expression by Writing it as Division

We have already seen this complex rational expression earlier in this chapter.

Equation:

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}$$

We noted that fraction bars tell us to divide, so rewrote it as the division problem

Equation:

$$\left(\frac{6x^2-7x+2}{4x-8}\right) \div \left(\frac{2x^2-8x+3}{x^2-5x+6}\right)$$

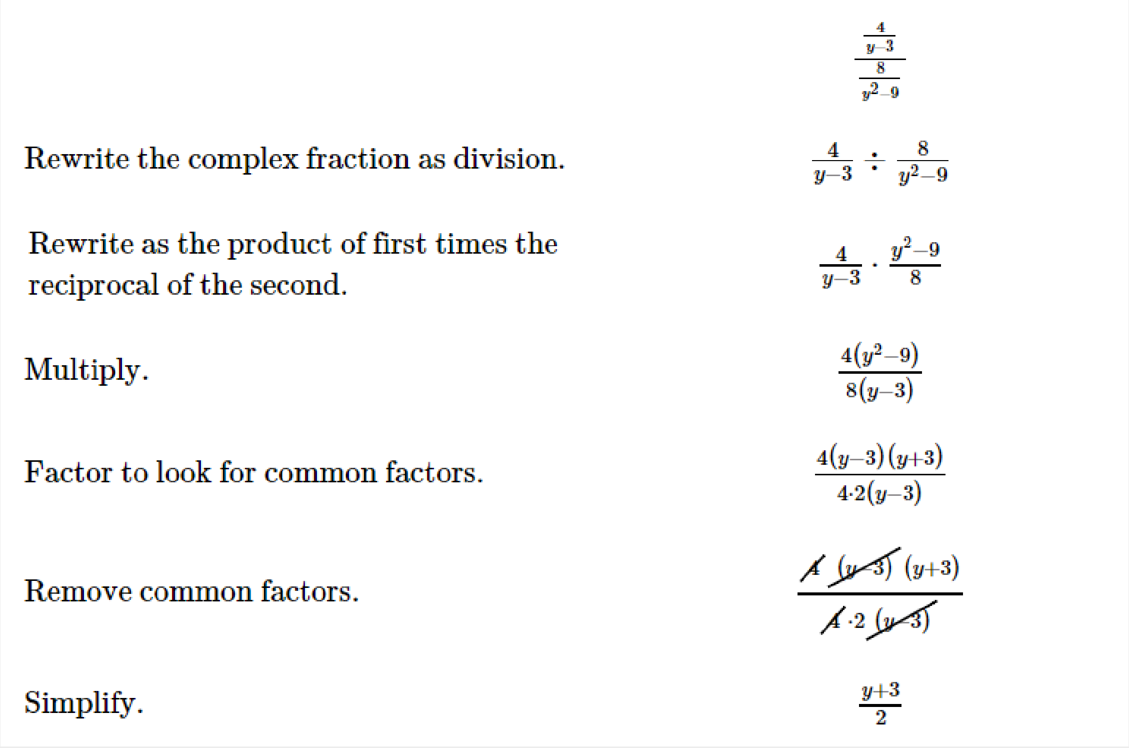
Then we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify rational expressions. We write it as if we were dividing two fractions.

Example:
Exercise:

Problem: Simplify: $\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}}$.

Solution:
Solution



	$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}}$
Rewrite the complex fraction as division.	$\frac{4}{y-3} \div \frac{8}{y^2-9}$
Rewrite as the product of first times the reciprocal of the second.	$\frac{4}{y-3} \cdot \frac{y^2-9}{8}$
Multiply.	$\frac{4(y^2-9)}{8(y-3)}$
Factor to look for common factors.	$\frac{4(y-3)(y+3)}{4 \cdot 2(y-3)}$
Remove common factors.	$\frac{\cancel{4}\ \cancel{(y-3)}\ (y+3)}{\cancel{4} \cdot 2\ \cancel{(y-3)}}$
Simplify.	$\frac{y+3}{2}$

Are there any value(s) of y that should not be allowed? The simplified rational expression has just a constant in the denominator. But the original complex rational expression had denominators of $y-3$ and y^2-9. This expression would be undefined if $y=3$ or $y=-3$.

Note:
Exercise:

Problem: Simplify: $\frac{\frac{2}{x^2-1}}{\frac{3}{x+1}}$.

Solution:

$\frac{2}{3(x-1)}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}$.

Solution:

$\frac{1}{2(x-3)}$

Fraction bars act as grouping symbols. So to follow the Order of Operations, we simplify the numerator and denominator as much as possible before we can do the division.

Example:
Exercise:

Problem: Simplify: $\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Solution:
Solution

	$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
Simplify the numerator and denominator.	
Find the LCD and add the fractions in the numerator. Find the LCD and add the fractions in the denominator.	$\frac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$
Simplify the numerator and denominator.	$\frac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$
Simplify the numerator and denominator, again.	$\frac{\frac{3}{6}}{\frac{1}{6}}$

Rewrite the complex rational expression as a division problem.	$\frac{3}{6} \div \frac{1}{6}$
Multiply the first times by the reciprocal of the second.	$\frac{3}{6} \cdot \frac{6}{1}$
Simplify.	3

Note:
Exercise:

Problem: Simplify: $\frac{\frac{1}{2}+\frac{2}{3}}{\frac{5}{6}+\frac{1}{12}}$.

Solution:

$\frac{14}{11}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{3}{4}-\frac{1}{3}}{\frac{1}{8}+\frac{5}{6}}$.

Solution:

$\frac{10}{23}$

Example:

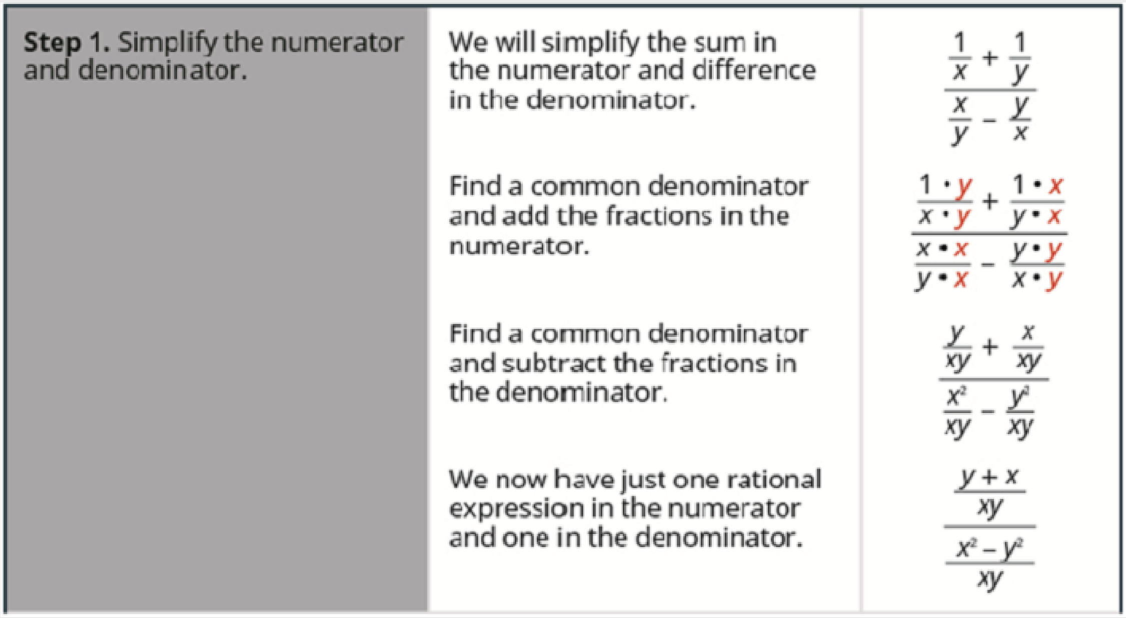
How to Simplify a Complex Rational Expression by Writing it as Division

Exercise:

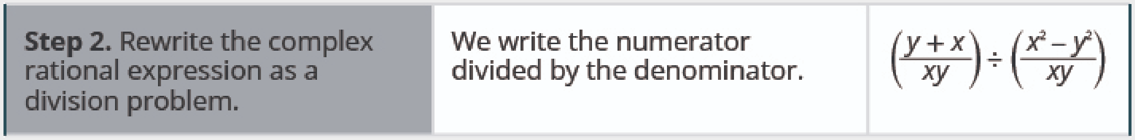
Problem: Simplify: $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Solution:

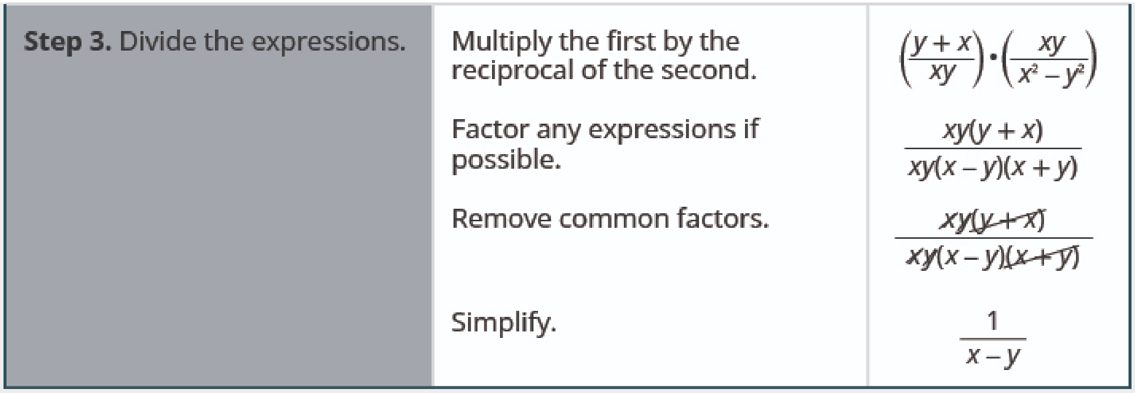
Solution



Step 1. Simplify the numerator and denominator.	We will simplify the sum in the numerator and difference in the denominator.	$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
	Find a common denominator and add the fractions in the numerator.	$\frac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$
	Find a common denominator and subtract the fractions in the denominator.	$\frac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$
	We now have just one rational expression in the numerator and one in the denominator.	$\frac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$



Step 2. Rewrite the complex rational expression as a division problem.	We write the numerator divided by the denominator.	$\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$



Step 3. Divide the expressions.	Multiply the first by the reciprocal of the second.	$\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$
	Factor any expressions if possible.	$\frac{xy(y+x)}{xy(x-y)(x+y)}$
	Remove common factors.	$\frac{\cancel{xy}\cancel{(y+x)}}{\cancel{xy}(x-y)\cancel{(x+y)}}$
	Simplify.	$\frac{1}{x-y}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$.

Solution:

$\frac{y+x}{y-x}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}$.

Solution:

$\frac{ab}{b-a}$

Note:
Simplify a complex rational expression by writing it as division.

Simplify the numerator and denominator.
Rewrite the complex rational expression as a division problem.
Divide the expressions.

Example:
Exercise:

Problem: Simplify: $\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$.

Solution:
Solution

	$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$
Simplify the numerator and denominator.	
Find the LCD and add the fractions in the numerator. Find the LCD and add the fractions in the denominator.	$\frac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$
Simplify the numerators.	

	$\dfrac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$
Subtract the rational expressions in the numerator and add in the denominator. Simplify.	$\dfrac{\frac{n^2+n}{n+5}}{\frac{2n}{(n+5)(n-5)}}$
Rewrite as fraction division.	$\frac{n^2+n}{n+5}\div\frac{2n}{(n+5)(n-5)}$
Multiply the first times the reciprocal of the second.	$\frac{n^2+n}{n+5}\cdot\frac{(n+5)(n-5)}{2n}$
Factor any expressions if possible.	$\frac{n(n+1)(n+5)(n-5)}{(n+5)2n}$
Remove common factors.	$\frac{\cancel{n}(n+1)\cancel{(n+5)}(n-5)}{\cancel{(n+5)}2\cancel{n}}$
Simplify.	$\frac{(n+1)(n-5)}{2}$

Note:
Exercise:

Problem: Simplify: $\dfrac{b-\frac{3b}{b+4}}{\frac{1}{b+4}+\frac{1}{b-4}}$.

Solution:

$\frac{(b+1)(b-4)}{2}$

Note:
Exercise:

Problem: Simplify: $\frac{1-\frac{3}{c+4}}{\frac{1}{c+4}+\frac{c}{3}}$.

Solution:

$\frac{3}{c+3}$

Simplify a Complex Rational Expression by Using the LCD

We “cleared” the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by LCD of all the rational expressions.

Let’s look at the complex rational expression we simplified one way in [link]. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{\text{LCD}}{\text{LCD}}$ we are multiplying by 1, so the value stays the same.

Example:
Exercise:

Problem: Simplify: $\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Solution:
Solution

	$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
The LCD of all the fractions in the whole expression is 6.	
Clear the fractions by multiplying the numerator and denominator by that LCD.	$\frac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$
Distribute.	$\frac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$
Simplify.	$\frac{2+1}{3-2}$
	$\frac{3}{1}$
	3

Note:
Exercise:

Problem: Simplify: $\frac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{10}+\frac{1}{5}}$.

Solution:

$\frac{7}{3}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{1}{4}+\frac{3}{8}}{\frac{1}{2}-\frac{5}{16}}$.

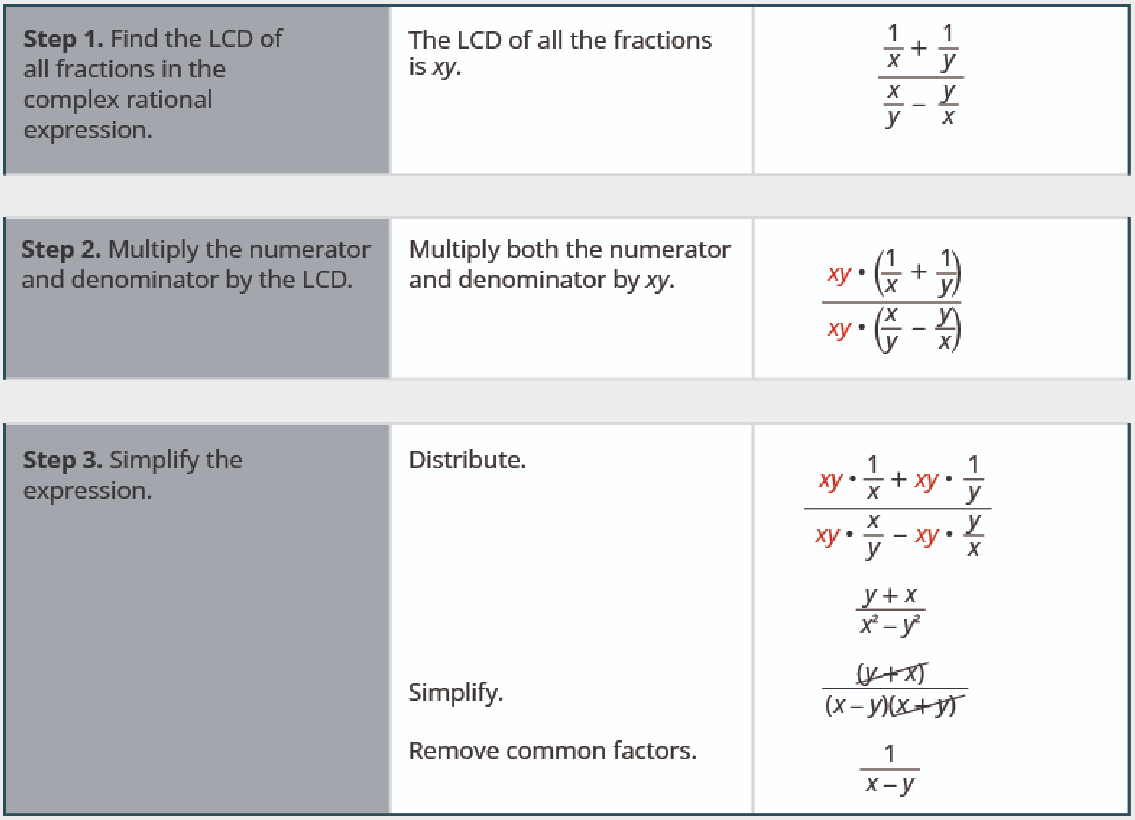
Solution:

$\frac{10}{3}$

Example:
How to Simplify a Complex Rational Expression by Using the LCD
Exercise:

Problem: Simplify: $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Solution:
Solution



Step 1. Find the LCD of all fractions in the complex rational expression.	The LCD of all the fractions is *xy*.	$\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
Step 2. Multiply the numerator and denominator by the LCD.	Multiply both the numerator and denominator by *xy*.	$\dfrac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$
Step 3. Simplify the expression.	Distribute.	$\dfrac{xy\cdot\frac{1}{x}+xy\cdot\frac{1}{y}}{xy\cdot\frac{x}{y}-xy\cdot\frac{y}{x}}$
		$\dfrac{y+x}{x^2-y^2}$
	Simplify.	$\dfrac{\cancel{(y+x)}}{(x-y)\cancel{(x+y)}}$
	Remove common factors.	$\dfrac{1}{x-y}$

Note:
Exercise:

Problem: Simplify: $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}+\frac{b}{a}}$.

Solution:

$\dfrac{b+a}{a^2+b^2}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}$.

Solution:

$\frac{y-x}{xy}$

Note:
Simplify a complex rational expression by using the LCD.

Find the LCD of all fractions in the complex rational expression.
Multiply the numerator and denominator by the LCD.
Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

Example:
Exercise:

Problem: Simplify: $\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$.

Solution:
Solution

	$\dfrac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$
Find the LCD of all fractions in the complex rational expression. The LCD is $(x+6)(x-6)$.	
Multiply the numerator and denominator by the LCD.	$\dfrac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}-\frac{4}{(x+6)(x-6)}\right)}$
Simplify the expression.	
Distribute in the denominator.	$\dfrac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}\right)-(x+6)(x-6)\left(\frac{4}{(x+6)(x-6)}\right)}$
Simplify.	$\dfrac{\cancel{(x+6)}(x-6)\frac{2}{\cancel{x+6}}}{(x+6)\cancel{(x-6)}\left(\frac{4}{\cancel{x-6}}\right)-\cancel{(x+6)(x-6)}\left(\frac{4}{\cancel{(x+6)(x-6)}}\right)}$
Simplify.	$\dfrac{2(x-6)}{4(x+6)-4}$
To simplify the denominator, distribute and combine like terms.	$\dfrac{2(x-6)}{4x+20}$
Remove common factors.	$\dfrac{\cancel{2}(x-6)}{\cancel{2}(2x+10)}$
Simplify.	$\dfrac{x-6}{2x+10}$

Notice that there are no more factors common to the numerator and denominator.

Note:
Exercise:

Problem: Simplify: $\frac{\frac{3}{x+2}}{\frac{5}{x-2}-\frac{3}{x^2-4}}$.

Solution:

$\frac{3x-6}{5x+7}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}}$.

Solution:

$\frac{x+21}{6x-43}$

Example:
Exercise:

Problem: Simplify: $\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$.

Solution:
Solution

	$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$
Find the LCD of all fractions in the complex rational expression. The LCD is $(m-3)(m-4)$.	
Multiply the numerator and denominator by the LCD.	$\frac{(m-3)(m-4)\frac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\frac{3}{m-3}-\frac{2}{m-4}\right)}$
Simplify.	$\frac{\cancel{(m-3)(m-4)}\frac{4}{\cancel{(m-3)(m-4)}}}{\cancel{(m-3)}(m-4)\left(\frac{3}{\cancel{m-3}}\right)-(m-3)\cancel{(m-4)}\left(\frac{2}{\cancel{m-4}}\right)}$
Simplify.	$\frac{4}{3(m-4)-2(m-3)}$
Distribute.	$\frac{4}{3m-12-2m+6}$
Combine like terms.	$\frac{4}{m-6}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{3}{x^2+7x+10}}{\frac{4}{x+2}+\frac{1}{x+5}}$.

Solution:

$\frac{3}{5x+22}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{4}{y+5}+\frac{2}{y+6}}{\frac{3y}{y^2+11y+30}}$.

Solution:

$\frac{6y+34}{3y}$

Example:
Exercise:

Problem: Simplify: $\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$.

Solution:

Solution

	$\dfrac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$
Find the LCD of all fractions in the complex rational expression.	
The LCD is $(y+1)(y-1)$.	
Multiply the numerator and denominator by the LCD.	$\dfrac{(y+1)(y-1)\frac{y}{y+1}}{(y+1)(y-1)\left(1+\frac{1}{y-1}\right)}$
Distribute in the denominator and simplify.	$\dfrac{\cancel{(y+1)}(y-1)\left(\frac{y}{\cancel{y+1}}\right)}{(y+1)(y-1)(1)+(y+1)\cancel{(y-1)}\left(\frac{1}{\cancel{y-1}}\right)}$
Simplify.	$\dfrac{(y-1)y}{(y+1)(y-1)+(y+1)}$
Simplify the denominator, and leave the numerator factored.	$\dfrac{y(y-1)}{y^2-1+y+1}$
	$\dfrac{y(y-1)}{y^2+y}$
Factor the denominator, and remove	$\dfrac{\cancel{y}(y-1)}{\cancel{y}(y+1)}$

factors common with the numerator.	
Simplify.	$\frac{y-1}{y+1}$

Note:
Exercise:

Problem: Simplify: $\frac{\frac{x}{x+3}}{1+\frac{1}{x+3}}$.

Solution:

$\frac{x}{x+4}$

Note:
Exercise:

Problem: Simplify: $\frac{1+\frac{1}{x-1}}{\frac{3}{x+1}}$.

Solution:

$\frac{x(x+1)}{3(x-1)}$

Key Concepts

- **To Simplify a Rational Expression by Writing it as Division**

  Simplify the numerator and denominator.
  Rewrite the complex rational expression as a division problem.
  Divide the expressions.

- **To Simplify a Complex Rational Expression by Using the LCD**

  Find the LCD of all fractions in the complex rational expression.
  Multiply the numerator and denominator by the LCD.
  Simplify the expression.

Practice Makes Perfect

Simplify a Complex Rational Expression by Writing It as Division

In the following exercises, simplify.

Exercise:

Problem: $\frac{\frac{2a}{a+4}}{\frac{4a^2}{a^2-16}}$

Solution:

$\frac{a-4}{2a}$

Exercise:

Problem: $\frac{\frac{3b}{b-5}}{\frac{b^2}{b^2-25}}$

Exercise:

Problem: $\frac{\frac{5}{c^2+5c-14}}{\frac{10}{c+7}}$

Solution:

$$\frac{1}{2(c-2)}$$

Exercise:

Problem: $\frac{\frac{8}{d^2+9d+18}}{\frac{12}{d+6}}$

Exercise:

Problem: $\frac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}}$

Solution:

$$\frac{24}{26}$$

Exercise:

Problem: $\frac{\frac{1}{2}+\frac{3}{4}}{\frac{3}{5}+\frac{7}{10}}$

Exercise:

Problem: $\frac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}}$

Solution:

$$\frac{20}{57}$$

Exercise:

Problem: $\frac{\frac{1}{2}-\frac{1}{6}}{\frac{2}{3}+\frac{3}{4}}$

Exercise:

Problem: $\frac{\frac{n}{m}+\frac{1}{n}}{\frac{1}{n}-\frac{n}{m}}$

Solution:

$\frac{n^2+m}{m-n^2}$

Exercise:

Problem: $\frac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}}$

Exercise:

Problem: $\frac{\frac{1}{r}+\frac{1}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

Solution:

$\frac{rt}{t-r}$

Exercise:

Problem: $\frac{\frac{2}{v}+\frac{2}{w}}{\frac{1}{v^2}-\frac{1}{w^2}}$

Exercise:

Problem: $\frac{x-\frac{2x}{x+3}}{\frac{1}{x+3}+\frac{1}{x-3}}$

Solution:

$\frac{(x+1)(x-3)}{2}$

Exercise:

Problem: $\frac{y-\frac{2y}{y-4}}{\frac{2}{y-4}-\frac{2}{y+4}}$

Exercise:

Problem: $\frac{2-\frac{2}{a+3}}{\frac{1}{a+3}+\frac{a}{2}}$

Solution:

$\frac{4}{a+1}$

Exercise:

Problem: $\frac{4-\frac{4}{b-5}}{\frac{1}{b-5}+\frac{b}{4}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify.

Exercise:

Problem: $\frac{\frac{1}{3}+\frac{1}{8}}{\frac{1}{4}+\frac{1}{12}}$

Solution:

$\frac{11}{8}$

Exercise:

Problem: $\frac{\frac{1}{4}+\frac{1}{9}}{\frac{1}{6}+\frac{1}{12}}$

Exercise:

Problem: $\frac{\frac{5}{6}+\frac{2}{9}}{\frac{7}{18}-\frac{1}{3}}$

Solution:

19

Exercise:

Problem: $\frac{\frac{1}{6}+\frac{4}{15}}{\frac{3}{5}-\frac{1}{2}}$

Exercise:

Problem: $\frac{\frac{c}{d}+\frac{1}{d}}{\frac{1}{d}-\frac{d}{c}}$

Solution:

$\frac{c^2+c}{c-d^2}$

Exercise:

Problem: $\frac{\frac{1}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

Exercise:

Problem: $\frac{\frac{1}{p}+\frac{1}{q}}{\frac{1}{p^2}-\frac{1}{q^2}}$

Solution:

$\frac{pq}{q-p}$

Exercise:

Problem: $\frac{\frac{2}{r}+\frac{2}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

Exercise:

Problem: $\frac{\frac{2}{x+5}}{\frac{3}{x-5}+\frac{1}{x^2-25}}$

Solution:

$\frac{2x-10}{3x+16}$

Exercise:

Problem: $\frac{\frac{5}{y-4}}{\frac{3}{y+4}+\frac{2}{y^2-16}}$

Exercise:

Problem: $\frac{\frac{5}{z^2-64}+\frac{3}{z+8}}{\frac{1}{z+8}+\frac{2}{z-8}}$

Solution:

$\frac{3z-19}{3z+8}$

Exercise:

Problem: $\frac{\frac{3}{s+6}+\frac{5}{s-6}}{\frac{1}{s^2-36}+\frac{4}{s+6}}$

Exercise:

Problem: $\frac{\frac{4}{a^2-2a-15}}{\frac{1}{a-5}+\frac{2}{a+3}}$

Solution:

$$\frac{4}{3a-7}$$

Exercise:

Problem: $\frac{\frac{5}{b^2-6b-27}}{\frac{3}{b-9}+\frac{1}{b+3}}$

Exercise:

Problem: $\frac{\frac{5}{c+2}-\frac{3}{c+7}}{\frac{5c}{c^2+9c+14}}$

Solution:

$$\frac{2c+29}{5c}$$

Exercise:

Problem: $\frac{\frac{6}{d-4}-\frac{2}{d+7}}{\frac{2d}{d^2+3d-28}}$

Exercise:

Problem: $\frac{2+\frac{1}{p-3}}{\frac{5}{p-3}}$

Solution:

$$\frac{(2p-5)}{5}$$

Exercise:

Problem: $\frac{\frac{n}{n-2}}{3+\frac{5}{n-2}}$

Exercise:

Problem: $\frac{\frac{m}{m+5}}{4+\frac{1}{m-5}}$

Solution:

$\frac{m(m-5)}{4m^2+m-95}$

Exercise:

Problem: $\frac{7+\frac{2}{q-2}}{\frac{1}{q+2}}$

Simplify

In the following exercises, use either method.

Exercise:

Problem: $\frac{\frac{3}{4}-\frac{2}{7}}{\frac{1}{2}+\frac{5}{14}}$

Solution:

$\frac{13}{24}$

Exercise:

Problem: $\frac{\frac{v}{w}+\frac{1}{v}}{\frac{1}{v}-\frac{v}{w}}$

Exercise:

Problem: $\frac{\frac{2}{a+4}}{\frac{1}{a^2-16}}$

Solution:

$2(a-4)$

Exercise:

Problem: $\frac{\frac{3}{b^2-3b-40}}{\frac{5}{b+5}-\frac{2}{b-8}}$

Exercise:

Problem: $\frac{\frac{3}{m}+\frac{3}{n}}{\frac{1}{m^2}-\frac{1}{n^2}}$

Solution:

$\frac{3mn}{n-m}$

Exercise:

Problem: $\frac{\frac{2}{r-9}}{\frac{1}{r+9}+\frac{3}{r^2-81}}$

Exercise:

Problem: $\frac{x-\frac{3x}{x+2}}{\frac{3}{x+2}+\frac{3}{x-2}}$

Solution:

$\frac{(x-1)(x-2)}{6}$

Exercise:

Problem: $\frac{\frac{y}{y+3}}{2+\frac{1}{y-3}}$

Everyday Math

Exercise:

Problem:

Electronics The resistance of a circuit formed by connecting two resistors in parallel is $\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$.

ⓐ Simplify the complex fraction $\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$.
ⓑ Find the resistance of the circuit when $R_1 = 8$ and $R_2 = 12$.

Solution:

$\frac{R_1R_2}{R_2+R_1}$
$\frac{24}{5}$

Exercise:

Problem:

Ironing Lenore can do the ironing for her family’s business in h hours. Her daughter would take $h + 2$ hours to get the ironing done. If Lenore and her daughter work together, using 2 irons, the number of hours it would take them to do all the ironing is $\frac{1}{\frac{1}{h}+\frac{1}{h+2}}$.

ⓐ Simplify the complex fraction $\frac{1}{\frac{1}{h}+\frac{1}{h+2}}$.
ⓑ Find the number of hours it would take Lenore and her daughter, working together, to get the ironing done if $h = 4$.

Writing Exercises

Exercise:

Problem:

In this section, you learned to simplify the complex fraction $\frac{\frac{3}{x+2}}{\frac{x}{x^2-4}}$ two ways:

rewriting it as a division problem

multiplying the numerator and denominator by the LCD

Which method do you prefer? Why?

Solution:

Answers will vary.

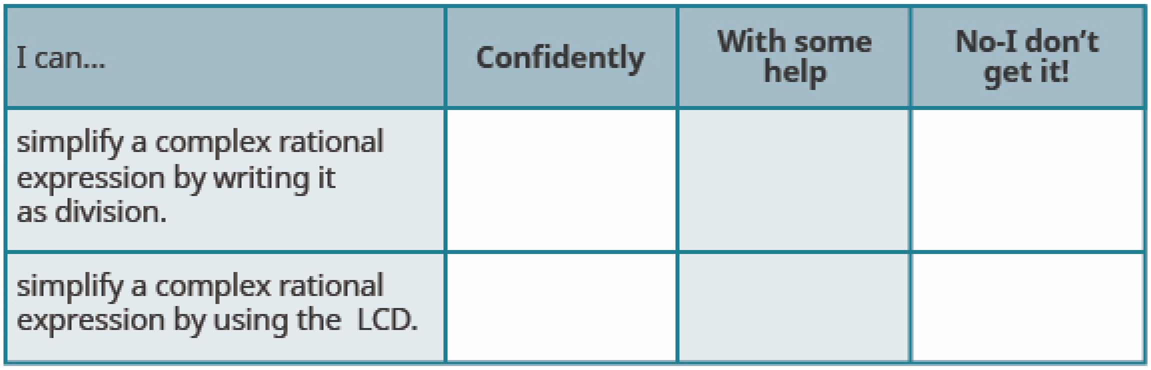
Exercise:

Problem:

Efraim wants to start simplifying the complex fraction $\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$ by cancelling the variables from the numerator and denominator. Explain what is wrong with Efraim’s plan.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don’t get it!
simplify a complex rational expression by writing it as division.			
simplify a complex rational expression by using the LCD.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

complex rational expression

A complex rational expression is a rational expression in which the numerator or denominator contains a rational expression.

Solve Rational Equations

By the end of this section, you will be able to:

- Solve rational equations
- Solve a rational equation for a specific variable

Note:
Before you get started, take this readiness quiz.
If you miss a problem, go back to the section listed and review the material.

1. Solve: $\frac{1}{6}x + \frac{1}{2} = \frac{1}{3}$.
   If you missed this problem, review [link].
2. Solve: $n^2 - 5n - 36 = 0$.
   If you missed this problem, review [link].
3. Solve for y in terms of x: $5x + 2y = 10$.
   If you missed this problem, review [link].

After defining the terms *expression* and *equation* early in Foundations, we have used them throughout this book. We have *simplified* many kinds of *expressions* and *solved* many kinds of *equations*. We have simplified many rational expressions so far in this chapter. Now we will solve rational equations.

The definition of a rational equation is similar to the definition of equation we used in Foundations.

Note:
Rational Equation
A **rational equation** is two rational expressions connected by an equal sign.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.
Equation:

Rational Expression	**Rational Equation**
$\frac{1}{8}x+\frac{1}{2}$	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36}=y+1$
$\frac{1}{n-3}+\frac{1}{n+4}$	$\frac{1}{n-3}+\frac{1}{n+4}=\frac{15}{n^2+n-12}$

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to “clear” the fractions.

Here is an example we did when we worked with linear equations:

	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$	LCD = 8
We multiplied both sides by the LCD.	$8\left(\frac{1}{8}x+\frac{1}{2}\right)=8\left(\frac{1}{4}\right)$	
Then we distributed.	$8\cdot\frac{1}{8}x+8\cdot\frac{1}{2}=8\cdot\frac{1}{4}$	
We simplified—and then we had an equation with no fractions.	$x+4=2$	

Finally, we solved that equation.	$x+4-4=2-4$	
	$x=-2$	

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then we will have an equation that does not contain rational expressions and thus is much easier for us to solve.

But because the original equation may have a variable in a denominator we must be careful that we don’t end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an *extraneous solution*.

Note:
Extraneous Solution to a Rational Equation
An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, c, by writing $x \neq c$ next to the equation.

Example:

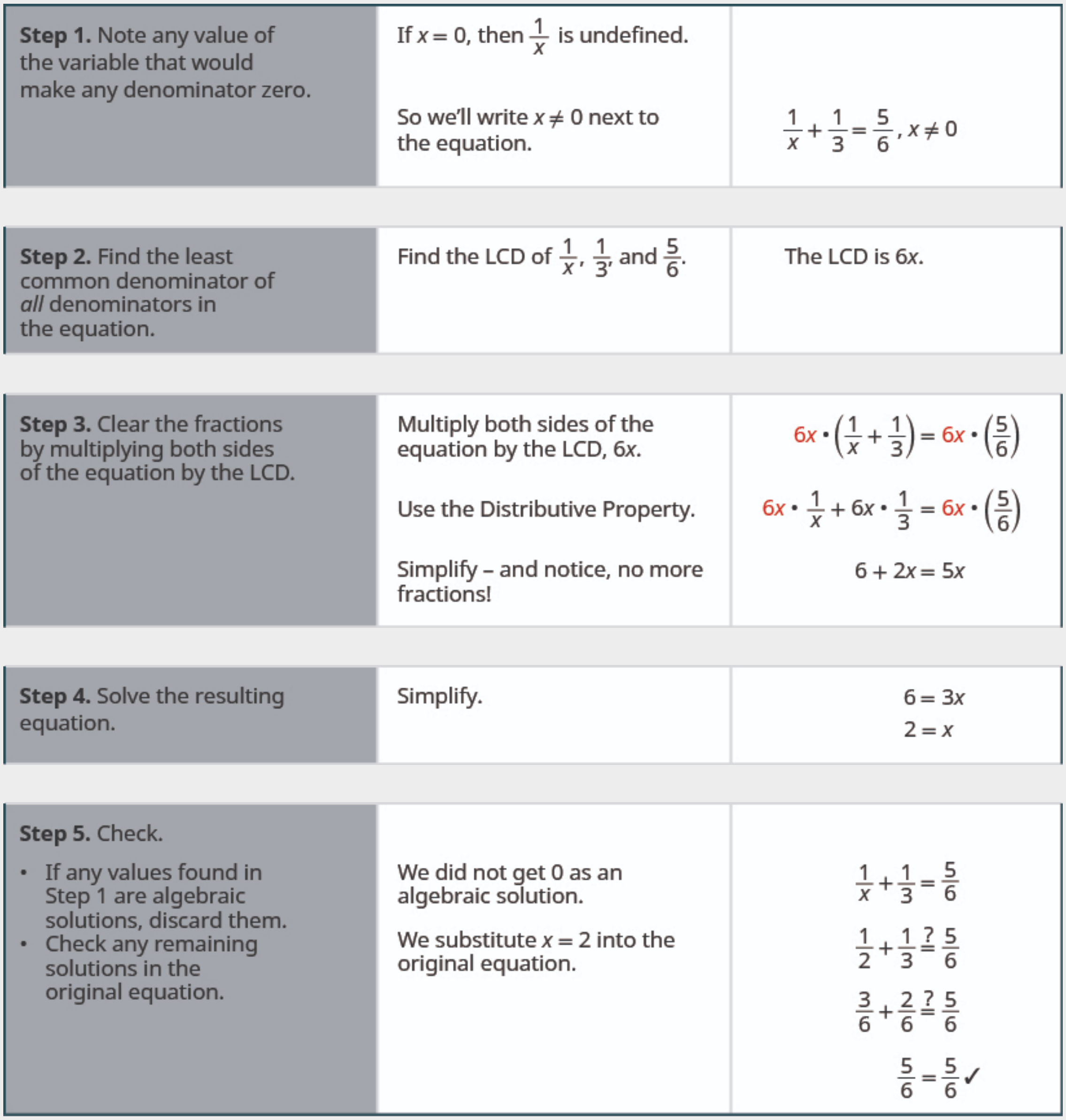
How to Solve Equations with Rational Expressions

Exercise:

Problem: Solve: $\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$.

Solution:

Solution



Step 1. Note any value of the variable that would make any denominator zero.	If $x=0$, then $\frac{1}{x}$ is undefined. So we'll write $x\neq 0$ next to the equation.	$\frac{1}{x}+\frac{1}{3}=\frac{5}{6},\ x\neq 0$
Step 2. Find the least common denominator of *all* denominators in the equation.	Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$.	The LCD is $6x$.
Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.	Multiply both sides of the equation by the LCD, $6x$. Use the Distributive Property. Simplify – and notice, no more fractions!	$6x\cdot\left(\frac{1}{x}+\frac{1}{3}\right)=6x\cdot\left(\frac{5}{6}\right)$ $6x\cdot\frac{1}{x}+6x\cdot\frac{1}{3}=6x\cdot\left(\frac{5}{6}\right)$ $6+2x=5x$
Step 4. Solve the resulting equation.	Simplify.	$6=3x$ $2=x$
Step 5. Check. • If any values found in Step 1 are algebraic solutions, discard them. • Check any remaining solutions in the original equation.	We did not get 0 as an algebraic solution. We substitute $x=2$ into the original equation.	$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$ $\frac{1}{2}+\frac{1}{3}\overset{?}{=}\frac{5}{6}$ $\frac{3}{6}+\frac{2}{6}\overset{?}{=}\frac{5}{6}$ $\frac{5}{6}=\frac{5}{6}$ ✓

Note:
Exercise:

Problem: Solve: $\frac{1}{y} + \frac{2}{3} = \frac{1}{5}$.

Solution:

$-\frac{15}{7}$

Note:
Exercise:

Problem: Solve: $\frac{2}{3} + \frac{1}{5} = \frac{1}{x}$.

Solution:

$\frac{15}{13}$

The steps of this method are shown below.

Note:
Solve equations with rational expressions.

Note any value of the variable that would make any denominator zero.
Find the least common denominator of *all* denominators in the equation.
Clear the fractions by multiplying both sides of the equation by the LCD.
Solve the resulting equation.
Check.

  - If any values found in Step 1 are algebraic solutions, discard them.

  - Check any remaining solutions in the original equation.

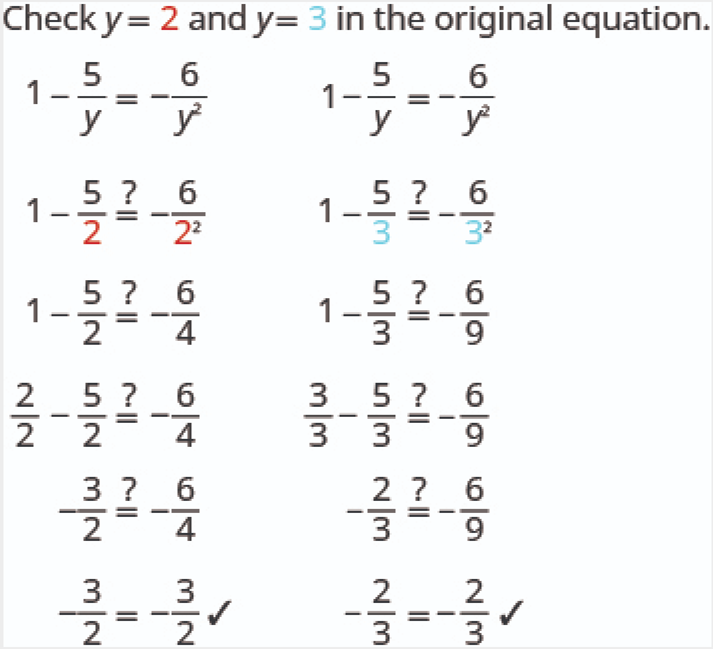
We always start by noting the values that would cause any denominators to be zero.

Example:
Exercise:

Problem: Solve: $1 - \frac{5}{y} = -\frac{6}{y^2}$.

Solution:
Solution

	$1-\frac{5}{y}=-\frac{6}{y^2}$
Note any value of the variable that would make any denominator zero.	$1-\frac{5}{y}=-\frac{6}{y^2}, y \neq 0$
Find the least common denominator of all denominators in the equation. The LCD is y^2.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$

Distribute.	$y^2 \cdot 1 - y^2\left(\frac{5}{y}\right) = y^2\left(-\frac{6}{y^2}\right)$
Multiply.	$y^2 - 5y = -6$
Solve the resulting equation. First write the quadratic equation in standard form.	$y^2 - 5y + 6 = 0$
Factor.	$(y-2)(y-3) = 0$
Use the Zero Product Property.	$y-2=0$ or $y-3=0$
Solve.	$y=2$ or $y=3$
Check.	
We did not get 0 as an algebraic solution.	
 Check $y=2$ and $y=3$ in the original equation. $1-\frac{5}{y}=-\frac{6}{y^2}$ $\quad$ $1-\frac{5}{y}=-\frac{6}{y^2}$ $1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{2^2}$ $\quad$ $1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{3^2}$ $1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4}$ $\quad$ $1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$ $\frac{2}{2}-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4}$ $\quad$ $\frac{3}{3}-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$ $-\frac{3}{2}\stackrel{?}{=}-\frac{6}{4}$ $\quad$ $-\frac{2}{3}\stackrel{?}{=}-\frac{6}{9}$ $-\frac{3}{2}=-\frac{3}{2}$ ✓ $\quad$ $-\frac{2}{3}=-\frac{2}{3}$ ✓	

Note:
Exercise:

Problem: Solve: $1 - \frac{2}{a} = \frac{15}{a^2}$.

Solution:

$5, -3$

Note:
Exercise:

Problem: Solve: $1 - \frac{4}{b} = \frac{12}{b^2}$.

Solution:

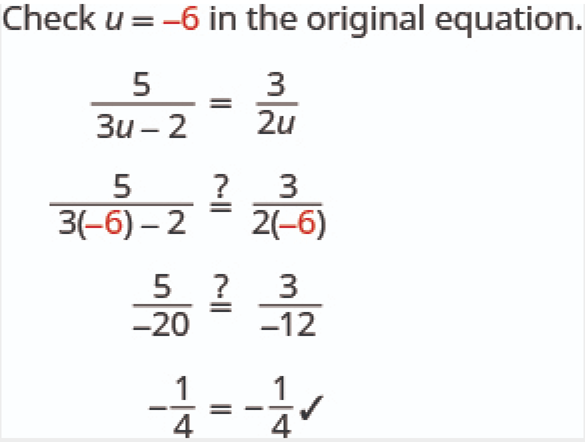
$6, -2$

Example:
Exercise:

Problem: Solve: $\frac{5}{3u-2} = \frac{3}{2u}$.

Solution:
Solution

	$\frac{5}{3u-2}=\frac{3}{2u}$
Note any value of the variable that would make any denominator zero.	$\frac{5}{3u-2}=\frac{3}{2u},\ u\neq\frac{2}{3},\ u\neq 0$
Find the least common denominator of all denominators in the equation. The LCD is $2u(3u-2)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$2u\,(3u-2)\left(\frac{5}{3u-2}\right)=2u\,(3u-2)\left(\frac{3}{2u}\right)$
Remove common factors.	$2u\,\cancel{(3u-2)}\left(\frac{5}{\cancel{3u-2}}\right)=\cancel{2u}\,(3u-2)\left(\frac{3}{\cancel{2u}}\right)$
Simplify.	$2u\,(5)=(3u-2)(3)$
Multiply.	$10u=9u-6$
Solve the resulting equation.	$u=-6$
We did not get 0 or $\frac{2}{3}$ as algebraic solutions.	



Check $u = -6$ in the original equation.

$$\frac{5}{3u-2} = \frac{3}{2u}$$

$$\frac{5}{3(-6)-2} \stackrel{?}{=} \frac{3}{2(-6)}$$

$$\frac{5}{-20} \stackrel{?}{=} \frac{3}{-12}$$

$$-\frac{1}{4} = -\frac{1}{4} \checkmark$$

Note:
Exercise:

Problem: Solve: $\frac{1}{x-1} = \frac{2}{3x}$.

Solution:

-2

Note:
Exercise:

Problem: Solve: $\frac{3}{5n+1} = \frac{2}{3n}$.

Solution:

-2

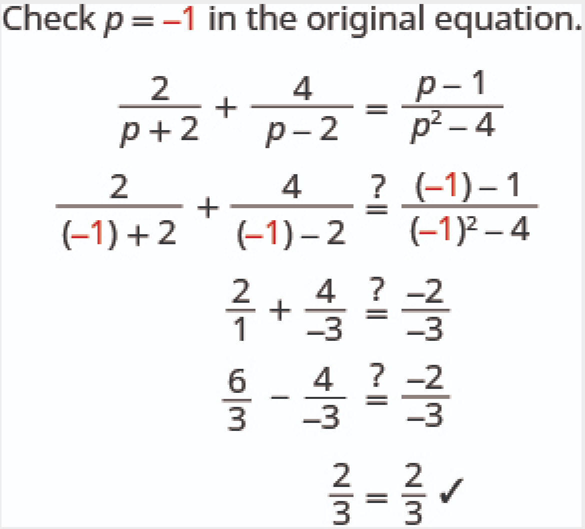
When one of the denominators is a quadratic, remember to factor it first to find the LCD.

Example:
Exercise:

Problem: Solve: $\frac{2}{p+2}+\frac{4}{p-2}=\frac{p-1}{p^2-4}$.

Solution:
Solution

	$\frac{2}{p+2}+\frac{4}{p-2}=\frac{p-1}{p^2-4}$
Note any value of the variable that would make any denominator zero.	$\frac{2}{p+2}+\frac{4}{p-2}=\frac{p-1}{(p+2)(p+2)}, p\neq-2, p\neq2$
Find the least common denominator of all denominators in the equation. The LCD is$(p+2)(p-2)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(p+2)(p-2)\left(\frac{2}{p+2}+\frac{4}{p-2}\right)=(p+2)(p-2)\left(\frac{p-1}{p^2-4}\right)$
Distribute.	$(p+2)(p-2)\frac{2}{p+2}+(p+2)(p-2)\frac{4}{p-2}=(p+2)(p-2)\left(\frac{p-1}{p^2-4}\right)$
Remove common factors.	

	$\cancel{(p+2)}(p-2)\frac{2}{\cancel{p+2}}+(p+2)\cancel{(p-2)}\frac{4}{\cancel{p-2}}=\cancel{(p+2)(p-2)}\left(\frac{p-1}{\cancel{p^2-4}}\right)$
Simplify.	$2(p-2)+4(p+2)=p-1$
Distribute.	$2p-4+4p+8=p-1$
Solve.	$6p+4=p-1$
	$5p=-5$
	$p=-1$
We did not get 2 or -2 as algebraic solutions.	
 Check $p=-1$ in the original equation. $\frac{2}{p+2}+\frac{4}{p-2}=\frac{p-1}{p^2-4}$ $\frac{2}{(-1)+2}+\frac{4}{(-1)-2}\overset{?}{=}\frac{(-1)-1}{(-1)^2-4}$ $\frac{2}{1}+\frac{4}{-3}\overset{?}{=}\frac{-2}{-3}$ $\frac{6}{3}-\frac{4}{-3}\overset{?}{=}\frac{-2}{-3}$ $\frac{2}{3}=\frac{2}{3}$ ✓	

Note:
Exercise:

Problem: Solve: $\frac{2}{x+1} + \frac{1}{x-1} = \frac{1}{x^2-1}$.

Solution:

$\frac{2}{3}$

Note:
Exercise:

Problem: Solve: $\frac{5}{y+3} + \frac{2}{y-3} = \frac{5}{y^2-9}$.

Solution:

2

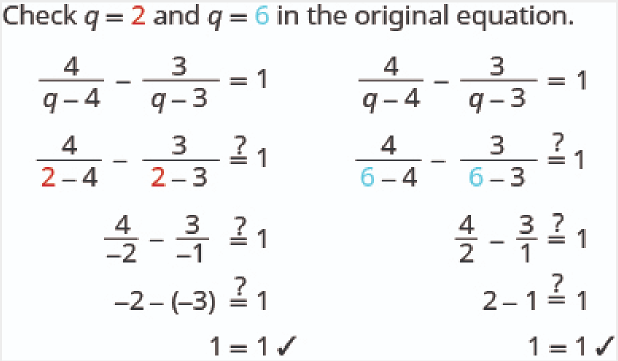
Example:
Exercise:

Problem: Solve: $\frac{4}{q-4} - \frac{3}{q-3} = 1$.

Solution:
Solution

	$\frac{4}{q-4} - \frac{3}{q-3} = 1$
Note any value of the variable that would make any denominator zero.	$\frac{4}{q-4} + \frac{3}{q-3} = 1, \quad q \neq 4, q \neq 3$
Find the least common denominator of all denominators in the equation. The LCD is $(q-4)(q-3)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(q-4)(q-3)\left(\frac{4}{q-4} - \frac{3}{q-3}\right) = (q-4)(q-3)(1)$
Distribute.	$(q-4)(q-3)\left(\frac{4}{q-4}\right) - (q-4)(q-3)\left(\frac{3}{q-3}\right) = (q-4)(q-3)(1)$
Remove common factors.	$\cancel{(q-4)}(q-3)\left(\frac{4}{\cancel{q-4}}\right) - (q-4)\cancel{(q-3)}\left(\frac{3}{\cancel{q-3}}\right) = (q-4)(q-3)(1)$
Simplify.	$4(q-3) - 3(q-4) = (q-4)(q-3)$
Simplify.	$4q - 12 - 3q + 12 = q^2 - 7q + 12$
Combine like terms.	$q = q^2 - 7q + 12$
Solve. First write in standard form.	$0 = q^2 - 8q + 12$
Factor.	$0 = (q-2)(q-6)$
Use the Zero Product Property.	$q = 2$ or $q = 6$

We did not get 4 or 3 as algebraic solutions.



Check $q = 2$ and $q = 6$ in the original equation.

$$\frac{4}{q-4} - \frac{3}{q-3} = 1 \qquad \frac{4}{q-4} - \frac{3}{q-3} = 1$$

$$\frac{4}{2-4} - \frac{3}{2-3} \overset{?}{=} 1 \qquad \frac{4}{6-4} - \frac{3}{6-3} \overset{?}{=} 1$$

$$\frac{4}{-2} - \frac{3}{-1} \overset{?}{=} 1 \qquad \frac{4}{2} - \frac{3}{1} \overset{?}{=} 1$$

$$-2-(-3) \overset{?}{=} 1 \qquad 2-1 \overset{?}{=} 1$$

$$1 = 1 \checkmark \qquad 1 = 1 \checkmark$$

Note:
Exercise:

Problem: Solve: $\frac{2}{x+5} - \frac{1}{x-1} = 1$.

Solution:

$-1, -2$

Note:
Exercise:

Problem: Solve: $\frac{3}{x+8} - \frac{2}{x-2} = 1$.

Solution:

$-2, -3$

Example:
Exercise:

Problem: Solve: $\frac{m+11}{m^2-5m+4} = \frac{5}{m-4} - \frac{3}{m-1}$.

Solution:
Solution

	$\frac{m+11}{m^2-5m+4} = \frac{5}{m-4} = \frac{3}{m-1}$
Factor all the denominators, so we can note any value of the variable the would make any denominator zero.	$\frac{m+11}{(m-4)(m-1)} = \frac{5}{m-4} - \frac{3}{m-1}, \; m \neq 4, m \neq 1$
Find the least common denominator of all denominators in the equation. The LCD is $(m-4)(m-1)$.	
Clear the fractions.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right) = (m-4)(m-1)\left(\frac{5}{m-4} - \frac{3}{m-1}\right)$
Distribute.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right) = (m-4)(m-1)\frac{5}{m-4} - (m-4)(m-1)\frac{3}{m-1}$
Remove common factors.	$\cancel{(m-4)}\cancel{(m-1)}\left(\frac{m+11}{\cancel{(m-4)}\cancel{(m-1)}}\right) = \cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}} - (m-4)\cancel{(m-1)}\frac{3}{\cancel{m-1}}$

Simplify.	$m+11=5(m-1)-3(m-4)$
Solve the resulting equation.	$m+11=5m-5-3m+12$
	$4=m$
Check. The only algebraic solution was 4, but we said that 4 would make a denominator equal to zero. The algebraic solution is an extraneous solution. There is no solution to this equation.	

Note:
Exercise:

Problem: Solve: $\frac{x+13}{x^2-7x+10}=\frac{6}{x-5}-\frac{4}{x-2}$.

Solution:

no solution

Note:
Exercise:

Problem: Solve: $\frac{y-14}{y^2+3y-4}=\frac{2}{y+4}+\frac{7}{y-1}$.

Solution:

-5

The equation we solved in [link] had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. Some equations have no solution.

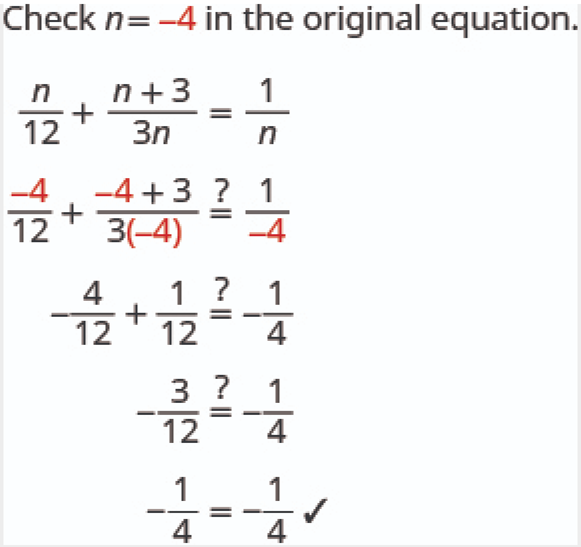
Example:
Exercise:

Problem: Solve: $\frac{n}{12}+\frac{n+3}{3n}=\frac{1}{n}$.

Solution:
Solution

	$\frac{n}{12}+\frac{n+3}{3n}=\frac{1}{n}$
Note any value of the variable that would make any denominator zero.	$\frac{n}{12}+\frac{n+3}{3n}=\frac{1}{n}, n\neq 0$
Find the least common denominator of all denominators in the equation. The LCD is $12n$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$12n\left(\frac{n}{12}+\frac{n+3}{3n}\right)=12n\left(\frac{1}{n}\right)$

Distribute.	$12n\left(\frac{n}{12}\right)+12n\left(\frac{n+3}{3n}\right)=12n\left(\frac{1}{n}\right)$
Remove common factors.	$\cancel{12}n\left(\frac{n}{\cancel{12}}\right)+4\cdot\cancel{3n}\left(\frac{n+3}{\cancel{3n}}\right)=12\cancel{n}\left(\frac{1}{\cancel{n}}\right)$
Simplify.	$n\cdot n+4(n+3)=12\cdot 1$
Solve the resulting equation.	$n^2+4n+12=12$
	$n^2+4n=0$
	$n(n+4)=0$
	$n=0$ or $n=-4$
Check.	
$n=0$ is an extraneous solution.	



Check $n = -4$ in the original equation.

$$\frac{n}{12} + \frac{n+3}{3n} = \frac{1}{n}$$

$$\frac{-4}{12} + \frac{-4+3}{3(-4)} \overset{?}{=} \frac{1}{-4}$$

$$-\frac{4}{12} + \frac{1}{12} \overset{?}{=} -\frac{1}{4}$$

$$-\frac{3}{12} \overset{?}{=} -\frac{1}{4}$$

$$-\frac{1}{4} = -\frac{1}{4} ✓$$

Note:
Exercise:

Problem: Solve: $\frac{x}{18} + \frac{x+6}{9x} = \frac{2}{3x}$.

Solution:

-2

Note:
Exercise:

Problem: Solve: $\frac{y+5}{5y} + \frac{y}{15} = \frac{1}{y}$.

Solution:

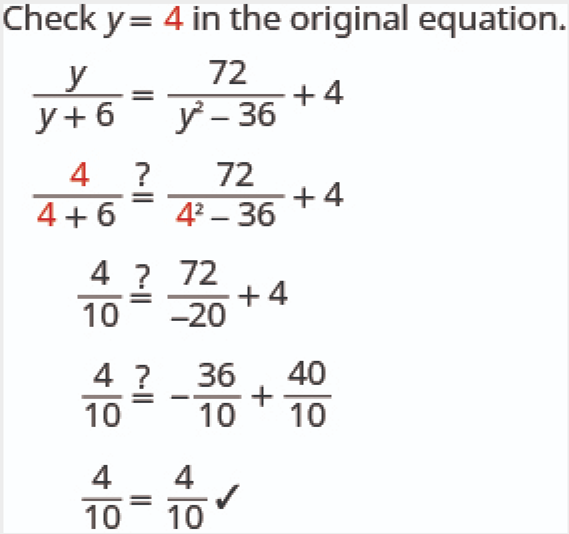
-3

Example:
Exercise:

Problem: Solve: $\frac{y}{y+6}=\frac{72}{y^2-36}+4$.

Solution:
Solution

	$\frac{y}{y+6}=\frac{72}{y^2-36}+4$
Factor all the denominators, so we can note any value of the variable that would make any denominator zero.	$\frac{y}{y+6}=\frac{72}{(y-6)(y+6)}+4, y\neq 6, y\neq -6$
Find the least common denominator. The LCD is$(y-6)(y+6)$.	
Clear the fractions.	$(y-6)(y+6)\left(\frac{y}{y+6}\right)=(y-6)(y+6)\left(\frac{72}{(y-6)(y+6)}+4\right)$
Simplify.	$(y-6)\cdot y=72+(y-6)(y+6)\cdot 4$
Simplify.	$y(y-6)=72+4(y^2-36)$

Solve the resulting equation.	$y^2 - 6y = 72 + 4y^2 - 144$
	$0 = 3y^2 + 6y - 72$
	$0 = 3(y^2 + 2y - 24)$
	$0 = 3(y + 6)(y - 4)$
	$y = -6,\ y = 4$
Check.	
$y = -6$ is an extraneous solution.	
 Check $y = 4$ in the original equation. $\frac{y}{y+6} = \frac{72}{y^2 - 36} + 4$ $\frac{4}{4+6} \overset{?}{=} \frac{72}{4^2 - 36} + 4$ $\frac{4}{10} \overset{?}{=} \frac{72}{-20} + 4$ $\frac{4}{10} \overset{?}{=} -\frac{36}{10} + \frac{40}{10}$ $\frac{4}{10} = \frac{4}{10}$ ✓	

Note:

Exercise:

Problem: Solve: $\frac{x}{x+4} = \frac{32}{x^2-16} + 5.$

Solution:

3

Note:
Exercise:

Problem: Solve: $\frac{y}{y+8} = \frac{128}{y^2-64} + 9.$

Solution:

-7

Example:
Exercise:

Problem: Solve: $\frac{x}{2x-2} - \frac{2}{3x+3} = \frac{5x^2-2x+9}{12x^2-12}.$

Solution:
Solution

	$\frac{x}{2x-2} - \frac{2}{3x+3} = \frac{5x^2-2x+9}{12x^2-12}$

We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.	$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2-2x+9}{12(x-1)(x+1)}$
Note any value of the variable that would make any denominator zero.	$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2-2x+9}{12(x-1)(x+1)}, x \neq 1, x \neq -1$
Find the least common denominator.The LCD is $12(x-1)(x+1)$	
Clear the fractions.	$12(x-1)(x+1)\left(\frac{x}{2(x-1)} - \frac{2}{3(x+1)}\right) = 12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$
Simplify.	$6(x+1) \cdot x - 4(x-1) \cdot 2 = 5x^2 - 2x + 9$
Simplify.	$6x(x+1) - 4 \cdot 2(x-1) = 5x^2 - 2x + 9$
Solve the resulting equation.	$6x^2 + 6x - 8x + 8 = 5x^2 - 2x + 9$
	$x^2 - 1 = 0$
	$(x-1)(x+1) = 0$
	$x = 1$ or $x = -1$
Check.	
$x = 1$ and $x = -1$ are extraneous	

solutions. The equation has no solution.	

Note:
Exercise:

Problem: Solve: $\frac{y}{5y-10} - \frac{5}{3y+6} = \frac{2y^2-19y+54}{15y^2-60}$.

Solution:

no solution

Note:
Exercise:

Problem: Solve: $\frac{z}{2z+8} - \frac{3}{4z-8} = \frac{3z^2-16z-6}{8z^2+8z-64}$.

Solution:

no solution

Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

We’ll start with a formula relating distance, rate, and time. We have used it many times before, but not usually in this form.

Example:
Exercise:

Problem: Solve: $\frac{D}{T} = R$ for T.

Solution:
Solution

	$\frac{D}{T} = R \text{ for } T$
Note any value of the variable that would make any denominator zero.	$\frac{D}{T} = R, T \neq 0$
Clear the fractions by multiplying both sides of the equations by the LCD, *T*.	$T\left(\frac{D}{T}\right) = T(R)$
Simplify.	$D = T \cdot R$
Divide both sides by *R* to isolate *T*.	$\frac{D}{R} = \frac{RT}{R}$
Simplify.	$\frac{D}{R} = T$

Note:
Exercise:

Problem: Solve: $\frac{A}{L} = W$ for L.

Solution:

$L = \frac{A}{W}$

Note:
Exercise:

Problem: Solve: $\frac{F}{A} = M$ for A.

Solution:

$A = \frac{F}{M}$

[link] uses the formula for slope that we used to get the point-slope form of an equation of a line.

Example:
Exercise:

Problem: Solve: $m = \frac{x-2}{y-3}$ for y.

Solution:
Solution

	$m = \frac{x-2}{y-3}$ for y
Note any value of the variable that would make any denominator zero.	$m = \frac{x-2}{y-3},\ y \neq 3$
Clear the fractions by multiplying both sides of the equations by the LCD, $y-3$.	$(y-3)m = (y-3)\left(\frac{x-2}{y-3}\right)$
Simplify.	$ym - 3m = x - 2$
Isolate the term with y.	$ym = x - 2 + 3m$
Divide both sides by m to isolate y.	$\frac{ym}{m} = \frac{x-2+3m}{m}$
Simplify.	$y = \frac{x-2+3m}{m}$

Note:
Exercise:

Problem: Solve: $\frac{y-2}{x+1} = \frac{2}{3}$ for x.

Solution:

$x = \frac{3y-8}{2}$

Note:
Exercise:

Problem: Solve: $5 = \frac{x}{1-y}$ for y.

Solution:

$y = \frac{5-x}{5}$

Be sure to follow all the steps in [link]. It may look like a very simple formula, but we cannot solve it instantly for either denominator.

Example:
Exercise:

Problem: Solve $\frac{1}{c} + \frac{1}{m} = 1$ for c.

Solution:
Solution

	$\frac{1}{c}+\frac{1}{m}=1$ for c
Note any value of the variable that would make any denominator zero.	$\frac{1}{c}+\frac{1}{m}=1, c\neq 0, m\neq 0$
Clear the fractions by multiplying both sides of the equations by the LCD, cm.	$cm\left(\frac{1}{c}+\frac{1}{m}\right)=cm(1)$
Distribute.	$cm\left(\frac{1}{c}\right)+cm\,\frac{1}{m}=cm(1)$
Simplify.	$m+c=cm$
Collect the terms with c to the right.	$m=cm-c$
Factor the expression on the right.	$m=c(m-1)$
To isolate c, divide both sides by $m-1$.	$\frac{m}{m-1}=\frac{c(m-1)}{m-1}$
Simplify by removing common factors.	$\frac{m}{m-1}=c$

Notice that even though we excluded $c=0$ and $m=0$ from the original equation, we must also now state that $m\neq 1$.

Note:

Exercise:

Problem: Solve: $\frac{1}{a}+\frac{1}{b}=c$ for a.

Solution:

$a=\frac{b}{cb-1}$

Note:
Exercise:

Problem: Solve: $\frac{2}{x}+\frac{1}{3}=\frac{1}{y}$ for y.

Solution:

$y=\frac{3x}{6+x}$

Key Concepts

- **Strategy to Solve Equations with Rational Expressions**

  Note any value of the variable that would make any denominator zero.
  Find the least common denominator of*all*denominators in the equation.
  Clear the fractions by multiplying both sides of the equation by the LCD.
  Solve the resulting equation.
  Check.

    - If any values found in Step 1 are algebraic solutions, discard them.
    - Check any remaining solutions in the original equation.

Practice Makes Perfect

Solve Rational Equations

In the following exercises, solve.

Exercise:

Problem: $\frac{1}{a} + \frac{2}{5} = \frac{1}{2}$

Solution:

10

Exercise:

Problem: $\frac{5}{6} + \frac{3}{b} = \frac{1}{3}$

Exercise:

Problem: $\frac{5}{2} - \frac{1}{c} = \frac{3}{4}$

Solution:

$\frac{4}{7}$

Exercise:

Problem: $\frac{6}{3} - \frac{2}{d} = \frac{4}{9}$

Exercise:

Problem: $\frac{4}{5} + \frac{1}{4} = \frac{2}{v}$

Solution:

$\frac{40}{21}$

Exercise:

Problem: $\frac{3}{7}+\frac{2}{3}=\frac{1}{w}$

Exercise:

Problem: $\frac{7}{9}+\frac{1}{x}=\frac{2}{3}$

Solution:

-9

Exercise:

Problem: $\frac{3}{8}+\frac{2}{y}=\frac{1}{4}$

Exercise:

Problem: $1-\frac{2}{m}=\frac{8}{m^2}$

Solution:

$-2, 4$

Exercise:

Problem: $1+\frac{4}{n}=\frac{21}{n^2}$

Exercise:

Problem: $1+\frac{9}{p}=\frac{-20}{p^2}$

Solution:

$-5, -4$

Exercise:

Problem: $1-\frac{7}{q}=\frac{-6}{q^2}$

Exercise:

Problem: $\frac{1}{r+3}=\frac{4}{2r}$

Solution:

-6

Exercise:

Problem: $\frac{3}{t-6}=\frac{1}{t}$

Exercise:

Problem: $\frac{5}{3v-2}=\frac{7}{4v}$

Solution:

14

Exercise:

Problem: $\frac{8}{2w+1}=\frac{3}{w}$

Exercise:

Problem: $\frac{3}{x+4}+\frac{7}{x-4}=\frac{8}{x^2-16}$

Solution:

$-\frac{4}{5}$

Exercise:

Problem: $\frac{5}{y-9}+\frac{1}{y+9}=\frac{18}{y^2-81}$

Exercise:

Problem: $\frac{8}{z-10}+\frac{7}{z+10}=\frac{5}{z^2-100}$

Solution:

$-\frac{1}{3}$

Exercise:

Problem: $\frac{9}{a+11}+\frac{6}{a-11}=\frac{7}{a^2-121}$

Exercise:

Problem: $\frac{1}{q+4}-\frac{2}{q-2}=1$

Solution:

$-2, -1$

Exercise:

Problem: $\frac{3}{r+10}-\frac{4}{r-4}=1$

Exercise:

Problem: $\frac{1}{t+7}-\frac{5}{t-5}=1$

Solution:

$-5, -1$

Exercise:

Problem: $\frac{2}{s+7} - \frac{3}{s-3} = 1$

Exercise:

Problem: $\frac{v-10}{v^2-5v+4} = \frac{3}{v-1} - \frac{6}{v-4}$

Solution:

no solution

Exercise:

Problem: $\frac{w+8}{w^2-11w+28} = \frac{5}{w-7} + \frac{2}{w-4}$

Exercise:

Problem: $\frac{x-10}{x^2+8x+12} = \frac{3}{x+2} + \frac{4}{x+6}$

Solution:

no solution

Exercise:

Problem: $\frac{y-3}{y^2-4y-5} = \frac{1}{y+1} + \frac{8}{y-5}$

Exercise:

Problem: $\frac{z}{16} + \frac{z+2}{4z} = \frac{1}{2z}$

Solution:

-4

Exercise:

Problem: $\frac{a}{9} + \frac{a+3}{3a} = \frac{1}{a}$

Exercise:

Problem: $\frac{b+3}{3b} + \frac{b}{24} = \frac{1}{b}$

Solution:

-8

Exercise:

Problem: $\frac{c+3}{12c} + \frac{c}{36} = \frac{1}{4c}$

Exercise:

Problem: $\frac{d}{d+3} = \frac{18}{d^2-9} + 4$

Solution:

2

Exercise:

Problem: $\frac{m}{m+5} = \frac{50}{m^2-25} + 6$

Exercise:

Problem: $\frac{n}{n+2} = \frac{8}{n^2-4} + 3$

Solution:

1

Exercise:

Problem: $\frac{p}{p+7} = \frac{98}{p^2-49} + 8$

Exercise:

Problem: $\frac{q}{3q-9} - \frac{3}{4q+12} = \frac{7q^2+6q+63}{24q^2-216}$

Solution:

no solution

Exercise:

Problem: $\frac{r}{3r-15} - \frac{1}{4r+20} = \frac{3r^2+17r+40}{12r^2-300}$

Exercise:

Problem: $\frac{s}{2s+6} - \frac{2}{5s+5} = \frac{5s^2-3s-7}{10s^2+40s+30}$

Solution:

$\frac{5}{4}$

Exercise:

Problem: $\frac{t}{6t-12} - \frac{5}{2t+10} = \frac{t^2-23t+70}{12t^2+36t-120}$

Solve a Rational Equation for a Specific Variable

In the following exercises, solve.

Exercise:

Problem: $\frac{C}{r} = 2\pi \text{ for } r$

Solution:

$r = \frac{C}{2\pi}$

Exercise:

Problem: $\frac{I}{r} = P \text{ for } r$

Exercise:

Problem: $\frac{V}{h} = lw \text{ for } h$

Solution:

$h = \frac{v}{lw}$

Exercise:

Problem: $\frac{2A}{b} = h \text{ for } b$

Exercise:

Problem: $\frac{v+3}{w-1} = \frac{1}{2} \text{ for } w$

Solution:

$w = 2v + 7$

Exercise:

Problem: $\frac{x+5}{2-y} = \frac{4}{3} \text{ for } y$

Exercise:

Problem: $a = \frac{b+3}{c-2}$ for c

Solution:

$c = \frac{b+3+2a}{a}$

Exercise:

Problem: $m = \frac{n}{2-n}$ for n

Exercise:

Problem: $\frac{1}{p} + \frac{2}{q} = 4$ for p

Solution:

$p = \frac{q}{4q-2}$

Exercise:

Problem: $\frac{3}{s} + \frac{1}{t} = 2$ for s

Exercise:

Problem: $\frac{2}{v} + \frac{1}{5} = \frac{1}{2}$ for v

Solution:

$v = \frac{20}{3}$

Exercise:

Problem: $\frac{6}{x} + \frac{2}{3} = \frac{1}{y}$ for y

Exercise:

Problem: $\frac{m+3}{n-2} = \frac{4}{5}$ for n

Solution:

$n = \frac{5m+23}{4}$

Exercise:

Problem: $\frac{E}{c} = m^2$ for c

Exercise:

Problem: $\frac{3}{x} - \frac{5}{y} = \frac{1}{4}$ for y

Solution:

$y = \frac{20x}{12-x}$

Exercise:

Problem: $\frac{R}{T} = W$ for T

Exercise:

Problem: $r = \frac{s}{3-t}$ for t

Solution:

$t = \frac{3r-s}{r}$

Exercise:

Problem: $c = \frac{2}{a} + \frac{b}{5}$ for a

Everyday Math

Exercise:

Problem:

House Painting Alain can paint a house in 4 days. Spiro would take 7 days to paint the same house. Solve the equation $\frac{1}{4}+\frac{1}{7}=\frac{1}{t}$ for t to find the number of days it would take them to paint the house if they worked together.

Solution:

$2\frac{6}{11}$ days

Exercise:

Problem:

Boating Ari can drive his boat 18 miles with the current in the same amount of time it takes to drive 10 miles against the current. If the speed of the boat is 7 knots, solve the equation $\frac{18}{7+c}=\frac{10}{7-c}$ for c to find the speed of the current.

Writing Exercises

Exercise:

Problem: Why is there no solution to the equation $\frac{3}{x-2}=\frac{5}{x-2}$?

Solution:

Answers will vary.

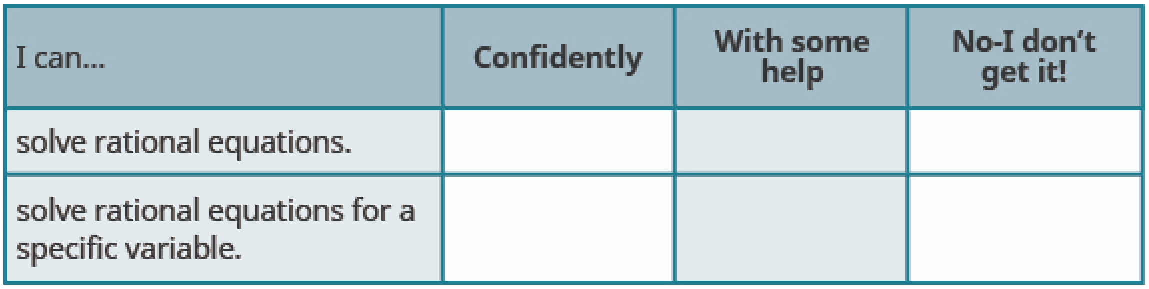
Exercise:

Problem:

Pete thinks the equation $\frac{y}{y+6} = \frac{72}{y^2-36} + 4$ has two solutions, $y = -6$ and $y = 4$. Explain why Pete is wrong.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
solve rational equations.			
solve rational equations for a specific variable.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

rational equation
A rational equation is two rational expressions connected by an equal sign.

extraneous solution to a rational equation
An extraneous solution to a rational equation is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

Solve Proportion and Similar Figure Applications
By the end of this section, you will be able to:

- Solve proportions
- Solve similar figure applications

Note:
Before you get started, take this readiness quiz.
If you miss a problem, go back to the section listed and review the material.

1. Solve $\frac{n}{3}=30$.
   If you missed this problem, review [link].
2. The perimeter of a triangular window is 23 feet. The lengths of two sides are ten feet and six feet. How long is the third side?
   If you missed this problem, review [link].

Solve Proportions

When two rational expressions are equal, the equation relating them is called a *proportion*.

Note:
Proportion
A **proportion** is an equation of the form $\frac{a}{b}=\frac{c}{d}$, where $b\ne 0, d\ne 0$.
The proportion is read "a is to b, as c is to d."

The equation $\frac{1}{2}=\frac{4}{8}$ is a proportion because the two fractions are equal. The proportion $\frac{1}{2}=\frac{4}{8}$ is read "1 is to 2 as 4 is to 8."

Proportions are used in many applications to 'scale up' quantities. We'll start with a very simple example so you can see how proportions work. Even if you can figure out the answer to the example right away, make sure you also learn to solve it using proportions.

Suppose a school principal wants to have 1 teacher for 20 students. She could use proportions to find the number of teachers for 60 students. We let *x* be the number of teachers for 60 students and then set up the proportion:

Equation:

$$\frac{1\text{ teacher}}{20\text{ students}} = \frac{x\text{ teachers}}{60\text{ students}}$$

We are careful to match the units of the numerators and the units of the denominators—teachers in the numerators, students in the denominators.

Since a proportion is an equation with rational expressions, we will solve proportions the same way we solved equations in Solve Rational Equations. We'll multiply both sides of the equation by the LCD to clear the fractions and then solve the resulting equation.

So let's finish solving the principal's problem now. We will omit writing the units until the last step.

	$\frac{1}{20} = \frac{x}{60}$
Multiply both sides by the LCD, 60.	$\frac{1}{20} \cdot 60 = \frac{x}{60} \cdot 60$
Simplify.	$3 = x$
	The principal needs 3 teachers for 60 students.

Now we'll do a few examples of solving numerical proportions without any units. Then we will solve applications using proportions.

Example:

Exercise:

Problem: Solve the proportion: $\frac{x}{63} = \frac{4}{7}$.

Solution:
Solution

		$\frac{x}{63} = \frac{4}{7}$
To isolate x, multiply both sides by the LCD, 63.		$63\left(\frac{x}{63}\right) = 63\left(\frac{4}{7}\right)$
Simplify.		$x = \frac{9 \cdot \cancel{7} \cdot 4}{\cancel{7}}$
Divide the common factors.		$x = 36$
Check. To check our answer, we substitute into the original proportion.		
	$\frac{x}{63} = \frac{4}{7}$	
Substitute $x = 36$.	$\frac{36}{63} \stackrel{?}{=} \frac{4}{7}$	
Show common factors.	$\frac{4 \cdot 9}{7 \cdot 9} \stackrel{?}{=} \frac{4}{7}$	
Simplify.	$\frac{4}{7} = \frac{4}{7}$ ✓	

Note:
Exercise:

Problem: Solve the proportion: $\frac{n}{84} = \frac{11}{12}$.

Solution:

77

Note:
Exercise:

Problem: Solve the proportion: $\frac{y}{96} = \frac{13}{12}$.

Solution:

104

When we work with proportions, we exclude values that would make either denominator zero, just like we do for all rational expressions. What value(s) should be excluded for the proportion in the next example?

Example:
Exercise:

Problem: Solve the proportion: $\frac{144}{a} = \frac{9}{4}$.

Solution:
Solution

		$\frac{144}{a}=\frac{9}{4}$
Multiply both sides by the LCD.		$\frac{144}{a}\cdot 4a=\frac{9}{4}\cdot 4a$
Remove common factors on each side.		$4\cdot 144=a\cdot 9$
Simplify.		$576=9a$
Divide both sides by 9.		$\frac{576}{9}=\frac{9a}{9}$
Simplify.		$64=a$
Check.		
	$\frac{144}{a}=\frac{9}{4}$	
Substitute $a=64$.	$\frac{144}{64}\overset{?}{=}\frac{9}{4}$	
Show common factors.	$\frac{9\cdot 16}{4\cdot 16}\overset{?}{=}\frac{9}{4}$	
Simplify.	$\frac{9}{4}=\frac{9}{4}$ ✓	

Note:
Exercise:

Problem: Solve the proportion: $\frac{91}{b} = \frac{7}{5}$.

Solution:

65

Note:
Exercise:

Problem: Solve the proportion: $\frac{39}{c} = \frac{13}{8}$.

Solution:

24

Example:
Exercise:

Problem: Solve the proportion: $\frac{n}{n+14} = \frac{5}{7}$.

Solution:
Solution

		$\frac{n}{n+14} = \frac{5}{7}$
Multiply both sides by the LCD.		$7(n+14)\left(\frac{n}{n+14}\right) = 7(n+14)\left(\frac{5}{7}\right)$

Remove common factors on each side.		$7n = 5(n + 14)$
Simplify.		$7n = 5n + 70$
Solve for n.		$2n = 70$
		$n = 35$
Check.		
	$\frac{n}{n + 14} = \frac{5}{7}$	
Substitute $n = 35$.	$\frac{35}{35 + 14} \overset{?}{=} \frac{5}{7}$	
Simplify.	$\frac{35}{49} \overset{?}{=} \frac{5}{7}$	
Show common factors.	$\frac{5 \cdot 7}{7 \cdot 7} \overset{?}{=} \frac{5}{7}$	
Simplify.	$\frac{5}{7} = \frac{5}{7}$ ✓	

Note:
Exercise:

Problem: Solve the proportion: $\frac{y}{y+55} = \frac{3}{8}$.

Solution:

33

Note:
Exercise:

Problem: Solve the proportion: $\frac{z}{z-84} = -\frac{1}{5}$.

Solution:

14

Example:
Exercise:

Problem: Solve: $\frac{p+12}{9} = \frac{p-12}{6}$.

Solution:
Solution

		$\frac{p+12}{9} = \frac{p-12}{6}$
Multiply both sides by the LCD, 18.		$18\left(\frac{p+12}{9}\right) = 18\left(\frac{p-12}{6}\right)$
Simplify.		$2(p+12) = 3(p-12)$

Distribute.		$2p+24=3p-36$
Solve for *p*.		$60=p$
Check.		
	$\frac{p+12}{9}=\frac{p-12}{6}$	
Substitute $p=60$.	$\frac{60+12}{9}\overset{?}{=}\frac{60-12}{6}$	
Simplify.	$\frac{72}{9}\overset{?}{=}\frac{48}{6}$	
Divide.	$8=8$ ✓	

Note:
Exercise:

Problem: Solve: $\frac{v+30}{8}=\frac{v+66}{12}$.

Solution:

42

Note:
Exercise:

Problem: Solve: $\frac{2x+15}{9}=\frac{7x+3}{15}$.

Solution:

6

To solve applications with proportions, we will follow our usual strategy for solving applications. But when we set up the proportion, we must make sure to have the units correct—the units in the numerators must match and the units in the denominators must match.

Example:
Exercise:

Problem:

When pediatricians prescribe acetaminophen to children, they prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of the child’s weight. If Zoe weighs 80 pounds, how many milliliters of acetaminophen will her doctor prescribe?

Solution:
Solution

Identify what we are asked to find, and choose a variable to represent it.		How many ml of acetaminophen will the doctor prescribe?
		Let $a =$ ml of acetaminophen.
Write a sentence that gives the information to find it.		If 5 ml is prescribed for every 25 pounds, how much will be prescribed for 80 pounds?
Translate into a proportion–be careful of the units. $\frac{\text{ml}}{\text{pounds}} = \frac{\text{ml}}{\text{pounds}}$		

		$\frac{5}{25} = \frac{a}{80}$
Multiply both sides by the LCD, 400.		$400\left(\frac{5}{25}\right) = 400\left(\frac{a}{80}\right)$
Remove common factors on each side.		$\cancel{25} \cdot 16\left(\frac{5}{\cancel{25}}\right) = \cancel{80} \cdot 5\left(\frac{a}{\cancel{80}}\right)$
Simplify, but don't multiply on the left. Notice what the next step will be.		$16 \cdot 5 = 5a$
Solve for a.		$\frac{16 \cdot 5}{5} = \frac{5a}{5}$
Check.		$16 = a$
Is the answer reasonable?		
Yes, since 80 is about 3 times 25, the medicine should be about 3 times 5. So 16 ml makes sense.		
Substitute $a = 16$ in the original proportion. $\frac{5}{25} = \frac{a}{80}$ $\frac{5}{25} \stackrel{?}{=} \frac{16}{80}$ $\frac{1}{5} = \frac{1}{5}$ ✓		
Write a complete sentence.		The pediatrician would prescribe 16 ml of acetaminophen to Zoe.

Note:
Exercise:

Problem:

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child’s weight. How many milliliters of acetaminophen will the doctor prescribe for Emilia, who weighs 60 pounds?

Solution:

12 ml

Note:
Exercise:

Problem:

For every 1 kilogram (kg) of a child’s weight, pediatricians prescribe 15 milligrams (mg) of a fever reducer. If Isabella weighs 12 kg, how many milligrams of the fever reducer will the pediatrician prescribe?

Solution:

180 ml

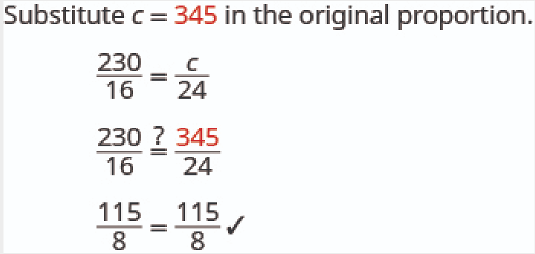
Example:
Exercise:

Problem:

A 16-ounce iced caramel macchiato has 230 calories. How many calories are there in a 24-ounce iced caramel macchiato?

Solution:
Solution

Identify what we are asked to find, and choose a variable to represent it.		How many calories are in a 24 ounce iced caramel macchiato?
		Let $c =$ calories in 24 ounces.
Write a sentence that gives the information to find it.		If there are 230 calories in 16 ounces, then how many calories are in 24 ounces?
Translate into a proportion–be careful of the units. $\frac{\text{calories}}{\text{ounce}} = \frac{\text{calories}}{\text{ounce}}$		$\frac{230}{16} = \frac{c}{24}$
Multiply both sides by the LCD, 48.		$48\left(\frac{230}{16}\right) = 48\left(\frac{c}{24}\right)$
Remove common factors on each side.		$\cancel{16} \cdot 3\left(\frac{230}{\cancel{16}}\right) = \cancel{24} \cdot 2\left(\frac{c}{\cancel{24}}\right)$
Simplify.		$690 = 2c$
Solve for *c*.		$\frac{690}{2} = \frac{2c}{2}$
		$345 = c$
Check.		
Is the answer reasonable?		
Yes, 345 calories for 24 ounces is more than 290 calories for 16 ounces, but not too much more.		

 Substitute $c = 345$ in the original proportion. $\frac{230}{16} = \frac{c}{24}$ $\frac{230}{16} \overset{?}{=} \frac{345}{24}$ $\frac{115}{8} = \frac{115}{8}$ ✓		
Write a complete sentence.		There are 345 calories in a 24-ounce iced caramel macchiato.

Note:
Exercise:

Problem:

At a fast-food restaurant, a 22-ounce chocolate shake has 850 calories. How many calories are in their 12-ounce chocolate shake? Round your answer to nearest whole number.

Solution:

464 calories

Note:
Exercise:

Problem:

Yaneli loves Starburst candies, but wants to keep her snacks to 100 calories. If the candies have 160 calories for 8 pieces, how many pieces can she have in her snack?

Solution:

5 pieces

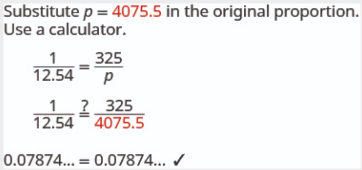
Example:
Exercise:

Problem:

Josiah went to Mexico for spring break and changed $325 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 12.54 Mexican pesos. How many Mexican pesos did he get for his trip?

Solution:
Solution

What are you asked to find?		How many Mexican pesos did Josiah get?
Assign a variable.		Let $p =$ the number of Mexican pesos.
Write a sentence that gives the information to find it.		If $1 US is equal to 12.54 Mexican pesos, then $325 is how many pesos?
Translate into a proportion–be careful of the units.		
$\frac{\$}{\text{pesos}} = \frac{\$}{\text{pesos}}$		$\frac{1}{12.54} = \frac{325}{p}$
Multiply both sides by the LCD, 12.54p.		$12.54p\left(\frac{1}{12.54}\right) = 12.54p\left(\frac{325}{p}\right)$
Remove common factors on each side.		$\cancel{12.54}p\left(\frac{1}{\cancel{12.54}}\right) = 12.54\cancel{p}\left(\frac{325}{\cancel{p}}\right)$
Simplify.		$p = 4075.5$

Check.		
Is the answer reasonable?		
Yes, $100 would be 1,254 pesos. $325 is a little more than 3 times this amount, so our answer of 4075.5 pesos makes sense.		
 Substitute $p = 4075.5$ in the original proportion. Use a calculator. $\frac{1}{12.54} = \frac{325}{p}$ $\frac{1}{12.54} \stackrel{?}{=} \frac{325}{4075.5}$ $0.07874... = 0.07874...$ ✓		
Write a complete sentence.		Josiah got 4075.5 pesos for his spring break trip.

Note:
Exercise:

Problem:

Yurianna is going to Europe and wants to change $800 dollars into Euros. At the current exchange rate, $1 US is equal to 0.738 Euro. How many Euros will she have for her trip?

Solution:

590.4 Euros

Note:
Exercise:

Problem:

Corey and Nicole are traveling to Japan and need to exchange $600 into Japanese yen. If each dollar is 94.1 yen, how many yen will they get?

Solution:

$56,460$ yen

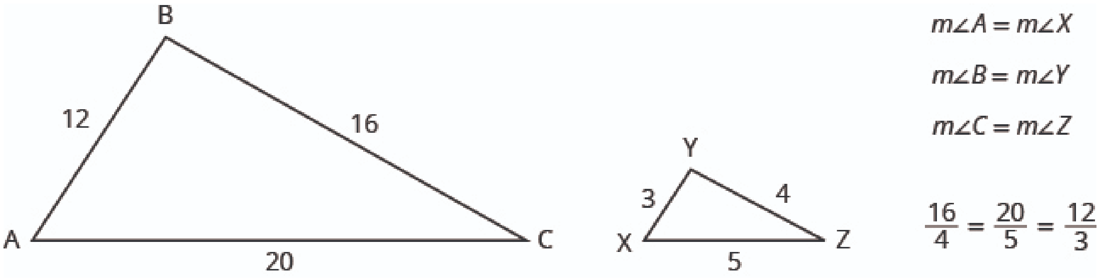
In the example above, we related the number of pesos to the number of dollars by using a proportion. We could say the number of pesos *is proportional to* the number of dollars. If two quantities are related by a proportion, we say that they are proportional.

Solve Similar Figure Applications

When you shrink or enlarge a photo on a phone or tablet, figure out a distance on a map, or use a pattern to build a bookcase or sew a dress, you are working with **similar figures**. If two figures have exactly the same shape, but different sizes, they are said to be *similar.* One is a scale model of the other. All their corresponding angles have the same measures and their corresponding sides are in the same ratio.

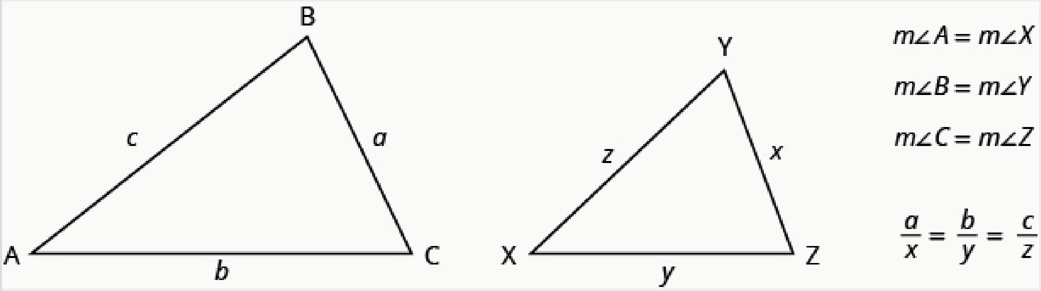
Note:
Similar Figures
Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides are in the same ratio.

For example, the two triangles in [link] are similar. Each side of ΔABC is 4 times the length of the corresponding side of ΔXYZ.



This is summed up in the Property of Similar Triangles.

Note:
Property of Similar Triangles
If ΔABC is similar to ΔXYZ, then their corresponding angle measure are equal and their corresponding sides are in the same ratio.



To solve applications with similar figures we will follow the Problem-Solving Strategy for Geometry Applications we used earlier.

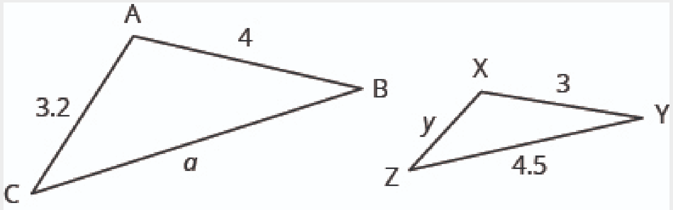
Note:
Solve geometry applications.

Read the problem and make all the words and ideas are understood. Draw the figure and label it with the given information.
Identify what we are looking for.
Name what we are looking for by choosing a variable to represent it.
Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
Solve the equation using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

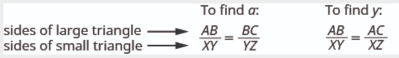
Example:
Exercise:

Problem:

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given. Find the lengths of the third sides.



Solution:
Solution

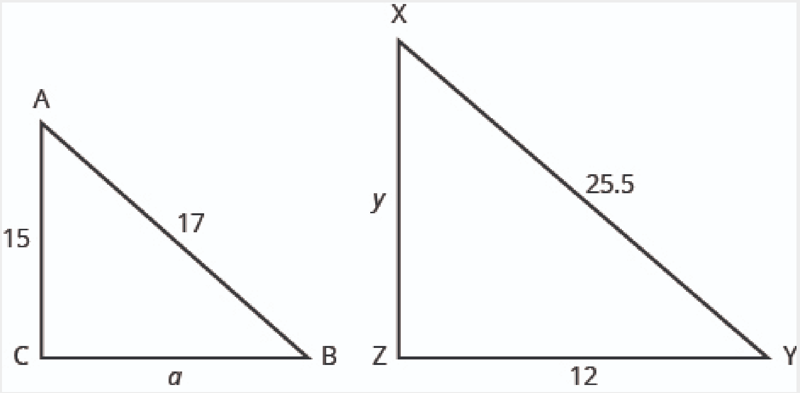
Step 1. Read the problem. Draw the figure and label it with the given information.	Figure is given.
Step 2. Identify what we are looking for.	the length of the sides of similar triangles
Step 3. Name the variables.	Let $a =$ length of the third side of ΔABC. $y =$ length of the third side of ΔXYZ
Step 4. Translate.	Since the triangles are similar, the corresponding sides are proportional.
We need to write an equation that compares the side we are looking for to a known ratio. Since the side *AB* = 4 corresponds to the side *XY* = 3	$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}.$  To find a: $\frac{AB}{XY} = \frac{BC}{YZ}$ To find y: $\frac{AB}{XY} = \frac{AC}{XZ}$ sides of large triangle → sides of small triangle →

we know $\frac{AB}{XY} = \frac{4}{3}$. So we write equations with $\frac{AB}{XY}$ to find the sides we are looking for. Be careful to match up corresponding sides correctly.	
Substitute.	$\frac{4}{3} = \frac{a}{4.5}$ $\frac{4}{3} = \frac{3.2}{y}$
Step 5. Solve the equation.	$3a = 4(4.5)$ $4y = 3(3.2)$
	$a = 6$ $y = 2.4$
Step 6. Check. $\begin{aligned} \frac{4}{3} &\overset{?}{=} \frac{6}{4.5} \\ 4(4.5) &\overset{?}{=} 6(3) \\ 18 &= 18\checkmark \end{aligned}$ $\begin{aligned} \frac{4}{3} &\overset{?}{=} \frac{3.2}{2.4} \\ 4(2.4) &\overset{?}{=} 3.2(3) \\ 9.6 &= 9.6\checkmark \end{aligned}$	
Step 7. Answer the question.	The third side of ΔABC is 6 and the third side of ΔXYZ is 2.4.

Note:
Exercise:

Problem:

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure.



Find the length of side a.

Solution:

8

Note:
Exercise:

Problem:

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure.

Find the length of side y.

Solution:

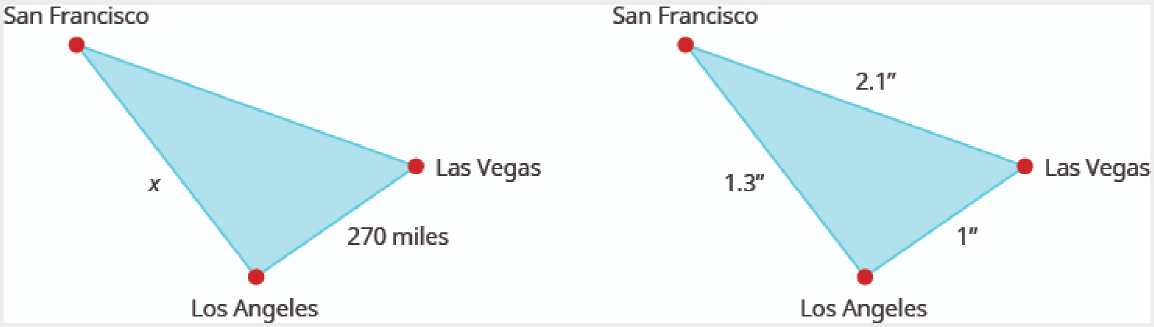
22.5

The next example shows how similar triangles are used with maps.

Example:
Exercise:

Problem:

On a map, San Francisco, Las Vegas, and Los Angeles form a triangle whose sides are shown in the figure below. If the actual distance from Los Angeles to Las Vegas is 270 miles find the distance from Los Angeles to San Francisco.



Solution:
Solution

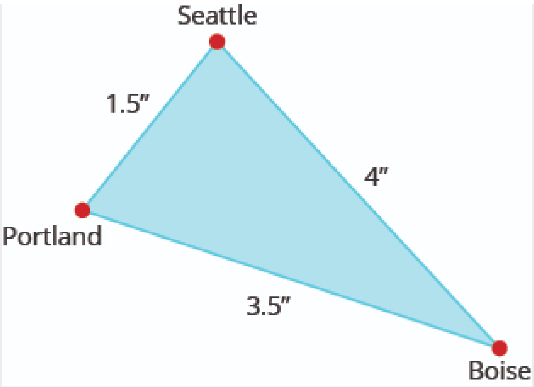
Read the problem. Draw the figures and label with the given information.	The figures are shown above.
Identify what we are looking for.	The actual distance from Los Angeles to San Francisco.
Name the variables.	Let $x =$ distance from Los Angeles to San Francisco.
Translate into an equation. Since the triangles are similar, the corresponding sides are proportional. We'll make the numerators "miles" and the denominators "inches."	$\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$
Solve the equation.	

	$\color{red}{1.3}\left(\frac{x}{1.3}\right) = \color{red}{1.3}\left(\frac{270}{1}\right)$
	$x = 351$
Check.	
On the map, the distance from Los Angeles to San Francisco is more than the distance from Los Angeles to Las Vegas. Since 351 is more than 270 the answer makes sense.	
Check $x = \color{red}{351}$ in the original proportion. Use a calculator. $\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$ $\frac{\color{red}{351} \text{ miles}}{1.3 \text{ inches}} \overset{?}{=} \frac{270 \text{ miles}}{1 \text{ inch}}$ $\frac{270 \text{ miles}}{1 \text{ inch}} = \frac{270 \text{ miles}}{1 \text{ inch}}$ ✓	
Answer the question.	The distance from Los Angeles to San Francisco is 351 miles.

Note:
Exercise:

Problem:

On the map, Seattle, Portland, and Boise form a triangle whose sides are shown in the figure below. If the actual distance from Seattle to Boise is 400 miles, find the distance from Seattle to Portland.



Solution:

150 miles

Note:
Exercise:

Problem: Using the map above, find the distance from Portland to Boise.

Solution:

350 miles

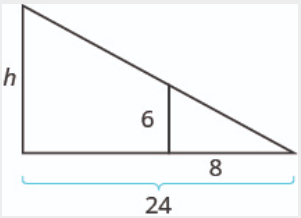
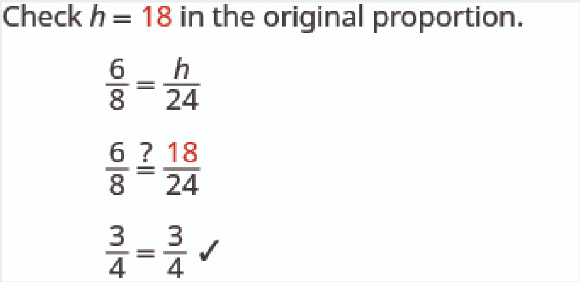
We can use similar figures to find heights that we cannot directly measure.

Example:
Exercise:

Problem:

Tyler is 6 feet tall. Late one afternoon, his shadow was 8 feet long. At the same time, the shadow of a tree was 24 feet long. Find the height of the tree.

Solution:
Solution

Read the problem and draw a figure.	
We are looking for *h*, the height of the tree.	
We will use similar triangles to write an equation.	
The small triangle is similar to the large triangle.	$\frac{h}{24}=\frac{6}{8}$
Solve the proportion.	$24\left(\frac{6}{8}\right)=24\left(\frac{h}{24}\right)$
Simplify.	$18=h$
Check.	
Tyler's height is less than his shadow's length so it makes sense that the tree's height is less than the length of its shadow.	
Check $h=18$ in the original proportion. $\frac{6}{8}=\frac{h}{24}$ $\frac{6}{8}\overset{?}{=}\frac{18}{24}$ $\frac{3}{4}=\frac{3}{4}$ ✓	

Note:
Exercise:

Problem:

A telephone pole casts a shadow that is 50 feet long. Nearby, an 8 foot tall traffic sign casts a shadow that is 10 feet long. How tall is the telephone pole?

Solution:

40 feet

Note:
Exercise:

Problem:

A pine tree casts a shadow of 80 feet next to a 30-foot tall building which casts a 40 feet shadow. How tall is the pine tree?

Solution:

60 feet

Key Concepts

- **Property of Similar Triangles**
  - If ΔABC is similar to ΔXYZ, then their corresponding angle measures are equal and their corresponding sides are in the same ratio.
- **Problem Solving Strategy for Geometry Applications**

  Read the problem and make sure all the words and ideas are understood. Draw the figure and label it with the given information.
  Identify what we are looking for.
  Name what we are looking for by choosing a variable to represent it.
  Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  Solve the equation using good algebra techniques.
  Check the answer in the problem and make sure it makes sense.

Answerthe question with a complete sentence.

Practice Makes Perfect

Solve Proportions

In the following exercises, solve.

Exercise:

Problem: $\frac{x}{56} = \frac{7}{8}$

Solution:

49

Exercise:

Problem: $\frac{n}{91} = \frac{8}{13}$

Exercise:

Problem: $\frac{49}{63} = \frac{z}{9}$

Solution:

7

Exercise:

Problem: $\frac{56}{72} = \frac{y}{9}$

Exercise:

Problem: $\frac{5}{a} = \frac{65}{117}$

Solution:

9

Exercise:

Problem: $\frac{4}{b} = \frac{64}{144}$

Exercise:

Problem: $\frac{98}{154} = \frac{-7}{p}$

Solution:

-11

Exercise:

Problem: $\frac{72}{156} = \frac{-6}{q}$

Exercise:

Problem: $\frac{a}{-8} = \frac{-42}{48}$

Solution:

7

Exercise:

Problem: $\frac{b}{-7} = \frac{-30}{42}$

Exercise:

Problem: $\frac{2.7}{j} = \frac{0.9}{0.2}$

Solution:

0.6

Exercise:

Problem: $\frac{2.8}{k} = \frac{2.1}{1.5}$

Exercise:

Problem: $\frac{a}{a+12} = \frac{4}{7}$

Solution:

16

Exercise:

Problem: $\frac{b}{b-16} = \frac{11}{9}$

Exercise:

Problem: $\frac{c}{c-104} = -\frac{5}{8}$

Solution:

40

Exercise:

Problem: $\frac{d}{d-48} = -\frac{13}{3}$

Exercise:

Problem: $\frac{m+90}{25} = \frac{m+30}{15}$

Solution:

60

Exercise:

Problem: $\frac{n+10}{4} = \frac{40-n}{6}$

Exercise:

Problem: $\frac{2p+4}{8} = \frac{p+18}{6}$

Solution:

30

Exercise:

Problem: $\frac{q-2}{2} = \frac{2q-7}{18}$

Exercise:

Problem:

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child’s weight. How many milliliters of acetaminophen will the doctor prescribe for Jocelyn, who weighs 45 pounds?

Solution:

9 ml

Exercise:

Problem:

Brianna, who weighs 6 kg, just received her shots and needs a pain killer. The pain killer is prescribed for children at 15 milligrams (mg) for every 1 kilogram (kg) of the child’s weight. How many milligrams will the doctor prescribe?

Exercise:

Problem:

A veterinarian prescribed Sunny, a 65 pound dog, an antibacterial medicine in case an infection emerges after her teeth were cleaned. If the dosage is 5 mg for every pound, how much medicine was Sunny given?

Solution:

325 mg

Exercise:

Problem:

Belle, a 13 pound cat, is suffering from joint pain. How much medicine should the veterinarian prescribe if the dosage is 1.8 mg per pound?

Exercise:

Problem:

A new energy drink advertises 106 calories for 8 ounces. How many calories are in 12 ounces of the drink?

Solution:

159 calories

Exercise:

Problem:

One 12 ounce can of soda has 150 calories. If Josiah drinks the big 32 ounce size from the local mini-mart, how many calories does he get?

Exercise:

Problem:

A new 7 ounce lemon ice drink is advertised for having only 140 calories. How many ounces could Sally drink if she wanted to drink just 100 calories?

Solution:

5 oz

Exercise:

Problem:

Reese loves to drink healthy green smoothies. A 16 ounce serving of smoothie has 170 calories. Reese drinks 24 ounces of these smoothies in one day. How many calories of smoothie is he consuming in one day?

Exercise:

Problem:

Janice is traveling to Canada and will change $250 US dollars into Canadian dollars. At the current exchange rate, $1 US is equal to $1.01 Canadian. How many Canadian dollars will she get for her trip?

Solution:

$252.5\text{ Canadian dollars}$

Exercise:

Problem:

Todd is traveling to Mexico and needs to exchange $450 into Mexican pesos. If each dollar is worth 12.29 pesos, how many pesos will he get for his trip?

Exercise:

Problem:

Steve changed $600 into 480 Euros. How many Euros did he receive for each US dollar?

Solution:

0.80 Euros

Exercise:

Problem:

Martha changed $350 US into 385 Australian dollars. How many Australian dollars did she receive for each US dollar?

Exercise:

Problem:

When traveling to Great Britain, Bethany exchanged her $900 into 570 British pounds. How many pounds did she receive for each American dollar?

Solution:

0.63 British pounds

Exercise:

Problem:

A missionary commissioned to South Africa had to exchange his $500 for the South African Rand which is worth 12.63 for every dollar. How many Rand did he have after the exchange?

Exercise:

Problem:

Ronald needs a morning breakfast drink that will give him at least 390 calories. Orange juice has 130 calories in one cup. How many cups does he need to drink to reach his calorie goal?

Solution:

3 cups

Exercise:

Problem:

Sarah drinks a 32-ounce energy drink containing 80 calories per 12 ounce. How many calories did she drink?

Exercise:

Problem:

Elizabeth is returning to the United States from Canada. She changes the remaining 300 Canadian dollars she has to $230.05 in American dollars. What was $1 worth in Canadian dollars?

Solution:

1.30 Canadian dollars

Exercise:

Problem:

Ben needs to convert $1000 to the Japanese Yen. One American dollar is worth 123.3 Yen. How much Yen will he have?

Exercise:

Problem:

A golden retriever weighing 85 pounds has diarrhea. His medicine is prescribed as 1 teaspoon per 5 pounds. How much medicine should he be given?

Solution:

17 tsp

Exercise:

Problem:

Five-year-old Lacy was stung by a bee. The dosage for the anti-itch liquid is 150 mg for her weight of 40 pounds. What is the dosage per pound?

Exercise:

Problem:

Karen eats $\frac{1}{2}$ cup of oatmeal that counts for 2 points on her weight loss program. Her husband, Joe, can have 3 points of oatmeal for breakfast. How much oatmeal can he have?

Solution:

$\frac{3}{4}$ cup

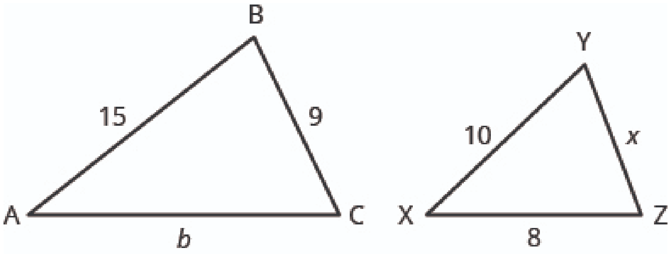
Exercise:

Problem:

An oatmeal cookie recipe calls for $\frac{1}{2}$ cup of butter to make 4 dozen cookies. Hilda needs to make 10 dozen cookies for the bake sale. How many cups of butter will she need?

Solve Similar Figure Applications

In the following exercises, ΔABC is similar to ΔXYZ. Find the length of the indicated side.



Exercise:

Problem: side *b*

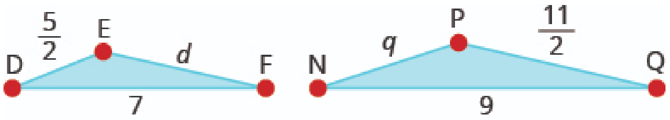
Solution:

12

Exercise:

Problem: side *x*

In the following exercises, ΔDEF is similar to ΔNPQ.



Exercise:

Problem: Find the length of side *d*.

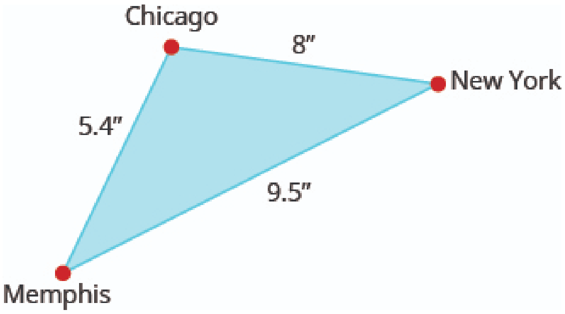
Solution:

$\frac{77}{18}$

Exercise:

Problem: Find the length of side *q*.

In the following two exercises, use the map shown. On the map, New York City, Chicago, and Memphis form a triangle whose sides are shown in the figure below. The actual distance from New York to Chicago is 800 miles.



Exercise:

Problem: Find the actual distance from New York to Memphis.

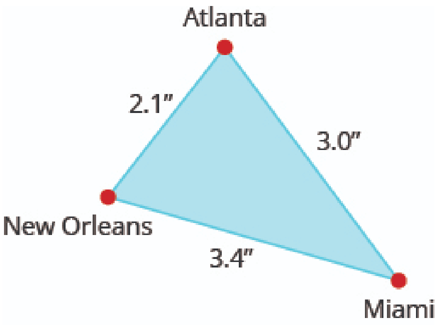
Solution:

950 miles

Exercise:

Problem: Find the actual distance from Chicago to Memphis.

In the following two exercises, use the map shown. On the map, Atlanta, Miami, and New Orleans form a triangle whose sides are shown in the figure below. The actual distance from Atlanta to New Orleans is 420 miles.



Exercise:

Problem: Find the actual distance from New Orleans to Miami.

Solution:

680 miles

Exercise:

Problem: Find the actual distance from Atlanta to Miami.

Exercise:

Problem:

A 2 foot tall dog casts a 3 foot shadow at the same time a cat casts a one foot shadow. How tall is the cat?

Solution:

$\frac{2}{3}$ foot (8 in)

Exercise:

Problem:

Larry and Tom were standing next to each other in the backyard when Tom challenged Larry to guess how tall he was. Larry knew his own height is 6.5 feet and when they measured their shadows, Larry's shadow was 8 feet and Tom's was 7.75 feet long. What is Tom's height?

Exercise:

Problem:

The tower portion of a windmill is 212 feet tall. A six foot tall person standing next to the tower casts a seven foot shadow. How long is the windmill's shadow?

Solution:

247.3 feet

Exercise:

Problem:

The height of the Statue of Liberty is 305 feet. Nicole, who is standing next to the statue, casts a 6 foot shadow and she is 5 feet tall. How long should the shadow of the statue be?

Everyday Math

Exercise:

Problem:

Heart Rate At the gym, Carol takes her pulse for 10 seconds and counts 19 beats.

ⓐ How many beats per minute is this?
ⓑ Has Carol met her target heart rate of 140 beats per minute?

Solution:

114 beats per minute
no

Exercise:

Problem:

Heart Rate Kevin wants to keep his heart rate at 160 beats per minute while training. During his workout he counts 27 beats in 10 seconds.

ⓐ How many beats per minute is this?
ⓑ Has Kevin met his target heart rate?

Exercise:

Problem: Cost of a Road Trip Jesse’s car gets 30 miles per gallon of gas.

ⓐ If Las Vegas is 285 miles away, how many gallons of gas are needed to get there and then home?
ⓑ If gas is $3.09 per gallon, what is the total cost of the gas for the trip?

Solution:

19 gallons
$58.71

Exercise:

Problem:

Cost of a Road Trip Danny wants to drive to Phoenix to see his grandfather. Phoenix is 370 miles from Danny’s home and his car gets 18.5 miles per gallon.

ⓐ How many gallons of gas will Danny need to get to and from Phoenix?
ⓑ If gas is $3.19 per gallon, what is the total cost for the gas to drive to see his grandfather?

Exercise:

Problem:

Lawn Fertilizer Phil wants to fertilize his lawn. Each bag of fertilizer covers about 4,000 square feet of lawn. Phil's lawn is approximately 13,500 square feet. How many bags of fertilizer will he have to buy?

Solution:

4 bags

Exercise:

Problem:

House Paint April wants to paint the exterior of her house. One gallon of paint covers about 350 square feet, and the exterior of the house measures approximately 2000 square feet. How many gallons of paint will she have to buy?

Exercise:

Problem:

Cooking Natalia's pasta recipe calls for 2 pounds of pasta for 1 quart of sauce. How many pounds of pasta should Natalia cook if she has 2.5 quarts of sauce?

Solution:

5

Exercise:

Problem:

Heating Oil A 275 gallon oil tank costs $400 to fill. How much would it cost to fill a 180 gallon oil tank?

Writing Exercises

Exercise:

Problem:

Marisol solves the proportion $\frac{144}{a} = \frac{9}{4}$ by ‘cross multiplying’, so her first step looks like $4 \cdot 144 = 9 \cdot a$. Explain how this differs from the method of solution shown in [link].

Solution:

Answers will vary.

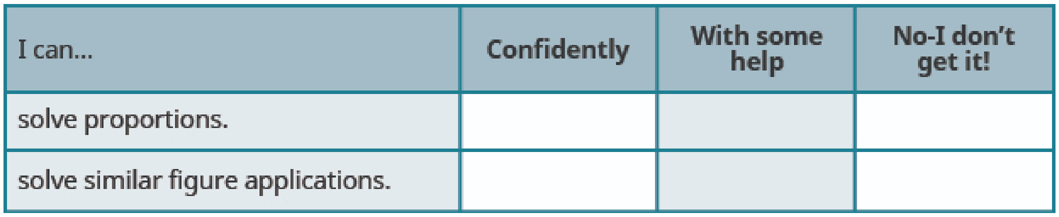
Exercise:

Problem:

Find a printed map and then write and solve an application problem similar to [link].

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don’t get it!
solve proportions.			
solve similar figure applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

proportion

A proportion is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$. The proportion is read “a is to b, as c is to d.”

similar figures

Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides are in the same ratio.

Solve Uniform Motion and Work Applications

By the end of this section, you will be able to:

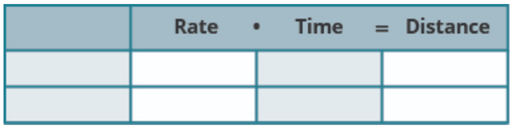
- Solve uniform motion applications
- Solve work applications

Note:
Before you get started, take this readiness quiz.
If you miss a problem, go back to the section listed and review the material.

1. An express train and a local bus leave Chicago to travel to Champaign. The express bus can make the trip in 2 hours and the local bus takes 5 hours for the trip. The speed of the express bus is 42 miles per hour faster than the speed of the local bus. Find the speed of the local bus.
   If you missed this problem, review [link].
2. Solve $\frac{1}{3}x+\frac{1}{4}x=\frac{5}{6}$.
   If you missed this problem, review [link].
3. Solve: $18t^2-30=-33t$.
   If you missed this problem, review [link].

Solve Uniform Motion Applications

We have solved uniform motion problems using the formula $D=rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.



	Rate •	Time =	Distance

The formula $D=rt$ assumes we know *r* and *t* and use them to find D. If we know D and *r* and need to find *t*, we would solve the equation for *t* and get the formula $t=\frac{D}{r}$.

We have also explained how flying with or against a current affects the speed of a vehicle. We will revisit that idea in the next example.

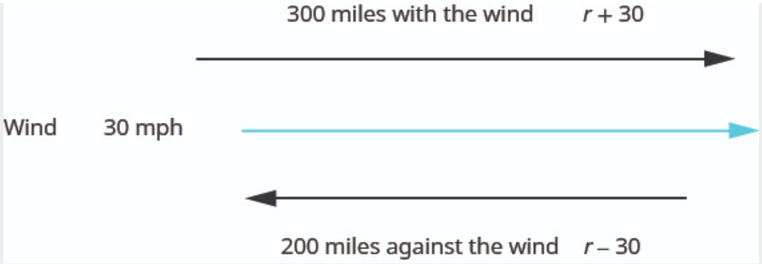
Example:
Exercise:

Problem:

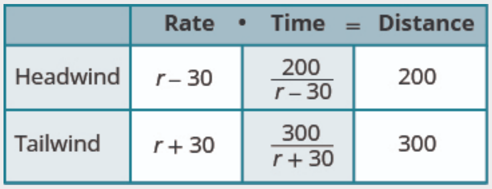
An airplane can fly 200 miles into a 30 mph headwind in the same amount of time it takes to fly 300 miles with a 30 mph tailwind. What is the speed of the airplane?

Solution:
Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

<table>
<tr><td>We are looking for the speed of the airplane.</td><td>Let $r =$ the speed of the airplane.</td></tr>
<tr><td>When the plane flies with the wind, the wind increases its speed and the rate is $r + 30$.</td><td></td></tr>
<tr><td>When the plane flies against the wind, the wind decreases its speed and the rate is $r - 30$.</td><td></td></tr>
<tr><td>Write in the rates.
Write in the distances.
Since $D = r \bullet t$, we solve for t and get $\frac{D}{r}$.
We divide the distance by the rate in each row, and place the expression in the time column.</td><td>
<table>
<tr><th></th><th>Rate</th><th>• Time</th><th>= Distance</th></tr>
<tr><td>Headwind</td><td>$r - 30$</td><td>$\frac{200}{r - 30}$</td><td>200</td></tr>
<tr><td>Tailwind</td><td>$r + 30$</td><td>$\frac{300}{r + 30}$</td><td>300</td></tr>
</table></td></tr>
<tr><td>We know the times are equal and so we write our equation.</td><td>$\frac{200}{r-30} = \frac{300}{r+30}$</td></tr>
<tr><td>We multiply both sides by the LCD.
$200(r + 30) = 300(r - 30)$</td><td>$(r+30)(r-30)\left(\frac{200}{r-30}\right) = (r+30)(r-30)\left(\frac{300}{r+30}\right)$</td></tr>
<tr><td>Simplify.</td><td>$(r+30)(200) = (r-30)(300)$</td></tr>
<tr><td></td><td>$200r + 6000 = 300r - 9000$</td></tr>
<tr><td>Solve.</td><td>$15000 = 100r$
$150 = r$</td></tr>
<tr><td>Check.</td><td></td></tr>
<tr><td>Is 150 mph a reasonable speed for an airplane? Yes. If the plane is traveling 150 mph and the wind is 30 mph:</td><td></td></tr>
<tr><td>Tailwind</td><td></td></tr>
</table>

$150+30=180\text{mph}$ $\frac{300}{180}=\frac{5}{3}$ hours	
Headwind $150-30=120\text{mph}$ $\frac{200}{120}=\frac{5}{3}$ hours	
The times are equal, so it checks.	The plane was traveling 150 mph.

Note:
Exercise:

Problem:

Link can ride his bike 20 miles into a 3 mph headwind in the same amount of time he can ride 30 miles with a 3 mph tailwind. What is Link’s biking speed?

Solution:

15 mph

Note:
Exercise:

Problem:

Judy can sail her boat 5 miles into a 7 mph headwind in the same amount of time she can sail 12 miles with a 7 mph tailwind. What is the speed of Judy’s boat without a wind?

Solution:

17 mph

In the next example, we will know the total time resulting from travelling different distances at different speeds.

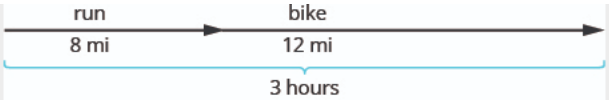
Example:
Exercise:

Problem:

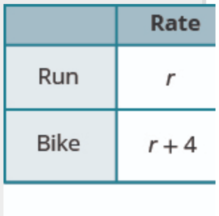
Jazmine trained for 3 hours on Saturday. She ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. What is her running speed?

Solution:
Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Jazmine's running speed.	Let $r =$ Jazmine's r
Her biking speed is 4 miles faster than her running speed.	$r + 4 =$ her biking
The distances are given, enter them into the chart.	
Since $D = r \bullet t$, we solve for t and get $t = \frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	 Rate Run r Bike $r + 4$
Write a word sentence.	Her time running pl hours.
Translate the sentence to get the equation.	$\frac{8}{r} + \frac{24}{r+4}$
Solve.	$r(r+4)\left(\frac{8}{r} + \frac{24}{r+}\right.$ $8(r+4) + 2$ $8r + 32 + 2$ $32 + 3$
	$(3r + 4) = 0 \quad (r$
	$r = -\frac{4}{3} \quad r = 8$
Check. $\cancel{r = -\frac{4}{3}}$ $r = 8$	
A negative speed does not make sense in this problem, so $r = 8$ is the solution.	
Is 8 mph a reasonable running speed? Yes.	
Run 8 mph $\frac{8\text{ miles}}{8\text{ mph}} = 1\text{ hour}$ Bike 12 mph $\frac{24\text{ miles}}{12\text{ mph}} = 2\text{ hours}$ Total 3 hours Jazmine's running speed is 8 mph.	

Note:
Exercise:

Problem:

Dennis went cross-country skiing for 6 hours on Saturday. He skied 20 mile uphill and then 20 miles back downhill, returning to his starting point. His uphill speed was 5 mph slower than his downhill speed. What was Dennis' speed going uphill and his speed going downhill?

Solution:

downhill: 10 mph, uphill: 5 mph

Note:
Exercise:

Problem:

Tony drove 4 hours to his home, driving 208 miles on the interstate and 40 miles on country roads. If he drove 15 mph faster on the interstate than on the country roads, what was his rate on the country roads?

Solution:

50 mph

Once again, we will use the uniform motion formula solved for the variable *t*.

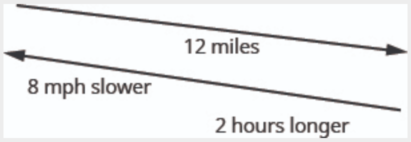
Example:
Exercise:

Problem:

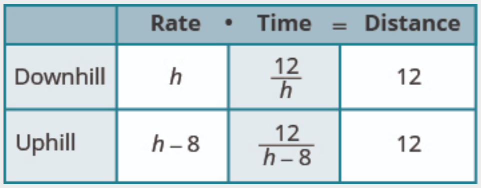
Hamilton rode his bike downhill 12 miles on the river trail from his house to the ocean and then rode uphill to return home. His uphill speed was 8 miles per hour slower than his downhill speed. It took him 2 hours longer to get home than it took him to get to the ocean. Find Hamilton's downhill speed.

Solution:
Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

<table>
<tr><td>We are looking for Hamilton’s downhill speed.</td><td>Let $r =$ Hamilton’s downhill speed.</td></tr>
<tr><td>His uplhill speed is 8 miles per hour slower.
Enter the rates into the chart.</td><td>$h - 8 =$ Hamilton’s uphill speed</td></tr>
<tr><td>The distance is the same in both directions, 12 miles.
Since $D = r \bullet t$, we solve for t and get $t = \frac{D}{r}$.
We divide the distance by the rate in each row, and place the expression in the time column.</td><td>
<table>
<tr><th></th><th>Rate</th><th>• Time</th><th>= Distance</th></tr>
<tr><td>Downhill</td><td>h</td><td>$\frac{12}{h}$</td><td>12</td></tr>
<tr><td>Uphill</td><td>$h-8$</td><td>$\frac{12}{h-8}$</td><td>12</td></tr>
</table></td></tr>
<tr><td>Write a word sentence about the time.</td><td>He took 2 hours longer uphill than downhill. The uphill time is 2 more than the downhill time.</td></tr>
<tr><td>Translate the sentence to get the equation.

Solve.</td><td>$$\begin{aligned} \frac{12}{h-8} &= \frac{12}{h}+2 \\ h(h-8)\left(\frac{12}{h-8}\right) &= h(h-8)\left(\frac{12}{h}+2\right) \\ 12h &= 12(h-8)+2h(h-8) \\ 12h &= 12h-96+2h^2-16h \\ 0 &= 2h^2-16h-96 \\ 0 &= 2\left(h^2-8h-48\right) \\ 0 &= 2(h-12)(h+4) \\ h-12 &= 0 \quad h+4=0 \\ h &= 12 \qquad \cancel{h=4} \end{aligned}$$</td></tr>
<tr><td>Check. Is 12 mph a reasonable speed for biking downhill? Yes.</td><td></td></tr>
<tr><td>Downhill 12 mph $\frac{12\text{ miles}}{12\text{ mph}} = 1\text{ hour}$</td><td></td></tr>
<tr><td>Uphill
$12 - 8 = 4\text{ mph}$ $\frac{12\text{ miles}}{4\text{ mph}} = 3\text{ hours}$</td><td></td></tr>
<tr><td></td><td>The uphill time is 2 hours more than the downhill time. Hamilton’s downhill speed is 12 mph.</td></tr>
</table>

Note:
Exercise:

Problem:

Kayla rode her bike 75 miles home from college one weekend and then rode the bus back to college. It took her 2 hours less to ride back to college on the bus than it took her to ride home on her bike, and the average speed of the bus was 10 miles per hour faster than Kayla’s biking speed. Find Kayla’s biking speed.

Solution:

15 mph

Note:
Exercise:

Problem:

Victoria jogs 12 miles to the park along a flat trail and then returns by jogging on an 18 mile hilly trail. She jogs 1 mile per hour slower on the hilly trail than on the flat trail, and her return trip takes her 3 hours longer. Find her rate of jogging on the flat trail.

Solution:

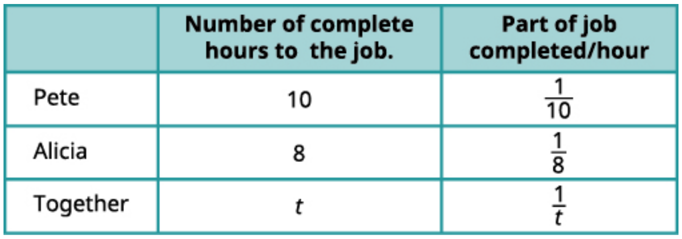
4 mph

Solve Work Applications

Suppose Pete can paint a room in 10 hours. If he works at a steady pace, in 1 hour he would paint $\frac{1}{10}$ of the room. If Alicia takes 8 hours to paint the same room, then in 1 hour she would paint $\frac{1}{8}$ of the room. How long would it take Pete and Alicia to paint the room if they worked together (and didn't interfere with each other's progress)?

This is a typical 'work' application. There are three quantities involved here – the time it would take each of the two people to do the job alone and the time it would take for them to do the job together.

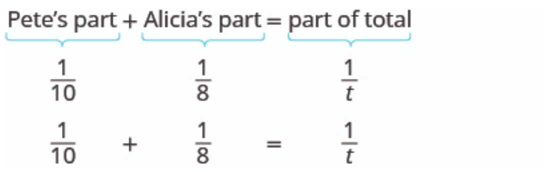
Let's get back to Pete and Alicia painting the room. We will let *t* be the number of hours it would take them to paint the room together. So in 1 hour working together they have completed $\frac{1}{t}$ of the job.



	Number of complete hours to the job.	Part of job completed/hour
Pete	10	$\frac{1}{10}$
Alicia	8	$\frac{1}{8}$
Together	t	$\frac{1}{t}$

In one hour Pete did $\frac{1}{10}$ of the job. Alicia did $\frac{1}{8}$ of the job. And together they did $\frac{1}{t}$ of the job.

We can model this with the word equation and then translate to a rational equation. To find the time it would take them if they worked together, we solve for *t*.



$$\underbrace{\text{Pete's part}}_{\frac{1}{10}} + \underbrace{\text{Alicia's part}}_{\frac{1}{8}} = \underbrace{\text{part of total}}_{\frac{1}{t}}$$

$$\frac{1}{10} + \frac{1}{8} = \frac{1}{t}$$

	$\frac{1}{10}+\frac{1}{8}=\frac{1}{t}$
Multiply by the LCD,40t.	$40t\left(\frac{1}{10}+\frac{1}{8}\right)=40t\left(\frac{1}{t}\right)$
Distribute.	$40t\cdot\frac{1}{10}+40t\cdot\frac{1}{8}=40t\left(\frac{1}{t}\right)$
Simplify and solve.	$4t+5t=40$
	$9t=40$
	$t=\frac{40}{9}$
We’ll write as a mixed number so that we can convert it to hours and minutes.	$t=4\frac{4}{9}\text{hours}$
Remember, 1 hour = 60 minutes.	$t=4\text{ hours}+\frac{4}{9}(60\text{ minutes})$
Multiply, and then round to the nearest minute.	$t=4\text{ hours}+27\text{ minutes}$
	It would take Pete and Alica about 4 hours and 27 minutes to paint the room.

Keep in mind, it should take less time for two people to complete a job working together than for either person to do it alone.

Example:
Exercise:

Problem:

The weekly gossip magazine has a big story about the Princess’ baby and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours to do the job and Press #2 takes 12 hours to do the job. How long will it take the printer to get the magazine printed with both presses running together?

Solution:
Solution

This is a work problem. A chart will help us organize the information.

<table>
<tr><td>Let $t =$ the number of hours needed to complete the job together.</td><td></td></tr>
<tr><td>Enter the hours per job for Press #1, Press #2 and when they work together.
If a job on Press #1 takes 6 hours, then in 1 hour $\frac{1}{6}$ of the job is completed.
Similarly find the part of the job completed/hours for Press #2 and when they both work together.</td><td><table>
<tr><th></th><th>Number of hours to complete the job</th><th>Part of job completed/ hour</th></tr>
<tr><td>Press #1</td><td>6</td><td>$\frac{1}{6}$</td></tr>
<tr><td>Press #2</td><td>12</td><td>$\frac{1}{12}$</td></tr>
<tr><td>Together</td><td>t</td><td>$\frac{1}{t}$</td></tr>
</table></td></tr>
<tr><td>Write a word sentence.</td><td></td></tr>
<tr><td>The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.</td><td></td></tr>
<tr><td>Translate to an equation.</td><td>Work completed by
$\underbrace{\text{Press #1} + \text{Press #2} = \text{Together}}$
$\frac{1}{6} \quad + \quad \frac{1}{12} \quad = \quad \frac{1}{t}$</td></tr>
<tr><td>Solve.</td><td>$\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$</td></tr>
<tr><td>Multiply by the LCD, $12t$.</td><td>$12t\left(\frac{1}{6} + \frac{1}{12}\right) = 12t\left(\frac{1}{t}\right)$</td></tr>
<tr><td>Simplify.</td><td>$2t + t = 12$</td></tr>
<tr><td></td><td>$3t = 12$</td></tr>
<tr><td></td><td>$t = 4$</td></tr>
<tr><td></td><td>When both presses are running it takes 4 hours to do the job.</td></tr>
</table>

Note:
Exercise:

Problem:

One gardener can mow a golf course in 4 hours, while another gardener can mow the same golf course in 6 hours. How long would it take if the two gardeners worked together to mow the golf course?

Solution:

2 hours and 24 minutes

Note:
Exercise:

Problem:

Carrie can weed the garden in 7 hours, while her mother can do it in 3. How long will it take the two of them working together?

Solution:

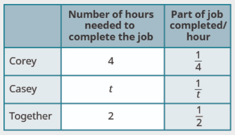
2 hours and 6 minutes

Example:
Exercise:

Problem:

Corey can shovel all the snow from the sidewalk and driveway in 4 hours. If he and his twin Casey work together, they can finish shoveling the snow in 2 hours. How many hours would it take Casey to do the job by himself?

Solution:
Solution

This is a work application. A chart will help us organize the information.	
We are looking for how many hours it would take Casey to complete the job by himself.	
Let $t =$ the number of hours needed for Casey to complete.	
Enter the hours per job for Corey, Casey, and when they work together. If Corey takes 4 hours, then in 1 hour $\frac{1}{4}$ of the job is completed. Similarly find the part of the job completed/hours for Casey and when they both work together.	

	Number of hours needed to complete the job	Part of job completed/ hour
Corey	4	$\frac{1}{4}$
Casey	t	$\frac{1}{t}$
Together	2	$\frac{1}{2}$

Write a word sentence.	
The part completed by Corey plus the part completed by Casey equals the amount completed together.	
Translate to an equation:	$\underbrace{\text{Corey} + \text{Casey} = \text{Together}}$ $\frac{1}{4} + \frac{1}{t} = \frac{1}{2}$
Solve.	$\frac{1}{4} + \frac{1}{t} = \frac{1}{2}$
Multiply by the LCD, $4t$.	$4t\left(\frac{1}{4} + \frac{1}{t}\right) = 4t\left(\frac{1}{2}\right)$
Simplify.	$t + 4 = 2t$
	$4 = t$
	It would take Casey 4 hours to do the job alone.

Note:
Exercise:

Problem:

Two hoses can fill a swimming pool in 10 hours. It would take one hose 26 hours to fill the pool by itself. How long would it take for the other hose, working alone, to fill the pool?

Solution:

16.25 hours

Note:
Exercise:

Problem:

Cara and Cindy, working together, can rake the yard in 4 hours. Working alone, it takes Cindy 6 hours to rake the yard. How long would it take Cara to rake the yard alone?

Solution:

12 hours

Practice Makes Perfect

Solve Uniform Motion Applications

In the following exercises, solve uniform motion applications

Exercise:

Problem:

Mary takes a sightseeing tour on a helicopter that can fly 450 miles against a 35 mph headwind in the same amount of time it can travel 702 miles with a 35 mph tailwind. Find the speed of the helicopter.

Solution:

160 mph

Exercise:

Problem:

A private jet can fly 1210 miles against a 25 mph headwind in the same amount of time it can fly 1694 miles with a 25 mph tailwind. Find the speed of the jet.

Exercise:

Problem:

A boat travels 140 miles downstream in the same time as it travels 92 miles upstream. The speed of the current is 6mph. What is the speed of the boat?

Solution:

29 mph

Exercise:

Problem:

Darrin can skateboard 2 miles against a 4 mph wind in the same amount of time he skateboards 6 miles with a 4 mph wind. Find the speed Darrin skateboards with no wind.

Exercise:

Problem:

Jane spent 2 hours exploring a mountain with a dirt bike. When she rode the 40 miles uphill, she went 5 mph slower than when she reached the peak and rode for 12 miles along the summit. What was her rate along the summit?

Solution:

30 mph

Exercise:

Problem:

Jill wanted to lose some weight so she planned a day of exercising. She spent a total of 2 hours riding her bike and jogging. She biked for 12 miles and jogged for 6 miles. Her rate for jogging was 10 mph less than biking rate. What was her rate when jogging?

Exercise:

Problem:

Bill wanted to try out different water craft. He went 62 miles downstream in a motor boat and 27 miles downstream on a jet ski. His speed on the jet ski was 10 mph faster than in the motor boat. Bill spent a total of 4 hours on the water. What was his rate of speed in the motor boat?

Solution:

20 mph

Exercise:

Problem:

Nancy took a 3 hour drive. She went 50 miles before she got caught in a storm. Then she drove 68 miles at 9 mph less than she had driven when the weather was good. What was her speed driving in the storm?

Exercise:

Problem:

Chester rode his bike uphill 24 miles and then back downhill at 2 mph faster than his uphill. If it took him 2 hours longer to ride uphill than downhill, l, what was his uphill rate?

Solution:

4 mph

Exercise:

Problem:

Matthew jogged to his friend's house 12 miles away and then got a ride back home. It took him 2 hours longer to jog there than ride back. His jogging rate was 25 mph slower than the rate when he was riding. What was his jogging rate?

Exercise:

Problem:

Hudson travels 1080 miles in a jet and then 240 miles by car to get to a business meeting. The jet goes 300 mph faster than the rate of the car, and the car ride takes 1 hour longer than the jet. What is the speed of the car?

Solution:

60 mph

Exercise:

Problem:

Nathan walked on an asphalt pathway for 12 miles. He walked the 12 miles back to his car on a gravel road through the forest. On the asphalt he walked 2 miles per hour faster than on the gravel. The walk on the gravel took one hour longer than the walk on the asphalt. How fast did he walk on the gravel?

Exercise:

Problem:

John can fly his airplane 2800 miles with a wind speed of 50 mph in the same time he can travel 2400 miles against the wind. If the speed of the wind is 50 mph, find the speed of his airplane.

Solution:

650 mph

Exercise:

Problem:

Jim’s speedboat can travel 20 miles upstream against a 3 mph current in the same amount of time it travels 22 miles downstream with a 3 mph current speed. Find the speed of Jim’s boat.

Exercise:

Problem:

Hazel needs to get to her granddaughter’s house by taking an airplane and a rental car. She travels 900 miles by plane and 250 miles by car. The plane travels 250 mph faster than the car. If she drives the rental car for 2 hours more than she rode the plane, find the speed of the car.

Solution:

50 mph

Exercise:

Problem:

Stu trained for 3 hours yesterday. He ran 14 miles and then biked 40 miles. His biking speed is 6 mph faster than his running speed. What is his running speed?

Exercise:

Problem:

When driving the 9 hour trip home, Sharon drove 390 miles on the interstate and 150 miles on country roads. Her speed on the interstate was 15 more than on country roads. What was her speed on country roads?

Solution:

50 mph

Exercise:

Problem:

Two sisters like to compete on their bike rides. Tamara can go 4 mph faster than her sister, Samantha. If it takes Samantha 1 hour longer than Tamara to go 80 miles, how fast can Samantha ride her bike?

Solve Work Applications

In the following exercises, solve work applications.

Exercise:

Problem:

Mike, an experienced bricklayer, can build a wall in 3 hours, while his son, who is learning, can do the job in 6 hours. How long does it take for them to build a wall together?

Solution:

2 hours

Exercise:

Problem:

It takes Sam 4 hours to rake the front lawn while his brother, Dave, can rake the lawn in 2 hours. How long will it take them to rake the lawn working together?

Exercise:

Problem:

Mary can clean her apartment in 6 hours while her roommate can clean the apartment in 5 hours. If they work together, how long would it take them to clean the apartment?

Solution:

2 hours and 44 minutes

Exercise:

Problem:

Brian can lay a slab of concrete in 6 hours, while Greg can do it in 4 hours. If Brian and Greg work together, how long will it take?

Exercise:

Problem:

Leeson can proofread a newspaper copy in 4 hours. If Ryan helps, they can do the job in 3 hours. How long would it take for Ryan to do his job alone?

Solution:

12 hours

Exercise:

Problem:

Paul can clean a classroom floor in 3 hours. When his assistant helps him, the job takes 2 hours. How long would it take the assistant to do it alone?

Exercise:

Problem:

Josephine can correct her students' test papers in 5 hours, but if her teacher's assistant helps, it would take them 3 hours. How long would it take the assistant to do it alone?

Solution:

7 hours and 30 minutes

Exercise:

Problem:

Washing his dad's car alone, eight year old Levi takes 2.5 hours. If his dad helps him, then it takes 1 hour. How long does it take the Levi's dad to wash the car by himself?

Exercise:

Problem:

Jackson can remove the shingles off of a house in 7 hours, while Martin can remove the shingles in 5 hours. How long will it take them to remove the shingles if they work together?

Solution:

2 hours and 55 minutes

Exercise:

Problem:

At the end of the day, Dodie can clean her hair salon in 15 minutes. Ann, who works with her, can clean the salon in 30 minutes. How long would it take them to clean the shop if they work together?

Exercise:

Problem:

Ronald can shovel the driveway in 4 hours, but if his brother Donald helps it would take 2 hours. How long would it take Donald to shovel the driveway alone?

Solution:

4 hours

Exercise:

Problem:

It takes Tina 3 hours to frost her holiday cookies, but if Candy helps her, it takes 2 hours. How long would it take Candy to frost the holiday cookies by herself?

Everyday Math

Exercise:

Problem:

Dana enjoys taking her dog for a walk, but sometimes her dog gets away and she has to run after him. Dana walked her dog for 7 miles but then had to run for 1 mile, spending a total time of 2.5 hours with her dog. Her running speed was 3 mph faster than her walking speed. Find her walking speed.

Solution:

3 mph

Exercise:

Problem:

Ken and Joe leave their apartment to go to a football game 45 miles away. Ken drives his car 30 mph faster than Joe can ride his bike. If it takes Joe 2 hours longer than Ken to get to the game, what is Joe's speed?

Writing Exercises

Exercise:

Problem: In [link], the solution $h = -4$ is crossed out. Explain why.

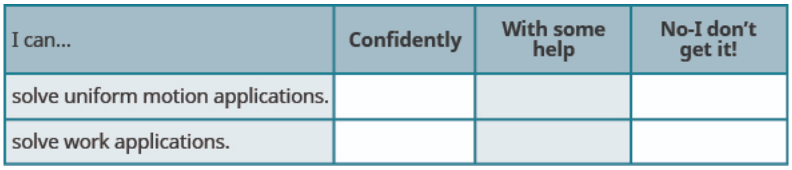
Exercise:

Problem:

Paula and Yuki are roommates. It takes Paula 3 hours to clean their apartment. It takes Yuki 4 hours to clean the apartment. The equation $\frac{1}{3}+\frac{1}{4}=\frac{1}{t}$ can be used to find t, the number of hours it would take both of them, working together, to clean their apartment. Explain how this equation models the situation.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
solve uniform motion applications.			
solve work applications.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Use Direct and Inverse Variation
By the end of this section, you will be able to:

- Solve direct variation problems
- Solve inverse variation problems

Note:
Before you get started, take this readiness quiz.
If you miss a problem, go back to the section listed and review the material.

1. Find the multiplicative inverse of -8.
   If you missed this problem, review [link].
2. Solve for n: $45 = 20n$.
   If you missed this problem, review [link].
3. Evaluate $5x^2$ when $x = 10$.
   If you missed this problem, review [link].

When two quantities are related by a proportion, we say they are *proportional* to each other. Another way to express this relation is to talk about the *variation* of the two quantities. We will discuss direct variation and inverse variation in this section.

Solve Direct Variation Problems

Lindsay gets paid $15 per hour at her job. If we let *s* be her salary and *h* be the number of hours she has worked, we could model this situation with the equation
Equation:

$$s = 15h$$

Lindsay’s salary is the product of a constant, 15, and the number of hours she works. We say that Lindsay’s salary *varies directly* with the number of hours she works. Two variables vary directly if one is the product of a constant and the other.

Note:
Direct Variation
For any two variables *x* and *y*, *y* varies directly with *x* if
Equation:

$$y = kx, \text{ where } k \neq 0$$

The constant *k* is called the constant of variation.

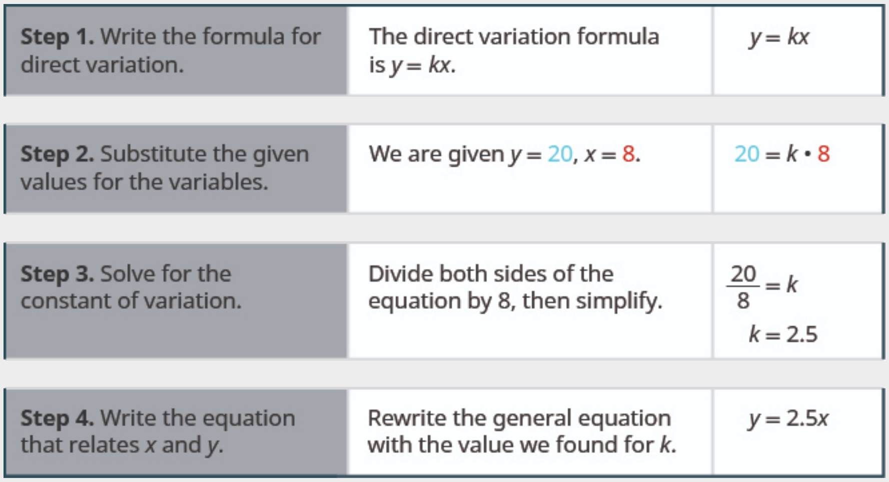
In applications using direct variation, generally we will know values of one pair of the variables and will be asked to find the equation that relates *x* and *y*. Then we can use that equation to find values of *y* for other values of *x*.

Example:

How to Solve Direct Variation Problems
Exercise:

Problem: If y varies directly with x and $y = 20$ when $x = 8$, find the equation that relates *x* and *y*.

Solution:
Solution



Step 1. Write the formula for direct variation.	The direct variation formula is $y = kx$.	$y = kx$
Step 2. Substitute the given values for the variables.	We are given $y = 20$, $x = 8$.	$20 = k \cdot 8$
Step 3. Solve for the constant of variation.	Divide both sides of the equation by 8, then simplify.	$\frac{20}{8} = k$ $k = 2.5$
Step 4. Write the equation that relates *x* and *y*.	Rewrite the general equation with the value we found for *k*.	$y = 2.5x$

Note:
Exercise:

Problem: If y varies directly as x and $y = 3,$ when $x = 10.$ find the equation that relates *x* and *y*.

Solution:

$y = \frac{3}{10}x$

Note:
Exercise:

Problem: If y varies directly as x and $y = 12$ when $x = 4$ find the equation that relates *x* and *y*.

Solution:

$y = 3x$

We’ll list the steps below.

Note:

Solve direct variation problems.

Write the formula for direct variation.
Substitute the given values for the variables.
Solve for the constant of variation.
Write the equation that relates x and y.

Now we’ll solve a few applications of direct variation.

Example:
Exercise:

Problem:

When Raoul runs on the treadmill at the gym, the number of calories, c, he burns varies directly with the number of minutes, m, he uses the treadmill. He burned 315 calories when he used the treadmill for 18 minutes.

ⓐ Write the equation that relates c and m.
ⓑ How many calories would he burn if he ran on the treadmill for 25 minutes?

Solution:
Solution

ⓐ

	The number of calories, c, varies directly with the number of minutes, m, on the treadmill, and $c = 315$ when $m = 18$.
Write the formula for direct variation.	$y = kx$
We will use c in place of y and m in place of x.	$c = km$
Substitute the given values for the variables.	$315 = k \cdot 18$
Solve for the constant of variation.	$\frac{315}{18} = \frac{k \cdot 18}{18}$
	$17.5 = k$

Write the equation that relates c and m.	$c = km$
Substitute in the constant of variation.	$c = 17.5m$

ⓑ

	Find c when $m = 25$.
Write the equation that relates c and$m$.	$c = 17.5m$
Substitute the given value for m.	$c = 17.5(25)$
Simplify.	$c = 437.5$
	Raoul would burn 437.5 calories if he used the treadmill for 25 minutes.

Note:
Exercise:

Problem:

The number of calories, c, burned varies directly with the amount of time, t, spent exercising. Arnold burned 312 calories in 65 minutes exercising.

ⓐ Write the equation that relates c and t.
ⓑ How many calories would he burn if he exercises for 90 minutes?

Solution:

ⓐ $c = 4.8t$ ⓑ 432 calories

Note:
Exercise:

Problem:

The distance a moving body travels, *d*, varies directly with time, *t*, it moves. A train travels 100 miles in 2 hours

ⓐ Write the equation that relates *d* and *t*.
ⓑ How many miles would it travel in 5 hours?

Solution:

ⓐ $d = 50t$ ⓑ 250 miles

In the previous example, the variables *c* and *m* were named in the problem. Usually that is not the case. We will have to name the variables in the next example as part of the solution, just like we do in most applied problems.

Example:
Exercise:

Problem:

The number of gallons of gas Eunice’s car uses varies directly with the number of miles she drives. Last week she drove 469.8 miles and used 14.5 gallons of gas.

ⓐ Write the equation that relates the number of gallons of gas used to the number of miles driven.
ⓑ How many gallons of gas would Eunice’s car use if she drove 1000 miles?

Solution:
Solution

	The number of gallons of gas varies directly with the number of miles driven.
First we will name the variables.	Let $g =$ number of gallons of gas. $m =$ number of miles driven
Write the formula for direct variation.	$y = kx$
We will use g in place of y and m in place of x.	$g = km$
Substitute the given values for the variables.	$g = 14.5$ when $m = 469.8$
	$14.5 = k(469.8)$

Solve for the constant of variation.	$\frac{14.5}{469.8} = \frac{k(469.8)}{469.8}$
We will round to the nearest thousandth.	$0.031 = k$
Write the equation that relates g and m.	$g = km$
Substitute in the constant of variation.	$g = 0.031m$

ⓑ

	Find g when $m = 1000$.
Write the equation that relates g and m.	$g = 0.031m$
Substitute the given value for m.	$g = 0.031\,(1000)$
Simplify.	$g = 31$
	Eunice's car would use 31 gallons of gas if she drove it 1,

Notice that in this example, the units on the constant of variation are gallons/mile. In everyday life, we usually talk about miles/gallon.

Note:
Exercise:

Problem:

The distance that Brad travels varies directly with the time spent traveling. Brad travelled 660 miles in 12 hours,

ⓐ Write the equation that relates the number of miles travelled to the time.
ⓑ How many miles could Brad travel in 4 hours?

Solution:

ⓐ $m = 55h$ ⓑ 220 miles

Note:
Exercise:

Problem:

The weight of a liquid varies directly as its volume. A liquid that weighs 24 pounds has a volume of 4 gallons.

ⓐ Write the equation that relates the weight to the volume.
ⓑ If a liquid has volume 13 gallons, what is its weight?

Solution:

ⓐ $w = 6v$ ⓑ 78 pounds

In some situations, one variable varies directly with the square of the other variable. When that happens, the equation of direct variation is $y = k\,x^2$. We solve these applications just as we did the previous ones, by substituting the given values into the equation to solve for k.

Example:
Exercise:

Problem:

The maximum load a beam will support varies directly with the square of the diagonal of the beam's cross-section. A beam with diagonal 4" will support a maximum load of 75 pounds.

ⓐ Write the equation that relates the maximum load to the cross-section.
ⓑ What is the maximum load that can be supported by a beam with diagonal 8"?

Solution:
Solution

	The maximum load varies directly with the square of the diagonal of the cross-section.
Name the variables.	Let $L =$ maximum load. $c =$ the diagonal of the cross-section
Write the formula for direct variation, where y varies directly with the square of x.	$y = kx^2$
We will use L in place of y and c in place of x.	$L = kc^2$
Substitute the given values for the variables.	$L = 75$ when $c = 4$
	$75 = k \cdot 4^2$
Solve for the constant of variation.	

	$\frac{75}{16} = \frac{k \cdot 16}{16}$
	$4.6875 = k$
Write the equation that relates L and c.	$L = kc^2$
Substitute in the constant of variation.	$L = 4.6875c^2$

ⓑ

	Find L when $c = 8$.
Write the equation that relates L and c.	$L = 4.6875c^2$
Substitute the given value for c.	$L = 4.6875(8)^2$
Simplify.	$L = 300$
	A beam with diagonal 8” could suppor a maximum load of 300 pounds.

Note:
Exercise:

Problem:

The distance an object falls is directly proportional to the square of the time it falls. A ball falls 144 feet in 3 seconds.

ⓐ Write the equation that relates the distance to the time.
ⓑ How far will an object fall in 4 seconds?

Solution:

ⓐ $d = 16t^2$ ⓑ 256 feet

Note:
Exercise:

Problem:

The area of a circle varies directly as the square of the radius. A circular pizza with a radius of 6 inches has an area of 113.04 square inches.

ⓐ Write the equation that relates the area to the radius.
ⓑ What is the area of a pizza with a radius of 9 inches?

Solution:

ⓐ $A = 3.14r^2$ ⓑ 254.34 square inches

Solve Inverse Variation Problems

Many applications involve two variable that *vary inversely*. As one variable increases, the other decreases. The equation that relates them is $y = \frac{k}{x}$.

Note:
Inverse Variation
For any two variables *x* and *y*, *y* varies inversely with *x* if
Equation:

$$y = \frac{k}{x}, \text{ where } k \neq 0$$

The constant *k* is called the constant of variation.

The word ‘inverse’ in inverse variation refers to the multiplicative inverse. The multiplicative inverse of *x* is $\frac{1}{x}$.

We solve inverse variation problems in the same way we solved direct variation problems. Only the general form of the equation has changed. We will copy the procedure box here and just change ‘direct’ to ‘inverse’.

Note:
Solve inverse variation problems.

Write the formula for inverse variation.
Substitute the given values for the variables.
Solve for the constant of variation.
Write the equation that relates x and y.

Example:
Exercise:

Problem: If *y* varies inversely with x and $y = 20$ when $x = 8$, find the equation that relates *x* and *y*.

Solution:
Solution

Write the formula for inverse variation.	$y = \frac{k}{x}$
Substitute the given values for the variables.	$y = 20$ when $x = 8$
	$20 = \frac{k}{8}$
Solve for the constant of variation.	$8(20) = 8\left(\frac{k}{8}\right)$
	$160 = k$
Write the equation that relates x and y.	$y = \frac{k}{x}$
Substitute in the constant of variation.	$y = \frac{160}{x}$

Note:
Exercise:

Problem: If p varies inversely with q and $p = 30$ when $q = 12$ find the equation that relates p and q.

Solution:

$p = \frac{360}{q}$

Note:
Exercise:

Problem: If y varies inversely with x and $y = 8$ when $x = 2$ find the equation that relates x and y.

Solution:

$y = \frac{16}{x}$

Example:
Exercise:

Problem:

The fuel consumption (mpg) of a car varies inversely with its weight. A car that weighs 3100 pounds gets 26 mpg on the highway.

ⓐ Write the equation of variation.
ⓑ What would be the fuel consumption of a car that weighs 4030 pounds?

Solution:
Solution

ⓐ

	The fuel consumption varies inversely with the weight.
First we will name the variables.	Let f = fuel consumption. w = weight
Write the formula for inverse variation.	$y = \frac{k}{x}$
We will use f in place of y and w in place of x.	$f = \frac{k}{w}$
Substitute the given values for the variables.	$f = 26$ when $w = 3100$
	$26 = \frac{k}{3100}$
Solve for the constant of variation.	$3100(26) = 3100\left(\frac{k}{3100}\right)$
	$80{,}600 = k$
Write the equation that relates f and w.	$f = \frac{k}{w}$
Substitute in the constant of variation.	$f = \frac{80{,}600}{w}$

ⓑ	Find f when $w = 4030$.
Write the equation that relates f and w.	$f = \frac{80{,}600}{w}$
Substitute the given value for w.	$f = \frac{80{,}600}{4030}$
Simplify.	$f = 20$
	A car that weighs 4030 pounds would have fuel consumption of 20 mpg.

Note:
Exercise:

Problem: A car's value varies inversely with its age. Elena bought a two-year-old car for $20,000.

ⓐ Write the equation of variation.
ⓑ What will be the value of Elena's car when it is 5 years old?

Solution:

ⓐ $v = \frac{40{,}000}{a}$ ⓑ $8,000

Note:
Exercise:

Problem:

The time required to empty a pool varies inversely as the rate of pumping. It took Lucy 2.5 hours to empty her pool using a pump that was rated at 400 gpm (gallons per minute).

ⓐ Write the equation of variation.
ⓑ How long will it take her to empty the pool using a pump rated at 500 gpm?

Solution:

ⓐ $t = \frac{1000}{r}$ ⓑ 2 hours

Example:
Exercise:

Problem:

The frequency of a guitar string varies inversely with its length. A 26” long string has a frequency of 440 vibrations per second.

ⓐ Write the equation of variation.
ⓑ How many vibrations per second will there be if the string's length is reduced to 20” by putting a finger on a fret?

Solution:
Solution

ⓐ

	The frequency varies inversely with the length.
Name the variables.	Let $f =$ frequency. $L =$ length
Write the formula for inverse variation.	$y = \frac{k}{x}$
We will use f in place of y and L in place of x.	$f = \frac{k}{L}$
Substitute the given values for the variables.	$f = 440$ when $L = 26$
	$440 = \frac{k}{26}$
Solve for the constant of variation.	$26(440) = 26\left(\frac{k}{26}\right)$
	$11{,}440 = k$
Write the equation that relates f and L.	$f = \frac{k}{L}$
Substitute in the constant of variation.	$f = \frac{11{,}440}{L}$

ⓑ

	Find f when $L = 20$.
Write the equation that relates f and L.	$f = \frac{11{,}440}{L}$
Substitute the given value for L.	$f = \frac{11{,}440}{20}$
Simplify.	$f = 572$
	A 20” guitar string has frequency of 572 vibrations per second.

Note:
Exercise:

Problem:

The number of hours it takes for ice to melt varies inversely with the air temperature. Suppose a block of ice melts in 2 hours when the temperature is 65 degrees.

ⓐ Write the equation of variation.
ⓑ How many hours would it take for the same block of ice to melt if the temperature was 78 degrees?

Solution:

ⓐ $h = \frac{130}{t}$ ⓑ $1\frac{2}{3}$ hours

Note:
Exercise:

Problem:

The force needed to break a board varies inversely with its length. Richard uses 24 pounds of pressure to break a 2-foot long board.

ⓐ Write the equation of variation.
ⓑ How many pounds of pressure is needed to break a 5-foot long board?

Solution:

ⓐ $F = \frac{48}{L}$ ⓑ 9.6 pounds

Practice Makes Perfect

Solve Direct Variation Problems

In the following exercises, solve.
Exercise:

Problem: If y varies directly as x and $y = 14,$ when $x = 3$, find the equation that relates x and y.

Solution:

$y=\frac{14}{3}x$

Exercise:

Problem: If p varies directly as q and $p=5,$ when $q=2$, find the equation that relates p and q.

Exercise:

Problem: If v varies directly as w and $v=24,$ when $w=8$, find the equation that relates v and w.

Solution:

$v=3w$

Exercise:

Problem: If a varies directly as b and $a=16,$ when $b=4$, find the equation that relates a and b.

Exercise:

Problem: If p varies directly as q and $p=9.6,$ when $q=3$, find the equation that relates p and q.

Solution:

$p=3.2q$

Exercise:

Problem: If y varies directly as x and $y=12.4,$ when $x=4$, find the equation that relates x and y

Exercise:

Problem: If a varies directly as b and $a=6,$ when $b=\frac{1}{3}$, find the equation that relates a and b.

Solution:

$a=18b$

Exercise:

Problem: If v varies directly as w and $v=8,$ when $w=\frac{1}{2}$, find the equation that relates v and w.

Exercise:

Problem:

The amount of money Sally earns, *P*, varies directly with the number, *n*, of necklaces she sells. When Sally sells 15 necklaces she earns $150.

ⓐ Write the equation that relates *P* and *n*.
ⓑ How much money would she earn if she sold 4 necklaces?

Solution:

ⓐ $P=10n$ ⓑ $40

Exercise:

Problem:

The price, *P*, that Eric pays for gas varies directly with the number of gallons, *g*, he buys. It costs him $50 to buy 20 gallons of gas.

ⓐ Write the equation that relates *P* and *g*.
ⓑ How much would 33 gallons cost Eric?

Exercise:

Problem:

Terri needs to make some pies for a fundraiser. The number of apples, *a*, varies directly with number of pies, *p*. It takes nine apples to make two pies.

ⓐ Write the equation that relates *a* and *p*.
ⓑ How many apples would Terri need for six pies?

Solution:

ⓐ $a = 4.5p$ ⓑ 27 apples

Exercise:

Problem:

Joseph is traveling on a road trip. The distance, *d*, he travels before stopping for lunch varies directly with the speed, *v*, he travels. He can travel 120 miles at a speed of 60 mph.

ⓐ Write the equation that relates *d* and *v*.
ⓑ How far would he travel before stopping for lunch at a rate of 65 mph?

Exercise:

Problem:

The price of gas that Jesse purchased varies directly to how many gallons he purchased. He purchased 10 gallons of gas for $39.80.

ⓐ Write the equation that relates the price to the number of gallons.
ⓑ How much will it cost Jesse for 15 gallons of gas?

Solution:

ⓐ $p = 3.98g$ ⓑ $\$59.70$

Exercise:

Problem:

The distance that Sarah travels varies directly to how long she drives. She travels 440 miles in 8 hours.

ⓐ Write the equation that relates the distance to the number of hours.
ⓑ How far can Sally travel in 6 hours?

Exercise:

Problem:

The mass of a liquid varies directly with its volume. A liquid with mass 16 kilograms has a volume of 2 liters.

ⓐ Write the equation that relates the mass to the volume.
ⓑ What is the volume of this liquid if its mass is 128 kilograms?

Solution:

ⓐ $m = 8v$ ⓑ 16 liters

Exercise:

Problem:

The length that a spring stretches varies directly with a weight placed at the end of the spring. When Sarah placed a 10 pound watermelon on a hanging scale, the spring stretched 5 inches.

ⓐ Write the equation that relates the length of the spring to the weight.
ⓑ What weight of watermelon would stretch the spring 6 inches?

Exercise:

Problem:

The distance an object falls varies directly to the square of the time it falls. A ball falls 45 feet in 3 seconds.

ⓐ Write the equation that relates the distance to the time.
ⓑ How far will the ball fall in 7 seconds?

Solution:

ⓐ $d = 5t^2$ ⓑ 245 feet

Exercise:

Problem:

The maximum load a beam will support varies directly with the square of the diagonal of the beam's cross-section. A beam with diagonal 6 inch will support a maximum load of 108 pounds.

ⓐ Write the equation that relates the load to the diagonal of the cross-section.
ⓑ What load will a beam with a 10 inch diagonal support?

Exercise:

Problem:

The area of a circle varies directly as the square of the radius. A circular pizza with a radius of 6 inches has an area of 113.04 square inches.

ⓐ Write the equation that relates the area to the radius.
ⓑ What is the area of a personal pizza with a radius 4 inches?

Solution:

ⓐ $A = 3.14r^2$ ⓑ 50.24 sq. in.

Exercise:

Problem:

The distance an object falls varies directly to the square of the time it falls. A ball falls 72 feet in 3 seconds,

ⓐ Write the equation that relates the distance to the time.
ⓑ How far will the ball have fallen in 8 seconds?

Solve Inverse Variation Problems

In the following exercises, solve.

Exercise:

Problem: If y varies inversely with x and $y=5$ when $x=4$ find the equation that relates x and y.

Solution:

$y=\frac{20}{x}$

Exercise:

Problem: If p varies inversely with q and $p=2$ when $q=1$ find the equation that relates p and q.

Exercise:

Problem: If v varies inversely with w and $v=6$ when $w=\frac{1}{2}$ find the equation that relates v and w.

Solution:

$v=\frac{3}{w}$

Exercise:

Problem: If a varies inversely with b and $a=12$ when $b=\frac{1}{3}$ find the equation that relates a and b.

Write an inverse variation equation to solve the following problems.

Exercise:

Problem:

The fuel consumption (mpg) of a car varies inversely with its weight. A Toyota Corolla weighs 2800 pounds and gets 33 mpg on the highway.

ⓐ Write the equation that relates the mpg to the car’s weight.
ⓑ What would the fuel consumption be for a Toyota Sequoia that weighs 5500 pounds?

Solution:

ⓐ $g=\frac{92{,}400}{w}$ ⓑ 16.8 mpg

Exercise:

Problem: A car’s value varies inversely with its age. Jackie bought a 10 year old car for $2,400.

ⓐ Write the equation that relates the car’s value to its age.
ⓑ What will be the value of Jackie’s car when it is 15 years old ?

Exercise:

Problem:

The time required to empty a tank varies inversely as the rate of pumping. It took Janet 5 hours to pump her flooded basement using a pump that was rated at 200 gpm (gallons per minute),

ⓐ Write the equation that relates the number of hours to the pump rate.
ⓑ How long would it take Janet to pump her basement if she used a pump rated at 400 gpm?

Solution:

ⓐ $t=\frac{1000}{r}$ ⓑ 2.5 hours

Exercise:

Problem:

The volume of a gas in a container varies inversely as pressure on the gas. A container of helium has a volume of 370 cubic inches under a pressure of 15 psi.

ⓐ Write the equation that relates the volume to the pressure.
ⓑ What would be the volume of this gas if the pressure was increased to 20 psi?

Exercise:

Problem:

On a string instrument, the length of a string varies inversely as the frequency of its vibrations. An 11-inch string on a violin has a frequency of 400 cycles per second.

ⓐ Write the equation that relates the string length to its frequency.
ⓑ What is the frequency of a 10-inch string?

Solution:

ⓐ $L=\frac{4{,}400}{f}$ ⓑ 440 cycles per second

Exercise:

Problem:

Paul, a dentist, determined that the number of cavities that develops in his patient’s mouth each year varies inversely to the number of minutes spent brushing each night. His patient, Lori, had 4 cavities when brushing her teeth 30 seconds (0.5 minutes) each night.

ⓐ Write the equation that relates the number of cavities to the time spent brushing.
ⓑ How many cavities would Paul expect Lori to have if she had brushed her teeth for 2 minutes each night?

Exercise:

Problem:

The number of tickets for a sports fundraiser varies inversely to the price of each ticket. Brianna can buy 25 tickets at $5 each.

ⓐ Write the equation that relates the number of tickets to the price of each ticket.
ⓑ How many tickets could Brianna buy if the price of each ticket was $2.50?

Solution:

ⓐ $t = \frac{125}{p}$ ⓑ 50 tickets

Exercise:

Problem:

Boyle's Law states that if the temperature of a gas stays constant, then the pressure varies inversely to the volume of the gas. Braydon, a scuba diver, has a tank that holds 6 liters of air under a pressure of 220 psi.

ⓐ Write the equation that relates pressure to volume.
ⓑ If the pressure increases to 330 psi, how much air can Braydon's tank hold?

Mixed Practice

Exercise:

Problem: If y varies directly as x and $y = 5,\ \text{when}\ x = 3.$, find the equation that relates x and y.

Solution:

$y = \frac{5}{3}x$

Exercise:

Problem: If v varies directly as w and $v = 21,\ \text{when}\ w = 8.$ find the equation that relates v and w.

Exercise:

Problem: If p varies inversely with q and $p = 5$ when $q = 6$, find the equation that relates p and q.

Solution:

$p = \frac{30}{q}$

Exercise:

Problem: If y varies inversely with x and $y = 11$ when $x = 3$ find the equation that relates x and y.

Exercise:

Problem: If p varies directly as q and $p = 10,\ \text{when}\ q = 2.$ find the equation that relates p and q.

Solution:

$p = 5q$

Exercise:

Problem: If v varies inversely with w and $v = 18$ when $w = \frac{1}{3}$ find the equation that relates v and w.

Exercise:

Problem:

The force needed to break a board varies inversely with its length. If Tom uses 20 pounds of pressure to break a 1.5-foot long board, how many pounds of pressure would he need to use to break a 6 foot long board?

Solution:

5 pounds

Exercise:

Problem:

The number of hours it takes for ice to melt varies inversely with the air temperature. A block of ice melts in 2.5 hours when the temperature is 54 degrees. How long would it take for the same block of ice to melt if the temperature was 45 degrees?

Exercise:

Problem:

The length a spring stretches varies directly with a weight placed at the end of the spring. When Meredith placed a 6-pound cantaloupe on a hanging scale, the spring stretched 2 inches. How far would the spring stretch if the cantaloupe weighed 9 pounds?

Solution:

3 inches

Exercise:

Problem:

The amount that June gets paid varies directly with the number of hours she works. When she worked 15 hours, she got paid $111. How much will she be paid for working 18 hours?

Exercise:

Problem:

The fuel consumption (mpg) of a car varies inversely with its weight. A Ford Focus weighs 3000 pounds and gets 28.7 mpg on the highway. What would the fuel consumption be for a Ford Expedition that weighs 5,500 pounds? Round to the nearest tenth.

Solution:

15.7 mpg

Exercise:

Problem:

The volume of a gas in a container varies inversely as the pressure on the gas. If a container of argon has a volume of 336 cubic inches under a pressure of 2,500 psi, what will be its volume if the pressure is decreased to 2,000 psi?

Exercise:

Problem:

The distance an object falls varies directly to the square of the time it falls. If an object falls 52.8 feet in 4 seconds, how far will it fall in 9 seconds?

Solution:

267.3 feet

Exercise:

Problem:

The area of the face of a Ferris wheel varies directly with the square of its radius. If the area of one face of a Ferris wheel with diameter 150 feet is 70,650 square feet, what is the area of one face of a Ferris wheel with diameter of 16 feet?

Everyday Math

Exercise:

Problem: Ride Service It costs $35 for a ride from the city center to the airport, 14 miles away.

ⓐ Write the equation that relates the cost, *c*, with the number of miles, *m*.
ⓑ What would it cost to travel 22 miles with this service?

Solution:

ⓐ $c=2.5m$ ⓑ $55

Exercise:

Problem:

Road Trip The number of hours it takes Jack to drive from Boston to Bangor is inversely proportional to his average driving speed. When he drives at an average speed of 40 miles per hour, it takes him 6 hours for the trip.

ⓐ Write the equation that relates the number of hours, *h*, with the speed, *s*.
ⓑ How long would the trip take if his average speed was 75 miles per hour?

Writing Exercises

Exercise:

Problem: In your own words, explain the difference between direct variation and inverse variation.

Solution:

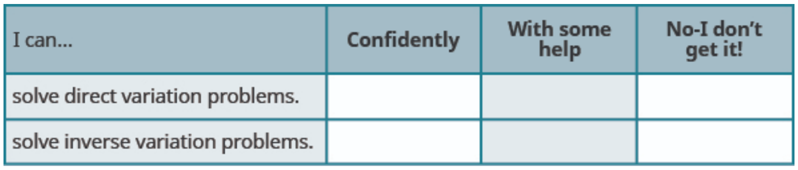
Answers will vary.

Exercise:

Problem: Make up an example from your life experience of inverse variation.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
solve direct variation problems.			
solve inverse variation problems.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next chapter? Why or why not?

Simplify Rational Expressions

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

Exercise:

Problem: $\frac{2a+1}{3a-2}$

Solution:

$a \neq \frac{2}{3}$

Exercise:

Problem: $\frac{b-3}{b^2-16}$

Exercise:

Problem: $\frac{3xy^2}{5y}$

Solution:

$y \neq 0$

Exercise:

Problem: $\frac{u-3}{u^2-u-30}$

Evaluate Rational Expressions

In the following exercises, evaluate the rational expressions for the given values.

Exercise:

Problem: $\frac{4p-1}{p^2+5}$ when $p = -1$

Solution:

$-\frac{5}{6}$

Exercise:

Problem: $\frac{q^2-5}{q+3}$ when q $= 7$

Exercise:

Problem: $\frac{y^2-8}{y^2-y-2}$ when $y = 1$

Solution:

$\frac{7}{2}$

Exercise:

Problem: $\frac{z^2+2}{4z-z^2}$ when z $= 3$

Simplify Rational Expressions

In the following exercises, simplify.

Exercise:

Problem: $\frac{10}{24}$

Solution:

$\frac{5}{12}$

Exercise:

Problem: $\frac{8m^4}{16mn^3}$

Exercise:

Problem: $\frac{14a-14}{a-1}$

Solution:

14

Exercise:

Problem: $\frac{b^2+7b+12}{b^2+8b+16}$

Simplify Rational Expressions with Opposite Factors

In the following exercises, simplify.

Exercise:

Problem: $\frac{c^2-c-2}{4-c^2}$

Solution:

$-\frac{c+1}{c+2}$

Exercise:

Problem: $\frac{d-16}{16-d}$

Exercise:

Problem: $\frac{7v-35}{25-v^2}$

Solution:

$-\frac{7}{5+v}$

Exercise:

Problem: $\frac{w^2-3w-28}{49-w^2}$

Multiply and Divide Rational Expressions

Multiply Rational Expressions

In the following exercises, multiply.

Exercise:

Problem: $\frac{3}{8} \cdot \frac{2}{15}$

Solution:

$\frac{1}{20}$

Exercise:

Problem: $\frac{2xy^2}{8y^3} \cdot \frac{16y}{24x}$

Exercise:

Problem: $\frac{3a^2+21a}{a^2+6a-7} \cdot \frac{a-1}{ab}$

Solution:

$\frac{3}{b}$

Exercise:

Problem: $\frac{5z^2}{5z^2+40z+35} \cdot \frac{z^2-1}{3z}$

Divide Rational Expressions

In the following exercises, divide.

Exercise:

Problem: $\frac{t^2-4t+12}{t^2+8t+12} \div \frac{t^2-36}{6t}$

Solution:

$\frac{6t}{(t+6)(t+6)}$

Exercise:

Problem: $\frac{r^2-16}{4} \div \frac{r^3-64}{2r^2-8r+32}$

Exercise:

Problem: $\frac{11+w}{w-9} \div \frac{121-w^2}{9-w}$

Solution:

$\frac{1}{w-11}$

Exercise:

Problem: $\frac{3y^2-12y-63}{4y+3} \div (6y^2-42y)$

Exercise:

Problem: $\frac{\frac{c^2-64}{3c^2+26c+16}}{\frac{c^2-4c-32}{15c+10}}$

Solution:

$\frac{5}{c+4}$

Exercise:

Problem: $\frac{8m^2-8m}{m-4} \cdot \frac{m^2+2m-24}{m^2+7m+10} \div \frac{2m^2-6m}{m+5}$

Add and Subtract Rational Expressions with a Common Denominator

Add Rational Expressions with a Common Denominator

In the following exercises, add.

Exercise:

Problem: $\frac{3}{5} + \frac{2}{5}$

Solution:

1

Exercise:

Problem: $\frac{4a^2}{2a-1} - \frac{1}{2a-1}$

Exercise:

Problem: $\frac{p^2+10p}{p+5} + \frac{25}{p+5}$

Solution:

$p+5$

Exercise:

Problem: $\frac{3x}{x-1} + \frac{2}{x-1}$

Subtract Rational Expressions with a Common Denominator

In the following exercises, subtract.

Exercise:

Problem: $\frac{d^2}{d+4} - \frac{3d+28}{d+4}$

Solution:

$d-7$

Exercise:

Problem: $\frac{z^2}{z+10} - \frac{100}{z+10}$

Exercise:

Problem: $\frac{4q^2-q+3}{q^2+6q+5} - \frac{3q^2+q+6}{q^2+6q+5}$

Solution:

$\frac{q-3}{q+5}$

Exercise:

Problem: $\frac{5t+4t+3}{t^2-25} - \frac{4t^2-8t-32}{t^2-25}$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add and subtract.

Exercise:

Problem: $\frac{18w}{6w-1} + \frac{3w-2}{1-6w}$

Solution:

$\frac{15w+2}{6w-1}$

Exercise:

Problem: $\frac{a^2+3a}{a^2-4} - \frac{3a-8}{4-a^2}$

Exercise:

Problem: $\frac{2b^2+3b+15}{b^2-49} - \frac{b^2+26b-1}{49-b^2}$

Solution:

$\frac{3b-2}{b+7}$

Exercise:

Problem: $\frac{8y^2-10y+7}{2y-5} + \frac{2y^2+7y+2}{5-2y}$

Add and Subtract Rational Expressions With Unlike Denominators

Find the Least Common Denominator of Rational Expressions

In the following exercises, find the LCD.

Exercise:

Problem: $\frac{4}{m^2-3m-10}, \frac{2m}{m^2-m-20}$

Solution:

$(m+2)(m-5)(m+4)$

Exercise:

Problem: $\frac{6}{n^2-4}, \frac{2n}{n^2-4n+4}$

Exercise:

Problem: $\frac{5}{3p^2+17p-6}, \frac{2m}{3p^2+23p-8}$

Solution:

$(3p-1)(p+6)(p+8)$

Find Equivalent Rational Expressions

In the following exercises, rewrite as equivalent rational expressions with the given denominator.

Exercise:

Problem: Rewrite as equivalent rational expressions with denominator $(m+2)(m-5)(m+4)$:

Equation:

$$\frac{4}{m^2-3m-10}, \frac{2m}{m^2-m-20}.$$

Exercise:

Problem: Rewrite as equivalent rational expressions with denominator $(n-2)(n-2)(n+2)$:
Equation:

$$\frac{6}{n^2-4n+4}, \frac{2n}{n^2-4}.$$

Solution:

$\frac{6n+12}{(n-2)(n-2)(n+2)}$,
$\frac{2n^2-4n}{(n-2)(n-2)(n+2)}$

Exercise:

Problem: Rewrite as equivalent rational expressions with denominator $(3p+1)(p+6)(p+8)$:
Equation:

$$\frac{5}{3p^2+19p+6}, \frac{7p}{3p^2+25p+8}$$

Add Rational Expressions with Different Denominators

In the following exercises, add.
Exercise:

Problem: $\frac{2}{3}+\frac{3}{5}$

Solution:

$\frac{19}{15}$

Exercise:

Problem: $\frac{7}{5a}+\frac{3}{2b}$

Exercise:

Problem: $\frac{2}{c-2}+\frac{9}{c+3}$

Solution:

$\frac{11c-12}{(c-2)(c+3)}$

Exercise:

Problem: $\frac{3d}{d^2-9}+\frac{5}{d^2+6d+9}$

Exercise:

Problem: $\frac{2x}{x^2+10x+24}+\frac{3x}{x^2+8x+16}$

Solution:

$\frac{5x^2+26x}{(x+4)(x+4)(x+6)}$

Exercise:

Problem: $\frac{5q}{p^2q-p^2}+\frac{4q}{q^2-1}$

Subtract Rational Expressions with Different Denominators

In the following exercises, subtract and add.

Exercise:

Problem: $\frac{3v}{v+2}-\frac{v+2}{v+8}$

Solution:

$\frac{2(v^2+10v-2)}{(v+2)(v+8)}$

Exercise:

Problem: $\frac{-3w-15}{w^2+w-20}-\frac{w+2}{4-w}$

Exercise:

Problem: $\frac{7m+3}{m+2}-5$

Solution:

$\frac{2m-7}{m+2}$

Exercise:

Problem: $\frac{n}{n+3}+\frac{2}{n-3}-\frac{n-9}{n^2-9}$

Exercise:

Problem: $\frac{8d}{d^2-64}-\frac{4}{d+8}$

Solution:

$\frac{4}{d-8}$

Exercise:

Problem: $\frac{5}{12x^2y}+\frac{7}{20xy^3}$

Simplify Complex Rational Expressions

Simplify a Complex Rational Expression by Writing it as Division

In the following exercises, simplify.

Exercise:

Problem: $\frac{\frac{5a}{a+2}}{\frac{10a^2}{a^2-4}}$

Solution:

$\frac{a-2}{2a}$

Exercise:

Problem: $\frac{\frac{2}{5}+\frac{5}{6}}{\frac{1}{3}+\frac{1}{4}}$

Exercise:

Problem: $\frac{x-\frac{3x}{x+5}}{\frac{1}{x+5}+\frac{1}{x-5}}$

Solution:

$\frac{(x+2)(x-5)}{2}$

Exercise:

Problem: $\frac{\frac{2}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify.

Exercise:

Problem: $\frac{1+\frac{2}{q-4}}{\frac{5}{q+4}}$

Solution:

$\frac{(q-2)(q+4)}{5(q-4)}$

Exercise:

Problem: $\frac{\frac{3}{a^2}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b^2}}$

Exercise:

Problem: $\frac{\frac{2}{z^2-49}+\frac{1}{z+7}}{\frac{9}{z+7}+\frac{12}{z-7}}$

Solution:

$\frac{z-5}{21z+21}$

Exercise:

Problem: $\frac{\frac{3}{y^2-4y-32}}{\frac{2}{y-8}+\frac{1}{y+4}}$

Solve Rational Equations

Solve Rational Equations

In the following exercises, solve.
Exercise:

Problem: $\frac{1}{2}+\frac{2}{3}=\frac{1}{x}$

Solution:

$\frac{6}{7}$

Exercise:

Problem: $1-\frac{2}{m}=\frac{8}{m^2}$

Exercise:

Problem: $\frac{1}{b-2}+\frac{1}{b+2}=\frac{3}{b^2-4}$

Solution:

$\frac{3}{2}$

Exercise:

Problem: $\frac{3}{q+8}-\frac{2}{q-2}=1$

Exercise:

Problem: $\frac{v-15}{v^2-9v+18}=\frac{4}{v-3}+\frac{2}{v-6}$

Solution:

no solution

Exercise:

Problem: $\frac{z}{12}+\frac{z+3}{3z}=\frac{1}{z}$

Solve a Rational Equation for a Specific Variable

In the following exercises, solve for the indicated variable.
Exercise:

Problem: $\frac{V}{l}=hw$ for l

Solution:

$l=\frac{V}{hw}$

Exercise:

Problem: $\frac{1}{x}-\frac{2}{y}=5$ for y

Exercise:

Problem: $x=\frac{y+5}{z-7}$ for z

Solution:

$z=\frac{y+5+7x}{x}$

Exercise:

Problem: $P=\frac{k}{V}$ for V

Solve Proportion and Similar Figure Applications

Solve Proportions

In the following exercises, solve.

Exercise:

Problem: $\frac{x}{4}=\frac{3}{5}$

Solution:

$\frac{12}{5}$

Exercise:

Problem: $\frac{3}{y}=\frac{9}{5}$

Exercise:

Problem: $\frac{s}{s+20}=\frac{3}{7}$

Solution:

15

Exercise:

Problem: $\frac{t-3}{5}=\frac{t+2}{9}$

In the following exercises, solve using proportions.

Exercise:

Problem:

Rachael had a 21 ounce strawberry shake that has 739 calories. How many calories are there in a 32 ounce shake?

Solution:

1126 calories

Exercise:

Problem:

Leo went to Mexico over Christmas break and changed $525 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 16.25 Mexican pesos. How many Mexican pesos did he get for his trip?

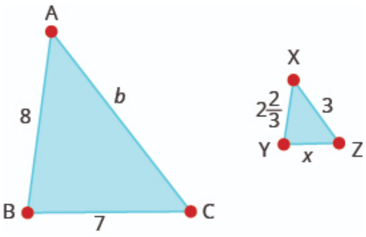
Solve Similar Figure Applications

In the following exercises, solve.

Exercise:

Problem:

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure. Find the lengths of the third sides.



Solution:

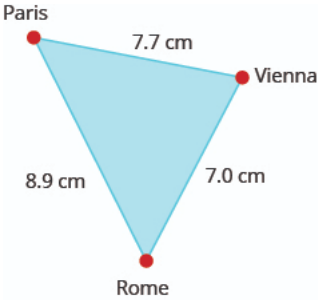
$b = 9; x = 2\frac{1}{3}$

Exercise:

Problem:

On a map of Europe, Paris, Rome, and Vienna form a triangle whose sides are shown in the figure below. If the actual distance from Rome to Vienna is 700 miles, find the distance from

ⓐ Paris to Rome
ⓑ Paris to Vienna



Exercise:

Problem:

Tony is 5.75 feet tall. Late one afternoon, his shadow was 8 feet long. At the same time, the shadow of a nearby tree was 32 feet long. Find the height of the tree.

Solution:

23 feet

Exercise:

Problem:

The height of a lighthouse in Pensacola, Florida is 150 feet. Standing next to the statue, 5.5 foot tall Natalie cast a 1.1 foot shadow. How long would the shadow of the lighthouse be?

Solve Uniform Motion and Work Applications Problems

Solve Uniform Motion Applications

In the following exercises, solve.

Exercise:

Problem:

When making the 5-hour drive home from visiting her parents, Lisa ran into bad weather. She was able to drive 176 miles while the weather was good, but then driving 10 mph slower, went 81 miles in the bad weather. How fast did she drive when the weather was bad?

Solution:

45 mph

Exercise:

Problem:

Mark is riding on a plane that can fly 490 miles with a tailwind of 20 mph in the same time that it can fly 350 miles against a tailwind of 20 mph. What is the speed of the plane?

Exercise:

Problem:

John can ride his bicycle 8 mph faster than Luke can ride his bike. It takes Luke 3 hours longer than John to ride 48 miles. How fast can John ride his bike?

Solution:

16 mph

Exercise:

Problem:

Mark was training for a triathlon. He ran 8 kilometers and biked 32 kilometers in a total of 3 hours. His running speed was 8 kilometers per hour less than his biking speed. What was his running speed?

Solve Work Applications

In the following exercises, solve.

Exercise:

Problem:

Jerry can frame a room in 1 hour, while Jake takes 4 hours. How long could they frame a room working together?

Solution:

$\frac{4}{5}$ hour

Exercise:

Problem:

Lisa takes 3 hours to mow the lawn while her cousin, Barb, takes 2 hours. How long will it take them working together?

Exercise:

Problem:

Jeffrey can paint a house in 6 days, but if he gets a helper, he can do it in 4 days. How long would it take the helper to paint the house alone?

Solution:

12 days

Exercise:

Problem:

Sue and Deb work together writing a book that takes them 90 days. If Sue worked alone it would take her 120 days. How long would it take Deb to write the book alone?

Use Direct and Inverse Variation

Solve Direct Variation Problems

In the following exercises, solve.

Exercise:

Problem: If y varies directly as x, when $y = 9$ and $x = 3$, find x when $y = 21$.

Solution:

7

Exercise:

Problem: If y varies directly as x, when $y = 20$ and $x = 2$ find y when $x = 4$.

Exercise:

Problem: If m varies directly with the square of n, when $m = 4$ and $n = 6$ find m when $n = 2$.

Solution:

$\frac{4}{9}$

Exercise:

Problem:

Vanessa is traveling to see her fiancé. The distance, d, varies directly with the speed, v, she drives. If she travels 258 miles driving 60 mph, how far would she travel going 70 mph?

Exercise:

Problem:

If the cost of a pizza varies directly with its diameter, and if an 8” diameter pizza costs \$12, how much would a 6” diameter pizza cost?

Solution:

$\$9$

Exercise:

Problem:

The distance to stop a car varies directly with the square of its speed. It takes 200 feet to stop a car going 50 mph. How many feet would it take to stop a car going 60 mph?

Solve Inverse Variation Problems

In the following exercises, solve.

Exercise:

Problem:

The number of tickets for a music fundraiser varies inversely with the price of the tickets. If Madelyn has just enough money to purchase 12 tickets for \$6, how many tickets can Madelyn afford to buy if the price increased to \$8?

Solution:

9 tickets

Exercise:

Problem:

On a string instrument, the length of a string varies inversely with the frequency of its vibrations. If an 11-inch string on a violin has a frequency of 360 cycles per second, what frequency does a 12 inch string have?

Practice Test

In the following exercises, simplify.

Exercise:

Problem: $\frac{3a^2b}{6ab^2}$

Solution:

$\frac{a}{2b}$

Exercise:

Problem: $\frac{5b-25}{b^2-25}$

In the following exercises, perform the indicated operation and simplify.

Exercise:

Problem: $\frac{4x}{x+2} \cdot \frac{x^2+5x+6}{12x^2}$

Solution:

$\frac{x+3}{3x}$

Exercise:

Problem: $\frac{5y}{4y-8} \cdot \frac{y^2-4}{10}$

Exercise:

Problem: $\frac{4}{pq} + \frac{5}{p}$

Solution:

$\frac{4+5q}{pq}$

Exercise:

Problem: $\frac{1}{z-9} - \frac{3}{z+9}$

Exercise:

Problem: $\frac{\frac{2}{3}+\frac{3}{5}}{\frac{2}{5}}$

Solution:

$\frac{19}{6}$

Exercise:

Problem: $\frac{\frac{1}{m}-\frac{1}{n}}{\frac{1}{n}+\frac{1}{m}}$

In the following exercises, solve each equation.

Exercise:

Problem: $\frac{1}{2} + \frac{2}{7} = \frac{1}{x}$

Solution:

$\frac{14}{11}$

Exercise:

Problem: $\frac{5}{y-6} = \frac{3}{y+6}$

Exercise:

Problem: $\frac{1}{z-5} + \frac{1}{z+5} = \frac{1}{z^2-25}$

Solution:

$\frac{1}{2}$

Exercise:

Problem: $\frac{t}{4} = \frac{3}{5}$

Exercise:

Problem: $\frac{2}{r-2} = \frac{3}{r-1}$

Solution:

4

In the following exercises, solve.

Exercise:

Problem: If y varies directly with x, and $x = 5$ when $y = 30$, find x when $y = 42$.

Exercise:

Problem: If y varies inversely with x and $x = 6$ when $y = 20$, find y when $x = 2$.

Solution:

60

Exercise:

Problem: If y varies inversely with the square of x and $x = 3$ when $y = 9$, find y when $x = 4$.

Exercise:

Problem:

The recommended erythromycin dosage for dogs, is 5 mg for every pound the dog weighs. If Daisy weighs 25 pounds, how many milligrams of erythromycin should her veterinarian prescribe?

Solution:

125 mg

Exercise:

Problem:

Julia spent 4 hours Sunday afternoon exercising at the gym. She ran on the treadmill for 10 miles and then biked for 20 miles. Her biking speed was 5 mph faster than her running speed on the treadmill. What was her running speed?

Exercise:

Problem:

Kurt can ride his bike for 30 miles with the wind in the same amount of time that he can go 21 miles against the wind. If the wind's speed is 6 mph, what is Kurt's speed on his bike?

Solution:

34 mph

Exercise:

Problem:

Amanda jogs to the park 8 miles using one route and then returns via a 14-mile route. The return trip takes her 1 hour longer than her jog to the park. Find her jogging rate.

Exercise:

Problem:

An experienced window washer can wash all the windows in Mike's house in 2 hours, while a new trainee can wash all the windows in 7 hours. How long would it take them working together?

Solution:

$1\frac{5}{9}$ hours

Exercise:

Problem:

Josh can split a truckload of logs in 8 hours, but working with his dad they can get it done in 3 hours. How long would it take Josh's dad working alone to split the logs?

Exercise:

Problem:

The price that Tyler pays for gas varies directly with the number of gallons he buys. If 24 gallons cost him \$59.76, what would 30 gallons cost?

Solution:

$\$74.70$

Exercise:

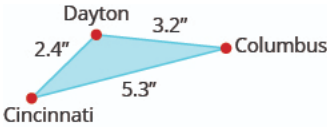
Problem:

The volume of a gas in a container varies inversely with the pressure on the gas. If a container of nitrogen has a volume of 29.5 liters with 2000 psi, what is the volume if the tank has a 14.7 psi rating? Round to the nearest whole number.

Exercise:

Problem:

The cities of Dayton, Columbus, and Cincinnati form a triangle in southern Ohio, as shown on the figure below, that gives the map distances between these cities in inches.



The actual distance from Dayton to Cincinnati is 48 miles. What is the actual distance between Dayton and Columbus?

Solution:

64 miles

Introduction class="introduction"

Square roots are used to determine the time it would take for a stone falling from the edge of this cliff to hit the land below.

Suppose a stone falls from the edge of a cliff. The number of feet the stone has dropped after seconds can be found by multiplying 16 times the square of . But to calculate the number of seconds it would take the stone to hit the land below, we need to use a square root. In this chapter, we will introduce and apply the properties of square roots, and extend these concepts to higher order roots and rational exponents.

Simplify and Use Square Roots

By the end of this section, you will be able to:

- Simplify expressions with square roots
- Estimate square roots
- Approximate square roots
- Simplify variable expressions with square roots

Note:
Before you get started, take this readiness quiz.

1. Simplify: ⓐ 9^2 ⓑ $(-9)^2$ ⓒ -9^2.
   If you missed this problem, review [link].
2. Round 3.846 to the nearest hundredth.
   If you missed this problem, review [link].
3. For each number, identify whether it is a real number or not a real number:
   ⓐ $-\sqrt{100}$ ⓑ $\sqrt{-100}$.
   If you missed this problem, review [link].

Simplify Expressions with Square Roots

Remember that when a number n is multiplied by itself, we write n^2 and read it “n squared.” For example, 15^2 reads as “15 squared,” and 225 is called the square of 15, since $15^2 = 225$.

Note:
Square of a Number
If $n^2 = m$, then m is the square of n.

Sometimes we will need to look at the relationship between numbers and their squares in reverse. Because 225 is the square of 15, we can also say that 15 is a square root of 225. A number whose square is m is called a *square root* of m.

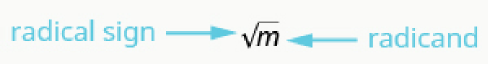
Note:
Square Root of a Number
If $n^2 = m$, then n is a square root of m.

Notice $(-15)^2 = 225$ also, so -15 is also a square root of 225. Therefore, both 15 and -15 are square roots of 225.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? The *radical sign*, $\sqrt{m}$, denotes the positive square root. The positive square root is also called the principal square root.

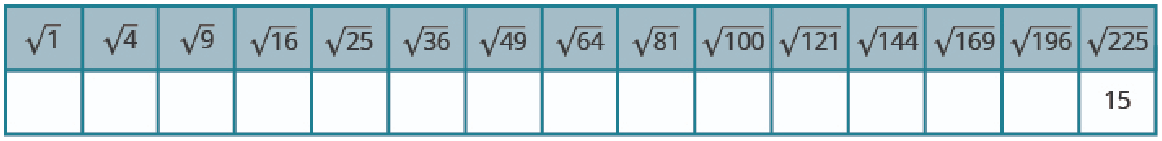
We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Note:
Square Root Notation



$\sqrt{m}$ is read as "the square root of m."
If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.
The square root of m, $\sqrt{m}$, is the positive number whose square is m.

Since 15 is the positive square root of 225, we write $\sqrt{225} = 15$. Fill in [link] to make a table of square roots you can refer to as you work this chapter.



$\sqrt{1}$	$\sqrt{4}$	$\sqrt{9}$	$\sqrt{16}$	$\sqrt{25}$	$\sqrt{36}$	$\sqrt{49}$	$\sqrt{64}$	$\sqrt{81}$	$\sqrt{100}$	$\sqrt{121}$	$\sqrt{144}$	$\sqrt{169}$	$\sqrt{196}$	$\sqrt{225}$
														15

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{225} = 15$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{225} = -15$.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{36}$ ⓑ $\sqrt{196}$ ⓒ $-\sqrt{81}$ ⓓ $-\sqrt{289}$.

Solution:
Solution

ⓐ

$$\sqrt{36}$$

Since $6^2 = 36$ $\quad 6$

ⓑ

$$\sqrt{196}$$

Since $14^2 = 196$ $\quad 14$

ⓒ

$$-\sqrt{81}$$

The negative is in front of the radical sign. $\quad -9$

ⓓ

$$-\sqrt{289}$$

The negative is in front of the radical sign. $\quad -17$

Note:
Exercise:

Problem: Simplify: ⓐ $-\sqrt{49}$ ⓑ $\sqrt{225}$.

Solution:

ⓐ -7 ⓑ 15

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{64}$ ⓑ $-\sqrt{121}$.

Solution:

ⓐ 8 ⓑ -11

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{-169}$ ⓑ $-\sqrt{64}$.

Solution:
Solution

ⓐ

$\sqrt{-169}$

There is no real number whose square is -169. $\sqrt{-169}$ is not a real number.

ⓑ

$-\sqrt{64}$

The negative is in front of the radical. -8

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{-196}$ ⓑ $-\sqrt{81}$.

Solution:

ⓐ not a real number ⓑ -9

Note:
Exercise:

Problem: Simplify: ⓐ $-\sqrt{49}$ ⓑ $\sqrt{-121}$.

Solution:

ⓐ -7 ⓑ not a real number

When using the order of operations to simplify an expression that has square roots, we treat the radical as a grouping symbol.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{25}+\sqrt{144}$ ⓑ $\sqrt{25+144}$.

Solution:
Solution

ⓐ

	$\sqrt{25}+\sqrt{144}$
Use the order of operations.	$5+12$
Simplify.	17

ⓑ

	$\sqrt{25+144}$
Simplify under the radical sign.	$\sqrt{169}$
Simplify.	13

Notice the different answers in parts ⓐ and ⓑ!

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{9}+\sqrt{16}$ ⓑ $\sqrt{9+16}$.

Solution:

ⓐ 7 ⓑ 5

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{64+225}$ ⓑ $\sqrt{64}+\sqrt{225}$.

Solution:

ⓐ 17 ⓑ 23

Estimate Square Roots

So far we have only considered square roots of perfect square numbers. The square roots of other numbers are not whole numbers. Look at [link] below.

Number	Square Root
4	$\sqrt{4} = 2$
5	$\sqrt{5}$
6	$\sqrt{6}$
7	$\sqrt{7}$
8	$\sqrt{8}$
9	$\sqrt{9} = 3$

The square roots of numbers between 4 and 9 must be between the two consecutive whole numbers 2 and 3, and they are not whole numbers. Based on the pattern in the table above, we could say that $\sqrt{5}$ must be between 2 and 3. Using inequality symbols, we write:
Equation:

$$2 < \sqrt{5} < 3$$

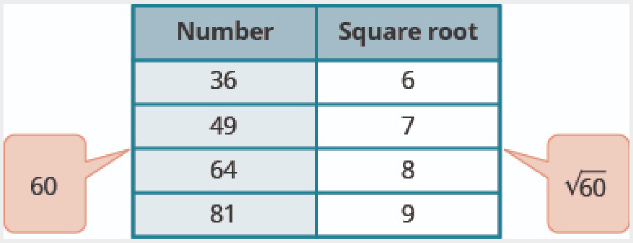
Example:
Exercise:

Problem: Estimate $\sqrt{60}$ between two consecutive whole numbers.

Solution:

Solution

Think of the perfect square numbers closest to 60. Make a small table of these perfect squares and their squares roots.

<table>
<tr><td>
<table>
<tr><th>Number</th><th>Square root</th></tr>
<tr><td>36</td><td>6</td></tr>
<tr><td>49</td><td>7</td></tr>
<tr><td>64</td><td>8</td></tr>
<tr><td>81</td><td>9</td></tr>
</table>
</td><td></td></tr>
<tr><td>Locate 60 between two consecutive perfect squares.</td><td>$49 < 60 < 64$</td></tr>
<tr><td>$\sqrt{60}$ is between their square roots.</td><td>$7 < \sqrt{60} < 8$</td></tr>
</table>

Note:
Exercise:

Problem: Estimate the square root $\sqrt{38}$ between two consecutive whole numbers.

Solution:

$6 < \sqrt{38} < 7$

Note:
Exercise:

Problem: Estimate the square root $\sqrt{84}$ between two consecutive whole numbers.

Solution:

$9 < \sqrt{84} < 10$

Approximate Square Roots

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find them. Find the $\sqrt{x}$ key on your calculator. You will use this key to approximate square roots.

When you use your calculator to find the square root of a number that is not a perfect square, the answer that you see is not the exact square root. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read 'approximately.'

Suppose your calculator has a 10-digit display. You would see that
Equation:

$$\sqrt{5} \approx 2.236067978$$

If we wanted to round $\sqrt{5}$ to two decimal places, we would say
Equation:

$$\sqrt{5} \approx 2.24$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:
Equation:

$$\begin{aligned} (2.236067978)^2 &= 5.000000002 \\ (2.24)^2 &= 5.0176 \end{aligned}$$

Their squares are close to 5, but are not exactly equal to 5.

Using the square root key on a calculator and then rounding to two decimal places, we can find:
Equation:

$$\begin{aligned}\sqrt{4} &= 2\\ \sqrt{5} &\approx 2.24\\ \sqrt{6} &\approx 2.45\\ \sqrt{7} &\approx 2.65\\ \sqrt{8} &\approx 2.83\\ \sqrt{9} &= 3\end{aligned}$$

Example:
Exercise:

Problem: Round $\sqrt{17}$ to two decimal places.

Solution:
Solution

	$\sqrt{17}$
Use the calculator square root key.	4.123105626...
Round to two decimal places.	4.12
	$\sqrt{17} \approx 4.12$

Note:
Exercise:

Problem: Round $\sqrt{11}$ to two decimal places.

Solution:

≈ 3.32

Note:
Exercise:

Problem: Round $\sqrt{13}$ to two decimal places.

Solution:

≈ 3.61

Simplify Variable Expressions with Square Roots

What if we have to find a square root of an expression with a variable? Consider $\sqrt{9x^2}$. Can you think of an expression whose square is $9x^2$?
Equation:

$$\begin{aligned}(?)^2 &= 9x^2 \\ (3x)^2 &= 9x^2, \qquad \text{so } \sqrt{9x^2} = 3x\end{aligned}$$

When we use the radical sign to take the square root of a variable expression, we should specify that $x \geq 0$ to make sure we get the *principal square root*.

However, in this chapter we will assume that each variable in a square-root expression represents a non-negative number and so we will not write $x \geq 0$ next to every radical.

What about square roots of higher powers of variables? Think about the Power Property of Exponents we used in Chapter 6.
Equation:

$$(a^m)^n = a^{m \cdot n}$$

If we square a^m, the exponent will become $2m$.
Equation:

$$(a^m)^2 = a^{2m}$$

How does this help us take square roots? Let’s look at a few:
Equation:

$$\begin{aligned}\sqrt{25u^8} = 5u^4 &\quad \text{because } \left(5u^4\right)^2 = 25u^8 \\ \sqrt{16r^{20}} = 4r^{10} &\quad \text{because } \left(4r^{10}\right)^2 = 16r^{20} \\ \sqrt{196q^{36}} = 14q^{18} &\quad \text{because } \left(14q^{18}\right)^2 = 196q^{36}\end{aligned}$$

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{x^6}$ ⓑ $\sqrt{y^{16}}$.

Solution:
Solution

ⓐ

	$\sqrt{x^6}$
Since $\left(x^3\right)^2 = x^6$.	x^3

ⓑ

	$\sqrt{y^{16}}$
Since $\left(y^8\right)^2 = y^{16}$.	y^8

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{y^8}$ ⓑ $\sqrt{z^{12}}$.

Solution:

ⓐ y^4 ⓑ z^6

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{m^4}$ ⓑ $\sqrt{b^{10}}$.

Solution:

ⓐ m^2 ⓑ b^5

Example:
Exercise:

Problem: Simplify: $\sqrt{16n^2}$.

Solution:
Solution

	$\sqrt{16n^2}$
Since $(4n)^2 = 16n^2$.	$4n$

Note:
Exercise:

Problem: Simplify: $\sqrt{64x^2}$.

Solution:

$8x$

Note:
Exercise:

Problem: Simplify: $\sqrt{169y^2}$.

Solution:

$13y$

Example:
Exercise:

Problem: Simplify: $-\sqrt{81c^2}$.

Solution:
Solution

	$-\sqrt{81c^2}$
Since $(9c)^2 = 81c^2$.	$-9c$

Note:
Exercise:

Problem: Simplify: $-\sqrt{121y^2}$.

Solution:

$-11y$

Note:
Exercise:

Problem: Simplify: $-\sqrt{100p^2}$.

Solution:

$-10p$

Example:
Exercise:

Problem: Simplify: $\sqrt{36x^2y^2}$.

Solution:
Solution

	$\sqrt{36x^2y^2}$
Since $(6xy)^2 = 36x^2y^2$.	$6xy$

Note:
Exercise:

Problem: Simplify: $\sqrt{100a^2b^2}$.

Solution:

$10ab$

Note:
Exercise:

Problem: Simplify: $\sqrt{225m^2n^2}$.

Solution:

$15mn$

Example:
Exercise:

Problem: Simplify: $\sqrt{64p^{64}}$.

Solution:
Solution

	$\sqrt{64p^{64}}$
Since $\left(8p^{32}\right)^2 = 64p^{64}$.	$8p^{32}$

Note:
Exercise:

Problem: Simplify: $\sqrt{49x^{30}}$.

Solution:

$7x^{15}$

Note:
Exercise:

Problem: Simplify: $\sqrt{81w^{36}}$.

Solution:

$9w^{18}$

Example:
Exercise:

Problem: Simplify: $\sqrt{121a^6b^8}$

Solution:
Solution

	$\sqrt{121a^6b^8}$
Since $\left(11a^3b^4\right)^2 = 121a^6b^8$.	$11a^3b^4$

Note:
Exercise:

Problem: Simplify: $\sqrt{169x^{10}y^{14}}$.

Solution:

$13x^5y^7$

Note:
Exercise:

Problem: Simplify: $\sqrt{144p^{12}q^{20}}$.

Solution:

$12p^6q^{10}$

Note:
Access this online resource for additional instruction and practice with square roots.

- Square Roots

Key Concepts

- Note that the square root of a negative number is not a real number.
- Every positive number has two square roots, one positive and one negative. The positive square root of a positive number is the principal square root.
- We can estimate square roots using nearby perfect squares.
- We can approximate square roots using a calculator.
- When we use the radical sign to take the square root of a variable expression, we should specify that $x \geq 0$ to make sure we get the principal square root.

Practice Makes Perfect

Simplify Expressions with Square Roots

In the following exercises, simplify.
Exercise:

Problem: $\sqrt{36}$

Solution:

6

Exercise:

Problem: $\sqrt{4}$

Exercise:

Problem: $\sqrt{64}$

Solution:

8

Exercise:

Problem: $\sqrt{169}$

Exercise:

Problem: $\sqrt{9}$

Solution:

3

Exercise:

Problem: $\sqrt{16}$

Exercise:

Problem: $\sqrt{100}$

Solution:

10

Exercise:

Problem: $\sqrt{144}$

Exercise:

Problem: $-\sqrt{4}$

Solution:

-2

Exercise:

Problem: $-\sqrt{100}$

Exercise:

Problem: $-\sqrt{1}$

Solution:

-1

Exercise:

Problem: $-\sqrt{121}$

Exercise:

Problem: $\sqrt{-121}$

Solution:

not a real number

Exercise:

Problem: $\sqrt{-36}$

Exercise:

Problem: $\sqrt{-9}$

Solution:

not a real number

Exercise:

Problem: $\sqrt{-49}$

Exercise:

Problem: $\sqrt{9+16}$

Solution:

5

Exercise:

Problem: $\sqrt{25+144}$

Exercise:

Problem: $\sqrt{9}+\sqrt{16}$

Solution:

7

Exercise:

Problem: $\sqrt{25} + \sqrt{144}$

Estimate Square Roots

In the following exercises, estimate each square root between two consecutive whole numbers.

Exercise:

Problem: $\sqrt{70}$

Solution:

$8 < \sqrt{70} < 9$

Exercise:

Problem: $\sqrt{55}$

Exercise:

Problem: $\sqrt{200}$

Solution:

$14 < \sqrt{200} < 15$

Exercise:

Problem: $\sqrt{172}$

Approximate Square Roots

In the following exercises, approximate each square root and round to two decimal places.

Exercise:

Problem: $\sqrt{19}$

Solution:

4.36

Exercise:

Problem: $\sqrt{21}$

Exercise:

Problem: $\sqrt{53}$

Solution:

7.28

Exercise:

Problem: $\sqrt{47}$

Simplify Variable Expressions with Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{y^2}$

Solution:

y

Exercise:

Problem: $\sqrt{b^2}$

Exercise:

Problem: $\sqrt{a^{14}}$

Solution:

a^7

Exercise:

Problem: $\sqrt{w^{24}}$

Exercise:

Problem: $\sqrt{49x^2}$

Solution:

$7x$

Exercise:

Problem: $\sqrt{100y^2}$

Exercise:

Problem: $\sqrt{121m^{20}}$

Solution:

$11m^{10}$

Exercise:

Problem: $\sqrt{25h^{44}}$

Exercise:

Problem: $\sqrt{81x^{36}}$

Solution:

$9x^{18}$

Exercise:

Problem: $\sqrt{144z^{84}}$

Exercise:

Problem: $-\sqrt{81x^{18}}$

Solution:

$-9x^9$

Exercise:

Problem: $-\sqrt{100m^{32}}$

Exercise:

Problem: $-\sqrt{64a^2}$

Solution:

$-8a$

Exercise:

Problem: $-\sqrt{25x^2}$

Exercise:

Problem: $\sqrt{144x^2y^2}$

Solution:

$12xy$

Exercise:

Problem: $\sqrt{196a^2b^2}$

Exercise:

Problem: $\sqrt{169w^8y^{10}}$

Solution:

$13w^4y^5$

Exercise:

Problem: $\sqrt{81p^{24}q^6}$

Exercise:

Problem: $\sqrt{9c^8d^{12}}$

Solution:

$3c^4d^6$

Exercise:

Problem: $\sqrt{36r^6s^{20}}$

Everyday Math

Exercise:

Problem:

Decorating Denise wants to have a square accent of designer tiles in her new shower. She can afford to buy 625 square centimeters of the designer tiles. How long can a side of the accent be?

Solution:

25 centimeters

Exercise:

Problem:

Decorating Morris wants to have a square mosaic inlaid in his new patio. His budget allows for 2025 square inch tiles. How long can a side of the mosaic be?

Writing Exercises

Exercise:

Problem: Why is there no real number equal to $\sqrt{-64}$?

Solution:

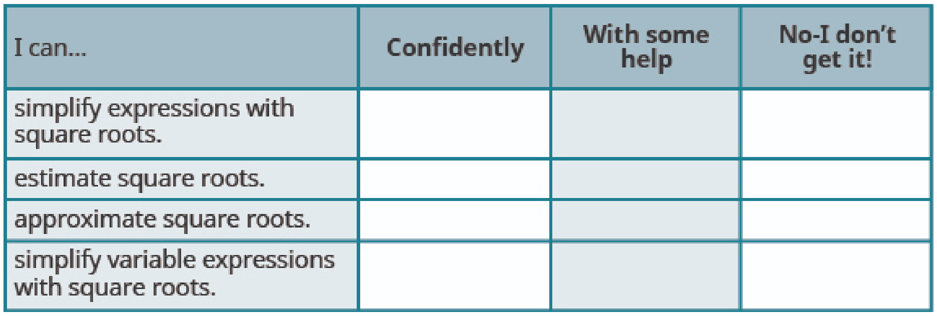
Answers will vary.

Exercise:

Problem: What is the difference between 9^2 and $\sqrt{9}$?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
simplify expressions with square roots.			
estimate square roots.			
approximate square roots.			
simplify variable expressions with square roots.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

square of a number

- If $n^2 = m$, then m is the square of n

square root of a number

- If $n^2 = m$, then n is a square root of m. The positive square root of m is called the principle square root of m.

square root notation

- If $m = n^2$, then $\sqrt{m} = n$. We read $\sqrt{m}$ as the principle square root of m, and the principle square root of a number is the positive square root of the number.

Simplify Square Roots

By the end of this section, you will be able to:

- Use the Product Property to simplify square roots
- Use the Quotient Property to simplify square roots

Note:
Before you get started take this readiness quiz.

1. Simplify: $\frac{80}{176}$.
   If you missed this problem, review [link].
2. Simplify: $\frac{n^9}{n^3}$.
   If you missed this problem, review [link].
3. Simplify: $\frac{q^4}{q^{12}}$.
   If you missed this problem, review [link].

In the last section, we estimated the square root of a number between two consecutive whole numbers. We can say that $\sqrt{50}$ is between 7 and 8. This is fairly easy to do when the numbers are small enough that we can use [link].

But what if we want to estimate $\sqrt{500}$? If we simplify the square root first, we'll be able to estimate it easily. There are other reasons, too, to simplify square roots as you'll see later in this chapter.

A square root is considered *simplified* when its **radicand**, the expression under the square root sign, contains no perfect square factors.

Note:
Simplified Square Root

$\sqrt{a}$ is considered simplified if a has no perfect square factors.

So $\sqrt{31}$ is simplified. But $\sqrt{32}$ is not simplified, because 16 is a perfect square factor of 32.

Use the Product Property to Simplify Square Roots

The properties we will use to simplify expressions with square roots are similar to the properties of exponents. We know that $(ab)^m = a^m b^m$. The corresponding property of square roots says that $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

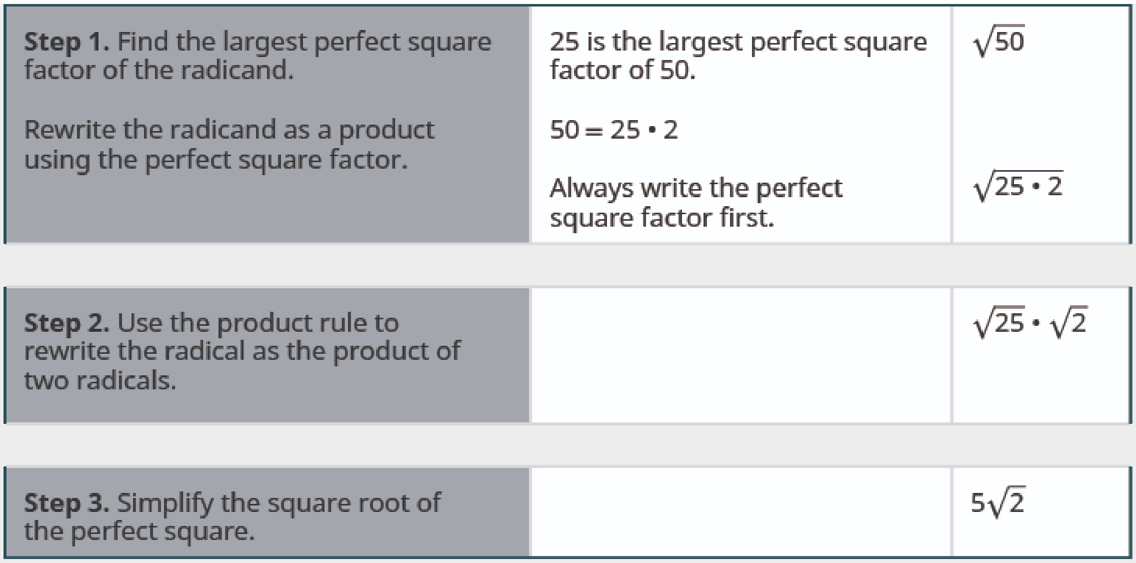
Note:
Product Property of Square Roots
If a, b are non-negative real numbers, then $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

We use the Product Property of Square Roots to remove all perfect square factors from a radical. We will show how to do this in [link].

Example:
How To Use the Product Property to Simplify a Square Root
Exercise:

Problem: Simplify: $\sqrt{50}$.

Solution:
Solution



Step 1. Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect square factor.	25 is the largest perfect square factor of 50. $50 = 25 \cdot 2$ Always write the perfect square factor first.	$\sqrt{50}$ $\sqrt{25 \cdot 2}$
Step 2. Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{25} \cdot \sqrt{2}$
Step 3. Simplify the square root of the perfect square.		$5\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $\sqrt{48}$.

Solution:

$4\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\sqrt{45}$.

Solution:

$3\sqrt{5}$

Notice in the previous example that the simplified form of $\sqrt{50}$ is $5\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

Note:
Simplify a square root using the product property.

Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect-square factor.
Use the product rule to rewrite the radical as the product of two radicals.
Simplify the square root of the perfect square.

Example:
Exercise:

Problem: Simplify: $\sqrt{500}$.

Solution:
Solution

	$\sqrt{500}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{100 \cdot 5}$
Rewrite the radical as the product of two radicals.	$\sqrt{100} \cdot \sqrt{5}$
Simplify.	$10\sqrt{5}$

Note:

Exercise:

Problem: Simplify: $\sqrt{288}$.

Solution:

$12\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $\sqrt{432}$.

Solution:

$12\sqrt{3}$

We could use the simplified form $10\sqrt{5}$ to estimate $\sqrt{500}$. We know 5 is between 2 and 3, and $\sqrt{500}$ is $10\sqrt{5}$. So $\sqrt{500}$ is between 20 and 30.

The next example is much like the previous examples, but with variables.

Example:
Exercise:

Problem: Simplify: $\sqrt{x^3}$.

Solution:
Solution

	$\sqrt{x^3}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{x^2 \cdot x}$
Rewrite the radical as the product of two radicals.	$\sqrt{x^2} \cdot \sqrt{x}$
Simplify.	$x\sqrt{x}$

Note:
Exercise:

Problem: Simplify: $\sqrt{b^5}$.

Solution:

$b^2\sqrt{b}$

Note:
Exercise:

Problem: Simplify: $\sqrt{p^9}$.

Solution:

$p^4\sqrt{p}$

We follow the same procedure when there is a coefficient in the radical, too.

Example:
Exercise:

Problem: Simplify: $\sqrt{25y^5}$.

Solution:
Solution

	$\sqrt{25y^5}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{25y^4 \cdot y}$
Rewrite the radical as the product of two radicals.	$\sqrt{25y^4} \cdot \sqrt{y}$
Simplify.	$5y^2\sqrt{y}$

Note:
Exercise:

Problem: Simplify: $\sqrt{16x^7}$.

Solution:

$4x^3\sqrt{x}$

Note:
Exercise:

Problem: Simplify: $\sqrt{49v^9}$.

Solution:

$7v^4\sqrt{v}$

In the next example both the constant and the variable have perfect square factors.

Example:
Exercise:

Problem: Simplify: $\sqrt{72n^7}$.

Solution:
Solution

	$\sqrt{72n^7}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{36n^6 \cdot 2n}$
Rewrite the radical as the product of two radicals.	$\sqrt{36n^6} \cdot \sqrt{2n}$
Simplify.	$6n^3\sqrt{2n}$

Note:
Exercise:

Problem: Simplify: $\sqrt{32y^5}$.

Solution:

$4y^2\sqrt{2y}$

Note:
Exercise:

Problem: Simplify: $\sqrt{75a^9}$.

Solution:

$5a^4\sqrt{3a}$

Example:
Exercise:

Problem: Simplify: $\sqrt{63u^3v^5}$.

Solution:
Solution

	$\sqrt{63u^3v^5}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{9u^2v^4 \cdot 7uv}$
Rewrite the radical as the product of two radicals.	$\sqrt{9u^2v^4} \cdot \sqrt{7uv}$
Simplify.	$3uv^2\sqrt{7uv}$

Note:

Exercise:

Problem: Simplify: $\sqrt{98a^7b^5}$.

Solution:

$7a^3b^2\sqrt{2ab}$

Note:
Exercise:

Problem: Simplify: $\sqrt{180m^9n^{11}}$.

Solution:

$6m^4n^5\sqrt{5mn}$

We have seen how to use the Order of Operations to simplify some expressions with radicals. To simplify $\sqrt{25}+\sqrt{144}$ we must simplify each square root separately first, then add to get the sum of 17.

The expression $\sqrt{17}+\sqrt{7}$ cannot be simplified—to begin we'd need to simplify each square root, but neither 17 nor 7 contains a perfect square factor.

In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer.

Example:

Exercise:

Problem: Simplify: $3 + \sqrt{32}$.

Solution:
Solution

	$3 + \sqrt{32}$
Rewrite the radicand as a product using the largest perfect square factor.	$3 + \sqrt{16 \cdot 2}$
Rewrite the radical as the product of two radicals.	$3 + \sqrt{16} \cdot \sqrt{2}$
Simplify.	$3 + 4\sqrt{2}$

The terms are not like and so we cannot add them. Trying to add an integer and a radical is like trying to add an integer and a variable—they are not like terms!

Note:
Exercise:

Problem: Simplify: $5 + \sqrt{75}$.

Solution:

$5 + 5\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $2+\sqrt{98}$.

Solution:

$2+7\sqrt{2}$

The next example includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.

Example:
Exercise:

Problem: Simplify: $\frac{4-\sqrt{48}}{2}$.

Solution:
Solution

	$\frac{4-\sqrt{48}}{2}$
Rewrite the radicand as a product using the largest perfect square factor.	$\frac{4-\sqrt{16\cdot 3}}{2}$
Rewrite the radical as the product of two radicals.	$\frac{4-\sqrt{16}\cdot\sqrt{3}}{2}$
Simplify.	$\frac{4-4\sqrt{3}}{2}$
Factor the common factor from the numerator.	$\frac{4\left(1-\sqrt{3}\right)}{2}$
Remove the common factor, 2, from the numerator and denominator.	$\frac{\cancel{2}\cdot 2\left(1-\sqrt{3}\right)}{\cancel{2}}$
Simplify.	$2\left(1-\sqrt{3}\right)$

Note:
Exercise:

Problem: Simplify: $\frac{10-\sqrt{75}}{5}$.

Solution:

$2-\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\frac{6-\sqrt{45}}{3}$.

Solution:

$2-\sqrt{5}$

Use the Quotient Property to Simplify Square Roots

Whenever you have to simplify a square root, the first step you should take is to determine whether the radicand is a perfect square. A *perfect square fraction* is a fraction in which both the numerator and the denominator are perfect squares.

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{9}{64}}$.

Solution:
Solution

	$\sqrt{\frac{9}{64}}$
Since $\left(\frac{3}{8}\right)^2=\frac{9}{64}$	$\frac{3}{8}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{25}{16}}$.

Solution:

$\frac{5}{4}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{49}{81}}$.

Solution:

$\frac{7}{9}$

If the numerator and denominator have any common factors, remove them. You may find a perfect square fraction!

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{45}{80}}$.

Solution:
Solution

	$\sqrt{\frac{45}{80}}$
Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.	$\sqrt{\frac{5\cdot 9}{5\cdot 16}}$
Simplify the fraction by removing common factors.	$\sqrt{\frac{9}{16}}$
Simplify. $\left(\frac{3}{4}\right)^{2}=\frac{9}{16}$	$\frac{3}{4}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{75}{48}}$.

Solution:

$\frac{5}{4}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{98}{162}}$.

Solution:

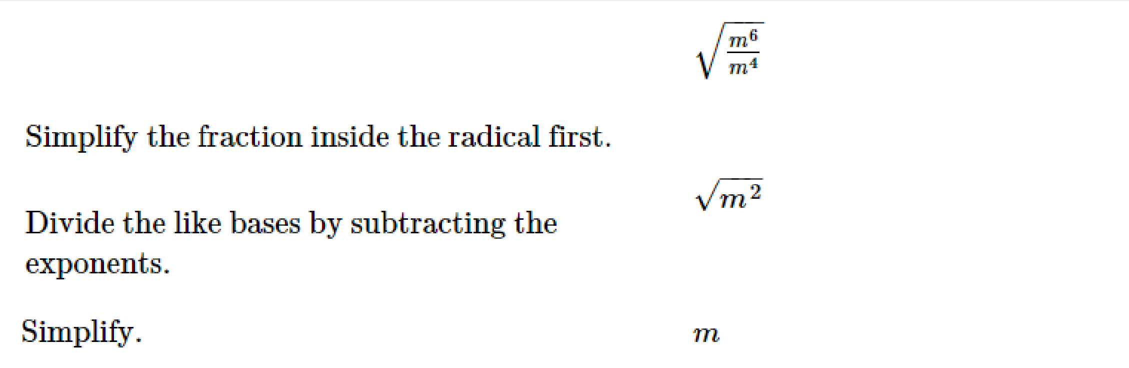
$\frac{7}{9}$

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the Quotient Property to simplify under the radical. We divide the like bases by subtracting their exponents, $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$.

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{m^6}{m^4}}$.

Solution:
Solution



	$\sqrt{\frac{m^6}{m^4}}$
Simplify the fraction inside the radical first.	
Divide the like bases by subtracting the exponents.	$\sqrt{m^2}$
Simplify.	m

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{a^8}{a^6}}$.

Solution:

a

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{x^{14}}{x^{10}}}$.

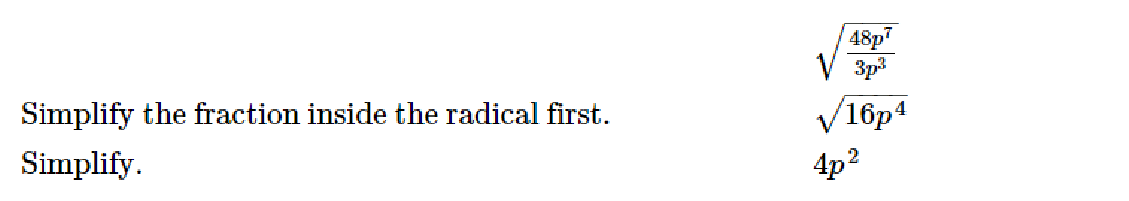
Solution:

x^2

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{48p^7}{3p^3}}$.

Solution:
Solution



	$\sqrt{\frac{48p^7}{3p^3}}$
Simplify the fraction inside the radical first.	$\sqrt{16p^4}$
Simplify.	$4p^2$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{75x^5}{3x}}$.

Solution:

$5x^2$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{72z^{12}}{2z^{10}}}$.

Solution:

$6z$

Remember the Quotient to a Power Property? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.
Equation:

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

We can use a similar property to simplify a square root of a fraction. After removing all common factors from the numerator and denominator, if the fraction is not a perfect square we simplify the numerator and denominator separately.

Note:
Quotient Property of Square Roots
If a, b are non-negative real numbers and $b \neq 0$, then
Equation:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{21}{64}}$.

Solution:
Solution

$$\sqrt{\frac{21}{64}}$$

We cannot simplify the fraction inside the radical. Rewrite using the quotient property.

$$\frac{\sqrt{21}}{\sqrt{64}}$$

Simplify the square root of 64. The numerator cannot be simplified.

$$\frac{\sqrt{21}}{8}$$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{19}{49}}$.

Solution:

$\frac{\sqrt{19}}{7}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{28}{81}}$.

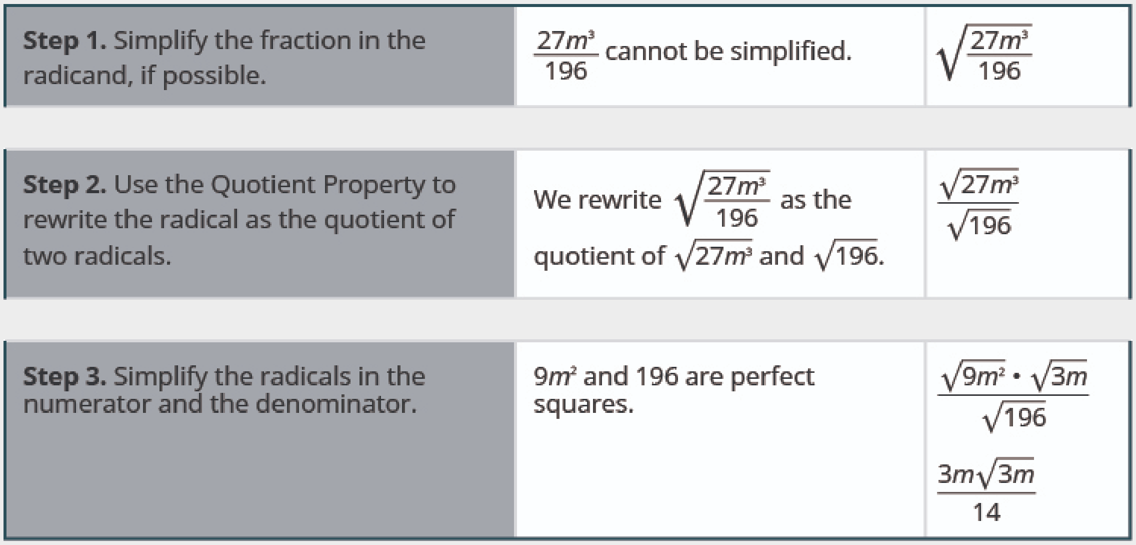
Solution:

$\frac{2\sqrt{7}}{9}$

Example:
How to Use the Quotient Property to Simplify a Square Root
Exercise:

Problem: Simplify: $\sqrt{\frac{27m^3}{196}}$.

Solution:
Solution



Step 1. Simplify the fraction in the radicand, if possible.	$\frac{27m^3}{196}$ cannot be simplified.	$\sqrt{\frac{27m^3}{196}}$
Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.	We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.	$\frac{\sqrt{27m^3}}{\sqrt{196}}$
Step 3. Simplify the radicals in the numerator and the denominator.	$9m^2$ and 196 are perfect squares.	$\frac{\sqrt{9m^2}\cdot\sqrt{3m}}{\sqrt{196}}$ $\frac{3m\sqrt{3m}}{14}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{24p^3}{49}}$.

Solution:

$\frac{2p\sqrt{6p}}{7}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{48x^5}{100}}$.

Solution:

$\frac{2x^2\sqrt{3x}}{5}$

Note:
Simplify a square root using the quotient property.

Simplify the fraction in the radicand, if possible.
Use the Quotient Property to rewrite the radical as the quotient of two radicals.
Simplify the radicals in the numerator and the denominator.

Example:

Exercise:

Problem: Simplify: $\sqrt{\frac{45x^5}{y^4}}$.

Solution:
Solution

	$\sqrt{\frac{45x^5}{y^4}}$
We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property.	$\frac{\sqrt{45x^5}}{\sqrt{y^4}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt{9x^4}\cdot\sqrt{5x}}{y^2}$
Simplify.	$\frac{3x^2\sqrt{5x}}{y^2}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{80m^3}{n^6}}$.

Solution:

$\frac{4m\sqrt{5m}}{n^3}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{54u^7}{v^8}}$.

Solution:

$\frac{3u^3\sqrt{6u}}{v^4}$

Be sure to simplify the fraction in the radicand first, if possible.

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{81d^9}{25d^4}}$.

Solution:
Solution

	$\sqrt{\frac{81d^9}{25d^4}}$
Simplify the fraction in the radicand.	$\sqrt{\frac{81d^5}{25}}$
Rewrite using the Quotient Property.	$\frac{\sqrt{81d^5}}{\sqrt{25}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt{81d^4}\cdot\sqrt{d}}{5}$
Simplify.	$\frac{9d^2\sqrt{d}}{5}$

Note:

Exercise:

Problem: Simplify: $\sqrt{\frac{64x^7}{9x^3}}$.

Solution:

$\frac{8x^2}{3}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{16a^9}{100a^5}}$.

Solution:

$\frac{2a^2}{5}$

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{18p^5q^7}{32pq^2}}$.

Solution:
Solution

	$\sqrt{\frac{18p^5q^7}{32pq^2}}$
Simplify the fraction in the radicand, if possible.	$\sqrt{\frac{9p^4q^5}{16}}$
Rewrite using the Quotient Property.	$\frac{\sqrt{9p^4q^5}}{\sqrt{16}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt{9p^4q^4}\cdot\sqrt{q}}{4}$
Simplify.	$\frac{3p^2q^2\sqrt{q}}{4}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{50x^5y^3}{72x^4y}}$.

Solution:

$\frac{5y\sqrt{x}}{6}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{48m^7n^2}{125m^5n^9}}$.

Solution:

$\frac{4m\sqrt{3}}{5n^3\sqrt{5n}}$

Key Concepts

- **Simplified Square Root** $\sqrt{a}$ is considered simplified if a has no perfect-square factors.
- **Product Property of Square Roots** If a, b are non-negative real numbers, then
  Equation:

  $$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

- **Simplify a Square Root Using the Product Property** To simplify a square root using the Product Property:

  Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect square factor.
  Use the product rule to rewrite the radical as the product of two radicals.
  Simplify the square root of the perfect square.

- **Quotient Property of Square Roots** If a, b are non-negative real numbers and $b \neq 0$, then
  Equation:

  $$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

- **Simplify a Square Root Using the Quotient Property** To simplify a square root using the Quotient Property:

  Simplify the fraction in the radicand, if possible.
  Use the Quotient Rule to rewrite the radical as the quotient of two radicals.
  Simplify the radicals in the numerator and the denominator.

Practice Makes Perfect

Use the Product Property to Simplify Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{27}$

Solution:

$3\sqrt{3}$

Exercise:

Problem: $\sqrt{80}$

Exercise:

Problem: $\sqrt{125}$

Solution:

$5\sqrt{5}$

Exercise:

Problem: $\sqrt{96}$

Exercise:

Problem: $\sqrt{200}$

Solution:

$10\sqrt{2}$

Exercise:

Problem: $\sqrt{147}$

Exercise:

Problem: $\sqrt{450}$

Solution:

$15\sqrt{2}$

Exercise:

Problem: $\sqrt{252}$

Exercise:

Problem: $\sqrt{800}$

Solution:

$20\sqrt{2}$

Exercise:

Problem: $\sqrt{288}$

Exercise:

Problem: $\sqrt{675}$

Solution:

$15\sqrt{3}$

Exercise:

Problem: $\sqrt{1250}$

Exercise:

Problem: $\sqrt{x^7}$

Solution:

$x^3\sqrt{x}$

Exercise:

Problem: $\sqrt{y^{11}}$

Exercise:

Problem: $\sqrt{p^3}$

Solution:

$p\sqrt{p}$

Exercise:

Problem: $\sqrt{q^5}$

Exercise:

Problem: $\sqrt{m^{13}}$

Solution:

$m^6\sqrt{m}$

Exercise:

Problem: $\sqrt{n^{21}}$

Exercise:

Problem: $\sqrt{r^{25}}$

Solution:

$r^{12}\sqrt{r}$

Exercise:

Problem: $\sqrt{s^{33}}$

Exercise:

Problem: $\sqrt{49n^{17}}$

Solution:

$7n^{8}\sqrt{n}$

Exercise:

Problem: $\sqrt{25m^{9}}$

Exercise:

Problem: $\sqrt{81r^{15}}$

Solution:

$9r^{7}\sqrt{r}$

Exercise:

Problem: $\sqrt{100s^{19}}$

Exercise:

Problem: $\sqrt{98m^{5}}$

Solution:

$7m^{2}\sqrt{2m}$

Exercise:

Problem: $\sqrt{32n^{11}}$

Exercise:

Problem: $\sqrt{125r^{13}}$

Solution:

$5r^{6}\sqrt{5r}$

Exercise:

Problem: $\sqrt{80s^{15}}$

Exercise:

Problem: $\sqrt{200p^{13}}$

Solution:

$10p^{6}\sqrt{2p}$

Exercise:

Problem: $\sqrt{128q^3}$

Exercise:

Problem: $\sqrt{242m^{23}}$

Solution:

$11m^{11}\sqrt{2m}$

Exercise:

Problem: $\sqrt{175n^{13}}$

Exercise:

Problem: $\sqrt{147m^7n^{11}}$

Solution:

$7m^3n^5\sqrt{3mn}$

Exercise:

Problem: $\sqrt{48m^7n^5}$

Exercise:

Problem: $\sqrt{75r^{13}s^9}$

Solution:

$5r^6s^4\sqrt{3rs}$

Exercise:

Problem: $\sqrt{96r^3s^3}$

Exercise:

Problem: $\sqrt{300p^9q^{11}}$

Solution:

$10p^4q^5\sqrt{3pq}$

Exercise:

Problem: $\sqrt{192q^3r^7}$

Exercise:

Problem: $\sqrt{242m^{13}n^{21}}$

Solution:

$11m^6n^{10}\sqrt{2mn}$

Exercise:

Problem: $\sqrt{150m^9n^3}$

Exercise:

Problem: $5+\sqrt{12}$

Solution:

$5+2\sqrt{3}$

Exercise:

Problem: $8+\sqrt{96}$

Exercise:

Problem: $1+\sqrt{45}$

Solution:

$1+3\sqrt{5}$

Exercise:

Problem: $3+\sqrt{125}$

Exercise:

Problem: $\frac{10-\sqrt{24}}{2}$

Solution:

$5-\sqrt{6}$

Exercise:

Problem: $\frac{8-\sqrt{80}}{4}$

Exercise:

Problem: $\frac{3+\sqrt{90}}{3}$

Solution:

$1+\sqrt{10}$

Exercise:

Problem: $\frac{15+\sqrt{75}}{5}$

Use the Quotient Property to Simplify Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{\frac{49}{64}}$

Solution:

$\frac{7}{8}$

Exercise:

Problem: $\sqrt{\frac{100}{36}}$

Exercise:

Problem: $\sqrt{\frac{121}{16}}$

Solution:

$\frac{11}{4}$

Exercise:

Problem: $\sqrt{\frac{144}{169}}$

Exercise:

Problem: $\sqrt{\frac{72}{98}}$

Solution:

$\frac{6}{7}$

Exercise:

Problem: $\sqrt{\frac{75}{12}}$

Exercise:

Problem: $\sqrt{\frac{45}{125}}$

Solution:

$\frac{3}{5}$

Exercise:

Problem: $\sqrt{\frac{300}{243}}$

Exercise:

Problem: $\sqrt{\frac{x^{10}}{x^6}}$

Solution:

x^2

Exercise:

Problem: $\sqrt{\frac{p^{20}}{p^{10}}}$

Exercise:

Problem: $\sqrt{\frac{y^4}{y^8}}$

Solution:

$\frac{1}{y^2}$

Exercise:

Problem: $\sqrt{\frac{q^8}{q^{14}}}$

Exercise:

Problem: $\sqrt{\frac{200x^7}{2x^3}}$

Solution:

$10x^2$

Exercise:

Problem: $\sqrt{\frac{98y^{11}}{2y^5}}$

Exercise:

Problem: $\sqrt{\frac{96p^9}{6p}}$

Solution:

$4p^4$

Exercise:

Problem: $\sqrt{\frac{108q^{10}}{3q^2}}$

Exercise:

Problem: $\sqrt{\frac{36}{35}}$

Solution:

$\frac{6}{\sqrt{35}}$

Exercise:

Problem: $\sqrt{\frac{144}{65}}$

Exercise:

Problem: $\sqrt{\frac{20}{81}}$

Solution:

$\frac{2\sqrt{5}}{9}$

Exercise:

Problem: $\sqrt{\frac{21}{196}}$

Exercise:

Problem: $\sqrt{\frac{96x^7}{121}}$

Solution:

$\frac{4x^3\sqrt{6x}}{11}$

Exercise:

Problem: $\sqrt{\frac{108y^4}{49}}$

Exercise:

Problem: $\sqrt{\frac{300m^5}{64}}$

Solution:

$\frac{10m^2\sqrt{3m}}{8}$

Exercise:

Problem: $\sqrt{\frac{125n^7}{169}}$

Exercise:

Problem: $\sqrt{\frac{98r^5}{100}}$

Solution:

$\frac{7r^2\sqrt{2r}}{10}$

Exercise:

Problem: $\sqrt{\frac{180s^{10}}{144}}$

Exercise:

Problem: $\sqrt{\frac{28q^6}{225}}$

Solution:

$\frac{2q^3\sqrt{7}}{15}$

Exercise:

Problem: $\sqrt{\frac{150r^3}{256}}$

Exercise:

Problem: $\sqrt{\frac{75r^9}{s^8}}$

Solution:

$\frac{5r^4\sqrt{3r}}{s^4}$

Exercise:

Problem: $\sqrt{\frac{72x^5}{y^6}}$

Exercise:

Problem: $\sqrt{\frac{28p^7}{q^2}}$

Solution:

$\frac{2p^3\sqrt{7p}}{q}$

Exercise:

Problem: $\sqrt{\frac{45r^3}{s^{10}}}$

Exercise:

Problem: $\sqrt{\frac{100x^5}{36x^3}}$

Solution:

$\frac{5x}{3}$

Exercise:

Problem: $\sqrt{\frac{49r^{12}}{16r^6}}$

Exercise:

Problem: $\sqrt{\frac{121p^5}{81p^2}}$

Solution:

$\frac{11p\sqrt{p}}{9}$

Exercise:

Problem: $\sqrt{\frac{25r^8}{64r}}$

Exercise:

Problem: $\sqrt{\frac{32x^5y^3}{18x^3y}}$

Solution:

$\frac{4xy}{3}$

Exercise:

Problem: $\sqrt{\frac{75r^6s^8}{48rs^4}}$

Exercise:

Problem: $\sqrt{\frac{27p^2q}{108p^5q^3}}$

Solution:

$\frac{1}{2pq\sqrt{p}}$

Exercise:

Problem: $\sqrt{\frac{50r^5s^2}{128r^2s^5}}$

Everyday Math

Exercise:

Problem:

ⓐ Elliott decides to construct a square garden that will take up 288 square feet of his yard. Simplify $\sqrt{288}$ to determine the length and the width of his garden. Round to the nearest tenth of a foot.
ⓑ Suppose Elliott decides to reduce the size of his square garden so that he can create a 5-foot-wide walking path on the north and east sides of the garden. Simplify $\sqrt{288} - 5$ to determine the length and width of the new garden. Round to the nearest tenth of a foot.

Solution:

ⓐ 17.0 feet ⓑ 12.0 feet

Exercise:

Problem:

ⓐ Melissa accidentally drops a pair of sunglasses from the top of a roller coaster, 64 feet above the ground. Simplify $\sqrt{\frac{64}{16}}$ to determine the number of seconds it takes for the sunglasses to reach the ground.
ⓑ Suppose the sunglasses in the previous example were dropped from a height of 144 feet. Simplify $\sqrt{\frac{144}{16}}$ to determine the number of seconds it takes for the sunglasses to reach the ground.

Writing Exercises

Exercise:

Problem: Explain why $\sqrt{x^4} = x^2$. Then explain why $\sqrt{x^{16}} = x^8$.

Solution:

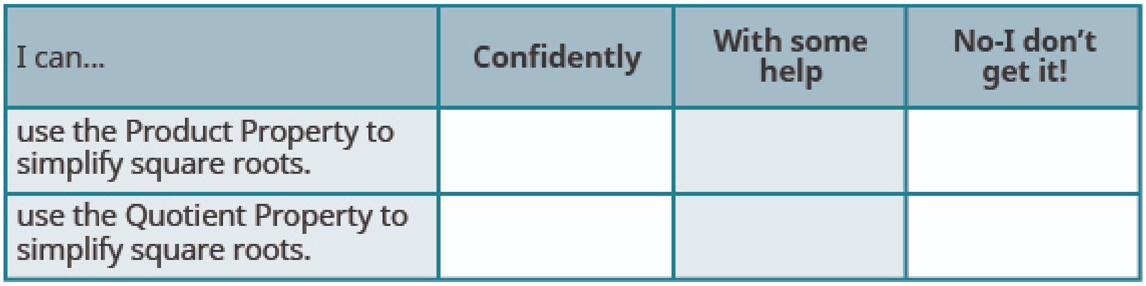
Answers will vary.

Exercise:

Problem: Explain why $7 + \sqrt{9}$ is not equal to $\sqrt{7 + 9}$.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
use the Product Property to simplify square roots.			
use the Quotient Property to simplify square roots.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Add and Subtract Square Roots
By the end of this section, you will be able to:

- Add and subtract like square roots
- Add and subtract square roots that need simplification

Note:
Before you get started, take this readiness quiz.

1. Add: ⓐ $3x+9x$ ⓑ $5m+5n$.
   If you missed this problem, review [link].
2. Simplify: $\sqrt{50x^3}$.
   If you missed this problem, review [link].

We know that we must follow the order of operations to simplify expressions with square roots. The radical is a grouping symbol, so we work inside the radical first. We simplify $\sqrt{2+7}$ in this way:

	$\sqrt{2+7}$
Add inside the radical.	$\sqrt{9}$
Simplify.	3

So if we have to add $\sqrt{2}+\sqrt{7}$, we must not combine them into one radical.
Equation:

$$\sqrt{2}+\sqrt{7}\neq\sqrt{2+7}$$

Trying to add square roots with different radicands is like trying to add unlike terms.

But, just like we can add	$x+x,$	we can add	$\sqrt{3}+\sqrt{3}.$
	$x+x=2x$		$\sqrt{3}+\sqrt{3}=2\sqrt{3}$

Adding square roots with the same radicand is just like adding like terms. We call square roots with the same radicand like square roots to remind us they work the same as like terms.

Note:
Like Square Roots
Square roots with the same radicand are called **like square roots**.

We add and subtract like square roots in the same way we add and subtract like terms. We know that $3x + 8x$ is $11x$. Similarly we add $3\sqrt{x} + 8\sqrt{x}$ and the result is $11\sqrt{x}$.

Add and Subtract Like Square Roots

Think about adding like terms with variables as you do the next few examples. When you have like radicands, you just add or subtract the coefficients. When the radicands are not like, you cannot combine the terms.

Example:
Exercise:

Problem: Simplify: $2\sqrt{2} - 7\sqrt{2}$.

Solution:
Solution

	$2\sqrt{2} - 7\sqrt{2}$
Since the radicals are like, we subtract the coefficients.	$-5\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $8\sqrt{2} - 9\sqrt{2}$.

Solution:

$-\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $5\sqrt{3} - 9\sqrt{3}$.

Solution:

$-4\sqrt{3}$

Example:
Exercise:

Problem: Simplify: $3\sqrt{y} + 4\sqrt{y}$.

Solution:
Solution

	$3\sqrt{y} + 4\sqrt{y}$
Since the radicals are like, we add the coefficients.	$7\sqrt{y}$

Note:
Exercise:

Problem: Simplify: $2\sqrt{x} + 7\sqrt{x}$.

Solution:

$9\sqrt{x}$

Note:
Exercise:

Problem: Simplify: $5\sqrt{u} + 3\sqrt{u}$.

Solution:

$8\sqrt{u}$

Example:
Exercise:

Problem: Simplify: $4\sqrt{x} - 2\sqrt{y}$.

Solution:
Solution

	$4\sqrt{x} - 2\sqrt{y}$
Since the radicals are not like, we cannot subtract them. We leave the expression as is.	$4\sqrt{x} - 2\sqrt{y}$

Note:
Exercise:

Problem: Simplify: $7\sqrt{p} - 6\sqrt{q}$.

Solution:

$7\sqrt{p} - 6\sqrt{q}$

Note:
Exercise:

Problem: Simplify: $6\sqrt{a} - 3\sqrt{b}$.

Solution:

$6\sqrt{a} - 3\sqrt{b}$

Example:
Exercise:

Problem: Simplify: $5\sqrt{13} + 4\sqrt{13} + 2\sqrt{13}$.

Solution:
Solution

	$5\sqrt{13} + 4\sqrt{13} + 2\sqrt{13}$
Since the radicals are like, we add the coefficients.	$11\sqrt{13}$

Note:
Exercise:

Problem: Simplify: $4\sqrt{11} + 2\sqrt{11} + 3\sqrt{11}$.

Solution:

$9\sqrt{11}$

Note:
Exercise:

Problem: Simplify: $6\sqrt{10} + 2\sqrt{10} + 3\sqrt{10}$.

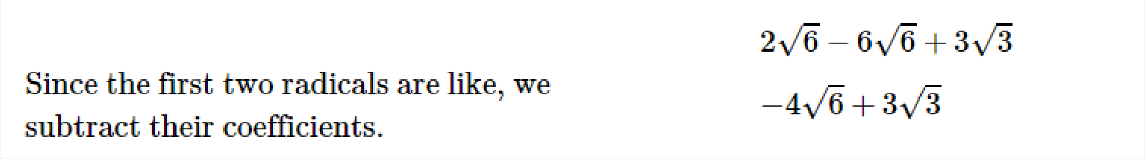
Solution:

$11\sqrt{10}$

Example:
Exercise:

Problem: Simplify: $2\sqrt{6} - 6\sqrt{6} + 3\sqrt{3}$.

Solution:
Solution



	$2\sqrt{6} - 6\sqrt{6} + 3\sqrt{3}$
Since the first two radicals are like, we subtract their coefficients.	$-4\sqrt{6} + 3\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $5\sqrt{5} - 4\sqrt{5} + 2\sqrt{6}$.

Solution:

$\sqrt{5} + 2\sqrt{6}$

Note:
Exercise:

Problem: Simplify: $3\sqrt{7} - 8\sqrt{7} + 2\sqrt{5}$.

Solution:

$-5\sqrt{7} + 2\sqrt{5}$

Example:
Exercise:

Problem: Simplify: $2\sqrt{5n} - 6\sqrt{5n} + 4\sqrt{5n}$.

Solution:
Solution

	$2\sqrt{5n} - 6\sqrt{5n} + 4\sqrt{5n}$
Since the radicals are like, we combine them.	$0\sqrt{5n}$
Simplify.	0

Note:
Exercise:

Problem: Simplify: $\sqrt{7x} - 7\sqrt{7x} + 4\sqrt{7x}$.

Solution:

$-2\sqrt{7x}$

Note:
Exercise:

Problem: Simplify: $4\sqrt{3y} - 7\sqrt{3y} + 2\sqrt{3y}$.

Solution:

$-\sqrt{3y}$

When radicals contain more than one variable, as long as all the variables and their exponents are identical, the radicals are like.

Example:
Exercise:

Problem: Simplify: $\sqrt{3xy} + 5\sqrt{3xy} - 4\sqrt{3xy}$.

Solution:
Solution

	$\sqrt{3xy} + 5\sqrt{3xy} - 4\sqrt{3xy}$
Since the radicals are like, we combine them.	$2\sqrt{3xy}$

Note:
Exercise:

Problem: Simplify: $\sqrt{5xy} + 4\sqrt{5xy} - 7\sqrt{5xy}$.

Solution:

$-2\sqrt{5xy}$

Note:
Exercise:

Problem: Simplify: $3\sqrt{7mn} + \sqrt{7mn} - 4\sqrt{7mn}$.

Solution:

0

Add and Subtract Square Roots that Need Simplification

Remember that we always simplify square roots by removing the largest perfect-square factor. Sometimes when we have to add or subtract square roots that do not appear to have like radicals, we find like radicals after simplifying the square roots.

Example:
Exercise:

Problem: Simplify: $\sqrt{20}+3\sqrt{5}$.

Solution:
Solution

	$\sqrt{20}+3\sqrt{5}$
Simplify the radicals, when possible.	$\sqrt{4}\cdot\sqrt{5}+3\sqrt{5}$
	$2\sqrt{5}+3\sqrt{5}$
Combine the like radicals.	$5\sqrt{5}$

Note:
Exercise:

Problem: Simplify: $\sqrt{18}+6\sqrt{2}$.

Solution:

$9\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $\sqrt{27}+4\sqrt{3}$.

Solution:

$7\sqrt{3}$

Example:
Exercise:

Problem: Simplify: $\sqrt{48} - \sqrt{75}$.

Solution:
Solution

	$\sqrt{48} - \sqrt{75}$
Simplify the radicals.	$\sqrt{16} \cdot \sqrt{3} - \sqrt{25} \cdot \sqrt{3}$
	$4\sqrt{3} - 5\sqrt{3}$
Combine the like radicals.	$-\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\sqrt{32} - \sqrt{18}$.

Solution:

$\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $\sqrt{20} - \sqrt{45}$.

Solution:

$-\sqrt{5}$

Just like we use the Associative Property of Multiplication to simplify $5\left(3x\right)$ and get $15x$, we can simplify $5\left(3\sqrt{x}\right)$ and get $15\sqrt{x}$. We will use the Associative Property to do this in the next example.

Example:
Exercise:

Problem: Simplify: $5\sqrt{18}-2\sqrt{8}$.

Solution:
Solution

	$5\sqrt{18}-2\sqrt{8}$
Simplify the radicals.	$5\cdot\sqrt{9}\cdot\sqrt{2}-2\cdot\sqrt{4}\cdot\sqrt{2}$
	$5\cdot3\cdot\sqrt{2}-2\cdot2\cdot\sqrt{2}$
	$15\sqrt{2}-4\sqrt{2}$
Combine the like radicals.	$11\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $4\sqrt{27}-3\sqrt{12}$.

Solution:

$6\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $3\sqrt{20}-7\sqrt{45}$.

Solution:

$-15\sqrt{5}$

Example:
Exercise:

Problem: Simplify: $\frac{3}{4}\sqrt{192}-\frac{5}{6}\sqrt{108}$.

Solution:
Solution

	$\frac{3}{4}\sqrt{192}-\frac{5}{6}\sqrt{108}$
Simplify the radicals.	$\frac{3}{4}\sqrt{64}\cdot\sqrt{3}-\frac{5}{6}\sqrt{36}\cdot\sqrt{3}$
	$\frac{3}{4}\cdot 8\cdot\sqrt{3}-\frac{5}{6}\cdot 6\cdot\sqrt{3}$
	$6\sqrt{3}-5\sqrt{3}$
Combine the like radicals.	$\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\frac{2}{3}\sqrt{108}-\frac{5}{7}\sqrt{147}$.

Solution:

$-\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\frac{3}{5}\sqrt{200} - \frac{3}{4}\sqrt{128}$.

Solution:

0

Example:
Exercise:

Problem: Simplify: $\frac{2}{3}\sqrt{48} - \frac{3}{4}\sqrt{12}$.

Solution:
Solution

	$\frac{2}{3}\sqrt{48} - \frac{3}{4}\sqrt{12}$
Simplify the radicals.	$\frac{2}{3}\sqrt{16}\cdot\sqrt{3} - \frac{3}{4}\sqrt{4}\cdot\sqrt{3}$ $\frac{2}{3}\cdot 4\cdot\sqrt{3} - \frac{3}{4}\cdot 2\cdot\sqrt{3}$ $\frac{8}{3}\sqrt{3} - \frac{3}{2}\sqrt{3}$
Find a common denominator to subtract the coefficients of the like radicals.	$\frac{16}{6}\sqrt{3} - \frac{9}{6}\sqrt{3}$
Simplify.	$\frac{7}{6}\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\frac{2}{5}\sqrt{32} - \frac{1}{3}\sqrt{8}$.

Solution:

$\frac{14}{15}\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $\frac{1}{3}\sqrt{80} - \frac{1}{4}\sqrt{125}$.

Solution:

$\frac{1}{12}\sqrt{5}$

In the next example, we will remove constant and variable factors from the square roots.

Example:
Exercise:

Problem: Simplify: $\sqrt{18n^5} - \sqrt{32n^5}$.

Solution:
Solution

	$\sqrt{18n^5} - \sqrt{32n^5}$
Simplify the radicals.	$\sqrt{9n^4} \cdot \sqrt{2n} - \sqrt{16n^4} \cdot \sqrt{2n}$ $3n^2\sqrt{2n} - 4n^2\sqrt{2n}$
Combine the like radicals.	$-n^2\sqrt{2n}$

Note:
Exercise:

Problem: Simplify: $\sqrt{32m^7} - \sqrt{50m^7}$.

Solution:

$-m^3\sqrt{2m}$

Note:
Exercise:

Problem: Simplify: $\sqrt{27p^3} - \sqrt{48p^3}$.

Solution:

$-p\sqrt{3p}$

Example:
Exercise:

Problem: Simplify: $9\sqrt{50m^2} - 6\sqrt{48m^2}$.

Solution:
Solution

	$9\sqrt{50m^2} - 6\sqrt{48m^2}$
Simplify the radicals.	$9\sqrt{25m^2} \cdot \sqrt{2} - 6\sqrt{16m^2} \cdot \sqrt{3}$
	$9 \cdot 5m \cdot \sqrt{2} - 6 \cdot 4m \cdot \sqrt{3}$
	$45m\sqrt{2} - 24m\sqrt{3}$
The radicals are not like and so cannot be combined.	

Note:
Exercise:

Problem: Simplify: $5\sqrt{32x^2} - 3\sqrt{48x^2}$.

Solution:

$20x\sqrt{2} - 12x\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $7\sqrt{48y^2} - 4\sqrt{72y^2}$.

Solution:

$28y\sqrt{3} - 24y\sqrt{2}$

Example:
Exercise:

Problem: Simplify: $2\sqrt{8x^2} - 5x\sqrt{32} + 5\sqrt{18x^2}$.

Solution:
Solution

	$2\sqrt{8x^2} - 5x\sqrt{32} + 5\sqrt{18x^2}$
Simplify the radicals.	$2\sqrt{4x^2} \cdot \sqrt{2} - 5x\sqrt{16} \cdot \sqrt{2} + 5\sqrt{9x^2} \cdot \sqrt{2}$
	$2 \cdot 2x \cdot \sqrt{2} - 5x \cdot 4 \cdot \sqrt{2} + 5 \cdot 3x \cdot \sqrt{2}$
	$4x\sqrt{2} - 20x\sqrt{2} + 15x\sqrt{2}$
Combine the like radicals.	$-x\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $3\sqrt{12x^2} - 2x\sqrt{48} + 4\sqrt{27x^2}$.

Solution:

$10x\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $3\sqrt{18x^2} - 6x\sqrt{32} + 2\sqrt{50x^2}$.

Solution:

$-5x\sqrt{2}$

Note:
Access this online resource for additional instruction and practice with the adding and subtracting square roots.

- Adding/Subtracting Square Roots

Key Concepts

- To add or subtract like square roots, add or subtract the coefficients and keep the like square root.
- Sometimes when we have to add or subtract square roots that do not appear to have like radicals, we find like radicals after simplifying the square roots.

Practice Makes Perfect

Add and Subtract Like Square Roots

In the following exercises, simplify.

Exercise:

Problem: $8\sqrt{2}-5\sqrt{2}$

Solution:

$3\sqrt{2}$

Exercise:

Problem: $7\sqrt{2}-3\sqrt{2}$

Exercise:

Problem: $3\sqrt{5}+6\sqrt{5}$

Solution:

$9\sqrt{5}$

Exercise:

Problem: $4\sqrt{5}+8\sqrt{5}$

Exercise:

Problem: $9\sqrt{7}-10\sqrt{7}$

Solution:

$-\sqrt{7}$

Exercise:

Problem: $11\sqrt{7}-12\sqrt{7}$

Exercise:

Problem: $7\sqrt{y}+2\sqrt{y}$

Solution:

$9\sqrt{y}$

Exercise:

Problem: $9\sqrt{n} + 3\sqrt{n}$

Exercise:

Problem: $\sqrt{a} - 4\sqrt{a}$

Solution:

$-3\sqrt{a}$

Exercise:

Problem: $\sqrt{b} - 6\sqrt{b}$

Exercise:

Problem: $5\sqrt{c} + 2\sqrt{c}$

Solution:

$7\sqrt{c}$

Exercise:

Problem: $7\sqrt{d} + 2\sqrt{d}$

Exercise:

Problem: $8\sqrt{a} - 2\sqrt{b}$

Solution:

$6\sqrt{b}$

Exercise:

Problem: $5\sqrt{c} - 3\sqrt{d}$

Exercise:

Problem: $5\sqrt{m}+\sqrt{n}$

Solution:

$5\sqrt{m}+\sqrt{n}$

Exercise:

Problem: $\sqrt{n}+3\sqrt{p}$

Exercise:

Problem: $8\sqrt{7}+2\sqrt{7}+3\sqrt{7}$

Solution:

$13\sqrt{7}$

Exercise:

Problem: $6\sqrt{5}+3\sqrt{5}+\sqrt{5}$

Exercise:

Problem: $3\sqrt{11}+2\sqrt{11}-8\sqrt{11}$

Solution:

$-3\sqrt{11}$

Exercise:

Problem: $2\sqrt{15}+5\sqrt{15}-9\sqrt{15}$

Exercise:

Problem: $3\sqrt{3}-8\sqrt{3}+7\sqrt{5}$

Solution:

$-5\sqrt{3}+7\sqrt{5}$

Exercise:

Problem: $5\sqrt{7} - 8\sqrt{7} + 6\sqrt{3}$

Exercise:

Problem: $6\sqrt{2} + 2\sqrt{2} - 3\sqrt{5}$

Solution:

$8\sqrt{2} - 3\sqrt{5}$

Exercise:

Problem: $7\sqrt{5} + \sqrt{5} - 8\sqrt{10}$

Exercise:

Problem: $3\sqrt{2a} - 4\sqrt{2a} + 5\sqrt{2a}$

Solution:

$4\sqrt{2a}$

Exercise:

Problem: $\sqrt{11b} - 5\sqrt{11b} + 3\sqrt{11b}$

Exercise:

Problem: $8\sqrt{3c} + 2\sqrt{3c} - 9\sqrt{3c}$

Solution:

$\sqrt{3c}$

Exercise:

Problem: $3\sqrt{5d} + 8\sqrt{5d} - 11\sqrt{5d}$

Exercise:

Problem: $5\sqrt{3ab} + \sqrt{3ab} - 2\sqrt{3ab}$

Solution:

$4\sqrt{3ab}$

Exercise:

Problem: $8\sqrt{11cd}+5\sqrt{11cd}-9\sqrt{11cd}$

Exercise:

Problem: $2\sqrt{pq}-5\sqrt{pq}+4\sqrt{pq}$

Solution:

$\sqrt{pq}$

Exercise:

Problem: $11\sqrt{2rs}-9\sqrt{2rs}+3\sqrt{2rs}$

Add and Subtract Square Roots that Need Simplification

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{50}+4\sqrt{2}$

Solution:

$9\sqrt{2}$

Exercise:

Problem: $\sqrt{48}+2\sqrt{3}$

Exercise:

Problem: $\sqrt{80}-3\sqrt{5}$

Solution:

$\sqrt{5}$

Exercise:

Problem: $\sqrt{28} - 4\sqrt{7}$

Exercise:

Problem: $\sqrt{27} - \sqrt{75}$

Solution:

$-2\sqrt{3}$

Exercise:

Problem: $\sqrt{72} - \sqrt{98}$

Exercise:

Problem: $\sqrt{48} + \sqrt{27}$

Solution:

$7\sqrt{3}$

Exercise:

Problem: $\sqrt{45} + \sqrt{80}$

Exercise:

Problem: $2\sqrt{50} - 3\sqrt{72}$

Solution:

$-8\sqrt{2}$

Exercise:

Problem: $3\sqrt{98} - \sqrt{128}$

Exercise:

Problem: $2\sqrt{12} + 3\sqrt{48}$

Solution:

$16\sqrt{3}$

Exercise:

Problem: $4\sqrt{75} + 2\sqrt{108}$

Exercise:

Problem: $\frac{2}{3}\sqrt{72} + \frac{1}{5}\sqrt{50}$

Solution:

$5\sqrt{2}$

Exercise:

Problem: $\frac{2}{5}\sqrt{75} + \frac{3}{4}\sqrt{48}$

Exercise:

Problem: $\frac{1}{2}\sqrt{20} - \frac{2}{3}\sqrt{45}$

Solution:

$-\sqrt{5}$

Exercise:

Problem: $\frac{2}{3}\sqrt{54} - \frac{3}{4}\sqrt{96}$

Exercise:

Problem: $\frac{1}{6}\sqrt{27} - \frac{3}{8}\sqrt{48}$

Solution:

$-\sqrt{3}$

Exercise:

Problem: $\frac{1}{8}\sqrt{32} - \frac{1}{10}\sqrt{50}$

Exercise:

Problem: $\frac{1}{4}\sqrt{98} - \frac{1}{3}\sqrt{128}$

Solution:

$-\frac{11}{12}\sqrt{2}$

Exercise:

Problem: $\frac{1}{3}\sqrt{24} + \frac{1}{4}\sqrt{54}$

Exercise:

Problem: $\sqrt{72a^5} - \sqrt{50a^5}$

Solution:

$-a^2\sqrt{2a}$

Exercise:

Problem: $\sqrt{48b^5} - \sqrt{75b^5}$

Exercise:

Problem: $\sqrt{80c^7} - \sqrt{20c^7}$

Solution:

$2c^3\sqrt{5c}$

Exercise:

Problem: $\sqrt{96d^9} - \sqrt{24d^9}$

Exercise:

Problem: $9\sqrt{80p^4} - 6\sqrt{98p^4}$

Solution:

$36p^2\sqrt{5} - 42p^2\sqrt{2}$

Exercise:

Problem: $8\sqrt{72q^6} - 3\sqrt{75q^6}$

Exercise:

Problem: $2\sqrt{50r^8} + 4\sqrt{54r^8}$

Solution:

$10r^4\sqrt{2} + 12r^4\sqrt{6}$

Exercise:

Problem: $5\sqrt{27s^6} + 2\sqrt{20s^6}$

Exercise:

Problem: $3\sqrt{20x^2} - 4\sqrt{45x^2} + 5x\sqrt{80}$

Solution:

$14x\sqrt{5}$

Exercise:

Problem: $2\sqrt{28x^2} - \sqrt{63x^2} + 6x\sqrt{7}$

Exercise:

Problem: $3\sqrt{128y^2} + 4y\sqrt{162} - 8\sqrt{98y^2}$

Solution:

$4y\sqrt{2}$

Exercise:

Problem: $3\sqrt{75y^2} + 8y\sqrt{48} - \sqrt{300y^2}$

Mixed Practice

Exercise:

Problem: $2\sqrt{8} + 6\sqrt{8} - 5\sqrt{8}$

Solution:

$3\sqrt{8}$

Exercise:

Problem: $\frac{2}{3}\sqrt{27} + \frac{3}{4}\sqrt{48}$

Exercise:

Problem: $\sqrt{175k^4} - \sqrt{63k^4}$

Solution:

$2k^2\sqrt{7}$

Exercise:

Problem: $\frac{5}{6}\sqrt{162} + \frac{3}{16}\sqrt{128}$

Exercise:

Problem: $2\sqrt{363} - 2\sqrt{300}$

Solution:

$2\sqrt{3}$

Exercise:

Problem: $\sqrt{150} + 4\sqrt{6}$

Exercise:

Problem: $9\sqrt{2}-8\sqrt{2}$

Solution:

$\sqrt{2}$

Exercise:

Problem: $5\sqrt{x}-8\sqrt{y}$

Exercise:

Problem: $8\sqrt{13}-4\sqrt{13}-3\sqrt{13}$

Solution:

$\sqrt{13}$

Exercise:

Problem: $5\sqrt{12c^4}-3\sqrt{27c^6}$

Exercise:

Problem: $\sqrt{80a^5}-\sqrt{45a^5}$

Solution:

$a^2\sqrt{5a}$

Exercise:

Problem: $\frac{3}{5}\sqrt{75}-\frac{1}{4}\sqrt{48}$

Exercise:

Problem: $21\sqrt{19}-2\sqrt{19}$

Solution:

$19\sqrt{19}$

Exercise:

Problem: $\sqrt{500}+\sqrt{405}$

Exercise:

Problem: $\frac{5}{6}\sqrt{27}+\frac{5}{8}\sqrt{48}$

Solution:

$5\sqrt{3}$

Exercise:

Problem: $11\sqrt{11}-10\sqrt{11}$

Exercise:

Problem: $\sqrt{75}-\sqrt{108}$

Solution:

$-\sqrt{3}$

Exercise:

Problem: $2\sqrt{98}-4\sqrt{72}$

Exercise:

Problem: $4\sqrt{24x^2}-\sqrt{54x^2}+3x\sqrt{6}$

Solution:

$8x\sqrt{6}$

Exercise:

Problem: $8\sqrt{80y^6}-6\sqrt{48y^6}$

Everyday Math

Exercise:

Problem:

A decorator decides to use square tiles as an accent strip in the design of a new shower, but she wants to rotate the tiles to look like diamonds. She will use 9 large tiles that measure 8 inches on a side and 8 small tiles that measure 2 inches on a side. Determine the width of the accent strip by simplifying the expression $9\left(8\sqrt{2}\right)+8\left(2\sqrt{2}\right)$. (Round to the nearest tenth of an inch.)

Solution:

124.5 inches

Exercise:

Problem:

Suzy wants to use square tiles on the border of a spa she is installing in her backyard. She will use large tiles that have area of 12 square inches, medium tiles that have area of 8 square inches, and small tiles that have area of 4 square inches. Once section of the border will require 4 large tiles, 8 medium tiles, and 10 small tiles to cover the width of the wall. Simplify the expression $4\sqrt{12}+8\sqrt{8}+10\sqrt{4}$ to determine the width of the wall.

Writing Exercises

Exercise:

Problem:

Explain the difference between like radicals and unlike radicals. Make sure your answer makes sense for radicals containing both numbers and variables.

Solution:

Answers will vary.

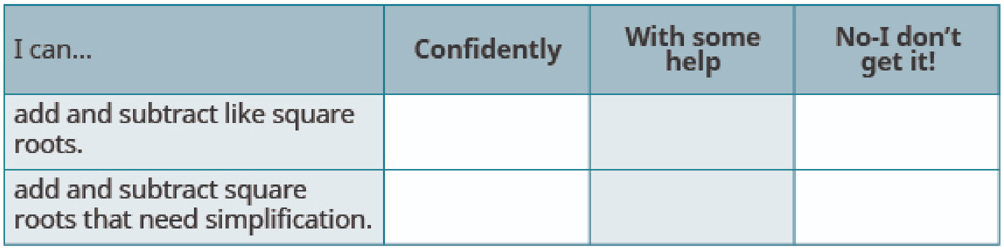
Exercise:

Problem:

Explain the process for determining whether two radicals are like or unlike. Make sure your answer makes sense for radicals containing both numbers and variables.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don't get it!
add and subtract like square roots.			
add and subtract square roots that need simplification.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

like square roots

Square roots with the same radicand are called like square roots.

Multiply Square Roots
By the end of this section, you will be able to:

- Multiply square roots
- Use polynomial multiplication to multiply square roots

Note:
Before you get started, take this readiness quiz.

1. Simplify: $(3u)(8v)$.
   If you missed this problem, review [link].
2. Simplify: $6(12-7n)$.
   If you missed this problem, review [link].
3. Simplify: $(2+a)(4-a)$.
   If you missed this problem, review [link].

Multiply Square Roots

We have used the Product Property of Square Roots to simplify square roots by removing the perfect square factors. The Product Property of Square Roots says
Equation:

$$\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}$$

We can use the Product Property of Square Roots ‘in reverse’ to multiply square roots.
Equation:

$$\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$$

Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Square Roots so we see both ways together.

Note:
Product Property of Square Roots
If *a*, *b* are nonnegative real numbers, then
Equation:

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \quad \text{and} \quad \sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

So we can multiply $\sqrt{3} \cdot \sqrt{5}$ in this way:
Equation:

$$\begin{gathered}\sqrt{3} \cdot \sqrt{5} \\ \sqrt{3 \cdot 5} \\ \sqrt{15}\end{gathered}$$

Sometimes the product gives us a perfect square:
Equation:

$$\begin{gathered}\sqrt{2} \cdot \sqrt{8} \\ \sqrt{2 \cdot 8} \\ \sqrt{16} \\ 4\end{gathered}$$

Even when the product is not a perfect square, we must look for perfect-square factors and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply $4x \cdot 3y$ we multiply the coefficients together and then the variables. The result is $12xy$. Keep this in mind as you do these examples.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{2} \cdot \sqrt{6}$ ⓑ $\left(4\sqrt{3}\right)\left(2\sqrt{12}\right)$.

Solution:
Solution

ⓐ

	$\sqrt{2} \cdot \sqrt{6}$
Multiply using the Product Property.	$\sqrt{12}$
Simplify the radical.	$\sqrt{4} \cdot \sqrt{3}$
Simplify.	$2\sqrt{3}$

ⓑ

	$\left(4\sqrt{3}\right)\left(2\sqrt{12}\right)$
Multiply using the Product Property.	$8\sqrt{36}$
Simplify the radical.	$8 \cdot 6$
Simplify.	48

Notice that in (b) we multiplied the coefficients and multiplied the radicals. Also, we did not simplify $\sqrt{12}$. We waited to get the product and then simplified.

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{3}\cdot\sqrt{6}$ ⓑ $\left(2\sqrt{6}\right)\left(3\sqrt{12}\right)$.

Solution:

ⓐ $3\sqrt{2}$ ⓑ $36\sqrt{2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{5}\cdot\sqrt{10}$ ⓑ $\left(6\sqrt{3}\right)\left(5\sqrt{6}\right)$.

Solution:

ⓐ $5\sqrt{2}$ ⓑ $90\sqrt{2}$

Example:
Exercise:

Problem: Simplify: $\left(6\sqrt{2}\right)\left(3\sqrt{10}\right)$.

Solution:
Solution

	$\left(6\sqrt{2}\right)\left(3\sqrt{10}\right)$
Multiply using the Product Property.	$18\sqrt{20}$
Simplify the radical.	$18\sqrt{4}\cdot\sqrt{5}$
Simplify.	$18\cdot 2\cdot\sqrt{5}$ $36\sqrt{5}$

Note:
Exercise:

Problem: Simplify: $\left(3\sqrt{2}\right)\left(2\sqrt{30}\right)$.

Solution:

$12\sqrt{15}$

Note:
Exercise:

Problem: Simplify: $\left(3\sqrt{3}\right)\left(3\sqrt{6}\right)$.

Solution:

$27\sqrt{2}$

When we have to multiply square roots, we first find the product and then remove any perfect square factors.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(\sqrt{8x^3}\right)\left(\sqrt{3x}\right)$ ⓑ $\left(\sqrt{20y^2}\right)\left(\sqrt{5y^3}\right)$.

Solution:
Solution

ⓐ

	$\left(\sqrt{8x^3}\right)\left(\sqrt{3x}\right)$
Multiply using the Product Property.	$\sqrt{24x^4}$
Simplify the radical.	$\sqrt{4x^4}\cdot\sqrt{6}$
Simplify.	$2x^2\sqrt{6}$

ⓑ

	$\left(\sqrt{20y^2}\right)\left(\sqrt{5y^3}\right)$
Multiply using the Product Property.	$\sqrt{100y^5}$
Simplify the radical.	$10y^2\sqrt{y}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\sqrt{6x^3}\right)\left(\sqrt{3x}\right)$ ⓑ $\left(\sqrt{2y^3}\right)\left(\sqrt{50y^2}\right)$.

Solution:

ⓐ $3x^2\sqrt{2}$ ⓑ $10y^2\sqrt{y}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\sqrt{6x^5}\right)\left(\sqrt{2x}\right)$ ⓑ $\left(\sqrt{12y^2}\right)\left(\sqrt{3y^5}\right)$.

Solution:

ⓐ $2x^3\sqrt{3}$ ⓑ $6y^3\sqrt{y}$

Example:
Exercise:

Problem: Simplify: $\left(10\sqrt{6p^3}\right)\left(3\sqrt{18p}\right)$.

Solution:
Solution

	$\left(10\sqrt{6p^3}\right)\left(3\sqrt{18p}\right)$
Multiply.	$30\sqrt{108p^4}$
Simplify the radical.	$30\sqrt{36p^4}\cdot\sqrt{3}$
	$30\cdot 6p^2\cdot\sqrt{3}$
	$180p^2\sqrt{3}$

Note:

Exercise:

Problem: Simplify: $\left(6\sqrt{2x^2}\right)\left(8\sqrt{45x^4}\right)$.

Solution:

$144x^3\sqrt{10}$

Note:
Exercise:

Problem: Simplify: $\left(2\sqrt{6y^4}\right)\left(12\sqrt{30y}\right)$.

Solution:

$144y^2\sqrt{5y}$

Example:
Exercise:

Problem: Simplify: ⓐ $\left(\sqrt{2}\right)^2$ ⓑ $\left(-\sqrt{11}\right)^2$.

Solution:
Solution

ⓐ

$$\left(\sqrt{2}\right)^2$$

Rewrite as a product.	$\left(\sqrt{2}\right)\left(\sqrt{2}\right)$
Multiply.	$\sqrt{4}$
Simplify.	2

ⓑ

$$\left(-\sqrt{11}\right)^2$$

Rewrite as a product.	$\left(-\sqrt{11}\right)\left(-\sqrt{11}\right)$
Multiply.	$\sqrt{121}$
Simplify.	11

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\sqrt{12}\right)^2$ ⓑ $\left(-\sqrt{15}\right)^2$.

Solution:

ⓐ 12 ⓑ 15

Note:
Exercise:

Problem: Simplify: ⓐ $\left(\sqrt{16}\right)^2$ ⓑ $\left(-\sqrt{20}\right)^2$.

Solution:

ⓐ 16 ⓑ 20

The results of the previous example lead us to this property.

Note:
Squaring a Square Root
If *a* is a nonnegative real number, then
Equation:

$$\left(\sqrt{a}\right)^2 = a$$

By realizing that squaring and taking a square root are ‘opposite’ operations, we can simplify $\left(\sqrt{2}\right)^2$ and get 2 right away. When we multiply the two like square roots in part (a) of the next example, it is the same as squaring.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(2\sqrt{3}\right)\left(8\sqrt{3}\right)$ ⓑ $\left(3\sqrt{6}\right)^2$.

Solution:
Solution

ⓐ

	$\left(2\sqrt{3}\right)\left(8\sqrt{3}\right)$
Multiply. Remember, $\left(\sqrt{3}\right)^2 = 3$.	$16 \cdot 3$
Simplify.	48

ⓑ

	$\left(3\sqrt{6}\right)^2$
Multiply.	$9 \cdot 6$
Simplify.	54

Note:
Exercise:

Problem: Simplify: ⓐ $\left(6\sqrt{11}\right)\left(5\sqrt{11}\right)$ ⓑ $\left(5\sqrt{8}\right)^2$.

Solution:

ⓐ 330 ⓑ 200

Note:
Exercise:

Problem: Simplify: ⓐ $\left(3\sqrt{7}\right)\left(10\sqrt{7}\right)$ ⓑ $\left(-4\sqrt{6}\right)^2$.

Solution:

ⓐ 210 ⓑ 96

Use Polynomial Multiplication to Multiply Square Roots

In the next few examples, we will use the Distributive Property to multiply expressions with square roots.

We will first distribute and then simplify the square roots when possible.

Example:
Exercise:

Problem: Simplify: ⓐ $3\left(5-\sqrt{2}\right)$ ⓑ $\sqrt{2}\left(4-\sqrt{10}\right)$.

Solution:
Solution

ⓐ

	$3\left(5-\sqrt{2}\right)$
Distribute.	$15-3\sqrt{2}$

ⓑ

	$\sqrt{2}\left(4-\sqrt{10}\right)$
Distribute.	$4\sqrt{2}-\sqrt{20}$
	$4\sqrt{2}-2\sqrt{5}$

Note:

Exercise:

Problem: Simplify: ⓐ $2\left(3-\sqrt{5}\right)$ ⓑ $\sqrt{3}\left(2-\sqrt{18}\right)$.

Solution:

ⓐ $6-2\sqrt{5}$ ⓑ $2\sqrt{3}-3\sqrt{6}$

Note:
Exercise:

Problem: Simplify: ⓐ $6\left(2+\sqrt{6}\right)$ ⓑ $\sqrt{7}\left(1+\sqrt{14}\right)$.

Solution:

ⓐ $12+6\sqrt{6}$ ⓑ $\sqrt{7}+7\sqrt{2}$

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{5}\left(7+2\sqrt{5}\right)$ ⓑ $\sqrt{6}\left(\sqrt{2}+\sqrt{18}\right)$.

Solution:
Solution

ⓐ

$$\sqrt{5}\left(7+2\sqrt{5}\right)$$

Multiply.	$7\sqrt{5}+2\cdot 5$
Simplify.	$7\sqrt{5}+10$
	$10+7\sqrt{5}$

ⓑ

$$\sqrt{6}\left(\sqrt{2}+\sqrt{18}\right)$$

Multiply.	$\sqrt{12}+\sqrt{108}$
Simplify.	$\sqrt{4}\cdot\sqrt{3}+\sqrt{36}\cdot\sqrt{3}$ $2\sqrt{3}+6\sqrt{3}$
Combine like radicals.	$8\sqrt{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{6}\left(1+3\sqrt{6}\right)$ ⓑ $\sqrt{12}\left(\sqrt{3}+\sqrt{24}\right)$.

Solution:

ⓐ $18+\sqrt{6}$ ⓑ $6+12\sqrt{2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{8}\left(2-5\sqrt{8}\right)$ ⓑ $\sqrt{14}\left(\sqrt{2}+\sqrt{42}\right)$.

Solution:

ⓐ $-40+4\sqrt{2}$ ⓑ $2\sqrt{7}+14\sqrt{3}$

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

Example:
Exercise:

Problem: Simplify: $\left(2+\sqrt{3}\right)\left(4-\sqrt{3}\right)$.

Solution:
Solution

	$\left(2+\sqrt{3}\right)\left(4-\sqrt{3}\right)$
Multiply.	$8-2\sqrt{3}+4\sqrt{3}-3$
Combine like terms.	$5+2\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\left(1+\sqrt{6}\right)\left(3-\sqrt{6}\right)$.

Solution:

$-3+2\sqrt{6}$

Note:
Exercise:

Problem: Simplify: $\left(4-\sqrt{10}\right)\left(2+\sqrt{10}\right)$.

Solution:

$-2+2\sqrt{10}$

Example:
Exercise:

Problem: Simplify: $\left(3-2\sqrt{7}\right)\left(4-2\sqrt{7}\right)$.

Solution:
Solution

	$\left(3-2\sqrt{7}\right)\left(4-2\sqrt{7}\right)$
Multiply.	$12-6\sqrt{7}-8\sqrt{7}+4\cdot 7$
Simplify.	$12-6\sqrt{7}-8\sqrt{7}+28$
Combine like terms.	$40-14\sqrt{7}$

Note:
Exercise:

Problem: Simplify: $\left(6-3\sqrt{7}\right)\left(3+4\sqrt{7}\right)$.

Solution:

$-66+15\sqrt{7}$

Note:
Exercise:

Problem: Simplify: $\left(2-3\sqrt{11}\right)\left(4-\sqrt{11}\right)$.

Solution:

$41-14\sqrt{11}$

Example:
Exercise:

Problem: Simplify: $\left(3\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+4\sqrt{5}\right)$.

Solution:
Solution

	$\left(3\sqrt{2} - \sqrt{5}\right)\left(\sqrt{2} + 4\sqrt{5}\right)$
Multiply.	$3 \cdot 2 + 12\sqrt{10} - \sqrt{10} - 4 \cdot 5$
Simplify.	$6 + 12\sqrt{10} - \sqrt{10} - 20$
Combine like terms.	$-14 + 11\sqrt{10}$

Note:
Exercise:

Problem: Simplify: $\left(5\sqrt{3} - \sqrt{7}\right)\left(\sqrt{3} + 2\sqrt{7}\right)$.

Solution:

$1 + 9\sqrt{21}$

Note:
Exercise:

Problem: Simplify: $\left(\sqrt{6} - 3\sqrt{8}\right)\left(2\sqrt{6} + \sqrt{8}\right)$

Solution:

$-12 - 20\sqrt{3}$

Example:
Exercise:

Problem: Simplify: $\left(4-2\sqrt{x}\right)\left(1+3\sqrt{x}\right)$.

Solution:
Solution

	$\left(4-2\sqrt{x}\right)\left(1+3\sqrt{x}\right)$
Multiply.	$4+12\sqrt{x}-2\sqrt{x}-6x$
Combine like terms.	$4+10\sqrt{x}-6x$

Note:
Exercise:

Problem: Simplify: $(6-5\sqrt{m})(2+3\sqrt{m})$.

Solution:

$12+8\sqrt{m}-15m$

Note:
Exercise:

Problem: Simplify: $(10+3\sqrt{n})(1-5\sqrt{n})$.

Solution:

$10-47\sqrt{n}-15n$

Note that some special products made our work easier when we multiplied binomials earlier. This is true when we multiply square roots, too. The special product formulas we used are shown below.

Note:
Special Product Formulas
Equation:

Binomial Squares	**Product of Conjugates**
$(a+b)^2 = a^2 + 2ab + b^2$	$(a-b)(a+b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	

We will use the special product formulas in the next few examples. We will start with the Binomial Squares formula.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(2+\sqrt{3}\right)^2$ ⓑ $\left(4-2\sqrt{5}\right)^2$.

Solution:
Solution

Be sure to include the $2ab$ term when squaring a binomial.

ⓐ

	$\overset{(a\ +\ b)^2}{\left(2+\sqrt{3}\right)^2}$
Multiply using the binomial square pattern.	$\overset{a^2}{2^2}\overset{+}{+}\overset{2ab}{2\cdot 2\cdot\sqrt{3}}\overset{+}{+}\overset{b^2}{\left(\sqrt{3}\right)^2}$
Simplify.	$4+4\sqrt{3}+3$
Combine like terms.	$7+4\sqrt{3}$

ⓑ

	$\overset{(a\ -\ b)^2}{\left(4-2\sqrt{5}\right)^2}$
Multiply using the binomial square pattern.	$\overset{a^2}{4^2}\overset{-}{-}\overset{2ab}{2\cdot 4\cdot 2\sqrt{5}}\overset{+}{+}\overset{b^2}{\left(2\sqrt{5}\right)^2}$
Simplify.	$16-16\sqrt{5}+4\cdot 5$ $16-16\sqrt{5}+20$
Combine like terms.	$36-16\sqrt{5}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(10+\sqrt{2}\right)^2$ ⓑ $\left(1+3\sqrt{6}\right)^2$.

Solution:

ⓐ $102+20\sqrt{2}$ ⓑ $55+6\sqrt{6}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(6-\sqrt{5}\right)^2$ ⓑ $\left(9-2\sqrt{10}\right)^2$.

Solution:

ⓐ $31-12\sqrt{5}$ ⓑ $121-36\sqrt{10}$

Example:
Exercise:

Problem: Simplify: $\left(1+3\sqrt{x}\right)^2$.

Solution:
Solution

	$(a + b)^2$ $\left(1+3\sqrt{x}\right)^2$
Multiply using the binomial square pattern.	$a^2 + 2ab + b^2$ $1^2 + 2\cdot 1\cdot 3\sqrt{x} + \left(3\sqrt{x}\right)^2$
Simplify.	$1+6\sqrt{x}+9x$

Note:
Exercise:

Problem: Simplify: $\left(2+5\sqrt{m}\right)^2$.

Solution:

$4+20\sqrt{m}+25m$

Note:
Exercise:

Problem: Simplify: $\left(3-4\sqrt{n}\right)^2$.

Solution:

$9-24\sqrt{n}+16n$

In the next two examples, we will find the product of conjugates.

Example:
Exercise:

Problem: Simplify: $\left(4-\sqrt{2}\right)\left(4+\sqrt{2}\right)$.

Solution:
Solution

	$\overset{(a\ -\ \ b)}{\left(4-\sqrt{2}\right)}\overset{(a\ +\ \ b)}{\left(4+\sqrt{2}\right)}$
Multiply using the binomial square pattern.	$\overset{a^2}{4^2}\overset{-}{-}\overset{b^2}{\left(\sqrt{2}\right)^2}$
Simplify.	$16-2$ 14

Note:
Exercise:

Problem: Simplify: $\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)$.

Solution:

1

Note:
Exercise:

Problem: Simplify: $\left(1+\sqrt{5}\right)\left(1-\sqrt{5}\right)$.

Solution:

-4

Example:
Exercise:

Problem: Simplify: $\left(5-2\sqrt{3}\right)\left(5+2\sqrt{3}\right)$.

Solution:
Solution

	$\overset{(a\ -\ \ b)}{\left(5-2\sqrt{3}\right)}\overset{(a\ +\ \ b)}{\left(5+2\sqrt{3}\right)}$
Multiply using the binomial square pattern.	

	$\overset{a^2}{5^2} \overset{-}{-} \overset{b^2}{\left(2\sqrt{3}\right)^2}$
Simplify.	$25 - 4 \cdot 3$ 13

Note:
Exercise:

Problem: Simplify: $\left(3 - 2\sqrt{5}\right)\left(3 + 2\sqrt{5}\right)$.

Solution:

-11

Note:
Exercise:

Problem: Simplify: $\left(4 + 5\sqrt{7}\right)\left(4 - 5\sqrt{7}\right)$.

Solution:

-159

Note:
Access these online resources for additional instruction and practice with multiplying square roots.

- Product Property
- Multiply Binomials with Square Roots

Key Concepts

- **Product Property of Square Roots** If a, b are nonnegative real numbers, then
  Equation:
  $$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \quad \text{and} \quad \sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$
- **Special formulas** for multiplying binomials and conjugates:
  Equation:
  $$\begin{array}{ll} (a+b)^2 = a^2 + 2ab + b^2 & \qquad (a-b)(a+b) = a^2 - b^2 \\ (a-b)^2 = a^2 - 2ab + b^2 & \end{array}$$
- The FOIL method can be used to multiply binomials containing radicals.

Practice Makes Perfect

Multiply Square Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\sqrt{2} \cdot \sqrt{8}$ ⓑ $\left(3\sqrt{3}\right)\left(2\sqrt{18}\right)$

Solution:

ⓐ 4 ⓑ $18\sqrt{6}$

Exercise:

ⓐ $\sqrt{6}\cdot\sqrt{6}$

Problem: ⓑ $\left(3\sqrt{2}\right)\left(2\sqrt{32}\right)$

Exercise:

ⓐ $\sqrt{7}\cdot\sqrt{14}$

Problem: ⓑ $\left(4\sqrt{8}\right)\left(5\sqrt{8}\right)$

Solution:

ⓐ $7\sqrt{2}$ ⓑ 160

Exercise:

ⓐ $\sqrt{6}\cdot\sqrt{12}$

Problem: ⓑ $\left(2\sqrt{5}\right)\left(2\sqrt{10}\right)$

Exercise:

Problem: $\left(5\sqrt{2}\right)\left(3\sqrt{6}\right)$

Solution:

$30\sqrt{3}$

Exercise:

Problem: $\left(2\sqrt{3}\right)\left(4\sqrt{6}\right)$

Exercise:

Problem: $\left(-2\sqrt{3}\right)\left(3\sqrt{18}\right)$

Solution:

$-18\sqrt{6}$

Exercise:

Problem: $\left(-4\sqrt{5}\right)\left(5\sqrt{10}\right)$

Exercise:

Problem: $\left(5\sqrt{6}\right)\left(-\sqrt{12}\right)$

Solution:

$-30\sqrt{2}$

Exercise:

Problem: $\left(6\sqrt{2}\right)\left(-\sqrt{10}\right)$

Exercise:

Problem: $\left(-2\sqrt{7}\right)\left(-2\sqrt{14}\right)$

Solution:

$28\sqrt{2}$

Exercise:

Problem: $\left(-2\sqrt{11}\right)\left(-4\sqrt{22}\right)$

Exercise:

ⓐ $\left(\sqrt{15y}\right)\left(\sqrt{5y^3}\right)$

Problem: ⓑ $\left(\sqrt{2n^2}\right)\left(\sqrt{18n^3}\right)$

Solution:

ⓐ $5y^2\sqrt{3}$ ⓑ $6n^2\sqrt{n}$

Exercise:

ⓐ $\left(\sqrt{14x^3}\right)\left(\sqrt{7x^3}\right)$

Problem: ⓑ $\left(\sqrt{3q^2}\right)\left(\sqrt{48q^3}\right)$

Exercise:

ⓐ $\left(\sqrt{16y^2}\right)\left(\sqrt{8y^4}\right)$

Problem: ⓑ $\left(\sqrt{11s^6}\right)\left(\sqrt{11s}\right)$

Solution:

ⓐ $8y^3\sqrt{2}$ ⓑ $11s^3\sqrt{s}$

Exercise:

ⓐ $\left(\sqrt{8x^3}\right)\left(\sqrt{3x}\right)$

Problem: ⓑ $\left(\sqrt{7r}\right)\left(\sqrt{7r^8}\right)$

Exercise:

Problem: $\left(2\sqrt{5b^3}\right)\left(4\sqrt{15b}\right)$

Solution:

$40b^2\sqrt{3}$

Exercise:

Problem: $\left(3\sqrt{8c^5}\right)\left(2\sqrt{6c^3}\right)$

Exercise:

Problem: $\left(6\sqrt{3d^3}\right)\left(4\sqrt{12d^5}\right)$

Solution:

$144d^4$

Exercise:

Problem: $\left(2\sqrt{5b^3}\right)\left(4\sqrt{15b}\right)$

Exercise:

Problem: $\left(3\sqrt{27\,\text{yd}^5}\right)\left(2\sqrt{3y^4}\right)$

Solution:

$54y^4\sqrt{y}$

Exercise:

Problem: $\left(-2\sqrt{7z^3}\right)\left(3\sqrt{14z^8}\right)$

Exercise:

Problem: $\left(4\sqrt{2k^5}\right)\left(-3\sqrt{32k^6}\right)$

Solution:

$-96k^5\sqrt{k}$

Exercise:

ⓐ $\left(\sqrt{7}\right)^2$

Problem: ⓑ $\left(-\sqrt{15}\right)^2$

Exercise:

ⓐ $\left(\sqrt{11}\right)^2$

Problem: ⓑ $\left(-\sqrt{21}\right)^2$

Solution:

ⓐ 11 ⓑ 21

Exercise:

ⓐ $\left(\sqrt{19}\right)^2$

Problem: ⓑ $\left(-\sqrt{5}\right)^2$

Exercise:

ⓐ $\left(\sqrt{23}\right)^2$

Problem: ⓑ $\left(-\sqrt{3}\right)^2$

Solution:

ⓐ 23 ⓑ 3

Exercise:

ⓐ $\left(4\sqrt{11}\right)\left(-3\sqrt{11}\right)$

Problem: ⓑ $\left(5\sqrt{3}\right)^2$

Exercise:

ⓐ $\left(2\sqrt{13}\right)\left(-9\sqrt{13}\right)$

Problem: ⓑ $\left(6\sqrt{5}\right)^2$

Solution:

ⓐ −234 ⓑ 180

Exercise:

ⓐ $\left(-3\sqrt{12}\right)\left(-2\sqrt{6}\right)$

Problem: ⓑ $\left(-4\sqrt{10}\right)^2$

Exercise:

Problem: ⓐ $\left(-7\sqrt{5}\right)\left(-3\sqrt{10}\right)$
ⓑ $\left(-2\sqrt{14}\right)^2$

Solution:

ⓐ $105\sqrt{2}$ ⓑ 56

Use Polynomial Multiplication to Multiply Square Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $3\left(4-\sqrt{3}\right)$
ⓑ $\sqrt{2}\left(4-\sqrt{6}\right)$

Exercise:

Problem: ⓐ $4\left(6-\sqrt{11}\right)$
ⓑ $\sqrt{2}\left(5-\sqrt{12}\right)$

Solution:

ⓐ $24-4\sqrt{11}$ ⓑ $5\sqrt{2}-2\sqrt{6}$

Exercise:

Problem: ⓐ $5\left(3-\sqrt{7}\right)$
ⓑ $\sqrt{3}\left(4-\sqrt{15}\right)$

Exercise:

Problem: ⓐ $7\left(-2-\sqrt{11}\right)$
ⓑ $\sqrt{7}\left(6-\sqrt{14}\right)$

Solution:

ⓐ $-14-7\sqrt{11}$ ⓑ $6\sqrt{7}-7\sqrt{2}$

Exercise:

Problem: ⓐ $\sqrt{7}\left(5+2\sqrt{7}\right)$
ⓑ $\sqrt{5}\left(\sqrt{10}+\sqrt{18}\right)$

Exercise:

Problem: ⓐ $\sqrt{11}\left(8+4\sqrt{11}\right)$
ⓑ $\sqrt{3}\left(\sqrt{12}+\sqrt{27}\right)$

Solution:

ⓐ $44+8\sqrt{11}$ ⓑ 15

Exercise:

Problem: ⓐ $\sqrt{11}\left(-3+4\sqrt{11}\right)$
ⓑ $\sqrt{3}\left(\sqrt{15}-\sqrt{18}\right)$

Exercise:

Problem: ⓐ $\sqrt{2}\left(-5+9\sqrt{2}\right)$
ⓑ $\sqrt{7}\left(\sqrt{3}-\sqrt{21}\right)$

Solution:

ⓐ $18-5\sqrt{2}$ ⓑ $\sqrt{21}-7\sqrt{3}$

Exercise:

Problem: $\left(8+\sqrt{3}\right)\left(2-\sqrt{3}\right)$

Exercise:

Problem: $\left(7+\sqrt{3}\right)\left(9-\sqrt{3}\right)$

Solution:

$60+2\sqrt{3}$

Exercise:

Problem: $\left(8-\sqrt{2}\right)\left(3+\sqrt{2}\right)$

Exercise:

Problem: $\left(9-\sqrt{2}\right)\left(6+\sqrt{2}\right)$

Solution:

$52+3\sqrt{2}$

Exercise:

Problem: $\left(3-\sqrt{7}\right)\left(5-\sqrt{7}\right)$

Exercise:

Problem: $\left(5-\sqrt{7}\right)\left(4-\sqrt{7}\right)$

Solution:

$27-9\sqrt{7}$

Exercise:

Problem: $\left(1+3\sqrt{10}\right)\left(5-2\sqrt{10}\right)$

Exercise:

Problem: $\left(7-2\sqrt{5}\right)\left(4+9\sqrt{5}\right)$

Solution:

$-62+55\sqrt{5}$

Exercise:

Problem: $\left(\sqrt{3}+\sqrt{10}\right)\left(\sqrt{3}+2\sqrt{10}\right)$

Exercise:

Problem: $\left(\sqrt{11}+\sqrt{5}\right)\left(\sqrt{11}+6\sqrt{5}\right)$

Solution:

$41+7\sqrt{55}$

Exercise:

Problem: $\left(2\sqrt{7}-5\sqrt{11}\right)\left(4\sqrt{7}+9\sqrt{11}\right)$

Exercise:

Problem: $\left(4\sqrt{6}+7\sqrt{13}\right)\left(8\sqrt{6}-3\sqrt{13}\right)$

Solution:

$-81+44\sqrt{78}$

Exercise:

Problem: $\left(5-\sqrt{u}\right)\left(3+\sqrt{u}\right)$

Exercise:

Problem: $\left(9-\sqrt{w}\right)\left(2+\sqrt{w}\right)$

Solution:

$18+7\sqrt{w}-w$

Exercise:

Problem: $\left(7+2\sqrt{m}\right)\left(4+9\sqrt{m}\right)$

Exercise:

Problem: $\left(6+5\sqrt{n}\right)\left(11+3\sqrt{n}\right)$

Solution:

$66+73\sqrt{n}+15n$

Exercise:

Problem: ⓐ $\left(3+\sqrt{5}\right)^2$
ⓑ $\left(2-5\sqrt{3}\right)^2$

Exercise:

Problem: ⓐ $\left(4+\sqrt{11}\right)^2$
ⓑ $\left(3-2\sqrt{5}\right)^2$

Solution:

ⓐ $27+8\sqrt{11}$ ⓑ $29-12\sqrt{5}$

Exercise:

Problem: ⓐ $\left(9-\sqrt{6}\right)^2$
ⓑ $\left(10+3\sqrt{7}\right)^2$

Exercise:

Problem: ⓐ $\left(5-\sqrt{10}\right)^2$
ⓑ $\left(8+3\sqrt{2}\right)^2$

Solution:

ⓐ $35-10\sqrt{10}$ ⓑ $82+48\sqrt{2}$

Exercise:

Problem: $\left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)$

Exercise:

Problem: $\left(10-\sqrt{3}\right)\left(10+\sqrt{3}\right)$

Solution:

97

Exercise:

Problem: $\left(4+\sqrt{2}\right)\left(4-\sqrt{2}\right)$

Exercise:

Problem: $\left(7+\sqrt{10}\right)\left(7-\sqrt{10}\right)$

Solution:

39

Exercise:

Problem: $\left(4+9\sqrt{3}\right)\left(4-9\sqrt{3}\right)$

Exercise:

Problem: $\left(1+8\sqrt{2}\right)\left(1-8\sqrt{2}\right)$

Solution:

-127

Exercise:

Problem: $\left(12-5\sqrt{5}\right)\left(12+5\sqrt{5}\right)$

Exercise:

Problem: $\left(9-4\sqrt{3}\right)\left(9+4\sqrt{3}\right)$

Solution:

33

Mixed Practice

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{3}\cdot\sqrt{21}$

Exercise:

Problem: $\left(4\sqrt{6}\right)\left(-\sqrt{18}\right)$

Solution:

$-24\sqrt{3}$

Exercise:

Problem: $\left(-5+\sqrt{7}\right)\left(6+\sqrt{21}\right)$

Exercise:

Problem: $\left(-5\sqrt{7}\right)\left(6\sqrt{21}\right)$

Solution:

$-210\sqrt{3}$

Exercise:

Problem: $\left(-4\sqrt{2}\right)\left(2\sqrt{18}\right)$

Exercise:

Problem: $\left(\sqrt{35y^3}\right)\left(\sqrt{7y^3}\right)$

Solution:

$7y^3\sqrt{5}$

Exercise:

Problem: $\left(4\sqrt{12x^5}\right)\left(2\sqrt{6x^3}\right)$

Exercise:

Problem: $\left(\sqrt{29}\right)^2$

Solution:

29

Exercise:

Problem: $\left(-4\sqrt{17}\right)\left(-3\sqrt{17}\right)$

Exercise:

Problem: $\left(-4+\sqrt{17}\right)\left(-3+\sqrt{17}\right)$

Solution:

$29-7\sqrt{17}$

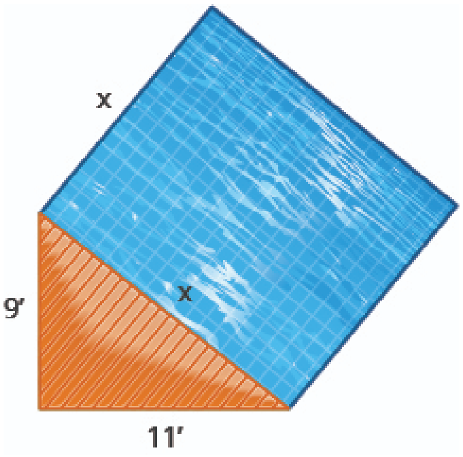
Everyday Math

Exercise:

Problem:

A landscaper wants to put a square reflecting pool next to a triangular deck, as shown below. The triangular deck is a right triangle, with legs of length 9 feet and 11 feet, and the pool will be adjacent to the hypotenuse.

ⓐ Use the Pythagorean Theorem to find the length of a side of the pool. Round your answer to the nearest tenth of a foot.
ⓑ Find the exact area of the pool.

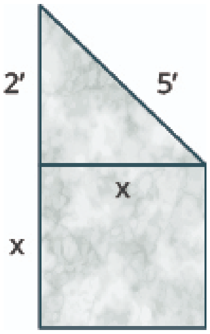


Exercise:

Problem:

An artist wants to make a small monument in the shape of a square base topped by a right triangle, as shown below. The square base will be adjacent to one leg of the triangle. The other leg of the triangle will measure 2 feet and the hypotenuse will be 5 feet.

ⓐ Use the Pythagorean Theorem to find the length of a side of the square base. Round your answer to the nearest tenth of a foot.



ⓑ Find the exact area of the face of the square base.

Solution:

ⓐ 4.6 feet ⓑ 21 sq. feet

Exercise:

Problem:

A square garden will be made with a stone border on one edge. If only $3+\sqrt{10}$ feet of stone are available, simplify $\left(3+\sqrt{10}\right)^2$ to determine the area of the largest such garden.

Exercise:

Problem:

A garden will be made so as to contain two square sections, one section with side length $\sqrt{5}+\sqrt{6}$ yards and one section with side length $\sqrt{2}+\sqrt{3}$ yards. Simplify $\left(\sqrt{5}+\sqrt{6}\right)\left(\sqrt{2}+\sqrt{3}\right)$ to determine the total area of the garden.

Solution:

$\sqrt{10}+\sqrt{15}+2\sqrt{3}+3\sqrt{2}$; approximately 14.74 square yards

Exercise:

Problem:

Suppose a third square section will be added to the garden in the previous exercise. The third section is to have a side length of $\sqrt{432}$ feet. Write an expression that gives the total area of the garden.

Writing Exercises

Exercise:

Problem:

ⓐ Explain why $\left(-\sqrt{n}\right)^2$ is always positive, for $n \geq 0$.
ⓑ Explain why $-\left(\sqrt{n}\right)^2$ is always negative, for $n \geq 0$.

Solution:

ⓐ when squaring a negative, it becomes a positive ⓑ since the negative is not included in the parentheses, it is not squared, and remains negative

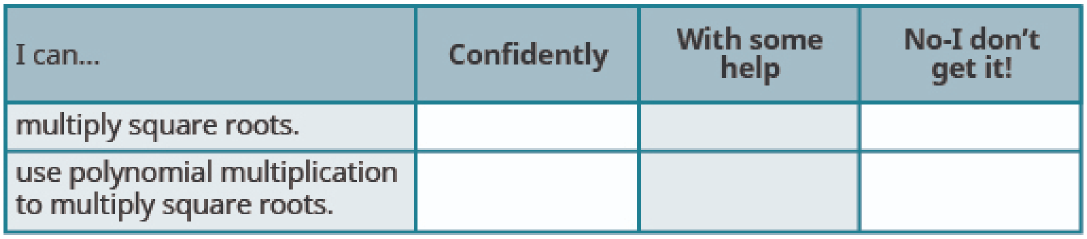
Exercise:

Problem:

Use the binomial square pattern to simplify $\left(3+\sqrt{2}\right)^2$. Explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don’t get it!
multiply square roots.			
use polynomial multiplication to multiply square roots.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Divide Square Roots
By the end of this section, you will be able to:

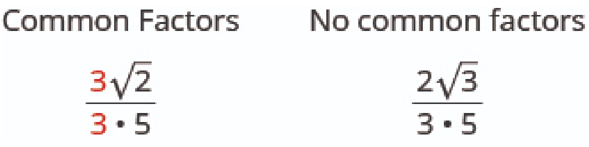
- Divide square roots
- Rationalize a one-term denominator
- Rationalize a two-term denominator

Note:
Before you get started, take this readiness quiz.

1. Find a fraction equivalent to $\frac{5}{8}$ with denominator 48.
   If you missed this problem, review [link].
2. Simplify: $\left(\sqrt{5}\right)^2$.
   If you missed this problem, review [link].
3. Multiply: $(7+3x)(7-3x)$.
   If you missed this problem, review [link].

Divide Square Roots

We know that we simplify fractions by removing factors common to the numerator and the denominator. When we have a fraction with a square root in the numerator, we first simplify the square root. Then we can look for common factors.



Common Factors	No common factors
$\frac{3\sqrt{2}}{3\cdot 5}$	$\frac{2\sqrt{3}}{3\cdot 5}$

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{54}}{6}$.

Solution:
Solution

	$\frac{\sqrt{54}}{6}$
Simplify the radical.	$\frac{\sqrt{9}\cdot\sqrt{6}}{6}$
Simplify.	$\frac{3\sqrt{6}}{6}$
Remove the common factors.	$\frac{\cancel{3}\sqrt{6}}{\cancel{3}\cdot 2}$
Simplify.	$\frac{\sqrt{6}}{2}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{32}}{8}$.

Solution:

$\frac{\sqrt{2}}{2}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{75}}{15}$.

Solution:

$\frac{\sqrt{3}}{3}$

Example:
Exercise:

Problem: Simplify: $\frac{6-\sqrt{24}}{12}$.

Solution:
Solution

	$\frac{6-\sqrt{24}}{12}$
Simplify the radical.	$\frac{6-\sqrt{4}\cdot\sqrt{6}}{12}$
Simplify.	$\frac{6-2\sqrt{6}}{12}$
Factor the common factor from the numerator.	$\frac{2\left(3-\sqrt{6}\right)}{2\cdot 6}$
Remove the common factors.	$\frac{\cancel{2}\left(3-\sqrt{6}\right)}{\cancel{2}\cdot 6}$
Simplify.	$\frac{3-\sqrt{6}}{6}$

Note:
Exercise:

Problem: Simplify: $\frac{8-\sqrt{40}}{10}$.

Solution:

$\frac{4-\sqrt{10}}{5}$

Note:
Exercise:

Problem: Simplify: $\frac{10-\sqrt{75}}{20}$.

Solution:

$\frac{2-\sqrt{3}}{4}$

We have used the Quotient Property of Square Roots to simplify square roots of fractions. The Quotient Property of Square Roots says

Equation:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}, b \neq 0$$

Sometimes we will need to use the Quotient Property of Square Roots ‘in reverse’ to simplify a fraction with square roots.

Equation:

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}, b \neq 0$$

We will rewrite the Quotient Property of Square Roots so we see both ways together. Remember: we assume all variables are greater than or equal to zero so that their square roots are real numbers.

Note:

Quotient Property of Square Roots

If a, b are non-negative real numbers and $b \neq 0$, then

Equation:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad \text{and} \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

We will use the Quotient Property of Square Roots ‘in reverse’ when the fraction we start with is the quotient of two square roots, and neither radicand is a perfect square. When we write the fraction in a single square root, we may find common factors in the numerator and denominator.

Example:

Exercise:

Problem: Simplify: $\frac{\sqrt{27}}{\sqrt{75}}$.

Solution:
Solution

	$\frac{\sqrt{27}}{\sqrt{75}}$
Neither radicand is a perfect square, so rewrite using the quotient property of square roots.	$\sqrt{\frac{27}{75}}$
Remove common factors in the numerator and denominator.	$\sqrt{\frac{\cancel{3}\cdot 9}{\cancel{3}\cdot 25}}$
Simplify.	$\sqrt{\frac{9}{25}}$
	$\frac{3}{5}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{48}}{\sqrt{108}}$.

Solution:

$\frac{2}{3}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{96}}{\sqrt{54}}$.

Solution:

$\frac{4}{3}$

We will use the Quotient Property for Exponents, $\frac{a^m}{a^n} = a^{m-n}$, when we have variables with exponents in the radicands.

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{6y^5}}{\sqrt{2y}}$.

Solution:
Solution

	$\frac{\sqrt{6y^5}}{\sqrt{2y}}$
Neither radicand is a perfect square, so rewrite using the quotient property of square roots.	$\sqrt{\frac{6y^5}{2y}}$
Remove common factors in the numerator and denominator.	$\sqrt{\frac{\cancel{2}\cdot 3\cdot y^4\cdot \cancel{y}}{\cancel{2}\cdot \cancel{y}}}$
Simplify.	$\sqrt{3y^4}$
Simplify the radical.	$y^2\sqrt{3}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{12r^3}}{\sqrt{6r}}$.

Solution:

$r\sqrt{2}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{14p^9}}{\sqrt{2p^5}}$.

Solution:

$p^2\sqrt{7}$

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{72x^3}}{\sqrt{162x}}$.

Solution:
Solution

	$\frac{\sqrt{72x^3}}{\sqrt{162x}}$
Rewrite using the quotient property of square roots.	$\sqrt{\frac{72x^3}{162x}}$
Remove common factors.	$\sqrt{\frac{\cancel{18}\cdot 4\cdot x^2\cdot \cancel{x}}{\cancel{18}\cdot 9\cdot \cancel{x}}}$
Simplify.	$\sqrt{\frac{4x^2}{9}}$
Simplify the radical.	$\frac{2x}{3}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{50s^3}}{\sqrt{128s}}$.

Solution:

$\frac{5s}{8}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{75q^5}}{\sqrt{108q}}$.

Solution:

$\frac{5q^2}{6}$

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$.

Solution:
Solution

$\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$

Rewrite using the quotient property of square roots.	$\sqrt{\frac{147ab^8}{3a^3b^4}}$
Remove common factors.	$\sqrt{\frac{49b^4}{a^2}}$
Simplify the radical.	$\frac{7b^2}{a}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$.

Solution:

$\frac{9x^2}{y^2}$

Note:
Exercise:

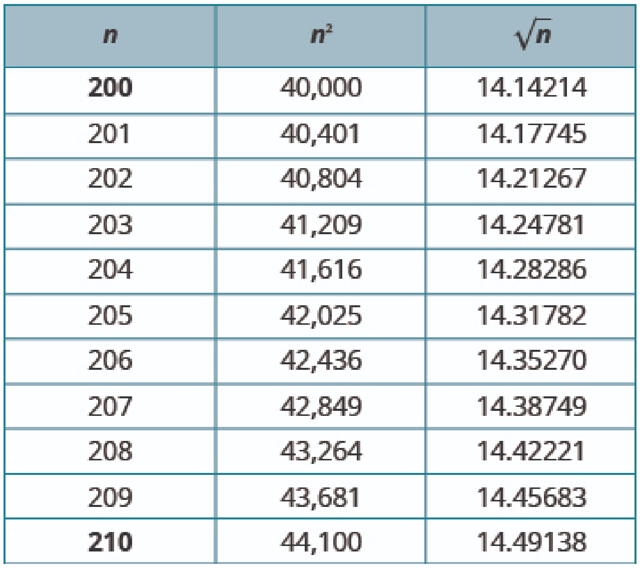
Problem: Simplify: $\frac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$.

Solution:

$$\frac{10n^3}{m}$$

Rationalize a One Term Denominator

Before the calculator became a tool of everyday life, tables of square roots were used to find approximate values of square roots. [link] shows a portion of a table of squares and square roots. Square roots are approximated to five decimal places in this table.



n	n^2	$\sqrt{n}$
200	40,000	14.14214
201	40,401	14.17745
202	40,804	14.21267
203	41,209	14.24781
204	41,616	14.28286
205	42,025	14.31782
206	42,436	14.35270
207	42,849	14.38749
208	43,264	14.42221
209	43,681	14.45683
210	44,100	14.49138

A table of square roots was used to find approximate values of square roots before there were calculators.

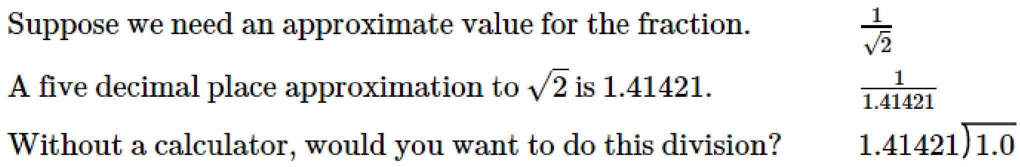
If someone needed to approximate a fraction with a square root in the denominator, it meant doing long division with a five decimal-place divisor. This was a very cumbersome process.

For this reason, a process called rationalizing the denominator was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. This process is still used today and is useful in other areas of mathematics, too.

Note:
Rationalizing the Denominator
The process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer is called **rationalizing the denominator**.

Square roots of numbers that are not perfect squares are irrational numbers. When we **rationalize the denominator**, we write an equivalent fraction with a rational number in the denominator.

Let’s look at a numerical example.

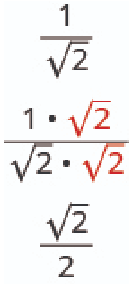


Suppose we need an approximate value for the fraction. $\frac{1}{\sqrt{2}}$

A five decimal place approximation to $\sqrt{2}$ is 1.41421. $\frac{1}{1.41421}$

Without a calculator, would you want to do this division? $1.41421\overline{)1.0}$

But we can find a fraction equivalent to $\frac{1}{\sqrt{2}}$ by multiplying the numerator and denominator by $\sqrt{2}$.



$$\frac{1}{\sqrt{2}}$$

$$\frac{1\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}$$

$$\frac{\sqrt{2}}{2}$$

Now if we need an approximate value, we divide $2\overline{)1.41421}$. This is much easier.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a square root.

Similarly, a square root is not considered simplified if the radicand contains a fraction.

Note:
Simplified Square Roots
A square root is considered simplified if there are

- no perfect-square factors in the radicand
- no fractions in the radicand
- no square roots in the denominator of a fraction

To rationalize a denominator, we use the property that $\left(\sqrt{a}\right)^2 = a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

Example:
Exercise:

Problem: Simplify: $\frac{4}{\sqrt{3}}$.

Solution:
Solution

To rationalize a denominator, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

	$\frac{4}{\sqrt{3}}$
Multiply both the numerator and denominator by $\sqrt{3}$.	$\frac{4\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}$
Simplify.	$\frac{4\sqrt{3}}{3}$

Note:
Exercise:

Problem: Simplify: $\frac{5}{\sqrt{3}}$.

Solution:

$\frac{5\sqrt{3}}{3}$

Note:
Exercise:

Problem: Simplify: $\frac{6}{\sqrt{5}}$.

Solution:

$\frac{6\sqrt{5}}{5}$

Example:
Exercise:

Problem: Simplify: $-\frac{8}{3\sqrt{6}}$.

Solution:
Solution

To remove the square root from the denominator, we multiply it by itself. To keep the fractions equivalent, we multiply both the numerator and denominator by $\sqrt{6}$.

	$-\frac{8}{3\sqrt{6}}$
Multiply both the numerator and the denominator by $\sqrt{6}$.	$-\frac{8\cdot\sqrt{6}}{3\sqrt{6}\cdot\sqrt{6}}$
Simplify.	$-\frac{8\sqrt{6}}{3\cdot 6}$
Remove common factors.	$-\frac{4\cdot\cancel{2}\sqrt{6}}{3\cdot\cancel{2}\cdot 3}$
Simplify.	$-\frac{4\sqrt{6}}{9}$

Note:
Exercise:

Problem: Simplify: $\frac{5}{2\sqrt{5}}$.

Solution:

$\frac{\sqrt{5}}{2}$

Note:
Exercise:

Problem: Simplify: $-\frac{9}{4\sqrt{3}}$.

Solution:

$-\frac{3\sqrt{3}}{4}$

Always simplify the radical in the denominator first, before you rationalize it. This way the numbers stay smaller and easier to work with.

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{5}{12}}$.

Solution:
Solution

	$\sqrt{\frac{5}{12}}$
The fraction is not a perfect square, so rewrite using the Quotient Property.	$\frac{\sqrt{5}}{\sqrt{12}}$
Simplify the denominator	$\frac{\sqrt{5}}{2\sqrt{3}}$

Rationalize the denominator.	$\frac{\sqrt{5}\cdot\sqrt{3}}{2\sqrt{3}\cdot\sqrt{3}}$
Simplify.	$\frac{\sqrt{15}}{2\cdot 3}$
Simplify.	$\frac{\sqrt{15}}{6}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{7}{18}}$.

Solution:

$\frac{\sqrt{14}}{6}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{3}{32}}$.

Solution:

$\frac{\sqrt{6}}{8}$

Example:
Exercise:

Problem: Simplify: $\sqrt{\frac{11}{28}}$.

Solution:
Solution

	$\sqrt{\frac{11}{28}}$
Rewrite using the Quotient Property.	$\frac{\sqrt{11}}{\sqrt{28}}$
Simplify the denominator.	$\frac{\sqrt{11}}{2\sqrt{7}}$
Rationalize the denominator.	$\frac{\sqrt{11}\cdot\sqrt{7}}{2\sqrt{7}\cdot\sqrt{7}}$
Simplify.	$\frac{\sqrt{77}}{2\cdot 7}$
Simplify.	$\frac{\sqrt{77}}{14}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{3}{27}}$.

Solution:

$\frac{1}{3}$

Note:
Exercise:

Problem: Simplify: $\sqrt{\frac{10}{50}}$.

Solution:

$\frac{\sqrt{5}}{5}$

Rationalize a Two-Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates pattern to rationalize the denominator.
Equation:

$$\begin{array}{cc} (a-b)(a+b) & (2-\sqrt{5})(2+\sqrt{5}) \\ a^2-b^2 & 2^2-\left(\sqrt{5}\right)^2 \\ & 4-5 \\ & -1 \end{array}$$

When we multiply a binomial that includes a square root by its conjugate, the product has no square roots.

Example:
Exercise:

Problem: Simplify: $\frac{4}{4+\sqrt{2}}$.

Solution:
Solution

	$\frac{4}{4+\sqrt{2}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{4(4-\sqrt{2})}{(4+\sqrt{2})(4-\sqrt{2})}$
Multiply the conjugates in the denominator.	$\frac{4(4-\sqrt{2})}{4^2-(\sqrt{2})^2}$
Simplify the denominator.	$\frac{4(4-\sqrt{2})}{16-2}$
Simplify the denominator.	$\frac{4(4-\sqrt{2})}{14}$
Remove common factors from the numerator and denominator.	$\frac{2(4-\sqrt{2})}{7}$
We leave the numerator in factored form to make it easier to look for common factors after we have simplified the	

denominator.

Note:
Exercise:

Problem: Simplify: $\frac{2}{2+\sqrt{3}}$.

Solution:

$\frac{2\left(2-\sqrt{3}\right)}{1}$

Note:
Exercise:

Problem: Simplify: $\frac{5}{5+\sqrt{3}}$.

Solution:

$\frac{5\left(5-\sqrt{3}\right)}{22}$

Example:
Exercise:

Problem: Simplify: $\frac{5}{2-\sqrt{3}}$.

Solution:
Solution

	$\frac{5}{2-\sqrt{3}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$
Multiply the conjugates in the denominator.	$\frac{5(2+\sqrt{3})}{2^2-(\sqrt{3})^2}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{4-3}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{1}$
Simplify.	$5(2+\sqrt{3})$

Note:
Exercise:

Problem: Simplify: $\frac{3}{1-\sqrt{5}}$.

Solution:

$$-\frac{3\left(1+\sqrt{5}\right)}{4}$$

Note:
Exercise:

Problem: Simplify: $\frac{2}{4-\sqrt{6}}$.

Solution:

$\frac{4+\sqrt{6}}{5}$

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Solution:
Solution

	$\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u}-\sqrt{6})(\sqrt{u}+\sqrt{6})}$
Multiply the conjugates in the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u})^2-(\sqrt{6})^2}$
Simplify the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{u-6}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

Solution:

$\frac{\sqrt{5}\left(\sqrt{x}-\sqrt{2}\right)}{x-2}$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{10}}{\sqrt{y}-\sqrt{3}}$.

Solution:

$\frac{\sqrt{10}\left(\sqrt{y}+\sqrt{3}\right)}{y-3}$

Example:
Exercise:

Problem: Simplify: $\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$.

Solution:
Solution

	$\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}$
Multiply the conjugates in the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x})^2-(\sqrt{7})^2}$
Simplify the denominator.	$\frac{(\sqrt{x}+\sqrt{7})^2}{x-7}$
We do not square the numerator. In factored form, we can see there are no common factors to remove from the numerator and denominator.	

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

Solution:

$$\frac{\left(\sqrt{p}+\sqrt{2}\right)^2}{p-2}$$

Note:
Exercise:

Problem: Simplify: $\frac{\sqrt{q}-\sqrt{10}}{\sqrt{q}+\sqrt{10}}$.

Solution:

$$\frac{\left(\sqrt{q}-\sqrt{10}\right)^2}{q-10}$$

Note:
Access this online resource for additional instruction and practice with dividing and rationalizing.

- Dividing and Rationalizing

Key Concepts

- **Quotient Property of Square Roots**
  - If a, b are non-negative real numbers and $b \neq 0$, then
    Equation:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad \text{and} \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

- **Simplified Square Roots**
  A square root is considered simplified if there are
  - no perfect square factors in the radicand
  - no fractions in the radicand
  - no square roots in the denominator of a fraction

Practice Makes Perfect

Divide Square Roots

In the following exercises, simplify.
Exercise:

Problem: $\frac{\sqrt{27}}{6}$

Solution:

$\frac{\sqrt{3}}{2}$

Exercise:

Problem: $\frac{\sqrt{50}}{10}$

Exercise:

Problem: $\frac{\sqrt{72}}{9}$

Solution:

$\frac{2\sqrt{2}}{3}$

Exercise:

Problem: $\frac{\sqrt{243}}{6}$

Exercise:

Problem: $\frac{2-\sqrt{32}}{8}$

Solution:

$\frac{1-2\sqrt{2}}{4}$

Exercise:

Problem: $\frac{3+\sqrt{27}}{9}$

Exercise:

Problem: $\frac{6+\sqrt{45}}{6}$

Solution:

$\frac{2+\sqrt{5}}{2}$

Exercise:

Problem: $\frac{10-\sqrt{200}}{20}$

Exercise:

Problem: $\frac{\sqrt{80}}{\sqrt{125}}$

Solution:

$\frac{4}{5}$

Exercise:

Problem: $\frac{\sqrt{72}}{\sqrt{200}}$

Exercise:

Problem: $\frac{\sqrt{128}}{\sqrt{72}}$

Solution:

$\frac{4}{3}$

Exercise:

Problem: $\frac{\sqrt{48}}{\sqrt{75}}$

Exercise:

Problem: ⓐ $\frac{\sqrt{8x^6}}{\sqrt{2x^2}}$ ⓑ $\frac{\sqrt{200m^5}}{\sqrt{98m}}$

Solution:

ⓐ $2x^2$ ⓑ $\frac{10m^2}{7}$

Exercise:

Problem: ⓐ $\frac{\sqrt{10y^3}}{\sqrt{5y}}$ ⓑ $\frac{\sqrt{108n^7}}{\sqrt{243n^3}}$

Exercise:

Problem: $\frac{\sqrt{75r^3}}{\sqrt{108r}}$

Solution:

$\frac{5r}{6}$

Exercise:

Problem: $\frac{\sqrt{196q^5}}{\sqrt{484q}}$

Exercise:

Problem: $\frac{\sqrt{1224p^5q^2}}{\sqrt{34p^3q^6}}$

Solution:

$\frac{6p}{q^2}$

Exercise:

Problem: $\frac{\sqrt{98rs^{10}}}{\sqrt{2r^3s^4}}$

Exercise:

Problem: $\frac{\sqrt{320mn^5}}{\sqrt{45m^7n^3}}$

Solution:

$\frac{8n}{3m^3}$

Exercise:

Problem: $\frac{\sqrt{810c^3d^7}}{\sqrt{1000c^5d}}$

Exercise:

Problem: $\frac{\sqrt{98}}{14}$

Solution:

$\frac{\sqrt{2}}{2}$

Exercise:

Problem: $\frac{\sqrt{72}}{18}$

Exercise:

Problem: $\frac{5+\sqrt{125}}{15}$

Solution:

$\frac{1+\sqrt{5}}{3}$

Exercise:

Problem: $\frac{6-\sqrt{45}}{12}$

Exercise:

Problem: $\frac{\sqrt{96}}{\sqrt{150}}$

Solution:

$\frac{4}{5}$

Exercise:

Problem: $\frac{\sqrt{28}}{\sqrt{63}}$

Exercise:

Problem: $\frac{\sqrt{26y^7}}{\sqrt{2y}}$

Solution:

$y^3\sqrt{13}$

Exercise:

Problem: $\frac{\sqrt{15x^3}}{\sqrt{3x}}$

Rationalize a One-Term Denominator

In the following exercises, simplify and rationalize the denominator.

Exercise:

Problem: $\frac{10}{\sqrt{6}}$

Solution:

$\frac{5\sqrt{6}}{3}$

Exercise:

Problem: $\frac{8}{\sqrt{3}}$

Exercise:

Problem: $\frac{6}{\sqrt{7}}$

Solution:

$\frac{6\sqrt{7}}{7}$

Exercise:

Problem: $\frac{4}{\sqrt{5}}$

Exercise:

Problem: $\frac{3}{\sqrt{13}}$

Solution:

$\frac{3\sqrt{13}}{13}$

Exercise:

Problem: $\frac{10}{\sqrt{11}}$

Exercise:

Problem: $\frac{10}{3\sqrt{10}}$

Solution:

$\frac{\sqrt{10}}{3}$

Exercise:

Problem: $\frac{2}{5\sqrt{2}}$

Exercise:

Problem: $\frac{4}{9\sqrt{5}}$

Solution:

$\frac{4\sqrt{5}}{45}$

Exercise:

Problem: $\frac{9}{2\sqrt{7}}$

Exercise:

Problem: $-\frac{9}{2\sqrt{3}}$

Solution:

$-\frac{3\sqrt{3}}{2}$

Exercise:

Problem: $-\frac{8}{3\sqrt{6}}$

Exercise:

Problem: $\sqrt{\frac{3}{20}}$

Solution:

$\frac{\sqrt{15}}{10}$

Exercise:

Problem: $\sqrt{\frac{4}{27}}$

Exercise:

Problem: $\sqrt{\frac{7}{40}}$

Solution:

$\frac{\sqrt{70}}{20}$

Exercise:

Problem: $\sqrt{\frac{8}{45}}$

Exercise:

Problem: $\sqrt{\frac{19}{175}}$

Solution:

$\frac{\sqrt{133}}{35}$

Exercise:

Problem: $\sqrt{\frac{17}{192}}$

Rationalize a Two-Term Denominator

In the following exercises, simplify by rationalizing the denominator.

Exercise:

Problem: ⓐ $\frac{3}{3+\sqrt{11}}$ ⓑ $\frac{8}{1-\sqrt{5}}$

Solution:

ⓐ $\frac{3\left(3-\sqrt{11}\right)}{-2}$ ⓑ $-2\left(1+\sqrt{5}\right)$

Exercise:

Problem: ⓐ $\frac{4}{4+\sqrt{7}}$ ⓑ $\frac{7}{2-\sqrt{6}}$

Exercise:

Problem: ⓐ $\frac{5}{5+\sqrt{6}}$ ⓑ $\frac{6}{3-\sqrt{7}}$

Solution:

ⓐ $\frac{5\left(5-\sqrt{6}\right)}{19}$ ⓑ $3\left(3+\sqrt{7}\right)$

Exercise:

Problem: ⓐ $\frac{6}{6+\sqrt{5}}$ ⓑ $\frac{5}{4-\sqrt{11}}$

Exercise:

Problem: $\frac{\sqrt{3}}{\sqrt{m}-\sqrt{5}}$

Solution:

$$\frac{\sqrt{3}\left(\sqrt{m}+\sqrt{5}\right)}{m-5}$$

Exercise:

Problem: $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

Exercise:

Problem: $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{6}}$

Solution:

$$\frac{\sqrt{2}\left(\sqrt{x}+\sqrt{6}\right)}{x-6}$$

Exercise:

Problem: $\frac{\sqrt{7}}{\sqrt{y}+\sqrt{3}}$

Exercise:

Problem: $\frac{\sqrt{r}+\sqrt{5}}{\sqrt{r}-\sqrt{5}}$

Solution:

$$\frac{\left(\sqrt{r}+\sqrt{5}\right)^{2}}{r-5}$$

Exercise:

Problem: $\frac{\sqrt{s}-\sqrt{6}}{\sqrt{s}+\sqrt{6}}$

Exercise:

Problem: $\frac{\sqrt{150x^{2}y^{6}}}{\sqrt{6x^{4}y^{2}}}$

Solution:

$\frac{5y^2}{x}$

Exercise:

Problem: $\frac{\sqrt{80p^3q}}{\sqrt{5pq^5}}$

Exercise:

Problem: $\frac{15}{\sqrt{5}}$

Solution:

$3\sqrt{5}$

Exercise:

Problem: $\frac{3}{5\sqrt{8}}$

Exercise:

Problem: $\sqrt{\frac{8}{54}}$

Solution:

$\frac{2\sqrt{3}}{9}$

Exercise:

Problem: $\sqrt{\frac{12}{20}}$

Exercise:

Problem: $\frac{3}{5+\sqrt{5}}$

Solution:

$$\frac{3\left(5-\sqrt{5}\right)}{20}$$

Exercise:

Problem: $\frac{20}{4-\sqrt{3}}$

Exercise:

Problem: $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{3}}$

Solution:

$$\frac{\sqrt{2}\left(\sqrt{x}+\sqrt{3}\right)}{x-3}$$

Exercise:

Problem: $\frac{\sqrt{5}}{\sqrt{y}-\sqrt{7}}$

Exercise:

Problem: $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

Solution:

$$\frac{\left(\sqrt{x}+2\sqrt{2}\right)^{2}}{x-8}$$

Exercise:

Problem: $\frac{\sqrt{m}-\sqrt{3}}{\sqrt{m}+\sqrt{3}}$

Everyday Math

Exercise:

Problem:

A supply kit is dropped from an airplane flying at an altitude of 250 feet. Simplify $\sqrt{\frac{250}{16}}$ to determine how many seconds it takes for the supply kit to reach the ground.

Solution:

$\frac{5\sqrt{10}}{4}$ seconds

Exercise:

Problem:

A flare is dropped into the ocean from an airplane flying at an altitude of 1,200 feet. Simplify $\sqrt{\frac{1200}{16}}$ to determine how many seconds it takes for the flare to reach the ocean.

Writing Exercises

Exercise:

ⓐ Simplify $\sqrt{\frac{27}{3}}$ and explain all your steps.

ⓑ Simplify $\sqrt{\frac{27}{5}}$ and explain all your steps.

Problem: ⓒ Why are the two methods of simplifying square roots different?

Solution:

Answers will vary.

Exercise:

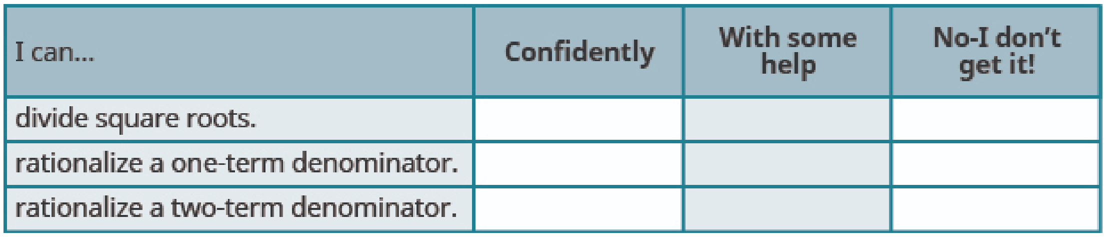
Problem:

ⓐ Approximate $\frac{1}{\sqrt{2}}$ by dividing $\frac{1}{1.414}$ using long division without a calculator.

ⓑ Rationalizing the denominator of $\frac{1}{\sqrt{2}}$ gives $\frac{\sqrt{2}}{2}$. Approximate $\frac{\sqrt{2}}{2}$ by dividing $\frac{1.414}{2}$ using long division without a calculator.
ⓒ Do you agree that rationalizing the denominator makes calculations easier? Why or why not?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
divide square roots.			
rationalize a one-term denominator.			
rationalize a two-term denominator.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

rationalizing the denominator

The process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer is called rationalizing the denominator.

Solve Equations with Square Roots
By the end of this section, you will be able to:

- Solve radical equations
- Use square roots in applications

Note:
Before you get started, take this readiness quiz.

1. Simplify: ⓐ $\sqrt{9}$ ⓑ 9^2.
   If you missed this problem, review [link] and [link].
2. Solve: $5\left(x+1\right)-4=3\left(2x-7\right)$.
   If you missed this problem, review [link].
3. Solve: $n^2-6n+8=0$.
   If you missed this problem, review [link].

Solve Radical Equations

In this section we will solve equations that have the variable in the radicand of a square root. Equations of this type are called radical equations.

Note:
Radical Equation
An equation in which the variable is in the radicand of a square root is called a **radical equation**.

As usual, in solving these equations, what we do to one side of an equation we must do to the other side as well. Since squaring a quantity and taking a square root are 'opposite' operations, we will square both sides in order to remove the radical sign and solve for the variable inside.

But remember that when we write $\sqrt{a}$ we mean the principal square root. So $\sqrt{a}\ge 0$ always. When we solve radical equations by squaring both sides we may get an algebraic solution that would make $\sqrt{a}$ negative. This algebraic solution would not be a solution to the original radical equation; it is an *extraneous solution.* We saw extraneous solutions when we solved rational equations, too.

Example:
Exercise:

Problem: For the equation $\sqrt{x+2}=x$:

ⓐ Is $x=2$ a solution? ⓑ Is $x=-1$ a solution?

Solution:

Solution

ⓐ Is $x = 2$ a solution?

	$\sqrt{x+2} = x$
Let $x = 2$.	$\sqrt{2+2} \stackrel{?}{=} 2$
Simplify.	$\sqrt{4} \stackrel{?}{=} 2$
	$2 = 2$ ✓
	2 is a solution.

ⓑ Is $x = -1$ a solution?

	$\sqrt{x+2} = x$
Let $x = -1$.	$\sqrt{-1+2} \stackrel{?}{=} -1$
Simplify.	$\sqrt{1} \stackrel{?}{=} -1$
	$1 \neq -1$
	−1 is not a solution.

	-1 is an extraneous solution to the equation.

Note:
Exercise:

Problem: For the equation $\sqrt{x+6}=x$:

ⓐ Is $x=-2$ a solution? ⓑ Is $x=3$ a solution?

Solution:

ⓐ no ⓑ yes

Note:
Exercise:

Problem: For the equation $\sqrt{-x+2}=x$:

ⓐ Is $x=-2$ a solution? ⓑ Is $x=1$ a solution?

Solution:

ⓐ no ⓑ yes

Now we will see how to solve a radical equation. Our strategy is based on the relation between taking a square root and squaring.

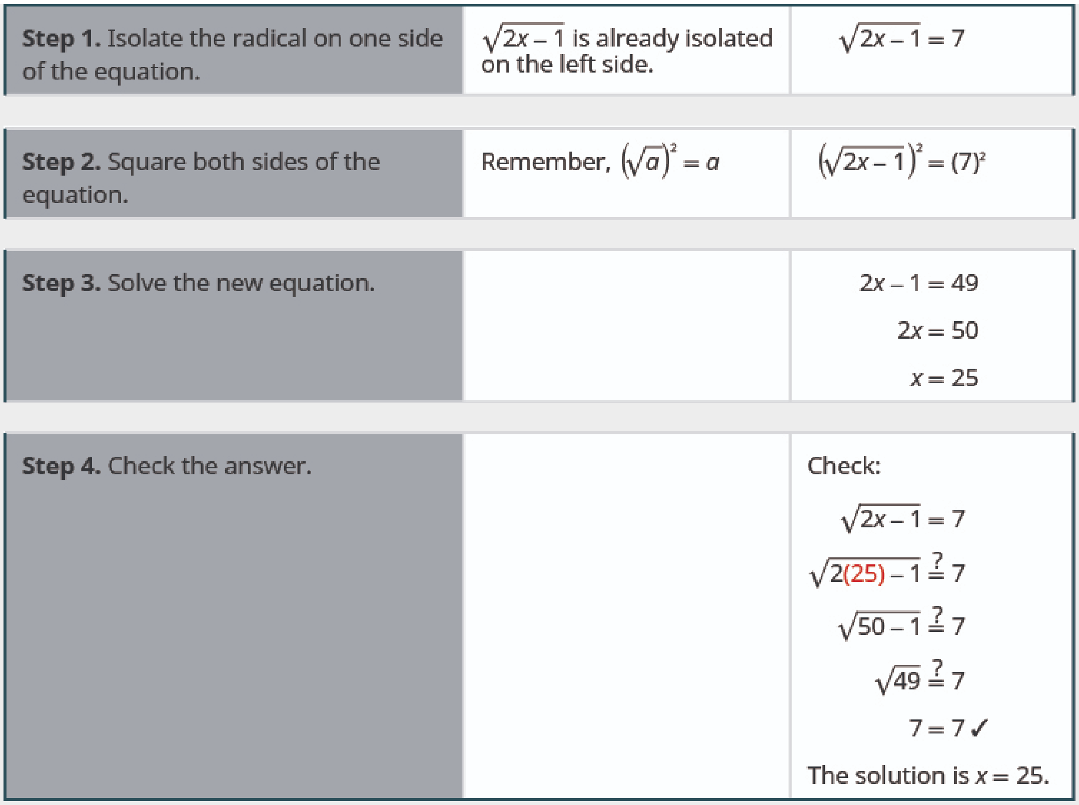
Equation:

$$\text{For } a \geq 0, \;\; \left(\sqrt{a}\right)^2 = a$$

Example:
How to Solve Radical Equations
Exercise:

Problem: Solve: $\sqrt{2x-1}=7$.

Solution:
Solution



Step 1. Isolate the radical on one side of the equation.	$\sqrt{2x-1}$ is already isolated on the left side.	$\sqrt{2x-1}=7$
Step 2. Square both sides of the equation.	Remember, $\left(\sqrt{a}\right)^2=a$	$\left(\sqrt{2x-1}\right)^2=(7)^2$
Step 3. Solve the new equation.		$2x-1=49$ $2x=50$ $x=25$
Step 4. Check the answer.		Check: $\sqrt{2x-1}=7$ $\sqrt{2(25)-1}\overset{?}{=}7$ $\sqrt{50-1}\overset{?}{=}7$ $\sqrt{49}\overset{?}{=}7$ $7=7$ ✓ The solution is $x=25$.

Note:
Exercise:

Problem: Solve: $\sqrt{3x-5}=5$.

Solution:

10

Note:
Exercise:

Problem: Solve: $\sqrt{4x+8}=6$.

Solution:

7

Note:
Solve a radical equation.

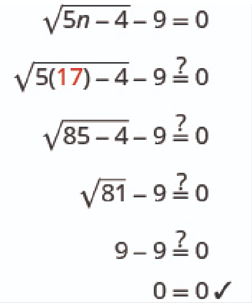
Isolate the radical on one side of the equation.
Square both sides of the equation.
Solve the new equation.
Check the answer.

Example:
Exercise:

Problem: Solve: $\sqrt{5n-4}-9=0$.

Solution:
Solution

	$\sqrt{5n-4}-9=0$
To isolate the radical, add 9 to both sides.	$\sqrt{5n-4}-9+9=0+9$
Simplify.	$\sqrt{5n-4}=9$
Square both sides of the equation.	$\left(\sqrt{5n-4}\right)^2=(9)^2$
Solve the new equation.	$5n-4=81$
	$5n=85$
	$n=17$
Check the answer.	

 $\sqrt{5n-4}-9=0$ $\sqrt{5(17)-4}-9\overset{?}{=}0$ $\sqrt{85-4}-9\overset{?}{=}0$ $\sqrt{81}-9\overset{?}{=}0$ $9-9\overset{?}{=}0$ $0=0$ ✓	
	The solution is *n* = 17.

Note:
Exercise:

Problem: Solve: $\sqrt{3m+2}-5=0$.

Solution:

$\frac{23}{3}$

Note:
Exercise:

Problem: Solve: $\sqrt{10z+1}-2=0$.

Solution:

$\frac{3}{10}$

Example:
Exercise:

Problem: Solve: $\sqrt{3y+5}+2=5$.

Solution:
Solution

	$\sqrt{3y+5}+2=5$
To isolate the radical, subtract 2 from both sides.	$\sqrt{3y+5}+2-2=5-2$
Simplify.	$\sqrt{3y+5}=3$
Square both sides of the equation.	$\left(\sqrt{3y+5}\right)^2=(3)^2$
Solve the new equation.	$3y+5=9$
	$3y=4$
	$y=\frac{4}{3}$
Check the answer.	
$\sqrt{3y+5}+2=5$ $\sqrt{3\left(\frac{4}{3}\right)+5}+2\overset{?}{=}5$ $\sqrt{4+5}+2\overset{?}{=}5$ $\sqrt{9}+2\overset{?}{=}5$ $3+2\overset{?}{=}5$ $5=5\checkmark$	
	The solution is $y=\frac{4}{3}$.

Note:
Exercise:

Problem: Solve: $\sqrt{3p+3}+3=5$.

Solution:

$\frac{1}{3}$

Note:
Exercise:

Problem: Solve: $\sqrt{5q+1}+4=6$.

Solution:

$\frac{3}{5}$

When we use a radical sign, we mean the principal or positive root. If an equation has a square root equal to a negative number, that equation will have no solution.

Example:
Exercise:

Problem: Solve: $\sqrt{9k-2}+1=0$.

Solution:
Solution

	$\sqrt{9k-2}+1=0$
To isolate the radical, subtract 1 from both sides.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$
Since the square root is equal to a negative number, the equation has no solution.	

Note:
Exercise:

Problem: Solve: $\sqrt{2r-3}+5=0$.

Solution:

no solution

Note:
Exercise:

Problem: Solve: $\sqrt{7s-3}+2=0$.

Solution:

no solution

If one side of the equation is a binomial, we use the binomial squares formula when we square it.

Note:
Binomial Squares
Equation:

$$(a+b)^2=a^2+2ab+b^2 \qquad (a-b)^2=a^2-2ab+b^2$$

Don’t forget the middle term!

Example:
Exercise:

Problem: Solve: $\sqrt{p-1}+1=p$.

Solution:
Solution

	$\sqrt{p-1}+1=p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2=(p-1)^2$
Simplify, then solve the new equation.	$p-1=p^2-2p+1$
It is a quadratic equation, so get zero on one side.	$0=p^2-3p+2$
Factor the right side.	$0=(p-1)(p-2)$
Use the zero product property.	$0=p-1 \quad 0=p-2$
Solve each equation.	$p=1 \quad p=2$
Check the answers.	
$p=1$ $\quad \sqrt{p-1}+1=p$ $\sqrt{1-1}+1\overset{?}{=}1$ $\sqrt{0}+1\overset{?}{=}1$ $1=1$ ✓ $p=2$ $\quad \sqrt{p-1}+1=p$ $\sqrt{2-1}+1\overset{?}{=}2$ $\sqrt{1}+1\overset{?}{=}2$ $2=2$ ✓	
	The solutions are $p=1$, $p=2$.

Note:
Exercise:

Problem: Solve: $\sqrt{x-2}+2=x$.

Solution:

$2, 3$

Note:
Exercise:

Problem: Solve: $\sqrt{y-5}+5=y$.

Solution:

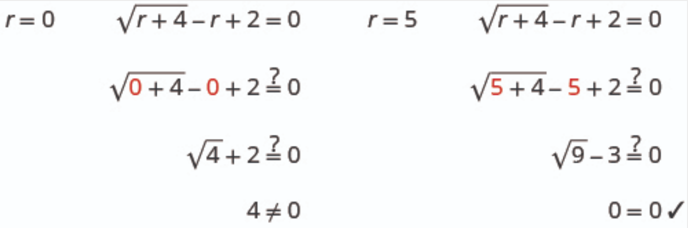
$5, 6$

Example:
Exercise:

Problem: Solve: $\sqrt{r+4}-r+2=0$.

Solution:
Solution

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	$\left(\sqrt{r+4}\right)^2=(r-2)^2$
Solve the new equation.	$r+4=r^2-4r+4$
It is a quadratic equation, so get zero on one side.	$0=r^2-5r$
Factor the right side.	$0=r(r-5)$
Use the zero product property.	$0=r \quad 0=r-5$
Solve the equation.	$r=0 \quad r=5$

Check the answer.	
 $r=0$ $\sqrt{r+4}-r+2=0$ $r=5$ $\sqrt{r+4}-r+2=0$ $\sqrt{0+4}-0+2\overset{?}{=}0$ $\sqrt{5+4}-5+2\overset{?}{=}0$ $\sqrt{4}+2\overset{?}{=}0$ $\sqrt{9}-3\overset{?}{=}0$ $4\neq 0$ $0=0$ ✓	The solution is $r=5$.
	$r=0$ is an extraneous solution.

Note:
Exercise:

Problem: Solve: $\sqrt{m+9}-m+3=0$.

Solution:

7

Note:
Exercise:

Problem: Solve: $\sqrt{n+1}-n+1=0$.

Solution:

3

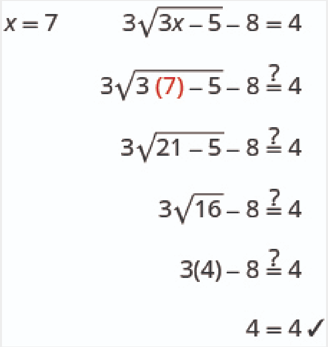
When there is a coefficient in front of the radical, we must square it, too.

Example:
Exercise:

Problem: Solve: $3\sqrt{3x-5}-8=4$.

Solution:

Solution

	$3\sqrt{3x-5}-8=4$
Isolate the radical.	$3\sqrt{3x-5}=12$
Square both sides of the equation.	$\left(3\sqrt{3x-5}\right)^2=(12)^2$
Simplify, then solve the new equation.	$9(3x-5)=144$
Distribute.	$27x-45=144$
Solve the equation.	$27x=189$
	$x=7$
Check the answer.	
 $x=7 \quad 3\sqrt{3x-5}-8=4$ $3\sqrt{3(7)-5}-8\overset{?}{=}4$ $3\sqrt{21-5}-8\overset{?}{=}4$ $3\sqrt{16}-8\overset{?}{=}4$ $3(4)-8\overset{?}{=}4$ $4=4$ ✓	The solution is $x=7$.

Note:
Exercise:

Problem: Solve: $2\sqrt{4a+2}-16=16$.

Solution:

$\frac{127}{2}$

Note:
Exercise:

Problem: Solve: $3\sqrt{6b+3}-25=50$.

Solution:

$\frac{311}{3}$

Example:
Exercise:

Problem: Solve: $\sqrt{4z-3}=\sqrt{3z+2}$.

Solution:
Solution

	$\sqrt{4z-3}=\sqrt{3z+2}$
The radical terms are isolated.	$\sqrt{4z-3}=\sqrt{3z+2}$
Square both sides of the equation.	$\left(\sqrt{4z-3}\right)^2=\left(\sqrt{3z+2}\right)^2$
Simplify, then solve the new equation.	$4z-3=3z+2$ $z-3=2$ $z=5$
Check the answer. We leave it to you to show that 5 checks!	The solution is $z=5$.

Note:
Exercise:

Problem: Solve: $\sqrt{4x-5}=\sqrt{2x+3}$.

Solution:

$x=4$

Note:
Exercise:

Problem: Solve: $\sqrt{7y-1}=\sqrt{2y-11}$.

Solution:

No solution; $y=-2$ is extraneous.

Sometimes after squaring both sides of an equation, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and square both sides of the equation again.

Example:
Exercise:

Problem: Solve: $\sqrt{m}+1=\sqrt{m+9}$.

Solution:
Solution

	$\sqrt{m}+1=\sqrt{m+9}$
The radical on the right side is isolated. Square both sides.	$\left(\sqrt{m}+1\right)^2=\left(\sqrt{m+9}\right)^2$
Simplify—be very careful as you multiply!	$m+2\sqrt{m}+1=m+9$
There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical.	$2\sqrt{m}=8$
Square both sides.	$\left(2\sqrt{m}\right)^2=(8)^2$
Simplify, then solve the new equation.	$4m=64$ $m=16$
Check the answer.	
We leave it to you to show that $m=16$ checks!	The solution is $m=16$.

Note:
Exercise:

Problem: Solve: $\sqrt{x}+3=\sqrt{x+5}$.

Solution:

no solution

Note:
Exercise:

Problem: Solve: $\sqrt{m} + 5 = \sqrt{m + 16}$.

Solution:

no solution

Example:
Exercise:

Problem: Solve: $\sqrt{q - 2} + 3 = \sqrt{4q + 1}$.

Solution:
Solution

	$\sqrt{q-2}+3=\sqrt{4q+1}$
The radical on the right side is isolated. Square both sides.	$\left(\sqrt{q-2}+3\right)^2=\left(\sqrt{4q+1}\right)^2$
Simplify.	$q-2+6\sqrt{q-2}+9=4q+1$
There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical.	$6\sqrt{q-2}=3q-6$
Square both sides.	$\left(6\sqrt{q-2}\right)^2=(3q-6)^2$
Simplify, then solve the new equation.	$36\left(q-2\right)=9q^2-36q+36$
Distribute.	$36q-72=9q^2-36q+36$
It is a quadratic equation, so get zero on one side.	$0=9q^2-72q+108$
Factor the right side.	$0=9\left(q^2-8q+12\right)$ $0=9\left(q-6\right)\left(q-2\right)$
Use the zero product property.	$q-6=0 \qquad q-2=0$ $q=6 \qquad q=2$
The checks are left to you. (Both solutions should work.)	The solutions are $q=6$ and $q=2$.

Note:
Exercise:

Problem: Solve: $\sqrt{y-3}+2=\sqrt{4y+2}$.

Solution:

no solution

Note:
Exercise:

Problem: Solve: $\sqrt{2n+2}+1=\sqrt{4n+5}$.

Solution:

$-1, 1$

Use Square Roots in Applications

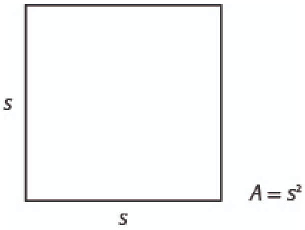
As you progress through your college courses, you'll encounter formulas that include square roots in many disciplines. We have already used formulas to solve geometry applications.

We will use our Problem Solving Strategy for Geometry Applications, with slight modifications, to give us a plan for solving applications with formulas from any discipline.

Note:
Solve applications with formulas.

Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
Identify what we are looking for.
Name what we are looking for by choosing a variable to represent it.
Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
Solve the equation using good algebra techniques.
Check the answer in the problem and make sure it makes sense.
Answer the question with a complete sentence.

We used the formula $A = L \cdot W$ to find the area of a rectangle with length L and width W. A square is a rectangle in which the length and width are equal. If we let s be the length of a side of a square, the area of the square is s^2.



The formula $A = s^2$ gives us the area of a square if we know the length of a side. What if we want to find the length of a side for a given area? Then we need to solve the equation for s.

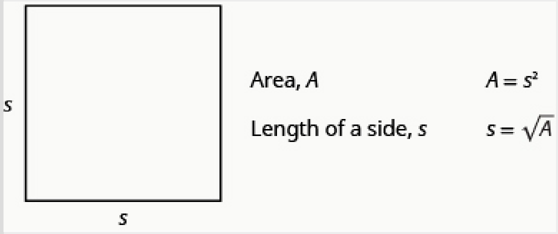
$$A = s^2$$

Take the square root of both sides. $\sqrt{A} = \sqrt{s^2}$

Simplify. $\sqrt{A} = s$

We can use the formula $s = \sqrt{A}$ to find the length of a side of a square for a given area.

Note:
Area of a Square



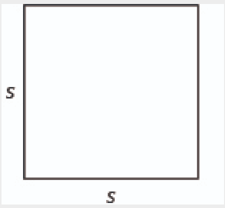
We will show an example of this in the next example.

Example:
Exercise:

Problem:

Mike and Lychelle want to make a square patio. They have enough concrete to pave an area of 200 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of the patio. Round your answer to the nearest tenth of a foot.

Solution:
Solution

Step 1. Read the problem. Draw a figure and label it with the given information.	
	A = 200 square feet
Step 2. Identify what you are looking for.	The length of a side of the square patio.
Step 3. Name what you are looking for by choosing a variable to represent it.	Let s = the length of a side.
Step 4. Translate into an equation by writing the	

appropriate formula or model for the situation. Substitute the given information.	$s = \sqrt{A}$, and $A = 200$ $s = \sqrt{200}$
Step 5. Solve the equation using good algebra techniques. Round to one decimal place.	$s = 14.14213\ldots$ $s \approx 14.1$
Step 6. Check the answer in the problem and make sure it makes sense.	
$14.1^2 \overset{?}{\approx} 200$ $14.1^2 \approx 198.81$ ✓	
This is close enough because we rounded the square root. Is a patio with side 14.1 feet reasonable? Yes.	
Step 7. Answer the question with a complete sentence.	Each side of the patio should be 14.1 feet.

Note:
Exercise:

Problem:

Katie wants to plant a square lawn in her front yard. She has enough sod to cover an area of 370 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of her lawn. Round your answer to the nearest tenth of a foot.

Solution:

19.2 feet

Note:
Exercise:

Problem:

Sergio wants to make a square mosaic as an inlay for a table he is building. He has enough tile to cover an area of 2704 square centimeters. Use the formula $s = \sqrt{A}$ to find the length of each side of his mosaic. Round your answer to the nearest tenth of a centimeter.

Solution:

52.0 cm

Another application of square roots has to do with gravity.

Note:
Falling Objects
On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula,
Equation:

$$t = \frac{\sqrt{h}}{4}$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h = 64$ into the formula.

	$t = \frac{\sqrt{h}}{4}$
	$t = \frac{\sqrt{64}}{4}$
Take the square root of 64.	$t = \frac{8}{4}$
Simplify the fraction.	$t = 2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

Example:

Exercise:

Problem:

Christy dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	The time it takes for the sunglasses to reach the river.
Step 3. Name what you are looking for by choosing a variable to represent it.	Let *t* = time.
Step 4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.	$t = \frac{\sqrt{h}}{4}$, and $h = 400$ $t = \frac{\sqrt{400}}{4}$
Step 5. Solve the equation using good algebra techniques.	$t = \frac{20}{4}$ $t = 5$
Step 6. Check the answer in the problem and make sure it makes sense.	
$5 \overset{?}{=} \frac{\sqrt{400}}{4}$ $5 \overset{?}{=} \frac{20}{4}$ $5 = 5✓$	

Does 5 seconds seem reasonable? Yes.	
Step 7. Answer the question with a complete sentence.	It will take 5 seconds for the sunglasses to hit the water.

Note:
Exercise:

Problem:

A helicopter dropped a rescue package from a height of 1,296 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

Solution:

9 seconds

Note:
Exercise:

Problem:

A window washer dropped a squeegee from a platform 196 feet above the sidewalk Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Solution:

3.5 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

Note:
Skid Marks and Speed of a Car
If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula,
Equation:

$$s = \sqrt{24d}$$

Example:
Exercise:

Problem:

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:
Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what we are looking for.	The speed of a car.
Step 3. **Name** what we are looking for.	Let s = the speed.
Step 4. **Translate** into an equation by writing the appropriate formula.	$s=\sqrt{24d}$, and $d=190$
Substitute the given information.	$s=\sqrt{24(190)}$
Step 5. **Solve the equation**.	$s=\sqrt{4560}$
	$s=67.52777...$
Round to 1 decimal place.	$s\approx 67.5$
Step 6. **Check** the answer in the problem. $67.5\overset{?}{\approx}\sqrt{24\,(190)}$ $67.5\overset{?}{\approx}\sqrt{4560}$ $67.5\overset{?}{\approx}67.5277...$	
Is 67.5 mph a reasonable speed?	Yes.
Step 7. **Answer** the question with a complete sentence.	The speed of the car was approximately 67.5 miles per hour.

Note:
Exercise:

Problem:

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:

42.7 feet

Note:
Exercise:

Problem:

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Solution:

54.1 feet

Key Concepts

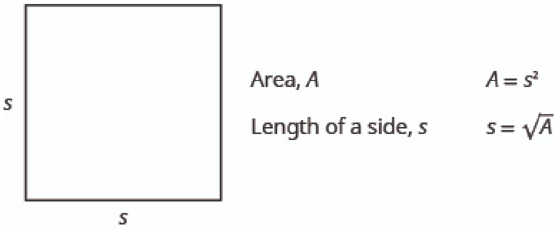
- **To Solve a Radical Equation:**

  Isolate the radical on one side of the equation.
  Square both sides of the equation.
  Solve the new equation.
  Check the answer. Some solutions obtained may not work in the original equation.

- **Solving Applications with Formulas**

  Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
  Identify what we are looking for.
  Name what we are looking for by choosing a variable to represent it.
  Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  Solve the equation using good algebra techniques.
  Check the answer in the problem and make sure it makes sense.
  Answer the question with a complete sentence.

- **Area of a Square**



- **Falling Objects**
    - On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.
- **Skid Marks and Speed of a Car**
    - If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

Practice Makes Perfect

Solve Radical Equations

In the following exercises, check whether the given values are solutions.

Exercise:

For the equation $\sqrt{x+12} = x$:
ⓐ Is $x = 4$ a solution?
Problem: ⓑ Is $x = -3$ a solution?

Solution:

ⓐ yes ⓑ no

Exercise:

For the equation $\sqrt{-y+20} = y$:
ⓐ Is $y = 4$ a solution?
Problem: ⓑ Is $y = -5$ a solution?

Exercise:

For the equation $\sqrt{t+6} = t$:
ⓐ Is $t = -2$ a solution?
Problem: ⓑ Is $t = 3$ a solution?

Solution:

ⓐ no ⓑ yes

Exercise:

For the equation $\sqrt{u+42}=u$:
ⓐ Is $u=-6$ a solution?
Problem: ⓑ Is $u=7$ a solution?

In the following exercises, solve.

Exercise:

Problem: $\sqrt{5y+1}=4$

Solution:

3

Exercise:

Problem: $\sqrt{7z+15}=6$

Exercise:

Problem: $\sqrt{5x-6}=8$

Solution:

14

Exercise:

Problem: $\sqrt{4x-3}=7$

Exercise:

Problem: $\sqrt{2m-3}-5=0$

Solution:

14

Exercise:

Problem: $\sqrt{2n-1}-3=0$

Exercise:

Problem: $\sqrt{6v-2}-10=0$

Solution:

17

Exercise:

Problem: $\sqrt{4u+2}-6=0$

Exercise:

Problem: $\sqrt{5q+3}-4=0$

Solution:

$\frac{13}{5}$

Exercise:

Problem: $\sqrt{4m+2}+2=6$

Exercise:

Problem: $\sqrt{6n+1}+4=8$

Solution:

$\frac{5}{2}$

Exercise:

Problem: $\sqrt{2u-3}+2=0$

Exercise:

Problem: $\sqrt{5v-2}+5=0$

Solution:

no solution

Exercise:

Problem: $\sqrt{3z-5}+2=0$

Exercise:

Problem: $\sqrt{2m+1}+4=0$

Solution:

no solution

Exercise:

Problem: ⓐ $\sqrt{u-3}+3=u$
ⓑ $\sqrt{x+1}-x+1=0$

Exercise:

Problem: ⓐ $\sqrt{v-10}+10=v$
ⓑ $\sqrt{y+4}-y+2=0$

Solution:

ⓐ $10, 11$ ⓑ 5

Exercise:

Problem: ⓐ $\sqrt{r-1}-r=-1$
ⓑ $\sqrt{z+100}-z+10=0$

Exercise:

Problem: ⓐ $\sqrt{s-8}-s=-8$
ⓑ $\sqrt{w+25}-w+5=0$

Solution:

ⓐ $8, 9$ ⓑ 11

Exercise:

Problem: $3\sqrt{2x-3}-20=7$

Exercise:

Problem: $2\sqrt{5x+1}-8=0$

Solution:

3

Exercise:

Problem: $2\sqrt{8r+1}-8=2$

Exercise:

Problem: $3\sqrt{7y+1}-10=8$

Solution:

5

Exercise:

Problem: $\sqrt{3u-2}=\sqrt{5u+1}$

Exercise:

Problem: $\sqrt{4v+3}=\sqrt{v-6}$

Solution:

not a real number

Exercise:

Problem: $\sqrt{8+2r}=\sqrt{3r+10}$

Exercise:

Problem: $\sqrt{12c+6}=\sqrt{10-4c}$

Solution:

$\frac{1}{4}$

Exercise:

Problem: ⓐ $\sqrt{a}+2=\sqrt{a+4}$
ⓑ $\sqrt{b-2}+1=\sqrt{3b+2}$

Exercise:

Problem: ⓐ $\sqrt{r}+6=\sqrt{r+8}$
ⓑ $\sqrt{s-3}+2=\sqrt{s+4}$

Solution:

ⓐ no solution ⓑ $\frac{57}{16}$

Exercise:

Problem: ⓐ $\sqrt{u}+1=\sqrt{u+4}$
ⓑ $\sqrt{n-5}+4=\sqrt{3n+7}$

Exercise:

Problem: ⓐ $\sqrt{x}+10=\sqrt{x+2}$
ⓑ $\sqrt{y-2}+2=\sqrt{2y+4}$

Solution:

ⓐ no solution ⓑ 6

Exercise:

Problem: $\sqrt{2y+4}+6=0$

Exercise:

Problem: $\sqrt{8u+1}+9=0$

Solution:

no solution

Exercise:

Problem: $\sqrt{a}+1=\sqrt{a+5}$

Exercise:

Problem: $\sqrt{d}-2=\sqrt{d-20}$

Solution:

36

Exercise:

Problem: $\sqrt{6s+4}=\sqrt{8s-28}$

Exercise:

Problem: $\sqrt{9p+9}=\sqrt{10p-6}$

Solution:

15

Use Square Roots in Applications

In the following exercises, solve. Round approximations to one decimal place.

Exercise:

Problem:

Landscaping Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

Exercise:

Problem:

Landscaping Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.

Solution:

11.4 feet

Exercise:

Problem:

Gravity While putting up holiday decorations, Renee dropped a light bulb from the top of a 64 foot tall tree. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the light bulb to reach the ground.

Exercise:

Problem:

Gravity An airplane dropped a flare from a height of 1024 feet above a lake. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the flare to reach the water.

Solution:

8 seconds

Exercise:

Problem:

Gravity A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.

Exercise:

Problem:

Gravity A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.

Solution:

15.8 seconds

Exercise:

Problem:

Accident investigation The skid marks for a car involved in an accident measured 54 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Exercise:

Problem:

Accident investigation The skid marks for a car involved in an accident measured 216 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution:

72 feet

Exercise:

Problem:

Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Exercise:

Problem:

Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 117 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Solution:

53.0 feet

Writing Exercises

Exercise:

Problem: Explain why an equation of the form $\sqrt{x} + 1 = 0$ has no solution.

Exercise:

Problem:

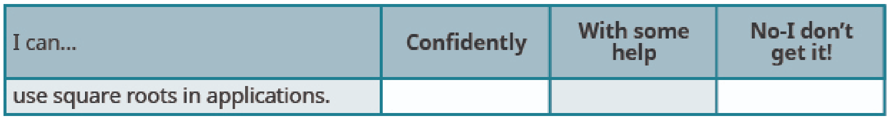
ⓐ Solve the equation $\sqrt{r+4} - r + 2 = 0$.
ⓑ Explain why one of the “solutions” that was found was not actually a solution to the equation.

Solution:

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don’t get it!
use square roots in applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

radical equation	An equation in which the variable is in the radicand of a square root is called a radical equation

Higher Roots

By the end of this section, you will be able to:

- Simplify expressions with higher roots
- Use the Product Property to simplify expressions with higher roots
- Use the Quotient Property to simplify expressions with higher roots
- Add and subtract higher roots

Note:
Before you get started, take this readiness quiz.

1. Simplify: y^5y^4.
   If you missed this problem, review [link].
2. Simplify: $\left(n^2\right)^6$.
   If you missed this problem, review [link].
3. Simplify: $\frac{x^8}{x^3}$.
   If you missed this problem, review [link].

Simplify Expressions with Higher Roots

Up to now, in this chapter we have worked with squares and square roots. We will now extend our work to include higher powers and higher roots.

Let's review some vocabulary first.

We write:	**We say:**
n^2	n squared
n^3	n cubed
n^4	n to the fourth
n^5	n to the fifth

The terms 'squared' and 'cubed' come from the formulas for area of a square and volume of a cube.

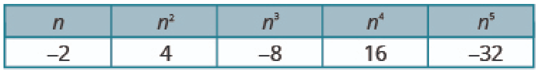
It will be helpful to have a table of the powers of the integers from -5 to 5. See [link].

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
–1	1	–1	1	–1
–2	4	–8	16	–32
–3	9	–27	81	–243
–4	16	–64	256	–1024
–5	25	–125	625	–3125

First through fifth powers of integers from -5 to 5.

Notice the signs in [link]. All powers of positive numbers are positive, of course. But when we have a negative number, the even powers are positive and the odd powers are negative. We'll copy the row with the powers of -2 below to help you see this.



n	n^2	n^3	n^4	n^5
–2	4	–8	16	–32

Earlier in this chapter we defined the square root of a number.

Equation:

$$\text{If } n^2 = m, \text{ then } n \text{ is a square root of } m.$$

And we have used the notation $\sqrt{m}$ to denote the **principal square root**. So $\sqrt{m} \geq 0$ always.

We will now extend the definition to higher roots.

Note:

*n*th Root of a Number

If $b^n = a$, then b is an ***n*th root of a number** a.

The principal *n*th root of a is written $\sqrt[n]{a}$.

n is called the **index** of the radical.

We do not write the index for a square root. Just like we use the word 'cubed' for b^3, we use the term 'cube root' for $\sqrt[3]{a}$.

We refer to [link] to help us find higher roots.

Equation:

$$\begin{array}{rclcrcl} 4^3 & = & 64 & \qquad & \sqrt[3]{64} & = & 4 \\ 3^4 & = & 81 & & \sqrt[4]{81} & = & 3 \\ (-2)^5 & = & -32 & & \sqrt[5]{-32} & = & -2 \end{array}$$

Could we have an even root of a negative number? No. We know that the square root of a negative number is not a real number. The same is true for any even root. Even roots of negative numbers are not real numbers. Odd roots of negative numbers are real numbers.

Note:

Properties of $\sqrt[n]{a}$

When n is an even number and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number

- $a < 0$, then $\sqrt[n]{a}$ is not a real number

When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{8}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{32}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{8}$
Since $(2)^3 = 8$.	2

ⓑ

	$\sqrt[4]{81}$
Since $(3)^4 = 81$.	3

ⓒ

	$\sqrt[5]{32}$
Since $(2)^5 = 32$.	2

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{27}$ ⓑ $\sqrt[4]{256}$ ⓒ $\sqrt[5]{243}$.

Solution:

ⓐ 3 ⓑ 4 ⓒ 3

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{1000}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{32}$.

Solution:

ⓐ 10 ⓑ 2 ⓒ 2

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-64}$ ⓑ $\sqrt[4]{-16}$ ⓒ $\sqrt[5]{-243}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{-64}$
Since $(-4)^3 = -64$.	-4

ⓑ

	$\sqrt[4]{-16}$
Think, $(?)^4 = -16.$ No real number raised to the fourth power is negative.	Not a real number.

ⓒ

	$\sqrt[5]{-243}$
Since $(-3)^5 = -243$.	-3

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-125}$ ⓑ $\sqrt[4]{-16}$ ⓒ $\sqrt[5]{-32}$.

Solution:

ⓐ -5 ⓑ not real ⓒ -2

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-216}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-1024}$.

Solution:

ⓐ -6 ⓑ not real ⓒ -4

When we worked with square roots that had variables in the radicand, we restricted the variables to non-negative values. Now we will remove this restriction.

The odd root of a number can be either positive or negative. We have seen that $\sqrt[3]{-64} = -4$.

But the even root of a non-negative number is always non-negative, because we take the principal *n*th root.

Suppose we start with $a = -5$.
Equation:

$$(-5)^4 = 625 \qquad \sqrt[4]{625} = 5$$

How can we make sure the fourth root of −5 raised to the fourth power, $(-5)^4$ is 5? We will see in the following property.

Note:
Simplifying Odd and Even Roots
For any integer $n \geq 2$,
Equation:

$$\begin{array}{ll} \text{when } n \text{ is odd} & \sqrt[n]{a^n} = a \\ \text{when } n \text{ is even} & \sqrt[n]{a^n} = |a| \end{array}$$

We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt{x^2}$ ⓑ $\sqrt[3]{n^3}$ ⓒ $\sqrt[4]{p^4}$ ⓓ $\sqrt[5]{y^5}$.

Solution:
Solution

We use the absolute value to be sure to get the positive root.

ⓐ

$$\begin{array}{ll} & \sqrt{x^2} \\ \text{Since } (x)^2 = x^2 \text{ and we want the positive root.} & |x| \end{array}$$

ⓑ

$$\begin{array}{ll} & \sqrt[3]{n^3} \\ \text{Since } (n)^3 = n^3\text{. It is an odd root so there is no need for an absolute value sign.} & n \end{array}$$

ⓒ

$$\begin{array}{ll} & \sqrt[4]{p^4} \\ \text{Since } (p)^4 = p^4 \text{ and we want the positive root.} & |p| \end{array}$$

ⓓ

$$\sqrt[5]{y^5}$$

Since $(y)^5 = y^5$. It is an odd root so there is no need for an absolute value sign.

$$y$$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{b^2}$ ⓑ $\sqrt[3]{w^3}$ ⓒ $\sqrt[4]{m^4}$ ⓓ $\sqrt[5]{q^5}$.

Solution:

ⓐ $|b|$ ⓑ w ⓒ $|m|$ ⓓ q

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt{y^2}$ ⓑ $\sqrt[3]{p^3}$ ⓒ $\sqrt[4]{z^4}$ ⓓ $\sqrt[5]{q^5}$.

Solution:

ⓐ $|y|$ ⓑ p ⓒ $|z|$ ⓓ q

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{y^{18}}$ ⓑ $\sqrt[4]{z^8}$.

Solution:
Solution

ⓐ

$$\sqrt[3]{y^{18}}$$

Since $\left(y^6\right)^3 = y^{18}$.

$$\sqrt[3]{\left(y^6\right)^3}$$

$$y^6$$

ⓑ

	$\sqrt[4]{z^8}$
Since $\left(z^2\right)^4 = z^8$.	$\sqrt[4]{\left(z^2\right)^4}$
Since z^2 is positive, we do not need an absolute value sign.	z^2

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[4]{u^{12}}$ ⓑ $\sqrt[3]{v^{15}}$.

Solution:

ⓐ u^3 ⓑ v^5

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[5]{c^{20}}$ ⓑ $\sqrt[6]{d^{24}}$.

Solution:

ⓐ c^4 ⓑ d^4

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{64p^6}$ ⓑ $\sqrt[4]{16q^{12}}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{64p^6}$
Rewrite $64p^6$ as $\left(4p^2\right)^3$.	$\sqrt[3]{\left(4p^2\right)^3}$
Take the cube root.	$4p^2$

ⓑ

	$\sqrt[4]{16q^{12}}$
Rewrite the radicand as a fourth power.	$\sqrt[4]{(2q^3)^4}$
Take the fourth root.	$2\left\|q^3\right\|$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{27x^{27}}$ ⓑ $\sqrt[4]{81q^{28}}$.

Solution:

ⓐ $3x^9$ ⓑ $3\left|q^7\right|$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{125p^9}$ ⓑ $\sqrt[5]{243q^{25}}$.

Solution:

ⓐ $5p^3$ ⓑ $3q^5$

Use the Product Property to Simplify Expressions with Higher Roots

We will simplify expressions with higher roots in much the same way as we simplified expressions with square roots. An *n*th root is considered simplified if it has no factors of m^n.

Note:
Simplified *n*th Root
$\sqrt[n]{a}$ is considered simplified if a has no factors of m^n.

We will generalize the Product Property of Square Roots to include any integer root $n \geq 2$.

Note:
Product Property of *n*th Roots
Equation:

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

when $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and for any integer $n \geq 2$

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{x^4}$ ⓑ $\sqrt[4]{x^7}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{x^4}$
Rewrite the radicand as a product using the largest perfect cube factor.	$\sqrt[3]{x^3 \cdot x}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{x^3} \cdot \sqrt[3]{x}$
Simplify.	$x\sqrt[3]{x}$

ⓑ

	$\sqrt[4]{x^7}$
Rewrite the radicand as a product using the greatest perfect fourth power factor.	$\sqrt[4]{x^4 \cdot x^3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$
Simplify.	$\lvert x\rvert\sqrt[4]{x^3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[4]{y^6}$ ⓑ $\sqrt[3]{z^5}$.

Solution:

ⓐ $\lvert y\rvert\sqrt[4]{y^2}$ ⓑ $z\sqrt[3]{z^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[5]{p^8}$ ⓑ $\sqrt[6]{q^{13}}$.

Solution:

ⓐ $p\sqrt[5]{p^3}$ ⓑ $q^2\sqrt[6]{q}$

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{16}$ ⓑ $\sqrt[4]{243}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{16}$
	$\sqrt[3]{2^4}$
Rewrite the radicand as a product using the greatest perfect cube factor.	$\sqrt[3]{2^3 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{2^3} \cdot \sqrt[3]{2}$
Simplify.	$2\sqrt[3]{2}$

ⓑ

	$\sqrt[4]{243}$
	$\sqrt[4]{3^5}$
Rewrite the radicand as a product using the greatest perfect fourth power factor.	$\sqrt[4]{3^4 \cdot 3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{3^4} \cdot \sqrt[4]{3}$
Simplify.	$3\sqrt[4]{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{81}$ ⓑ $\sqrt[4]{64}$.

Solution:

ⓐ $3\sqrt[3]{3}$ ⓑ $2\sqrt[4]{4}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[4]{729}$.

Solution:

ⓐ $5\sqrt[3]{5}$ ⓑ $3\sqrt[4]{9}$

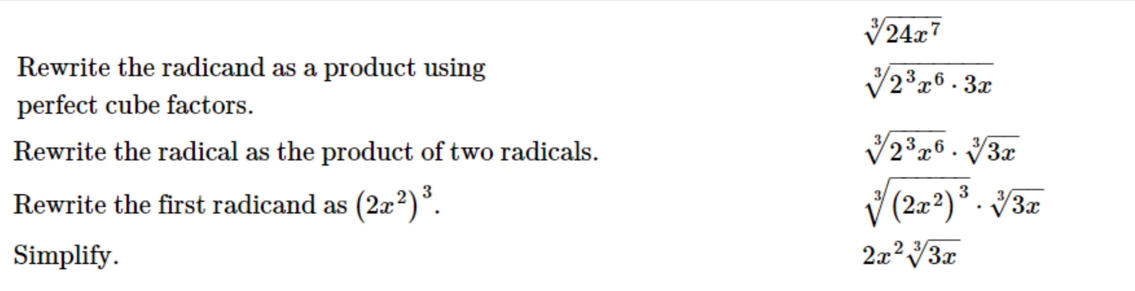
Don’t forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{24x^7}$ ⓑ $\sqrt[4]{80y^{14}}$.

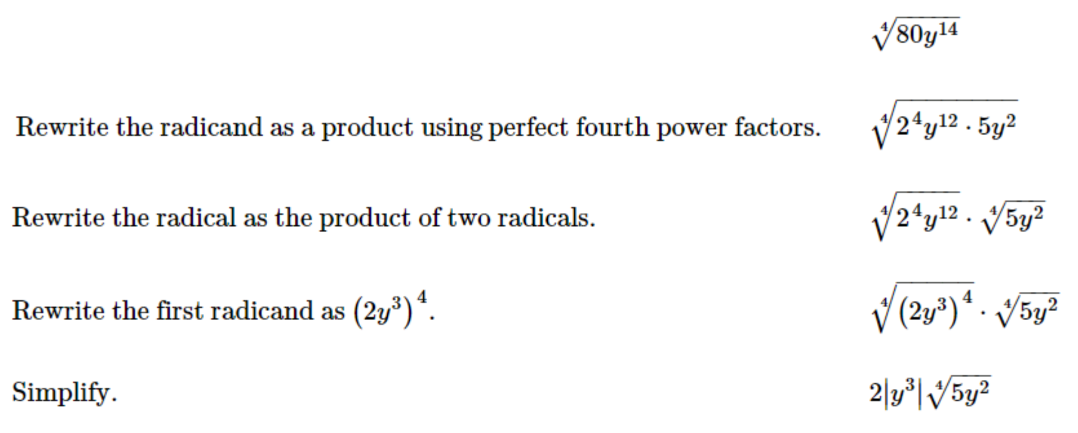
Solution:
Solution

ⓐ



	$\sqrt[3]{24x^7}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{2^3x^6\cdot 3x}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{2^3x^6}\cdot\sqrt[3]{3x}$
Rewrite the first radicand as $\left(2x^2\right)^3$.	$\sqrt[3]{\left(2x^2\right)^3}\cdot\sqrt[3]{3x}$
Simplify.	$2x^2\sqrt[3]{3x}$

ⓑ



	$\sqrt[4]{80y^{14}}$
Rewrite the radicand as a product using perfect fourth power factors.	$\sqrt[4]{2^4y^{12}\cdot 5y^2}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{2^4y^{12}}\cdot\sqrt[4]{5y^2}$
Rewrite the first radicand as $\left(2y^3\right)^4$.	$\sqrt[4]{\left(2y^3\right)^4}\cdot\sqrt[4]{5y^2}$
Simplify.	$2\lvert y^3\rvert\sqrt[4]{5y^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{54p^{10}}$ ⓑ $\sqrt[4]{64q^{10}}$.

Solution:

ⓐ $3p^3\sqrt[3]{2p}$ ⓑ $2q^2\sqrt[4]{4q^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{128m^{11}}$ ⓑ $\sqrt[4]{162n^7}$.

Solution:

ⓐ $4m^3\sqrt[3]{2m^2}$ ⓑ $3\left|n\right|\sqrt[4]{2n^3}$

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-16}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{-27}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3}$
Take the cube root.	-3

ⓑ

	$\sqrt[4]{-16}$
There is no real number n where $n^4 = -16$.	Not a real number.

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-108}$ ⓑ $\sqrt[4]{-48}$.

Solution:

ⓐ $-3\sqrt[3]{4}$ ⓑ not real

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{-625}$ ⓑ $\sqrt[4]{-324}$.

Solution:

ⓐ $-5\sqrt[3]{5}$ ⓑ not real

Use the Quotient Property to Simplify Expressions with Higher Roots

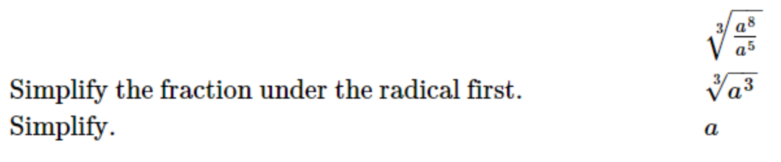
We can simplify higher roots with quotients in the same way we simplified square roots. First we simplify any fractions inside the radical.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{\frac{a^8}{a^5}}$ ⓑ $\sqrt[4]{\frac{a^{10}}{a^2}}$.

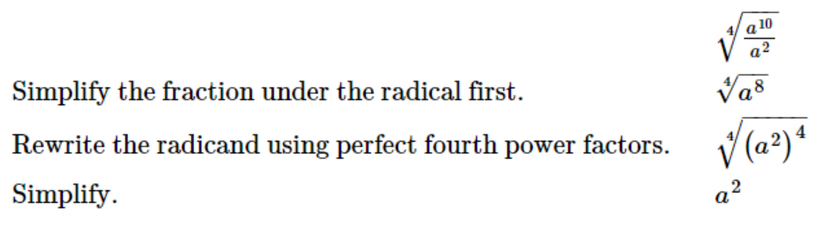
Solution:
Solution

ⓐ



	$\sqrt[3]{\frac{a^8}{a^5}}$
Simplify the fraction under the radical first.	$\sqrt[3]{a^3}$
Simplify.	a

ⓑ



	$\sqrt[4]{\frac{a^{10}}{a^2}}$
Simplify the fraction under the radical first.	$\sqrt[4]{a^8}$
Rewrite the radicand using perfect fourth power factors.	$\sqrt[4]{\left(a^2\right)^4}$
Simplify.	a^2

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[4]{\frac{x^7}{x^3}}$ ⓑ $\sqrt[4]{\frac{y^{17}}{y^5}}$.

Solution:

ⓐ $|x|$ ⓑ y^3

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{\frac{m^{13}}{m^7}}$ ⓑ $\sqrt[5]{\frac{n^{12}}{n^2}}$.

Solution:

ⓐ m^2 ⓑ n^2

Previously, we used the Quotient Property ‘in reverse’ to simplify square roots. Now we will generalize the formula to include higher roots.

Note:
Quotient Property of *n*th Roots
Equation:

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

when $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0,$ and for any integer $n \geq 2$

Example:

Exercise:

Problem: Simplify: ⓐ $\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$ ⓑ $\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$.

Solution:
Solution

ⓐ

	$\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$
Neither radicand is a perfect cube, so use the Quotient Property to write as one radical.	$\sqrt[3]{\frac{-108}{2}}$
Simplify the fraction under the radical.	$\sqrt[3]{-54}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{(-3)^3} \cdot \sqrt[3]{2}$
Simplify.	$-3\sqrt[3]{2}$

ⓑ

	$\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$
Neither radicand is a perfect fourth power, so use the Quotient Property to write as one radical.	$\sqrt[4]{\frac{96x^7}{3x^2}}$
Simplify the fraction under the radical.	$\sqrt[4]{32x^5}$
Rewrite the radicand as a product using perfect fourth power factors.	$\sqrt[4]{2^4x^4 \cdot 2x}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{(2x)^4} \cdot \sqrt[4]{2x}$
Simplify.	$2\left\|x\right\|\sqrt[4]{2x}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{\sqrt[3]{-532}}{\sqrt[3]{2}}$ ⓑ $\frac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}}$.

Solution:

ⓐ $-\sqrt[3]{266}$ ⓑ $3\,|m|\sqrt[4]{2m^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{\sqrt[3]{-192}}{\sqrt[3]{3}}$ ⓑ $\frac{\sqrt[4]{324n^7}}{\sqrt[4]{2n^3}}$.

Solution:

ⓐ -4 ⓑ $3\,|n|\sqrt[4]{2}$

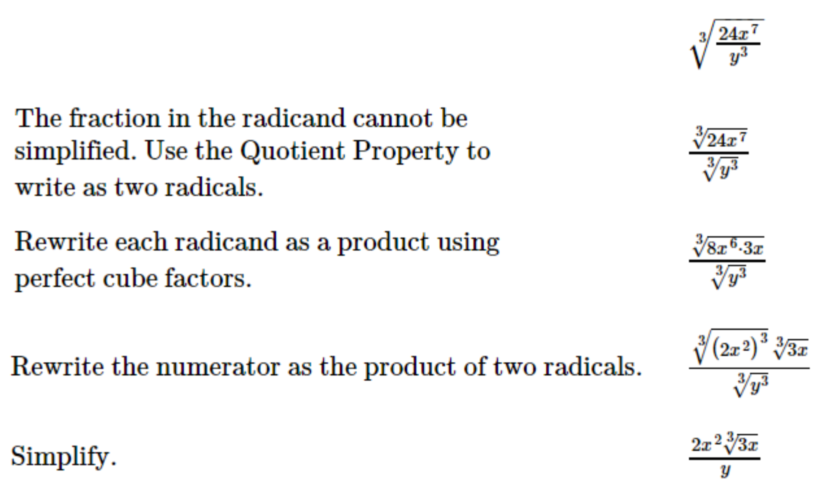
If the fraction inside the radical cannot be simplified, we use the first form of the Quotient Property to rewrite the expression as the quotient of two radicals.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{\frac{24x^7}{y^3}}$ ⓑ $\sqrt[4]{\frac{48x^{10}}{y^8}}$.

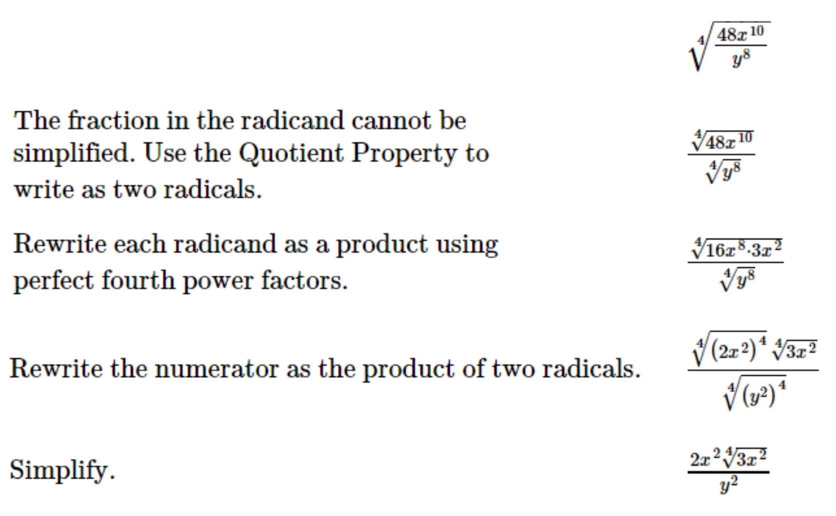
Solution:
Solution

ⓐ



	$\sqrt[3]{\frac{24x^7}{y^3}}$
The fraction in the radicand cannot be simplified. Use the Quotient Property to write as two radicals.	$\frac{\sqrt[3]{24x^7}}{\sqrt[3]{y^3}}$
Rewrite each radicand as a product using perfect cube factors.	$\frac{\sqrt[3]{8x^6\cdot 3x}}{\sqrt[3]{y^3}}$
Rewrite the numerator as the product of two radicals.	$\frac{\sqrt[3]{(2x^2)^3}\,\sqrt[3]{3x}}{\sqrt[3]{y^3}}$
Simplify.	$\frac{2x^2\sqrt[3]{3x}}{y}$

ⓑ



	$\sqrt[4]{\frac{48x^{10}}{y^8}}$
The fraction in the radicand cannot be simplified. Use the Quotient Property to write as two radicals.	$\frac{\sqrt[4]{48x^{10}}}{\sqrt[4]{y^8}}$
Rewrite each radicand as a product using perfect fourth power factors.	$\frac{\sqrt[4]{16x^8 \cdot 3x^2}}{\sqrt[4]{y^8}}$
Rewrite the numerator as the product of two radicals.	$\frac{\sqrt[4]{(2x^2)^4}\sqrt[4]{3x^2}}{\sqrt[4]{(y^2)^4}}$
Simplify.	$\frac{2x^2\sqrt[4]{3x^2}}{y^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{\frac{108c^{10}}{d^6}}$ ⓑ $\sqrt[4]{\frac{80x^{10}}{y^5}}$.

Solution:

ⓐ $\frac{3c^3\sqrt[3]{4c}}{d^2}$ ⓑ $\frac{2x^2}{|y|}\sqrt[4]{\frac{5x^2}{y}}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{\frac{40r^3}{s}}$ ⓑ $\sqrt[4]{\frac{162m^{14}}{n^{12}}}$.

Solution:

ⓐ $2r\sqrt[3]{\frac{5}{s}}$ ⓑ $\frac{3m^3\sqrt[4]{2m^2}}{|n^3|}$

Add and Subtract Higher Roots

We can add and subtract higher roots like we added and subtracted square roots. First we provide a formal definition of like radicals.

Note:
Like Radicals
Radicals with the same index and same radicand are called **like radicals**.

Like radicals have the same index and the same radicand.

- $9\sqrt[4]{42x}$ and $-2\sqrt[4]{42x}$ are like radicals.
- $5\sqrt[3]{125x}$ and $6\sqrt[3]{125y}$ are not like radicals. The radicands are different.
- $2\sqrt[5]{1000q}$ and $-4\sqrt[4]{1000q}$ are not like radicals. The indices are different.

We add and subtract like radicals in the same way we add and subtract like terms. We can add $9\sqrt[4]{42x} + \left(-2\sqrt[4]{42x}\right)$ and the result is $7\sqrt[4]{42x}$.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{4x} + \sqrt[3]{4x}$ ⓑ $4\sqrt[4]{8} - 2\sqrt[4]{8}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{4x} + \sqrt[3]{4x}$
The radicals are like, so we add the coefficients.	$2\sqrt[3]{4x}$

ⓑ

	$4\sqrt[4]{8} - 2\sqrt[4]{8}$
The radicals are like, so we subtract the coefficients.	$2\sqrt[4]{8}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[5]{3x} + \sqrt[5]{3x}$ ⓑ $3\sqrt[3]{9} - \sqrt[3]{9}$.

Solution:

ⓐ $2\sqrt[5]{3x}$ ⓑ $2\sqrt[3]{9}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[4]{10y} + \sqrt[4]{10y}$ ⓑ $5\sqrt[6]{32} - 3\sqrt[6]{32}$.

Solution:

ⓐ $2\sqrt[4]{10y}$ ⓑ $2\sqrt[6]{32}$

When an expression does not appear to have like radicals, we will simplify each radical first. Sometimes this leads to an expression with like radicals.

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{54} - \sqrt[3]{16}$ ⓑ $\sqrt[4]{48} + \sqrt[4]{243}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{54} - \sqrt[3]{16}$
Rewrite each radicand using perfect cube factors.	$\sqrt[3]{27} \cdot \sqrt[3]{2} - \sqrt[3]{8} \cdot \sqrt[3]{2}$
Rewrite the perfect cubes.	$\sqrt[3]{(3)^3}\,\sqrt[3]{2} - \sqrt[3]{(2)^3}\,\sqrt[3]{2}$
Simplify the radicals where possible.	$3\sqrt[3]{2} - 2\sqrt[3]{2}$
Combine like radicals.	$\sqrt[3]{2}$

ⓑ

	$\sqrt[4]{48} + \sqrt[4]{243}$
Rewrite using perfect fourth power factors.	$\sqrt[4]{16} \cdot \sqrt[4]{3} + \sqrt[4]{81} \cdot \sqrt[4]{3}$
Rewrite the perfect fourth powers.	$\sqrt[4]{(2)^4}\,\sqrt[4]{3} + \sqrt[4]{(3)^4}\,\sqrt[4]{3}$
Simplify the radicals where possible.	$2\sqrt[4]{3} + 3\sqrt[4]{3}$
Combine like radicals.	$5\sqrt[4]{3}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{192}-\sqrt[3]{81}$ ⓑ $\sqrt[4]{32}+\sqrt[4]{512}$.

Solution:

ⓐ $\sqrt[3]{3}$ ⓑ $6\sqrt[4]{2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{128}-\sqrt[3]{250}$ ⓑ $\sqrt[5]{64}+\sqrt[5]{486}$.

Solution:

ⓐ $-\sqrt[3]{2}$ ⓑ $5\sqrt[5]{2}$

Example:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{24x^4}-\sqrt[3]{-81x^7}$ ⓑ $\sqrt[4]{162y^9}+\sqrt[4]{516y^5}$.

Solution:
Solution

ⓐ

	$\sqrt[3]{24x^4}-\sqrt[3]{-81x^7}$
Rewrite each radicand using perfect cube factors.	$\sqrt[3]{8x^3}\cdot\sqrt[3]{3x}-\sqrt[3]{-27x^6}\cdot\sqrt[3]{3x}$
Rewrite the perfect cubes.	$\sqrt[3]{(2x)^3}\,\sqrt[3]{3x}-\sqrt[3]{(-3x^2)^3}\,\sqrt[3]{3x}$
Simplify the radicals where possible.	$2x\sqrt[3]{3x}-\left(-3x^2\sqrt[3]{3x}\right)$

ⓑ

	$\sqrt[4]{162y^9}+\sqrt[4]{516y^5}$
Rewrite each radicand using perfect fourth power factors.	$\sqrt[4]{81y^8}\cdot\sqrt[4]{2y}+\sqrt[4]{256y^4}\cdot\sqrt[4]{2y}$
Rewrite the perfect fourth powers.	$\sqrt[4]{(3y^2)^4}\cdot\sqrt[4]{2y}+\sqrt[4]{(4y)^4}\cdot\sqrt[4]{2y}$
Simplify the radicals where possible.	$3y^2\sqrt[4]{2y}+4\left\|y\right\|\sqrt[4]{2y}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{32y^5} - \sqrt[3]{-108y^8}$ ⓑ $\sqrt[4]{243r^{11}} + \sqrt[4]{768r^{10}}$.

Solution:

ⓐ $2y\sqrt[3]{4y^2} + 3y^2\sqrt[3]{4y^2}$ ⓑ $3r^2\sqrt[4]{3r^3} + 4r^2\sqrt[4]{3r^2}$

Note:
Exercise:

Problem: Simplify: ⓐ $\sqrt[3]{40z^7} - \sqrt[3]{-135z^4}$ ⓑ $\sqrt[4]{80s^{13}} + \sqrt[4]{1280s^6}$.

Solution:

ⓐ $2z^2\sqrt[3]{5z} + 3z\sqrt[3]{5z}$ ⓑ $2\left|s^3\right|\sqrt[4]{5s} + 4\left|s\right|\sqrt[4]{5s^2}$

Note:
Access these online resources for additional instruction and practice with simplifying higher roots.

- Simplifying Higher Roots
- Add/Subtract Roots with Higher Indices

Key Concepts

- **Properties of**
- $\sqrt[n]{a}$ when n is an even number and
    - $a \geq 0$, then $\sqrt[n]{a}$ is a real number
    - $a < 0$, then $\sqrt[n]{a}$ is not a real number
    - When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.
    - For any integer $n \geq 2$, when n is odd $\sqrt[n]{a^n} = a$
    - For any integer $n \geq 2$, when n is even $\sqrt[n]{a^n} = |a|$
- $\sqrt[n]{a}$ is considered simplified if a has no factors of m^n.
- **Product Property of *n*th Roots**
  Equation:

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \text{ and } \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

- **Quotient Property of *n*th Roots**
  Equation:

$$\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}} \text{ and } \frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$$

- To combine like radicals, simply add or subtract the coefficients while keeping the radical the same.

Practice Makes Perfect

Simplify Expressions with Higher Roots

In the following exercises, simplify.

Exercise:

ⓐ $\sqrt[3]{216}$
ⓑ $\sqrt[4]{256}$
Problem: ⓒ $\sqrt[5]{32}$

Exercise:

ⓐ $\sqrt[3]{27}$
ⓑ $\sqrt[4]{16}$
Problem: ⓒ $\sqrt[5]{243}$

Solution:

ⓐ 3 ⓑ 2 ⓒ 3

Exercise:

ⓐ $\sqrt[3]{512}$
ⓑ $\sqrt[4]{81}$
Problem: ⓒ $\sqrt[5]{1}$

Exercise:

ⓐ $\sqrt[3]{125}$
ⓑ $\sqrt[4]{1296}$
Problem: ⓒ $\sqrt[5]{1024}$

Solution:

ⓐ 5 ⓑ 6 ⓒ 4

Exercise:

ⓐ $\sqrt[3]{-8}$
ⓑ $\sqrt[4]{-81}$
Problem: ⓒ $\sqrt[5]{-32}$

Exercise:

ⓐ $\sqrt[3]{-64}$
ⓑ $\sqrt[4]{-16}$
Problem: ⓒ $\sqrt[5]{-243}$

Solution:

ⓐ -4 ⓑ not real ⓒ -3

Exercise:

ⓐ $\sqrt[3]{-125}$
ⓑ $\sqrt[4]{-1296}$
Problem: ⓒ $\sqrt[5]{-1024}$

Exercise:

ⓐ $\sqrt[3]{-512}$
ⓑ $\sqrt[4]{-81}$
Problem: ⓒ $\sqrt[5]{-1}$

Solution:

ⓐ -8 ⓑ not a real number ⓒ -1

Exercise:

ⓐ $\sqrt[5]{u^5}$
Problem: ⓑ $\sqrt[8]{v^8}$

Exercise:

ⓐ $\sqrt[3]{a^3}$
Problem: ⓑ

$\sqrt[12]{b^{12}}$

Solution:

ⓐ a ⓑ $|b|$

Exercise:

Problem:
ⓐ $\sqrt[4]{y^4}$
ⓑ $\sqrt[7]{m^7}$

Exercise:

Problem:
ⓐ $\sqrt[8]{k^8}$
ⓑ $\sqrt[6]{p^6}$

Solution:

ⓐ $|k|$ ⓑ $|p|$

Exercise:

Problem:
ⓐ $\sqrt[3]{x^9}$
ⓑ $\sqrt[4]{y^{12}}$

Exercise:

Problem:
ⓐ $\sqrt[5]{a^{10}}$
ⓑ $\sqrt[3]{b^{27}}$

Solution:

ⓐ a^2 ⓑ b^9

Exercise:

Problem:
ⓐ $\sqrt[4]{m^8}$
ⓑ $\sqrt[5]{n^{20}}$

Exercise:

Problem:
ⓐ $\sqrt[6]{r^{12}}$
ⓑ $\sqrt[3]{s^{30}}$

Solution:

ⓐ r^2 ⓑ s^{10}

Exercise:

Problem: ⓐ $\sqrt[4]{16x^8}$
ⓑ $\sqrt[6]{64y^{12}}$

Exercise:

Problem: ⓐ $\sqrt[3]{-8c^9}$
ⓑ $\sqrt[3]{125d^{15}}$

Solution:

ⓐ $-2c^3$ ⓑ $5d^5$

Exercise:

Problem: ⓐ $\sqrt[3]{216a^6}$
ⓑ $\sqrt[5]{32b^{20}}$

Exercise:

Problem: ⓐ $\sqrt[7]{128r^{14}}$
ⓑ $\sqrt[4]{81s^{24}}$

Solution:

ⓐ $2r^2$ ⓑ $3s^6$

Use the Product Property to Simplify Expressions with Higher Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\sqrt[3]{r^5}$ ⓑ $\sqrt[4]{s^{10}}$

Exercise:

Problem: ⓐ $\sqrt[5]{u^7}$ ⓑ $\sqrt[6]{v^{11}}$

Solution:

ⓐ $u\sqrt[5]{u^2}$ ⓑ $v\sqrt[6]{v^5}$

Exercise:

Problem: ⓐ $\sqrt[4]{m^5}$ ⓑ $\sqrt[8]{n^{10}}$

Exercise:

Problem: ⓐ $\sqrt[5]{p^8}$ ⓑ $\sqrt[3]{q^8}$

Solution:

ⓐ $p\sqrt[5]{p^3}$ ⓑ $q^2\sqrt[3]{q^2}$

Exercise:

Problem: ⓐ $\sqrt[4]{32}$ ⓑ $\sqrt[5]{64}$

Exercise:

Problem: ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[6]{128}$

Solution:

ⓐ $5\sqrt[3]{5}$ ⓑ $2\sqrt[6]{2}$

Exercise:

Problem: ⓐ $\sqrt[5]{64}$ ⓑ $\sqrt[3]{256}$

Exercise:

Problem: ⓐ $\sqrt[4]{3125}$ ⓑ $\sqrt[3]{81}$

Solution:

ⓐ $5\sqrt[4]{5}$ ⓑ $3\sqrt[3]{3}$

Exercise:

Problem: ⓐ $\sqrt[3]{108x^5}$ ⓑ $\sqrt[4]{48y^6}$

Exercise:

Problem: ⓐ $\sqrt[5]{96a^7}$ ⓑ $\sqrt[3]{375b^4}$

Solution:

ⓐ $2a\sqrt[5]{3a^2}$ ⓑ $5b\sqrt[3]{3b}$

Exercise:

Problem: ⓐ $\sqrt[4]{405m^{10}}$ ⓑ $\sqrt[5]{160n^8}$

Exercise:

Problem: ⓐ $\sqrt[3]{512p^5}$ ⓑ $\sqrt[4]{324q^7}$

Solution:

ⓐ $8p\sqrt[3]{p^2}$ ⓑ $3q\sqrt[4]{4q^3}$

Exercise:

Problem: ⓐ $\sqrt[3]{-864}$ ⓑ $\sqrt[4]{-256}$

Exercise:

Problem: ⓐ $\sqrt[5]{-486}$ ⓑ $\sqrt[6]{-64}$

Solution:

ⓐ $-3\sqrt[5]{2}$ ⓑ not real

Exercise:

Problem: ⓐ $\sqrt[5]{-32}$ ⓑ $\sqrt[8]{-1}$

Exercise:

Problem: ⓐ $\sqrt[3]{-8}$ ⓑ $\sqrt[4]{-16}$

Solution:

ⓐ -2 ⓑ not real

Use the Quotient Property to Simplify Expressions with Higher Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\sqrt[3]{\frac{p^{11}}{p^2}}$ ⓑ $\sqrt[4]{\frac{q^{17}}{q^{13}}}$

Exercise:

Problem: ⓐ $\sqrt[5]{\frac{d^{12}}{d^7}}$ ⓑ $\sqrt[8]{\frac{m^{12}}{m^4}}$

Solution:

ⓐ d ⓑ $|m|$

Exercise:

Problem: ⓐ $\sqrt[5]{\frac{u^{21}}{u^{11}}}$ ⓑ $\sqrt[6]{\frac{v^{30}}{v^{12}}}$

Exercise:

Problem: ⓐ $\sqrt[3]{\frac{r^{14}}{r^5}}$ ⓑ $\sqrt[4]{\frac{c^{21}}{c^9}}$

Solution:

ⓐ r^2 ⓑ $\left|c^3\right|$

Exercise:

Problem: ⓐ $\frac{\sqrt[4]{64}}{\sqrt[4]{2}}$ ⓑ $\frac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

Exercise:

Problem: ⓐ $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$ ⓑ $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

Solution:

ⓐ -5 ⓑ $2m\sqrt[4]{m^2}$

Exercise:

Problem: ⓐ $\sqrt[3]{\frac{1050}{2}}$ ⓑ $\sqrt[4]{\frac{486y^9}{2y^3}}$

Exercise:

Problem: ⓐ $\sqrt[3]{\frac{162}{6}}$ ⓑ $\sqrt[4]{\frac{160r^{10}}{5r^3}}$

Solution:

ⓐ $3\sqrt[3]{6}$ ⓑ $2\left|r\right|\sqrt[4]{2r^3}$

Exercise:

Problem: ⓐ $\sqrt[3]{\frac{54a^8}{b^3}}$ ⓑ $\sqrt[4]{\frac{64c^5}{d^{12}}}$

Exercise:

Problem: ⓐ $\sqrt[5]{\frac{96r^{11}}{s^{15}}}$ ⓑ $\sqrt[6]{\frac{128u^7}{v^6}}$

Solution:

ⓐ $\frac{2r^2\sqrt[5]{3r}}{s^3}$ ⓑ $\frac{2u\sqrt[6]{2u}}{v}$

Exercise:

Problem: ⓐ $\sqrt[3]{\frac{81s^8}{t^3}}$ ⓑ $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$

Exercise:

Problem: ⓐ $\sqrt[3]{\frac{625u^{10}}{v^3}}$ ⓑ $\sqrt[4]{\frac{729c^{21}}{d^8}}$

Solution:

ⓐ $\frac{5u^3\sqrt[3]{5u}}{v}$ ⓑ $\frac{3c^5\sqrt[4]{9c}}{d^2}$

Add and Subtract Higher Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\sqrt[7]{8p}+\sqrt[7]{8p}$
ⓑ $3\sqrt[3]{25}-\sqrt[3]{25}$

Exercise:

Problem: ⓐ $\sqrt[3]{15q}+\sqrt[3]{15q}$
ⓑ $2\sqrt[4]{27}-6\sqrt[4]{27}$

Solution:

ⓐ $2\sqrt[3]{15q}$ ⓑ $-4\sqrt[4]{27}$

Exercise:

Problem: ⓐ $3\sqrt[5]{9x}+7\sqrt[5]{9x}$
ⓑ $8\sqrt[7]{3q}-2\sqrt[7]{3q}$

Exercise:

Problem: ⓐ

$23\sqrt[12]{4y}+19\sqrt[12]{4y}$

ⓑ

$31\sqrt[10]{5z}-17\sqrt[10]{5z}$

Solution:

ⓐ

$42\sqrt[12]{4y}$

ⓑ

$14\sqrt[10]{5z}$

Exercise:

ⓐ $\sqrt[3]{81}-\sqrt[3]{192}$
Problem: ⓑ $\sqrt[4]{512}-\sqrt[4]{32}$

Exercise:

ⓐ $\sqrt[3]{250}-\sqrt[3]{54}$
Problem: ⓑ $\sqrt[4]{243}-\sqrt[4]{1875}$

Solution:

ⓐ $2\sqrt[3]{2}$ ⓑ $-2\sqrt[4]{3}$

Exercise:

ⓐ $\sqrt[3]{128}+\sqrt[3]{250}$
Problem: ⓑ $\sqrt[5]{729}+\sqrt[5]{96}$

Exercise:

ⓐ $\sqrt[4]{243}+\sqrt[4]{1250}$
Problem: ⓑ $\sqrt[3]{2000}+\sqrt[3]{54}$

Solution:

ⓐ $3\sqrt[4]{3}+5\sqrt[4]{2}$ ⓑ $13\sqrt[3]{2}$

Exercise:

ⓐ $\sqrt[3]{64a^{10}}-\sqrt[3]{-216a^{12}}$
Problem: ⓑ $\sqrt[4]{486u^{7}}+\sqrt[4]{768u^{3}}$

Exercise:

ⓐ $\sqrt[3]{80b^{5}}-\sqrt[3]{-270b^{3}}$
Problem: ⓑ $\sqrt[4]{160v^{10}}-\sqrt[4]{1280v^{3}}$

Solution:

ⓐ $2b\sqrt[3]{10b^{2}}+3b\sqrt[3]{10}$ ⓑ $2v^{2}\sqrt[4]{10v^{2}}-4\sqrt[4]{5v^{3}}$

Mixed Practice

In the following exercises, simplify.
Exercise:

Problem: $\sqrt[4]{16}$

Exercise:

Problem: $\sqrt[6]{64}$

Solution:

2

Exercise:

Problem: $\sqrt[3]{a^3}$

Exercise:

Problem:

$\sqrt[12]{b^{12}}$

Solution:

$|b|$

Exercise:

Problem: $\sqrt[3]{-8c^9}$

Exercise:

Problem: $\sqrt[3]{125d^{15}}$

Solution:

$5d^5$

Exercise:

Problem: $\sqrt[3]{r^5}$

Exercise:

Problem: $\sqrt[4]{s^{10}}$

Solution:

$s^2\sqrt[4]{s^2}$

Exercise:

Problem: $\sqrt[3]{108x^5}$

Exercise:

Problem: $\sqrt[4]{48y^6}$

Solution:

$2y\sqrt[4]{3y^2}$

Exercise:

Problem: $\sqrt[5]{-486}$

Exercise:

Problem: $\sqrt[6]{-64}$

Solution:

not real

Exercise:

Problem: $\frac{\sqrt[4]{64}}{\sqrt[4]{2}}$

Exercise:

Problem: $\frac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

Solution:

$2x\sqrt[5]{2x}$

Exercise:

Problem: $\sqrt[5]{\frac{96r^{11}}{s^3}}$

Exercise:

Problem: $\sqrt[6]{\frac{256u^{19}}{v^{18}}}$

Solution:

$\frac{2u^3\sqrt[6]{4u}}{v^3}$

Exercise:

Problem: $\sqrt[3]{81} - \sqrt[3]{192}$

Exercise:

Problem: $\sqrt[4]{512} - \sqrt[4]{32}$

Solution:

$2\sqrt[4]{2}$

Exercise:

Problem: $\sqrt[3]{64a^{10}} - \sqrt[3]{-216a^{12}}$

Exercise:

Problem: $\sqrt[4]{486u^7} + \sqrt[4]{768u^3}$

Solution:

$3u\sqrt[4]{6u^3} + 4\sqrt[4]{3u^3}$

Everyday Math

Exercise:

Problem:

Population growth The expression $10 \cdot x^n$ models the growth of a mold population after n generations. There were 10 spores at the start, and each had x offspring. So $10 \cdot x^n$ is the number of offspring at the fifth generation. At the fifth generation there were 10,240 offspring. Simplify the expression $\sqrt[5]{\frac{10{,}240}{10}}$ to determine the number of offspring of each spore.

Exercise:

Problem:

Spread of a virus The expression $3 \cdot x^n$ models the spread of a virus after n cycles. There were three people originally infected with the virus, and each of them infected x people. So $3 \cdot x^4$ is the number of people infected on the fourth cycle. At the fourth cycle 1875 people were infected. Simplify the expression $\sqrt[4]{\frac{1875}{3}}$ to determine the number of people each person infected.

Solution:

5

Writing Exercises

Exercise:

Problem: Explain how you know that $\sqrt[5]{x^{10}} = x^2$.

Exercise:

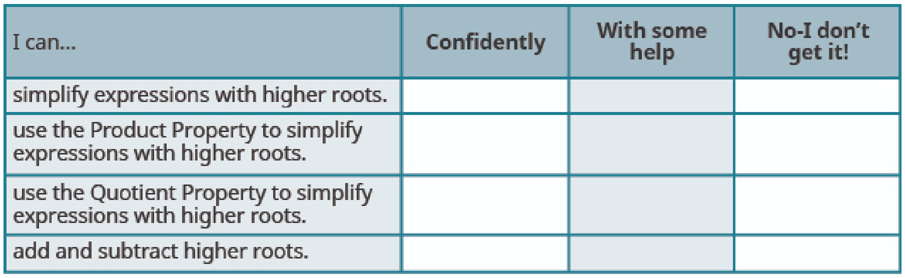
Problem: Explain why $\sqrt[4]{-64}$ is not a real number but $\sqrt[3]{-64}$ is.

Solution:

Answers may vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
simplify expressions with higher roots.			
use the Product Property to simplify expressions with higher roots.			
use the Quotient Property to simplify expressions with higher roots.			
add and subtract higher roots.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

nth root of a number
If $b^n = a$, then b is an nth root of a.

principal nth root
The principal nth root of a is written $\sqrt[n]{a}$.

index
$\sqrt[n]{a}$ n is called the *index* of the radical.

like radicals
Radicals with the same index and same radicand are called like radicals.

Rational Exponents

By the end of this section, you will be able to:

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the Laws of Exponents to simply expressions with rational exponents

Note:

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{15}+\frac{5}{12}$.
   If you missed this problem, review [link].
2. Simplify: $\left(4x^2y^5\right)^3$.
   If you missed this problem, review [link].
3. Simplify: 5^{-3}.
   If you missed this problem, review [link].

Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use **rational exponents**, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $\left(a^m\right)^n=a^{m\cdot n}$ when *m* and *n* are whole numbers. Let's assume we are now not limited to whole numbers.

Suppose we want to find a number *p* such that $\left(8^p\right)^3=8$. We will use the Power Property of Exponents to find the value of *p*.

$$\begin{aligned}
& & (8^p)^3 &= 8\\
&\text{Multiply the exponents on the left.} & 8^{3p} &= 8\\
&\text{Write the exponent 1 on the right.} & 8^{3p} &= 8^1\\
&\text{The exponents must be equal.} & 3p &= 1\\
&\text{Solve for } p. & p &= \frac{1}{3}
\end{aligned}$$

$$\text{So } \left(8^{\frac{1}{3}}\right)^3 = 8.$$

But we know also $\left(\sqrt[3]{8}\right)^3 = 8$. Then it must be that $8^{\frac{1}{3}} = \sqrt[3]{8}$.

This same logic can be used for any positive integer exponent *n* to show that $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Note:
Rational Exponent $a^{\frac{1}{n}}$
If $\sqrt[n]{a}$ is a real number and $n \geq 2$, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you’ll practice converting expressions between these two notations.

Example:
Exercise:

Problem: Write as a radical expression: ⓐ $x^{\frac{1}{2}}$ ⓑ $y^{\frac{1}{3}}$ ⓒ $z^{\frac{1}{4}}$.

Solution:

Solution

We want to write each expression in the form $\sqrt[n]{a}$.

ⓐ

$$x^{\frac{1}{2}}$$

The denominator of the exponent is 2, so the index of the radical is 2. We do not show the index when it is 2.

$$\sqrt{x}$$

ⓑ

$$y^{\frac{1}{3}}$$

The denominator of the exponent is 3, so the index is 3.

$$\sqrt[3]{y}$$

ⓒ

$$z^{\frac{1}{4}}$$

The denominator of the exponent is 4, so the index is 4.

$$\sqrt[4]{z}$$

Note:
Exercise:

Problem: Write as a radical expression: ⓐ $t^{\frac{1}{2}}$ ⓑ $m^{\frac{1}{3}}$ ⓒ $r^{\frac{1}{4}}$.

Solution:

ⓐ $\sqrt{t}$ ⓑ $\sqrt[3]{m}$ ⓒ $\sqrt[4]{r}$

Note:

Exercise:

Problem: Write as a radial expression: ⓐ $b^{\frac{1}{2}}$ ⓑ $z^{\frac{1}{3}}$ ⓒ $p^{\frac{1}{4}}$.

Solution:

ⓐ $\sqrt{b}$ ⓑ $\sqrt[3]{z}$ ⓒ $\sqrt[4]{p}$

Example:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{x}$ ⓑ $\sqrt[3]{y}$ ⓒ $\sqrt[4]{z}$.

Solution:
Solution

We want to write each radical in the form $a^{\frac{1}{n}}$.

ⓐ

$\sqrt{x}$

No index is shown, so it is 2.
The denominator of the exponent will be 2.

$x^{\frac{1}{2}}$

ⓑ

$\sqrt[3]{y}$

The index is 3, so the denominator of the exponent is 3.

$y^{\frac{1}{3}}$

ⓒ

$\sqrt[4]{z}$

The index is 4, so the denominator of the exponent is 4.

$z^{\frac{1}{4}}$

Note:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{s}$ ⓑ $\sqrt[3]{x}$ ⓒ $\sqrt[4]{b}$.

Solution:

ⓐ $s^{\frac{1}{2}}$ ⓑ $x^{\frac{1}{3}}$ ⓒ $b^{\frac{1}{4}}$

Note:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{v}$ ⓑ $\sqrt[3]{p}$ ⓒ $\sqrt[4]{p}$.

Solution:

ⓐ $v^{\frac{1}{2}}$ ⓑ $p^{\frac{1}{3}}$ ⓒ $p^{\frac{1}{4}}$

Example:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{5y}$ ⓑ $\sqrt[3]{4x}$ ⓒ $3\sqrt[4]{5z}$.

Solution:
Solution

We want to write each radical in the form $a^{\frac{1}{n}}$.

ⓐ

$\sqrt{5y}$

No index is shown, so it is 2.
The denominator of the exponent will be 2.

$(5y)^{\frac{1}{2}}$

ⓑ

$\sqrt[3]{4x}$

The index is 3, so the denominator of the exponent is 3.

$(4x)^{\frac{1}{3}}$

ⓒ

$3\sqrt[4]{5z}$

The index is 4, so the denominator of the exponent is 4.

$3(5z)^{\frac{1}{4}}$

Note:
Exercise:

Problem:

Write with a rational exponent: ⓐ $\sqrt{10m}$ ⓑ $\sqrt[5]{3n}$ ⓒ $3\sqrt[4]{6y}$.

Solution:

ⓐ $(10m)^{\frac{1}{2}}$ ⓑ $(3n)^{\frac{1}{5}}$ ⓒ $3(6y)^{\frac{1}{4}}$

Note:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt[7]{3k}$ ⓑ $\sqrt[4]{5j}$ ⓒ $8\sqrt[3]{2a}$.

Solution:

ⓐ $(3k)^{\frac{1}{7}}$ ⓑ $(5j)^{\frac{1}{4}}$ ⓒ $8(2a)^{\frac{1}{3}}$

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

Example:
Exercise:

Problem: Simplify: ⓐ $25^{\frac{1}{2}}$ ⓑ $64^{\frac{1}{3}}$ ⓒ $256^{\frac{1}{4}}$.

Solution:
Solution

ⓐ

	$25^{\frac{1}{2}}$
Rewrite as a square root.	$\sqrt{25}$
Simplify.	5

ⓑ

	$64^{\frac{1}{3}}$
Rewrite as a cube root.	$\sqrt[3]{64}$
Recognize 64 is a perfect cube.	$\sqrt[3]{4^3}$
Simplify.	4

ⓒ

	$256^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{256}$
Recognize 256 is a perfect fourth power.	$\sqrt[4]{4^4}$
Simplify.	4

Note:
Exercise:

Problem: Simplify: ⓐ $36^{\frac{1}{2}}$ ⓑ $8^{\frac{1}{3}}$ ⓒ $16^{\frac{1}{4}}$.

Solution:

ⓐ 6 ⓑ 2 ⓒ 2

Note:
Exercise:

Problem: Simplify: ⓐ $100^{\frac{1}{2}}$ ⓑ $27^{\frac{1}{3}}$ ⓒ $81^{\frac{1}{4}}$.

Solution:

ⓐ 10 ⓑ 3 ⓒ 3

Be careful of the placement of the negative signs in the next example. We will need to use the property $a^{-n} = \frac{1}{a^n}$ in one case.

Example:
Exercise:

Problem: Simplify: ⓐ $(-64)^{\frac{1}{3}}$ ⓑ $-64^{\frac{1}{3}}$ ⓒ $(64)^{-\frac{1}{3}}$.

Solution:
Solution

ⓐ

	$(-64)^{\frac{1}{3}}$
Rewrite as a cube root.	$\sqrt[3]{-64}$
Rewrite -64 as a perfect cube.	$\sqrt[3]{(-4)^3}$
Simplify.	-4

ⓑ

	$-64^{\frac{1}{3}}$
The exponent applies only to the 64.	$-\left(64^{\frac{1}{3}}\right)$
Rewrite as a cube root.	$-\sqrt[3]{64}$
Rewrite 64 as 4^3.	$-\sqrt[3]{4^3}$
Simplify.	-4

ⓒ

	$(64)^{-\frac{1}{3}}$
Rewrite as a fraction with a positive exponent, using the property, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\sqrt[3]{64}}$
Write as a cube root.	
Rewrite 64 as 4^3.	$\frac{1}{\sqrt[3]{4^3}}$
Simplify.	$\frac{1}{4}$

Note:
Exercise:

Problem: Simplify: ⓐ $(-125)^{\frac{1}{3}}$ ⓑ $-125^{\frac{1}{3}}$ ⓒ $(125)^{-\frac{1}{3}}$.

Solution:

ⓐ -5 ⓑ -5 ⓒ $\frac{1}{5}$

Note:
Exercise:

Problem: Simplify: ⓐ $(-32)^{\frac{1}{5}}$ ⓑ $-32^{\frac{1}{5}}$ ⓒ $(32)^{-\frac{1}{5}}$.

Solution:

ⓐ -2 ⓑ -2 ⓒ $\frac{1}{2}$

Example:
Exercise:

Problem: Simplify: ⓐ $(-16)^{\frac{1}{4}}$ ⓑ $-16^{\frac{1}{4}}$ ⓒ $(16)^{-\frac{1}{4}}$.

Solution:
Solution

ⓐ

$(-16)^{\frac{1}{4}}$

Rewrite as a fourth root. $\sqrt[4]{-16}$

There is no real number whose fourth power is -16.

ⓑ

	$-16^{\frac{1}{4}}$
The exponent only applies to the 16. Rewrite as a fourth root.	$-\sqrt[4]{16}$
Rewrite 16 as 2^4.	$-\sqrt[4]{2^4}$
Simplify.	-2

ⓒ

	$(16)^{-\frac{1}{4}}$
Rewrite using the property $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{(16)^{\frac{1}{4}}}$
Rewrite as a fourth root.	$\frac{1}{\sqrt[4]{16}}$
Rewrite 16 as 2^4.	$\frac{1}{\sqrt[4]{2^4}}$
Simplify.	$\frac{1}{2}$

Note:
Exercise:

Problem: Simplify: ⓐ $(-64)^{\frac{1}{2}}$ ⓑ $-64^{\frac{1}{2}}$ ⓒ $(64)^{-\frac{1}{2}}$.

Solution:

ⓐ not a real number ⓑ -8 ⓒ $\frac{1}{8}$

Note:
Exercise:

Problem: Simplify: ⓐ $(-256)^{\frac{1}{4}}$ ⓑ $-256^{\frac{1}{4}}$ ⓒ $(256)^{-\frac{1}{4}}$.

Solution:

ⓐ not a real number ⓑ -4 ⓒ $\frac{1}{4}$

Simplify Expressions with $a^{\frac{m}{n}}$

Let’s work with the Power Property for Exponents some more.

Suppose we raise $a^{\frac{1}{n}}$ to the power *m*.

$$\left(a^{\frac{1}{n}}\right)^{m}$$

Multiply the exponents. $a^{\frac{1}{n}\cdot m}$

Simplify. $a^{\frac{m}{n}}$

So $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^{m}$.

Now suppose we take a^{m} to the $\frac{1}{n}$ power.

$$(a^{m})^{\frac{1}{n}}$$

Multiply the exponents. $a^{m\cdot\frac{1}{n}}$

Simplify. $a^{\frac{m}{n}}$

So $a^{\frac{m}{n}} = \sqrt[n]{a^{m}}$ also.

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller.

Note:

Rational Exponent $a^{\frac{m}{n}}$

For any positive integers *m* and *n*,

Equation:

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$$
$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

Example:

Exercise:

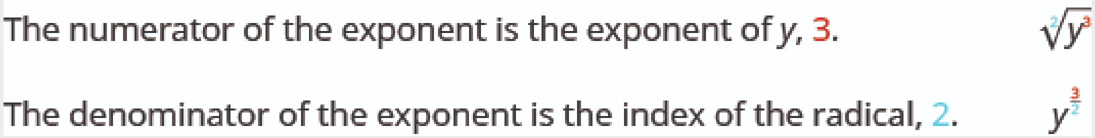
Problem: Write with a rational exponent: ⓐ $\sqrt{y^3}$ ⓑ $\sqrt[3]{x^2}$ ⓒ $\sqrt[4]{z^3}$.

Solution:

Solution

We want to use $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ to write each radical in the form $a^{\frac{m}{n}}$.

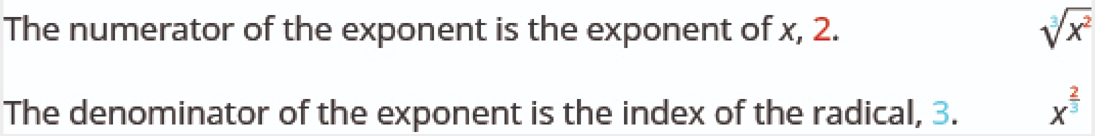
ⓐ



The numerator of the exponent is the exponent of *y*, 3. $\sqrt[2]{y^3}$

The denominator of the exponent is the index of the radical, 2. $y^{\frac{3}{2}}$

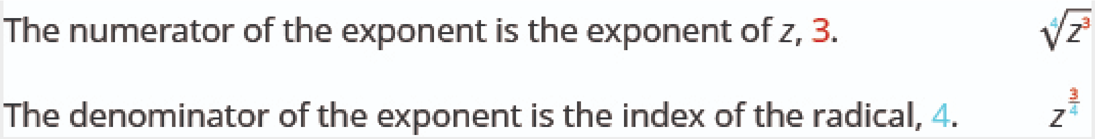
ⓑ



The numerator of the exponent is the exponent of *x*, 2. $\sqrt[3]{x^2}$

The denominator of the exponent is the index of the radical, 3. $x^{\frac{2}{3}}$

ⓒ



The numerator of the exponent is the exponent of z, 3. $\sqrt[4]{z^3}$

The denominator of the exponent is the index of the radical, 4. $z^{\frac{3}{4}}$

Note:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt{x^5}$ ⓑ $\sqrt[4]{z^3}$ ⓒ $\sqrt[5]{y^2}$.

Solution:

ⓐ $x^{\frac{5}{2}}$ ⓑ $z^{\frac{3}{4}}$ ⓒ $y^{\frac{2}{5}}$

Note:
Exercise:

Problem: Write with a rational exponent: ⓐ $\sqrt[5]{a^2}$ ⓑ $\sqrt[3]{b^7}$ ⓒ $\sqrt[4]{m^5}$.

Solution:

ⓐ $a^{\frac{2}{5}}$ ⓑ $b^{\frac{7}{3}}$ ⓒ $m^{\frac{5}{4}}$

Example:
Exercise:

Problem: Simplify: ⓐ $9^{\frac{3}{2}}$ ⓑ $125^{\frac{2}{3}}$ ⓒ $81^{\frac{3}{4}}$.

Solution:

Solution

We will rewrite each expression as a radical first using the property, $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

ⓐ

	$9^{\frac{3}{2}}$
The power of the radical is the numerator of the exponent, 3. Since the denominator of the exponent is 2, this is a square root.	$\left(\sqrt{9}\right)^3$
Simplify.	$(3)^3$ 27

ⓑ

	$125^{\frac{2}{3}}$
The power of the radical is the numerator of the exponent, 2. The index of the radical is the denominator of the exponent, 3.	$\left(\sqrt[3]{125}\right)^2$
Simplify.	$(5)^2$ 25

ⓒ

	$81^{\frac{3}{4}}$
The power of the radical is the numerator of the exponent, 3. The index of the radical is the denominator of the exponent, 4.	$\left(\sqrt[4]{81}\right)^3$
Simplify.	$(3)^3$ 27

Note:
Exercise:

Problem: Simplify: ⓐ $4^{\frac{3}{2}}$ ⓑ $27^{\frac{2}{3}}$ ⓒ $625^{\frac{3}{4}}$.

Solution:

ⓐ 8 ⓑ 9 ⓒ 125

Note:
Exercise:

Problem: Simplify: ⓐ $8^{\frac{5}{3}}$ ⓑ $81^{\frac{3}{2}}$ ⓒ $16^{\frac{3}{4}}$.

Solution:

ⓐ 32 ⓑ 729 ⓒ 8

Remember that $b^{-p} = \frac{1}{b^p}$. The negative sign in the exponent does not change the sign of the expression.

Example:
Exercise:

Problem: Simplify: ⓐ $16^{-\frac{3}{2}}$ ⓑ $32^{-\frac{2}{5}}$ ⓒ $4^{-\frac{5}{2}}$.

Solution:
Solution

We will rewrite each expression first using $b^{-p} = \frac{1}{b^p}$ and then change to radical form.

ⓐ

$$16^{-\frac{3}{2}}$$

Rewrite using $b^{-p} = \frac{1}{b^p}$.

$$\frac{1}{16^{\frac{3}{2}}}$$

Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2.

$$\frac{1}{\left(\sqrt{16}\right)^3}$$

Simplify.

$$\frac{1}{4^3}$$

$$\frac{1}{64}$$

ⓑ

$$32^{-\frac{2}{5}}$$

Rewrite using $b^{-p} = \frac{1}{b^p}$.

$$\frac{1}{32^{\frac{2}{5}}}$$

Change to radical form.

$$\frac{1}{\left(\sqrt[5]{32}\right)^2}$$

Rewrite the radicand as a power.

$$\frac{1}{\left(\sqrt[5]{2^5}\right)^2}$$

Simplify.

$$\frac{1}{2^2}$$

$$\frac{1}{4}$$

ⓒ

	$4^{-\frac{5}{2}}$
Rewrite using $b^{-p} = \frac{1}{b^p}$.	$\frac{1}{4^{\frac{5}{2}}}$
Change to radical form.	$\frac{1}{\left(\sqrt{4}\right)^5}$
Simplify.	$\frac{1}{2^5}$ $\frac{1}{32}$

Note:
Exercise:

Problem: Simplify: ⓐ $8^{-\frac{5}{3}}$ ⓑ $81^{-\frac{3}{2}}$ ⓒ $16^{-\frac{3}{4}}$.

Solution:

ⓐ $\frac{1}{32}$ ⓑ $\frac{1}{729}$ ⓒ $\frac{1}{8}$

Note:
Exercise:

Problem: Simplify: ⓐ $4^{-\frac{3}{2}}$ ⓑ $27^{-\frac{2}{3}}$ ⓒ $625^{-\frac{3}{4}}$.

Solution:

ⓐ $\frac{1}{8}$ ⓑ $\frac{1}{9}$ ⓒ $\frac{1}{125}$

Example:
Exercise:

Problem: Simplify: ⓐ $-25^{\frac{3}{2}}$ ⓑ $-25^{-\frac{3}{2}}$ ⓒ $(-25)^{\frac{3}{2}}$.

Solution:
Solution

ⓐ

	$-25^{\frac{3}{2}}$
Rewrite in radical form.	$-\left(\sqrt{25}\right)^{3}$
Simplify the radical.	$-(5)^{3}$
Simplify.	-125

ⓑ

	$-25^{-\frac{3}{2}}$
Rewrite using $b^{-p} = \frac{1}{b^{p}}$.	$-\left(\frac{1}{25^{\frac{3}{2}}}\right)$
Rewrite in radical form.	$-\left(\frac{1}{\left(\sqrt{25}\right)^{3}}\right)$
Simplify the radical.	$-\left(\frac{1}{(5)^{3}}\right)$
Simplify.	$-\frac{1}{125}$

ⓒ

	$(-25)^{\frac{3}{2}}$
Rewrite in radical form.	$\left(\sqrt{-25}\right)^3$
There is no real number whose square root is -25.	Not a real number.

Note:
Exercise:

Problem: Simplify: ⓐ $-16^{\frac{3}{2}}$ ⓑ $-16^{-\frac{3}{2}}$ ⓒ $(-16)^{-\frac{3}{2}}$.

Solution:

ⓐ -64 ⓑ $-\frac{1}{64}$ ⓒ not a real number

Note:
Exercise:

Problem: Simplify: ⓐ $-81^{\frac{3}{2}}$ ⓑ $-81^{-\frac{3}{2}}$ ⓒ $(-81)^{-\frac{3}{2}}$.

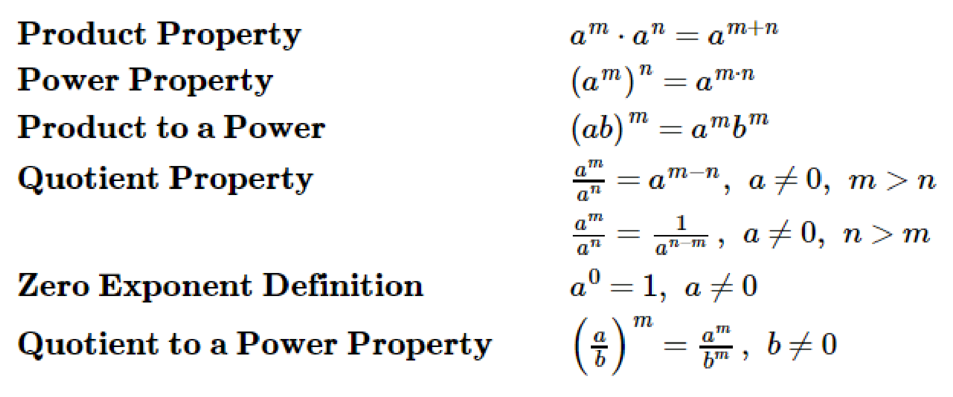
Solution:

ⓐ -729 ⓑ $-\frac{1}{729}$ ⓒ not a real number

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

The same laws of exponents that we already used apply to rational exponents, too. We will list the Exponent Properties here to have them for reference as we simplify expressions.

Note:
Summary of Exponent Properties
If a, b are real numbers and m, n are rational numbers, then



Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n},\ a \neq 0,\ m > n$
	$\frac{a^m}{a^n} = \frac{1}{a^{n-m}},\ a \neq 0,\ n > m$
Zero Exponent Definition	$a^0 = 1,\ a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0$

When we multiply the same base, we add the exponents.

Example:
Exercise:

Problem: Simplify: ⓐ $2^{\frac{1}{2}} \cdot 2^{\frac{5}{2}}$ ⓑ $x^{\frac{2}{3}} \cdot x^{\frac{4}{3}}$ ⓒ $z^{\frac{3}{4}} \cdot z^{\frac{5}{4}}$.

Solution:
Solution

ⓐ

	$2^{\frac{1}{2}} \cdot 2^{\frac{5}{2}}$
The bases are the same, so we add the exponents.	$2^{\frac{1}{2}+\frac{5}{2}}$
Add the fractions.	$2^{\frac{6}{2}}$
Simplify the exponent.	2^{3}
Simplify.	8

ⓑ

	$x^{\frac{2}{3}} \cdot x^{\frac{4}{3}}$
The bases are the same, so we add the exponents.	$x^{\frac{2}{3}+\frac{4}{3}}$
Add the fractions.	$x^{\frac{6}{3}}$
Simplify.	x^{2}

ⓒ

	$z^{\frac{3}{4}} \cdot z^{\frac{5}{4}}$
The bases are the same, so we add the exponents.	$z^{\frac{3}{4}+\frac{5}{4}}$
Add the fractions.	$z^{\frac{8}{4}}$
Simplify.	z^{2}

Note:
Exercise:

Problem: Simplify: ⓐ $3^{\frac{2}{3}} \cdot 3^{\frac{4}{3}}$ ⓑ $y^{\frac{1}{3}} \cdot y^{\frac{8}{3}}$ ⓒ $m^{\frac{1}{4}} \cdot m^{\frac{3}{4}}$.

Solution:

ⓐ 9 ⓑ y^3 ⓒ m

Note:
Exercise:

Problem: Simplify: ⓐ $5^{\frac{3}{5}} \cdot 5^{\frac{7}{5}}$ ⓑ $z^{\frac{1}{8}} \cdot z^{\frac{7}{8}}$ ⓒ $n^{\frac{2}{7}} \cdot n^{\frac{5}{7}}$.

Solution:

ⓐ 25 ⓑ z ⓒ n

We will use the Power Property in the next example.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(x^4\right)^{\frac{1}{2}}$ ⓑ $\left(y^6\right)^{\frac{1}{3}}$ ⓒ $\left(z^9\right)^{\frac{2}{3}}$.

Solution:
Solution

ⓐ

	$\left(x^4\right)^{\frac{1}{2}}$
To raise a power to a power, we multiply the exponents.	$x^{4 \cdot \frac{1}{2}}$
Simplify.	x^2

ⓑ

	$\left(y^6\right)^{\frac{1}{3}}$
To raise a power to a power, we multiply the exponents.	$y^{6\cdot\frac{1}{3}}$
Simplify.	y^2

ⓒ

	$\left(z^9\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$z^{9\cdot\frac{2}{3}}$
Simplify.	z^6

Note:
Exercise:

Problem: Simplify: ⓐ $\left(p^{10}\right)^{\frac{1}{5}}$ ⓑ $\left(q^8\right)^{\frac{3}{4}}$ ⓒ $\left(x^6\right)^{\frac{4}{3}}$.

Solution:

ⓐ p^2 ⓑ q^6 ⓒ x^8

Note:
Exercise:

Problem: Simplify: ⓐ $\left(r^6\right)^{\frac{5}{3}}$ ⓑ $\left(s^{12}\right)^{\frac{3}{4}}$ ⓒ $\left(m^9\right)^{\frac{2}{9}}$.

Solution:

ⓐ r^{10} ⓑ s^9 ⓒ m^2

The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{x^{\frac{4}{3}}}{x^{\frac{1}{3}}}$ ⓑ $\frac{y^{\frac{3}{4}}}{y^{\frac{1}{4}}}$ ⓒ $\frac{z^{\frac{2}{3}}}{z^{\frac{5}{3}}}$.

Solution:
Solution

ⓐ

	$\frac{x^{\frac{4}{3}}}{x^{\frac{1}{3}}}$
To divide with the same base, we subtract the exponents.	$x^{\frac{4}{3}-\frac{1}{3}}$
Simplify.	x

ⓑ

	$\frac{y^{\frac{3}{4}}}{y^{\frac{1}{4}}}$
To divide with the same base, we subtract the exponents.	$y^{\frac{3}{4}-\frac{1}{4}}$
Simplify.	$y^{\frac{1}{2}}$

ⓒ

	$\frac{z^{\frac{2}{3}}}{z^{\frac{5}{3}}}$
To divide with the same base, we subtract the exponents.	$z^{\frac{2}{3}-\frac{5}{3}}$
Rewrite without a negative exponent.	$\frac{1}{z}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{u^{\frac{5}{4}}}{u^{\frac{1}{4}}}$ ⓑ $\frac{v^{\frac{3}{5}}}{v^{\frac{2}{5}}}$ ⓒ $\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$.

Solution:

ⓐ u ⓑ $v^{\frac{1}{5}}$ ⓒ $\frac{1}{x}$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{c^{\frac{12}{5}}}{c^{\frac{2}{5}}}$ ⓑ $\frac{m^{\frac{5}{4}}}{m^{\frac{9}{4}}}$ ⓒ $\frac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$.

Solution:

ⓐ c^2 ⓑ $\frac{1}{m}$ ⓒ $\frac{1}{d}$

Sometimes we need to use more than one property. In the next two examples, we will use both the Product to a Power Property and then the Power Property.

Example:
Exercise:

Problem: Simplify: ⓐ $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ ⓑ $\left(8v^{\frac{1}{4}}\right)^{\frac{2}{3}}$.

Solution:
Solution

ⓐ

	$\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
First we use the Product to a Power Property.	$(27)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
Rewrite 27 as a power of 3.	$\left(3^{3}\right)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$\left(3^{2}\right)\left(u^{\frac{1}{3}}\right)$
Simplify.	$9u^{\frac{1}{3}}$

ⓑ

	$\left(8v^{\frac{1}{4}}\right)^{\frac{2}{3}}$
First we use the Product to a Power Property.	$(8)^{\frac{2}{3}}\left(v^{\frac{1}{4}}\right)^{\frac{2}{3}}$
Rewrite 8 as a power of 2.	$\left(2^{3}\right)^{\frac{2}{3}}\left(v^{\frac{1}{4}}\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$\left(2^{2}\right)\left(v^{\frac{1}{6}}\right)$
Simplify.	$4v^{\frac{1}{6}}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(32x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ ⓑ $\left(64y^{\frac{2}{3}}\right)^{\frac{1}{3}}$.

Solution:

ⓐ $8x^{\frac{1}{5}}$ ⓑ $4y^{\frac{2}{9}}$

Note:
Exercise:

Problem: Simplify: ⓐ $\left(16m^{\frac{1}{3}}\right)^{\frac{3}{2}}$ ⓑ $\left(81n^{\frac{2}{5}}\right)^{\frac{3}{2}}$.

Solution:

ⓐ $64m^{\frac{1}{2}}$ ⓑ $729n^{\frac{3}{5}}$

Example:
Exercise:

Problem: Simplify: ⓐ $\left(m^3n^9\right)^{\frac{1}{3}}$ ⓑ $\left(p^4q^8\right)^{\frac{1}{4}}$.

Solution:
Solution

ⓐ	
	$\left(m^3n^9\right)^{\frac{1}{3}}$
First we use the Product to a Power Property.	$\left(m^3\right)^{\frac{1}{3}}\left(n^9\right)^{\frac{1}{3}}$
To raise a power to a power, we multiply the exponents.	mn^3
ⓑ	
	$\left(p^4q^8\right)^{\frac{1}{4}}$
First we use the Product to a Power Property.	$\left(p^4\right)^{\frac{1}{4}}\left(q^8\right)^{\frac{1}{4}}$
To raise a power to a power, we multiply the exponents.	pq^2

We will use both the Product and Quotient Properties in the next example.

Example:
Exercise:

Problem: Simplify: ⓐ $\frac{x^{\frac{3}{4}}\cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$ ⓑ $\frac{y^{\frac{4}{3}}\cdot y}{y^{-\frac{2}{3}}}$.

Solution:
Solution

ⓐ

$$\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$$

Use the Product Property in the numerator, add the exponents.

$$\frac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$$

Use the Quotient Property, subtract the exponents.

$$x^{\frac{8}{4}}$$

Simplify.

$$x^2$$

ⓑ

$$\frac{y^{\frac{4}{3}} \cdot y}{y^{-\frac{2}{3}}}$$

Use the Product Property in the numerator, add the exponents.

$$\frac{y^{\frac{7}{3}}}{y^{-\frac{2}{3}}}$$

Use the Quotient Property, subtract the exponents.

$$y^{\frac{9}{3}}$$

Simplify.

$$y^3$$

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$ ⓑ $\frac{n^{\frac{1}{6}} \cdot n}{n^{-\frac{11}{6}}}$.

Solution:

ⓐ m^2 ⓑ n^3

Note:
Exercise:

Problem: Simplify: ⓐ $\frac{u^{\frac{4}{5}} \cdot u^{-\frac{2}{5}}}{u^{-\frac{13}{5}}}$ ⓑ $\frac{v^{\frac{1}{2}} \cdot v}{v^{-\frac{7}{2}}}$.

Solution:

ⓐ u^3 ⓑ v^5

Key Concepts

- **Summary of Exponent Properties**
- If a, b are real numbers and m, n are rational numbers, then
  - **Product Property** $a^m \cdot a^n = a^{m+n}$
  - **Power Property** $(a^m)^n = a^{m \cdot n}$
  - **Product to a Power** $(ab)^m = a^m b^m$
  - **Quotient Property**:
    Equation:

$$\frac{a^m}{a^n} = a^{m-n}, \;\; a \neq 0, \;\; m > n$$

    Equation:

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \;\; a \neq 0, \;\; n > m$$

  - **Zero Exponent Definition** $a^0 = 1$, $a \neq 0$
  - **Quotient to a Power Property** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \;\; b \neq 0$

Practice Makes Perfect

Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

Exercise:

ⓐ $x^{\frac{1}{2}}$
ⓑ $y^{\frac{1}{3}}$
Problem: ⓒ $z^{\frac{1}{4}}$

Exercise:

ⓐ $r^{\frac{1}{2}}$
ⓑ $s^{\frac{1}{3}}$
Problem: ⓒ $t^{\frac{1}{4}}$

Solution:

ⓐ $\sqrt{r}$ ⓑ $\sqrt[3]{s}$ ⓒ $\sqrt[4]{t}$

Exercise:

ⓐ $u^{\frac{1}{5}}$
ⓑ $v^{\frac{1}{9}}$
Problem: ⓒ $w^{\frac{1}{20}}$

Exercise:

ⓐ $g^{\frac{1}{7}}$
ⓑ $h^{\frac{1}{5}}$
Problem: ⓒ $j^{\frac{1}{25}}$

Solution:

ⓐ $\sqrt[7]{g}$ ⓑ $\sqrt[5]{h}$ ⓒ $\sqrt[25]{j}$

In the following exercises, write with a rational exponent.

Exercise:

Problem: ⓐ $-\sqrt[7]{x}$
ⓑ $\sqrt[9]{y}$
ⓒ $\sqrt[5]{f}$

Exercise:

Problem: ⓐ $\sqrt[8]{r}$
ⓑ

$\sqrt[10]{s}$

ⓒ $\sqrt[4]{t}$

Solution:

ⓐ $r^{\frac{1}{8}}$ ⓑ $s^{\frac{1}{10}}$ ⓒ $t^{\frac{1}{4}}$

Exercise:

Problem: ⓐ $\sqrt[3]{a}$
ⓑ

$\sqrt[12]{b}$

ⓒ $\sqrt{c}$

Exercise:

ⓐ $\sqrt[5]{u}$
ⓑ $\sqrt{v}$
Problem: ⓒ

$\sqrt[16]{w}$

Solution:

ⓐ $u^{\frac{1}{5}}$ ⓑ $v^{\frac{1}{2}}$ ⓒ $w^{\frac{1}{16}}$

Exercise:

ⓐ $\sqrt[3]{7c}$
ⓑ $\sqrt[7]{12d}$
Problem: ⓒ $3\sqrt[4]{5f}$

Exercise:

ⓐ $\sqrt[4]{5x}$
ⓑ $\sqrt[8]{9y}$
Problem: ⓒ $7\sqrt[5]{3z}$

Solution:

ⓐ $(5x)^{\frac{1}{4}}$ ⓑ $(9y)^{\frac{1}{8}}$ ⓒ $7(3z)^{\frac{1}{5}}$

Exercise:

ⓐ $\sqrt{21p}$
ⓑ $\sqrt[4]{8q}$
Problem: ⓒ $4\sqrt[6]{36r}$

Exercise:

ⓐ $\sqrt[3]{25a}$
ⓑ $\sqrt{3b}$
Problem: ⓒ

$\sqrt[10]{40c}$

Solution:

ⓐ $(25a)^{\frac{1}{3}}$ ⓑ $(3b)^{\frac{1}{2}}$ ⓒ $(40c)^{\frac{1}{10}}$

In the following exercises, simplify.

Exercise:

ⓐ $81^{\frac{1}{2}}$
ⓑ $125^{\frac{1}{3}}$
Problem: ⓒ $64^{\frac{1}{2}}$

Exercise:

ⓐ $625^{\frac{1}{4}}$
ⓑ $243^{\frac{1}{5}}$
Problem: ⓒ $32^{\frac{1}{5}}$

Solution:

ⓐ 5 ⓑ 3 ⓒ 2

Exercise:

ⓐ $16^{\frac{1}{4}}$
ⓑ $16^{\frac{1}{2}}$
Problem: ⓒ $3125^{\frac{1}{5}}$

Exercise:

ⓐ $216^{\frac{1}{3}}$
ⓑ $32^{\frac{1}{5}}$
Problem: ⓒ $81^{\frac{1}{4}}$

Solution:

ⓐ 6 ⓑ 2 ⓒ 3

Exercise:

ⓐ $(-216)^{\frac{1}{3}}$
ⓑ $-216^{\frac{1}{3}}$
Problem: ⓒ $(216)^{-\frac{1}{3}}$

Exercise:

ⓐ $(-243)^{\frac{1}{5}}$
ⓑ $-243^{\frac{1}{5}}$
Problem: ⓒ $(243)^{-\frac{1}{5}}$

Solution:

ⓐ -3 ⓑ -3 ⓒ $\frac{1}{3}$

Exercise:

ⓐ $(-1)^{\frac{1}{3}}$
ⓑ $-1^{\frac{1}{3}}$
Problem: ⓒ $(1)^{-\frac{1}{3}}$

Exercise:

ⓐ $(-1000)^{\frac{1}{3}}$
ⓑ $-1000^{\frac{1}{3}}$
Problem: ⓒ $(1000)^{-\frac{1}{3}}$

Solution:

ⓐ -10 ⓑ -10 ⓒ $\frac{1}{10}$

Exercise:

ⓐ $(-81)^{\frac{1}{4}}$
ⓑ $-81^{\frac{1}{4}}$
Problem: ⓒ $(81)^{-\frac{1}{4}}$

Exercise:

ⓐ $(-49)^{\frac{1}{2}}$
ⓑ $-49^{\frac{1}{2}}$
Problem: ⓒ $(49)^{-\frac{1}{2}}$

Solution:

ⓐ not a real number ⓑ -7 ⓒ $\frac{1}{7}$

Exercise:

ⓐ $(-36)^{\frac{1}{2}}$
ⓑ $-36^{\frac{1}{2}}$
Problem: ⓒ $(36)^{-\frac{1}{2}}$

Exercise:

ⓐ $(-1)^{\frac{1}{4}}$
ⓑ $(1)^{-\frac{1}{4}}$
Problem: ⓒ $-1^{\frac{1}{4}}$

Solution:

ⓐ not a real number ⓑ 1 ⓒ -1

Exercise:

ⓐ $(-100)^{\frac{1}{2}}$
ⓑ $-100^{\frac{1}{2}}$
Problem: ⓒ $(100)^{-\frac{1}{2}}$

Exercise:

ⓐ $(-32)^{\frac{1}{5}}$
ⓑ $(243)^{-\frac{1}{5}}$
Problem: ⓒ $-125^{\frac{1}{3}}$

Solution:

ⓐ -2 ⓑ $\frac{1}{3}$
ⓒ -5

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

Exercise:

ⓐ $\sqrt{m^5}$
ⓑ $\sqrt[3]{n^2}$
Problem: ⓒ $\sqrt[4]{p^3}$

Exercise:

ⓐ $\sqrt[4]{r^7}$
ⓑ $\sqrt[5]{s^3}$
Problem: ⓒ $\sqrt[3]{t^7}$

Solution:

ⓐ $r^{\frac{7}{4}}$ ⓑ $s^{\frac{3}{5}}$ ⓒ $t^{\frac{7}{3}}$

Exercise:

ⓐ $\sqrt[5]{u^2}$
ⓑ $\sqrt[5]{v^8}$
Problem: ⓒ $\sqrt[9]{w^4}$

Exercise:

ⓐ $\sqrt[3]{a}$
ⓑ $\sqrt{b^5}$
Problem: ⓒ $\sqrt[3]{c^5}$

Solution:

ⓐ $a^{\frac{1}{3}}$ ⓑ $b^{\frac{5}{2}}$ ⓒ $c^{\frac{5}{3}}$

In the following exercises, simplify.

Exercise:

ⓐ $16^{\frac{3}{2}}$
ⓑ $8^{\frac{2}{3}}$
Problem: ⓒ $10{,}000^{\frac{3}{4}}$

Exercise:

ⓐ $1000^{\frac{2}{3}}$
ⓑ $25^{\frac{3}{2}}$
Problem: ⓒ $32^{\frac{3}{5}}$

Solution:

ⓐ 100 ⓑ 125 ⓒ 8

Exercise:

Problem: ⓐ $27^{\frac{5}{3}}$
ⓑ $16^{\frac{5}{4}}$
ⓒ $32^{\frac{2}{5}}$

Exercise:

Problem: ⓐ $16^{\frac{3}{2}}$
ⓑ $125^{\frac{5}{3}}$
ⓒ $64^{\frac{4}{3}}$

Solution:

ⓐ 64 ⓑ 3125 ⓒ 256

Exercise:

Problem: ⓐ $32^{\frac{2}{5}}$
ⓑ $27^{-\frac{2}{3}}$
ⓒ $25^{-\frac{3}{2}}$

Exercise:

Problem: ⓐ $64^{\frac{5}{2}}$
ⓑ $81^{-\frac{3}{2}}$
ⓒ $27^{-\frac{4}{3}}$

Solution:

ⓐ 32,768 ⓑ $\frac{1}{729}$ ⓒ $\frac{1}{81}$

Exercise:

Problem: ⓐ $25^{\frac{3}{2}}$
ⓑ $9^{-\frac{3}{2}}$
ⓒ $(-64)^{\frac{2}{3}}$

Exercise:

Problem: ⓐ $100^{\frac{3}{2}}$
ⓑ $49^{-\frac{5}{2}}$
ⓒ $(-100)^{\frac{3}{2}}$

Solution:

ⓐ 1000 ⓑ $\frac{1}{16{,}807}$ ⓒ not a real number

Exercise:

Problem: ⓐ $-9^{\frac{3}{2}}$
ⓑ $-9^{-\frac{3}{2}}$
ⓒ $(-9)^{\frac{3}{2}}$

Exercise:

Problem: ⓐ $-64^{\frac{3}{2}}$
ⓑ $-64^{-\frac{3}{2}}$
ⓒ $(-64)^{\frac{3}{2}}$

Solution:

ⓐ -512
ⓑ $-\frac{1}{512}$ ⓒ not a real number

Exercise:

Problem: ⓐ $-100^{\frac{3}{2}}$
ⓑ $-100^{-\frac{3}{2}}$
ⓒ $(-100)^{\frac{3}{2}}$

Exercise:

Problem: ⓐ $-49^{\frac{3}{2}}$
ⓑ $-49^{-\frac{3}{2}}$
ⓒ $(-49)^{\frac{3}{2}}$

Solution:

ⓐ -343 ⓑ $-\frac{1}{343}$ ⓒ not a real number

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

Exercise:

Problem: ⓐ $4^{\frac{5}{8}} \cdot 4^{\frac{11}{8}}$
ⓑ $m^{\frac{7}{12}} \cdot m^{\frac{17}{12}}$
ⓒ $p^{\frac{3}{7}} \cdot p^{\frac{18}{7}}$

Exercise:

ⓐ $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$
ⓑ $n^{\frac{2}{10}} \cdot n^{\frac{8}{10}}$
Problem: ⓒ $q^{\frac{2}{5}} \cdot q^{\frac{13}{5}}$

Solution:

ⓐ 216 ⓑ n ⓒ q^3

Exercise:

ⓐ $5^{\frac{1}{2}} \cdot 5^{\frac{7}{2}}$
ⓑ $c^{\frac{3}{4}} \cdot c^{\frac{9}{4}}$
Problem: ⓒ $d^{\frac{3}{5}} \cdot d^{\frac{2}{5}}$

Exercise:

ⓐ $10^{\frac{1}{3}} \cdot 10^{\frac{5}{3}}$
ⓑ $x^{\frac{5}{6}} \cdot x^{\frac{7}{6}}$
Problem: ⓒ $y^{\frac{11}{8}} \cdot y^{\frac{21}{8}}$

Solution:

ⓐ 100 ⓑ x^2 ⓒ y^4

Exercise:

ⓐ $\left(m^6\right)^{\frac{5}{2}}$
ⓑ $\left(n^9\right)^{\frac{4}{3}}$
Problem: ⓒ $\left(p^{12}\right)^{\frac{3}{4}}$

Exercise:

ⓐ $\left(a^{12}\right)^{\frac{1}{6}}$

ⓑ $\left(b^{15}\right)^{\frac{3}{5}}$

Problem: ⓒ $\left(c^{11}\right)^{\frac{1}{11}}$

Solution:

ⓐ a^2 ⓑ b^9
ⓒ c

Exercise:

ⓐ $\left(x^{12}\right)^{\frac{2}{3}}$

ⓑ $\left(y^{20}\right)^{\frac{2}{5}}$

Problem: ⓒ $\left(z^{16}\right)^{\frac{1}{16}}$

Exercise:

ⓐ $\left(h^{6}\right)^{\frac{4}{3}}$

ⓑ $\left(k^{12}\right)^{\frac{3}{4}}$

Problem: ⓒ $\left(j^{10}\right)^{\frac{7}{5}}$

Solution:

ⓐ h^8 ⓑ k^9 ⓒ j^{14}

Exercise:

ⓐ $\frac{x^{\frac{7}{2}}}{x^{\frac{5}{2}}}$

ⓑ $\frac{y^{\frac{5}{2}}}{y^{\frac{1}{2}}}$

Problem: ⓒ $\frac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$

Exercise:

ⓐ $\frac{s^{\frac{11}{5}}}{s^{\frac{6}{5}}}$

ⓑ $\frac{z^{\frac{7}{3}}}{z^{\frac{1}{3}}}$

Problem: ⓒ $\frac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

Solution:

ⓐ s ⓑ z^2 ⓒ $\frac{1}{w}$

Exercise:

ⓐ $\frac{t^{\frac{12}{5}}}{t^{\frac{7}{5}}}$

ⓑ $\frac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}}$

Problem: ⓒ $\frac{m^{\frac{13}{8}}}{m^{\frac{5}{8}}}$

Exercise:

ⓐ $\frac{u^{\frac{13}{9}}}{u^{\frac{4}{9}}}$
ⓑ $\frac{r^{\frac{15}{7}}}{r^{\frac{8}{7}}}$
Problem: ⓒ $\frac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$

Solution:

ⓐ u ⓑ r ⓒ $\frac{1}{n}$

Exercise:

ⓐ $\left(9p^{\frac{2}{3}}\right)^{\frac{5}{2}}$
Problem: ⓑ $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$

Exercise:

ⓐ $\left(81r^{\frac{4}{5}}\right)^{\frac{1}{4}}$
Problem: ⓑ $\left(64s^{\frac{3}{7}}\right)^{\frac{1}{6}}$

Solution:

ⓐ $3r^{\frac{1}{5}}$ ⓑ $2s^{\frac{1}{14}}$

Exercise:

ⓐ $\left(16u^{\frac{1}{3}}\right)^{\frac{3}{4}}$

Problem: ⓑ $\left(100v^{\frac{2}{5}}\right)^{\frac{3}{2}}$

Exercise:

ⓐ $\left(27m^{\frac{3}{4}}\right)^{\frac{2}{3}}$

Problem: ⓑ $\left(625n^{\frac{8}{3}}\right)^{\frac{3}{4}}$

Solution:

ⓐ $9m^{\frac{1}{2}}$ ⓑ $125n^{2}$

Exercise:

ⓐ $\left(x^{8}y^{10}\right)^{\frac{1}{2}}$

Problem: ⓑ $\left(a^{9}b^{12}\right)^{\frac{1}{3}}$

Exercise:

ⓐ $\left(r^{8}s^{4}\right)^{\frac{1}{4}}$

Problem: ⓑ $\left(u^{15}v^{20}\right)^{\frac{1}{5}}$

Solution:

ⓐ $r^{2}s$ ⓑ $u^{3}v^{4}$

Exercise:

ⓐ $\left(a^{6}b^{16}\right)^{\frac{1}{2}}$

Problem: ⓑ $\left(j^{9}k^{6}\right)^{\frac{2}{3}}$

Exercise:

ⓐ $\left(r^{16}s^{10}\right)^{\frac{1}{2}}$

Problem: ⓑ $\left(u^{10}v^{5}\right)^{\frac{4}{5}}$

Solution:

ⓐ $r^{8}s^{5}$ ⓑ $u^{8}v^{4}$

Exercise:

ⓐ $\frac{r^{\frac{5}{2}}\cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$

Problem: ⓑ $\frac{s^{\frac{1}{5}}\cdot s}{s^{-\frac{9}{5}}}$

Exercise:

ⓐ $\frac{a^{\frac{3}{4}}\cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$

Problem: ⓑ $\frac{b^{\frac{2}{3}}\cdot b}{b^{-\frac{7}{3}}}$

Solution:

ⓐ a^{3} ⓑ b^{4}

Exercise:

ⓐ $\frac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$

Problem: ⓑ $\frac{d^{\frac{3}{5}} \cdot d}{d^{-\frac{2}{5}}}$

Exercise:

ⓐ $\frac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$

Problem: ⓑ $\frac{n^{\frac{3}{7}} \cdot n}{n^{-\frac{4}{7}}}$

Solution:

ⓐ m ⓑ n^2

Exercise:

Problem: $4^{\frac{5}{2}} \cdot 4^{\frac{1}{2}}$

Exercise:

Problem: $n^{\frac{2}{6}} \cdot n^{\frac{4}{6}}$

Solution:

n

Exercise:

Problem: $\left(a^{24}\right)^{\frac{1}{6}}$

Exercise:

Problem: $\left(b^{10}\right)^{\frac{3}{5}}$

Solution:

b^6

Exercise:

Problem: $\frac{w^{\frac{2}{5}}}{w^{\frac{7}{5}}}$

Exercise:

Problem: $\frac{z^{\frac{2}{3}}}{z^{\frac{8}{3}}}$

Solution:

$\frac{1}{z^2}$

Exercise:

Problem: $\left(27r^{\frac{3}{5}}\right)^{\frac{1}{3}}$

Exercise:

Problem: $\left(64s^{\frac{3}{5}}\right)^{\frac{1}{6}}$

Solution:

$2s^{\frac{1}{10}}$

Exercise:

Problem: $\left(r^9 s^{12}\right)^{\frac{1}{3}}$

Exercise:

Problem: $\left(u^{12} v^{18}\right)^{\frac{1}{6}}$

Solution:

$u^2 v^3$

Everyday Math

Exercise:

Problem:

Landscaping Joe wants to have a square garden plot in his backyard. He has enough compost to cover an area of 144 square feet. Simplify $144^{\frac{1}{2}}$ to find the length of each side of his garden.

Exercise:

Problem:

Landscaping Elliott wants to make a square patio in his yard. He has enough concrete to pave an area of 242 square feet. Simplify $242^{\frac{1}{2}}$ to find the length of each side of his patio.Round to the nearest tenth of a foot.

Solution:

15.6 feet

Exercise:

Problem:

Gravity While putting up holiday decorations, Bob dropped a decoration from the top of a tree that is 12 feet tall. Simplify $\frac{12^{\frac{1}{2}}}{16^{\frac{1}{2}}}$ to find how many seconds it took for the decoration to reach the ground. Round to the nearest tenth of a second.

Exercise:

Problem:

Gravity An airplane dropped a flare from a height of 1024 feet above a lake. Simplify $\frac{1024^{\frac{1}{2}}}{16^{\frac{1}{2}}}$ to find how many seconds it took for the flare to reach the water.

Solution:

8 seconds

Writing Exercises

Exercise:

Problem:

Show two different algebraic methods to simplify $4^{\frac{3}{2}}$. Explain all your steps.

Exercise:

Problem: Explain why the expression $(-16)^{\frac{3}{2}}$ cannot be evaluated.

Simplify and Use Square Roots

Simplify Expressions with Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{64}$

Exercise:

Problem: $\sqrt{144}$

Solution:

12

Exercise:

Problem: $-\sqrt{25}$

Exercise:

Problem: $-\sqrt{81}$

Solution:

-9

Exercise:

Problem: $\sqrt{-9}$

Exercise:

Problem: $\sqrt{-36}$

Solution:

not a real number

Exercise:

Problem: $\sqrt{64} + \sqrt{225}$

Exercise:

Problem: $\sqrt{64 + 225}$

Solution:

17

Estimate Square Roots

In the following exercises, estimate each square root between two consecutive whole numbers.

Exercise:

Problem: $\sqrt{28}$

Exercise:

Problem: $\sqrt{155}$

Solution:

$12 < \sqrt{155} < 13$

Approximate Square Roots

In the following exercises, approximate each square root and round to two decimal places.

Exercise:

Problem: $\sqrt{15}$

Exercise:

Problem: $\sqrt{57}$

Solution:

7.55

Simplify Variable Expressions with Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{q^2}$

Exercise:

Problem: $\sqrt{64b^2}$

Solution:

$8b$

Exercise:

Problem: $-\sqrt{121a^2}$

Exercise:

Problem: $\sqrt{225m^2n^2}$

Solution:

$15mn$

Exercise:

Problem: $-\sqrt{100q^2}$

Exercise:

Problem: $\sqrt{49y^2}$

Solution:

$7y$

Exercise:

Problem: $\sqrt{4a^2b^2}$

Exercise:

Problem: $\sqrt{121c^2d^2}$

Solution:

$11cd$

Simplify Square Roots

Use the Product Property to Simplify Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{300}$

Exercise:

Problem: $\sqrt{98}$

Solution:

$7\sqrt{2}$

Exercise:

Problem: $\sqrt{x^{13}}$

Exercise:

Problem: $\sqrt{y^{19}}$

Solution:

$y^9\sqrt{y}$

Exercise:

Problem: $\sqrt{16m^4}$

Exercise:

Problem: $\sqrt{36n^{13}}$

Solution:

$6n^6\sqrt{n}$

Exercise:

Problem: $\sqrt{288m^{21}}$

Exercise:

Problem: $\sqrt{150n^7}$

Solution:

$5n^3\sqrt{6n}$

Exercise:

Problem: $\sqrt{48r^5s^4}$

Exercise:

Problem: $\sqrt{108r^5s^3}$

Solution:

$6r^2s\sqrt{3rs}$

Exercise:

Problem: $\frac{10-\sqrt{50}}{5}$

Exercise:

Problem: $\frac{6+\sqrt{72}}{6}$

Solution:

$1+\sqrt{2}$

Use the Quotient Property to Simplify Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{\frac{16}{25}}$

Exercise:

Problem: $\sqrt{\frac{81}{36}}$

Solution:

$\frac{3}{2}$

Exercise:

Problem: $\sqrt{\frac{x^8}{x^4}}$

Exercise:

Problem: $\sqrt{\frac{y^6}{y^2}}$

Solution:

y^2

Exercise:

Problem: $\sqrt{\frac{98p^6}{2p^2}}$

Exercise:

Problem: $\sqrt{\frac{72q^8}{2q^4}}$

Solution:

$6q^2$

Exercise:

Problem: $\sqrt{\frac{65}{121}}$

Exercise:

Problem: $\sqrt{\frac{26}{169}}$

Solution:

$\frac{\sqrt{26}}{13}$

Exercise:

Problem: $\sqrt{\frac{64x^4}{25x^2}}$

Exercise:

Problem: $\sqrt{\frac{36r^{10}}{16r^5}}$

Solution:

$\frac{3r^2\sqrt{r}}{2}$

Exercise:

Problem: $\sqrt{\frac{48p^3q^5}{27pq}}$

Exercise:

Problem: $\sqrt{\frac{12r^5s^7}{75r^2s}}$

Solution:

$\frac{2rs^3\sqrt{r}}{5}$

Add and Subtract Square Roots

Add and Subtract Like Square Roots

In the following exercises, simplify.

Exercise:

Problem: $3\sqrt{2} + \sqrt{2}$

Exercise:

Problem: $5\sqrt{5} + 7\sqrt{5}$

Solution:

$12\sqrt{5}$

Exercise:

Problem: $4\sqrt{y} + 4\sqrt{y}$

Exercise:

Problem: $6\sqrt{m} - 2\sqrt{m}$

Solution:

$4\sqrt{m}$

Exercise:

Problem: $-3\sqrt{7} + 2\sqrt{7} - \sqrt{7}$

Exercise:

Problem: $8\sqrt{13} + 2\sqrt{3} + 3\sqrt{13}$

Solution:

$11\sqrt{13}+2\sqrt{3}$

Exercise:

Problem: $3\sqrt{5xy}-\sqrt{5xy}+3\sqrt{5xy}$

Exercise:

Problem: $2\sqrt{3rs}+\sqrt{3rs}-5\sqrt{rs}$

Solution:

$3\sqrt{3rs}-5\sqrt{rs}$

Add and Subtract Square Roots that Need Simplification

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{32}+3\sqrt{2}$

Exercise:

Problem: $\sqrt{8}+3\sqrt{2}$

Solution:

$5\sqrt{2}$

Exercise:

Problem: $\sqrt{72}+\sqrt{50}$

Exercise:

Problem: $\sqrt{48}+\sqrt{75}$

Solution:

$9\sqrt{3}$

Exercise:

Problem: $3\sqrt{32}+\sqrt{98}$

Exercise:

Problem: $\frac{1}{3}\sqrt{27}-\frac{1}{8}\sqrt{192}$

Solution:

0

Exercise:

Problem: $\sqrt{50y^5}-\sqrt{72y^5}$

Exercise:

Problem: $6\sqrt{18n^4}-3\sqrt{8n^4}+n^2\sqrt{50}$

Solution:

$17n^2\sqrt{2}$

Multiply Square Roots

Multiply Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{2} \cdot \sqrt{20}$

Exercise:

Problem: $2\sqrt{2} \cdot 6\sqrt{14}$

Solution:

$24\sqrt{7}$

Exercise:

Problem: $\sqrt{2m^2} \cdot \sqrt{20m^4}$

Exercise:

Problem: $\left(6\sqrt{2y}\right)\left(3\sqrt{50y^3}\right)$

Solution:

$180y^2$

Exercise:

Problem: $\left(6\sqrt{3v^4}\right)\left(5\sqrt{30v}\right)$

Exercise:

Problem: $\left(\sqrt{8}\right)^2$

Solution:

8

Exercise:

Problem: $\left(-\sqrt{10}\right)^2$

Exercise:

Problem: $\left(2\sqrt{5}\right)\left(5\sqrt{5}\right)$

Solution:

50

Exercise:

Problem: $\left(-3\sqrt{3}\right)\left(5\sqrt{18}\right)$

Use Polynomial Multiplication to Multiply Square Roots

In the following exercises, simplify.

Exercise:

Problem: $10(2-\sqrt{7})$

Solution:

$20-10\sqrt{7}$

Exercise:

Problem: $\sqrt{3}\left(4+\sqrt{12}\right)$

Exercise:

Problem: $\left(5+\sqrt{2}\right)\left(3-\sqrt{2}\right)$

Solution:

$13-2\sqrt{2}$

Exercise:

Problem: $\left(5-3\sqrt{7}\right)\left(1-2\sqrt{7}\right)$

Exercise:

Problem: $(1-3\sqrt{x})\ (5+2\sqrt{x})$

Solution:

$5-13\sqrt{x}-6x$

Exercise:

Problem: $(3+4\sqrt{y})\ (10-\sqrt{y})$

Exercise:

Problem: $(1+6\sqrt{p})^2$

Solution:

$1+12\sqrt{p}+36p$

Exercise:

Problem: $\left(2-6\sqrt{5}\right)^2$

Exercise:

Problem: $\left(3+2\sqrt{7}\right)\left(3-2\sqrt{7}\right)$

Solution:

-19

Exercise:

Problem: $\left(6-\sqrt{11}\right)\left(6+\sqrt{11}\right)$

Divide Square Roots

Divide Square Roots

In the following exercises, simplify.

Exercise:

Problem: $\frac{\sqrt{75}}{10}$

Solution:

$\frac{\sqrt{3}}{2}$

Exercise:

Problem: $\frac{2-\sqrt{12}}{6}$

Exercise:

Problem: $\frac{\sqrt{48}}{\sqrt{27}}$

Solution:

$\frac{4}{3}$

Exercise:

Problem: $\frac{\sqrt{75x^7}}{\sqrt{3x^3}}$

Exercise:

Problem: $\frac{\sqrt{20y^5}}{\sqrt{2y}}$

Solution:

$y^2\sqrt{10}$

Exercise:

Problem: $\frac{\sqrt{98p^6q^4}}{\sqrt{2p^4q^8}}$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

Exercise:

Problem: $\frac{10}{\sqrt{15}}$

Solution:

$\frac{2\sqrt{15}}{3}$

Exercise:

Problem: $\frac{6}{\sqrt{6}}$

Exercise:

Problem: $\frac{5}{3\sqrt{5}}$

Solution:

$\frac{\sqrt{5}}{3}$

Exercise:

Problem: $\frac{10}{2\sqrt{6}}$

Exercise:

Problem: $\sqrt{\frac{3}{28}}$

Solution:

$\frac{\sqrt{21}}{14}$

Exercise:

Problem: $\sqrt{\frac{9}{75}}$

Rationalize a Two Term Denominator

In the following exercises, rationalize the denominator.

Exercise:

Problem: $\frac{4}{4+\sqrt{27}}$

Solution:

$\frac{16-12\sqrt{3}}{-11}$

Exercise:

Problem: $\frac{5}{2-\sqrt{10}}$

Exercise:

Problem: $\frac{4}{2-\sqrt{5}}$

Solution:

$$-8-4\sqrt{5}$$

Exercise:

Problem: $\frac{5}{4-\sqrt{8}}$

Exercise:

Problem: $\frac{\sqrt{2}}{\sqrt{p}+\sqrt{3}}$

Solution:

$$\frac{\sqrt{2p}-\sqrt{6}}{p-3}$$

Exercise:

Problem: $\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x}+\sqrt{2}}$

Solve Equations with Square Roots

Solve Radical Equations

In the following exercises, solve the equation.

Exercise:

Problem: $\sqrt{7z+1}=6$

Solution:

5

Exercise:

Problem: $\sqrt{4u-2}-4=0$

Exercise:

Problem: $\sqrt{6m+4}-5=0$

Solution:

$\frac{7}{2}$

Exercise:

Problem: $\sqrt{2u-3}+2=0$

Exercise:

Problem: $\sqrt{u-4}+4=u$

Solution:

$4, 5$

Exercise:

Problem: $\sqrt{v-9}+9=0$

Exercise:

Problem: $\sqrt{r-4}-r=-10$

Solution:

13

Exercise:

Problem: $\sqrt{s-9}-s=-9$

Exercise:

Problem: $2\sqrt{2x-7}-4=8$

Solution:

$\frac{43}{2}$

Exercise:

Problem: $\sqrt{2-x}=\sqrt{2x-7}$

Exercise:

Problem: $\sqrt{a}+3=\sqrt{a+9}$

Solution:

0

Exercise:

Problem: $\sqrt{r}+3=\sqrt{r+4}$

Exercise:

Problem: $\sqrt{u}+2=\sqrt{u+5}$

Solution:

$\frac{1}{16}$

Exercise:

Problem: $\sqrt{n+11}-1=\sqrt{n+4}$

Exercise:

Problem: $\sqrt{y+5}+1=\sqrt{2y+3}$

Solution:

11

Use Square Roots in Applications

In the following exercises, solve. Round approximations to one decimal place.

Exercise:

Problem:

A pallet of sod will cover an area of about 600 square feet. Trinh wants to order a pallet of sod to make a square lawn in his backyard. Use the formula $s=\sqrt{A}$ to find the length of each side of his lawn.

Exercise:

Problem:

A helicopter dropped a package from a height of 900 feet above a stranded hiker. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the hiker.

Solution:

7.5 seconds

Exercise:

Problem:

Officer Morales measured the skid marks of one of the cars involved in an accident. The length of the skid marks was 245 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied.

Higher Roots

Simplify Expressions with Higher Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\sqrt[6]{64}$ ⓑ $\sqrt[3]{64}$

Solution:

ⓐ 2 ⓑ 4

Exercise:

Problem: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-64}$

Exercise:

ⓐ $\sqrt[9]{d^9}$

Problem: ⓑ $\sqrt[8]{v^8}$

Solution:

ⓐ d ⓑ $|v|$

Exercise:

ⓐ $\sqrt[5]{a^{10}}$

Problem: ⓑ $\sqrt[3]{b^{27}}$

Exercise:

ⓐ $\sqrt[4]{16x^8}$

Problem: ⓑ $\sqrt[6]{64y^{12}}$

Solution:

ⓐ $2x^2$ ⓑ $2y^2$

Exercise:

ⓐ $\sqrt[7]{128r^{14}}$

Problem: ⓑ $\sqrt[4]{81s^{24}}$

Use the Product Property to Simplify Expressions with Higher Roots

In the following exercises, simplify.

Exercise:

Problem: ⓐ $\sqrt[9]{d^9}$ ⓑ

$\sqrt[11]{m^{17}}$

Solution:

ⓐ d ⓑ

$m\sqrt[11]{m^{6}}$

Exercise:

Problem: ⓐ $\sqrt[3]{54}$ ⓑ $\sqrt[4]{128}$

Exercise:

Problem: ⓐ $\sqrt[5]{64c^8}$ ⓑ $\sqrt[4]{48d^7}$

Solution:

ⓐ $2c\sqrt[5]{2c^3}$ ⓑ $2d\sqrt[4]{3d^3}$

Exercise:

Problem: ⓐ $\sqrt[3]{343q^7}$ ⓑ $\sqrt[6]{192r^9}$

Exercise:

Problem: ⓐ $\sqrt[3]{-500}$ ⓑ $\sqrt[4]{-16}$

Solution:

ⓐ $-5\sqrt[3]{4}$ ⓑ not a real number

Use the Quotient Property to Simplify Expressions with Higher Roots

In the following exercises, simplify.

Exercise:

Problem: $\sqrt[5]{\frac{r^{10}}{r^5}}$

Exercise:

Problem: $\sqrt[3]{\frac{w^{12}}{w^2}}$

Solution:

$w^3\sqrt[3]{w}$

Exercise:

Problem: $\sqrt[4]{\frac{64y^8}{4y^5}}$

Exercise:

Problem: $\sqrt[3]{\frac{54z^9}{2z^3}}$

Solution:

$3z^2$

Exercise:

Problem: $\sqrt[6]{\frac{64a^7}{b^2}}$

Add and Subtract Higher Roots

In the following exercises, simplify.

Exercise:

Problem: $4\sqrt[5]{20} - 2\sqrt[5]{20}$

Solution:

$2\sqrt[5]{20}$

Exercise:

Problem: $4\sqrt[3]{18} + 3\sqrt[3]{18}$

Exercise:

Problem: $\sqrt[4]{1250} - \sqrt[4]{162}$

Solution:

$2\sqrt[4]{2}$

Exercise:

Problem: $\sqrt[3]{640c^5} - \sqrt[3]{-80c^3}$

Exercise:

Problem: $\sqrt[5]{96t^8} + \sqrt[5]{486t^4}$

Solution:

$2t\sqrt[5]{3t^3} + 3\sqrt[5]{2t^4}$

Rational Exponents

Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

Exercise:

Problem: $r^{\frac{1}{8}}$

Exercise:

Problem: $s^{\frac{1}{10}}$

Solution:

$\sqrt[10]{s}$

In the following exercises, write with a rational exponent.

Exercise:

Problem: $\sqrt[5]{u}$

Exercise:

Problem: $\sqrt[6]{v}$

Solution:

$v^{\frac{1}{6}}$

Exercise:

Problem: $\sqrt[3]{9m}$

Exercise:

Problem: $\sqrt[6]{10z}$

Solution:

$(10z)^{\frac{1}{6}}$

In the following exercises, simplify.

Exercise:

Problem: $16^{\frac{1}{4}}$

Exercise:

Problem: $32^{\frac{1}{5}}$

Solution:

2

Exercise:

Problem: $(-125)^{\frac{1}{3}}$

Exercise:

Problem: $(125)^{-\frac{1}{3}}$

Solution:

$\frac{1}{5}$

Exercise:

Problem: $(-9)^{\frac{1}{2}}$

Exercise:

Problem: $(36)^{-\frac{1}{2}}$

Solution:

$\frac{1}{6}$

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

Exercise:

Problem: $\sqrt[3]{q^5}$

Exercise:

Problem: $\sqrt[5]{n^8}$

Solution:

$n^{\frac{8}{5}}$

In the following exercises, simplify.

Exercise:

Problem: $27^{-\frac{2}{3}}$

Exercise:

Problem: $64^{\frac{5}{2}}$

Solution:

32,768

Exercise:

Problem: $36^{\frac{3}{2}}$

Exercise:

Problem: $81^{-\frac{5}{2}}$

Solution:

$\frac{1}{59{,}049}$

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

Exercise:

Problem: $3^{\frac{4}{5}} \cdot 3^{\frac{6}{5}}$

Exercise:

Problem: $\left(x^6\right)^{\frac{4}{3}}$

Solution:

x^8

Exercise:

Problem: $\frac{z^{\frac{5}{2}}}{z^{\frac{7}{5}}}$

Exercise:

Problem: $\left(16s^{\frac{9}{4}}\right)^{\frac{1}{4}}$

Solution:

$2s^{\frac{9}{16}}$

Exercise:

Problem: $\left(m^8n^{12}\right)^{\frac{1}{4}}$

Exercise:

Problem: $\frac{z^{\frac{2}{3}}\cdot z^{-\frac{1}{3}}}{z^{-\frac{5}{3}}}$

Solution:

z^2

Practice Test

In the following exercises, simplify.

Exercise:

Problem: $\sqrt{81+144}$

Exercise:

Problem: $\sqrt{169m^4n^2}$

Solution:

$13m^2\left|n\right|$

Exercise:

Problem: $\sqrt{36n^{13}}$

Exercise:

Problem: $3\sqrt{13}+5\sqrt{2}+\sqrt{13}$

Solution:

$4\sqrt{13}+5\sqrt{2}$

Exercise:

Problem: $5\sqrt{20}+2\sqrt{125}$

Exercise:

Problem: $\left(3\sqrt{6y}\right)\left(2\sqrt{50y^3}\right)$

Solution:

$180y^2\sqrt{3}$

Exercise:

Problem: $\left(2-5\sqrt{x}\right)\left(3+\sqrt{x}\right)$

Exercise:

Problem: $\left(1-2\sqrt{q}\right)^2$

Solution:

$1-4\sqrt{q}+4q$

Exercise:

ⓐ $\sqrt[4]{a^{12}}$

Problem: ⓑ $\sqrt[3]{b^{21}}$

Exercise:

ⓐ $\sqrt[4]{81x^{12}}$

Problem: ⓑ $\sqrt[6]{64y^{18}}$

Solution:

ⓐ $3x^3$ ⓑ $2y^3$

Exercise:

Problem: $\sqrt{\frac{64r^{12}}{25r^6}}$

Exercise:

Problem: $\sqrt{\frac{14y^3}{7y}}$

Solution:

$y\sqrt{2}$

Exercise:

Problem: $\frac{\sqrt[5]{256x^7}}{\sqrt[5]{4x^2}}$

Exercise:

Problem: $\sqrt[4]{512} - 2\sqrt[4]{32}$

Solution:

0

Exercise:

Problem: ⓐ $256^{\frac{1}{4}}$
ⓑ $243^{\frac{1}{5}}$

Exercise:

Problem: $49^{\frac{3}{2}}$

Solution:

343

Exercise:

Problem: $25^{-\frac{5}{2}}$

Exercise:

Problem: $\frac{w^{\frac{3}{4}}}{w^{\frac{7}{4}}}$

Solution:

$\frac{1}{w}$

Exercise:

Problem: $\left(27s^{\frac{3}{5}}\right)^{\frac{1}{3}}$

In the following exercises, rationalize the denominator.

Exercise:

Problem: $\frac{3}{2\sqrt{6}}$

Solution:

$\frac{\sqrt{6}}{4}$

Exercise:

Problem: $\frac{\sqrt{3}}{\sqrt{x}+\sqrt{5}}$

In the following exercises, solve.

Exercise:

Problem: $3\sqrt{2x-3}-20=7$

Solution:

42

Exercise:

Problem: $\sqrt{3u-2}=\sqrt{5u+1}$

In the following exercise, solve.

Exercise:

Problem:

A helicopter flying at an altitude of 600 feet dropped a package to a lifeboat. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the hiker. Round your answer to the nearest tenth of a second.

Solution:

6.1 seconds

Glossary

rational exponents

- If $\sqrt[n]{a}$ is a real number and $n \geq 2$, $a^{\frac{1}{n}} = \sqrt[n]{a}$.
- For any positive integers *m* and *n*, $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

Introduction class="introduction"

Fireworks accompany festive celebrations around the world. (Credit: modification of work by tlc, Flickr)

The trajectories of fireworks are modeled by quadratic equations. The equations can be used to predict the maximum height of a firework and the number of seconds it will take from launch to explosion. In this chapter, we will study the properties of quadratic equations, solve them, graph them, and see how they are applied as models of various situations.

Solve Quadratic Equations Using the Square Root Property

By the end of this section, you will be able to:

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x - h)^2 = k$ using the Square Root Property

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\sqrt{75}$.
   If you missed this problem, review [link].
2. Simplify: $\sqrt{\frac{64}{3}}$.
   If you missed this problem, review [link].
3. Factor: $4x^2 - 12x + 9$.
   If you missed this problem, review [link].

Quadratic equations are equations of the form $ax^2 + bx + c = 0$, where $a \neq 0$. They differ from linear equations by including a term with the variable raised to the second power. We use different methods to solve **quadratic equation**s than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will use three other methods to solve quadratic equations.

Solve Quadratic Equations of the Form *ax*^{2} = *k* Using the Square Root Property

We have already solved some quadratic equations by factoring. Let’s review how we used factoring to solve the quadratic equation $x^2 = 9$.

Equation:

	$x^2 = 9$
Put the equation in standard form.	$x^2 - 9 = 0$
Factor the left side.	$(x - 3)(x + 3) = 0$
Use the Zero Product Property.	$(x - 3) = 0, \quad (x + 3) = 0$
Solve each equation.	$x = 3, \quad x = -3$
Combine the two solutions into $\pm$ form.	$x = \pm 3$

(The solution is read $\backslash x$ is equal to positive or negative 3.’)

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

These equations are all of the form $x^2 = k$.
We defined the square root of a number in this way:
Equation:

$$\text{If } n^2 = m, \text{ then } n \text{ is a square root of } m.$$

This leads to the **Square Root Property**.

Note:
Square Root Property
If $x^2 = k$, and $k \geq 0$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$: the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$.

Now, we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.
Equation:

$$\begin{array}{lrcl} & x^2 & = & 9 \\ \text{Use the Square Root Property.} & x & = & \pm\sqrt{9} \\ \text{Simplify the radical.} & x & = & \pm 3 \\ \text{Rewrite to show the two solutions.} & x = 3, x & = & -3 \end{array}$$

What happens when the constant is not a perfect square? Let’s use the Square Root Property to solve the equation $x^2 = 7$.
Equation:

$$\begin{array}{lrcl} & x^2 & = & 7 \\ \text{Use the Square Root Property.} & x & = & \pm\sqrt{7} \\ \text{Rewrite to show two solutions.} & x = \sqrt{7}, & & x = -\sqrt{7} \\ \text{We cannot simplify } \sqrt{7}, \text{ so we leave the answer as a radical.} & & & \end{array}$$

Example:
Exercise:

Problem: Solve: $x^2 = 169$.

Solution:
Solution

	$x^2 = 169$
Use the Square Root Property.	$x = \pm\sqrt{169}$
Simplify the radical.	$x = \pm 13$
Rewrite to show two solutions.	$x = 13, \quad x = -13$

Note:
Exercise:

Problem: Solve: $x^2 = 81$.

Solution:

$x = 9, x = -9$

Note:
Exercise:

Problem: Solve: $y^2 = 121$.

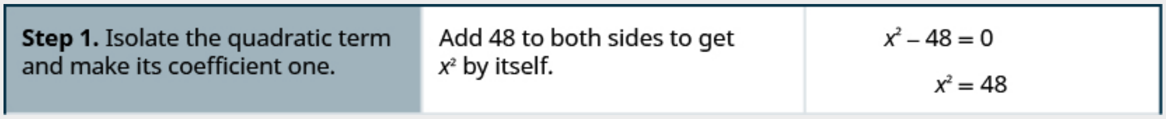
Solution:

$y = 11, y = -11$

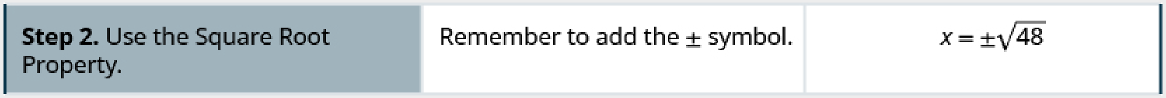
Example:
How to Solve a Quadratic Equation of the Form $ax^2 = k$ Using the Square Root Property
Exercise:

Problem: Solve: $x^2 - 48 = 0$.

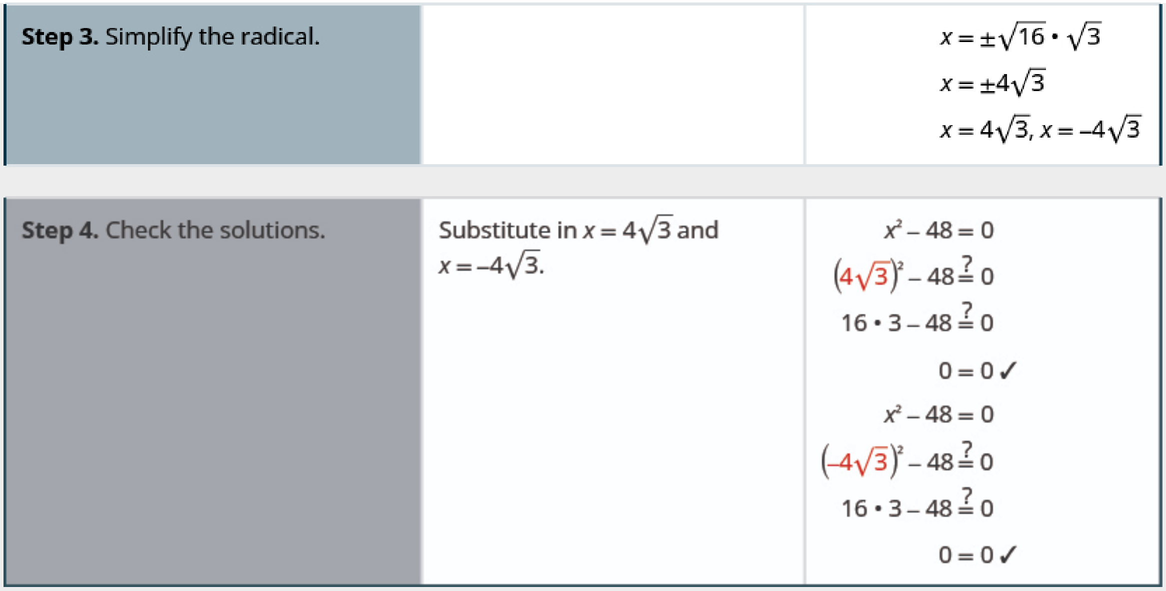
Solution:
Solution



Step 1. Isolate the quadratic term and make its coefficient one.	Add 48 to both sides to get x^2 by itself.	$x^2 - 48 = 0$ $x^2 = 48$



Step 2. Use the Square Root Property.	Remember to add the ± symbol.	$x = \pm\sqrt{48}$



Step 3. Simplify the radical.		$x=\pm\sqrt{16}\cdot\sqrt{3}$ $x=\pm4\sqrt{3}$ $x=4\sqrt{3}, x=-4\sqrt{3}$
Step 4. Check the solutions.	Substitute in $x=4\sqrt{3}$ and $x=-4\sqrt{3}$.	$x^2-48=0$ $(4\sqrt{3})^2-48\stackrel{?}{=}0$ $16\cdot3-48\stackrel{?}{=}0$ $0=0$ ✓ $x^2-48=0$ $(-4\sqrt{3})^2-48\stackrel{?}{=}0$ $16\cdot3-48\stackrel{?}{=}0$ $0=0$ ✓

Note:
Exercise:

Problem: Solve: $x^2-50=0$.

Solution:

$x=5\sqrt{2}, x=-5\sqrt{2}$

Note:
Exercise:

Problem: Solve: $y^2-27=0$.

Solution:

$y=3\sqrt{3}, y=-3\sqrt{3}$

Note:
Solve a quadratic equation using the Square Root Property.

Isolate the quadratic term and make its coefficient one.
Use Square Root Property.
Simplify the radical.

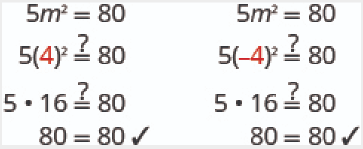
Check the solutions.

To use the Square Root Property, the coefficient of the variable term must equal 1. In the next example, we must divide both sides of the equation by 5 before using the Square Root Property.

Example:
Exercise:

Problem: Solve: $5m^2 = 80$.

Solution:
Solution

The quadratic term is isolated.	$5m^2 = 80$
Divide by 5 to make its cofficient 1.	$\frac{5m^2}{5} = \frac{80}{5}$
Simplify.	$m^2 = 16$
Use the Square Root Property.	$m = \pm\sqrt{16}$
Simplify the radical.	$m = \pm 4$
Rewrite to show two solutions.	$m = 4, m = -4$
Check the solutions.  $5m^2 = 80$ $\quad$ $5m^2 = 80$ $5(4)^2 \overset{?}{=} 80$ $\quad$ $5(-4)^2 \overset{?}{=} 80$ $5 \cdot 16 \overset{?}{=} 80$ $\quad$ $5 \cdot 16 \overset{?}{=} 80$ $80 = 80$ ✓ $\quad$ $80 = 80$ ✓	

Note:
Exercise:

Problem: Solve: $2x^2 = 98$.

Solution:

$x = 7, x = -7$

Note:
Exercise:

Problem: Solve: $3z^2 = 108$.

Solution:

$z = 6, z = -6$

The Square Root Property started by stating, ‘If $x^2 = k$, and $k \geq 0$’. What will happen if $k < 0$? This will be the case in the next example.

Example:
Exercise:

Problem: Solve: $q^2 + 24 = 0$.

Solution:
Solution

	$q^2 + 24 = 0$
Isolate the quadratic term.	$q^2 = -24$
Use the Square Root Property.	$q = \pm\sqrt{-24}$
The $\sqrt{-24}$ is not a real number.	There is no real solution.

Note:
Exercise:

Problem: Solve: $c^2 + 12 = 0$.

Solution:

no real solution

Note:
Exercise:

Problem: Solve: $d^2 + 81 = 0$.

Solution:

no real solution

Remember, we first isolate the quadratic term and then make the coefficient equal to one.

Example:
Exercise:

Problem: Solve: $\frac{2}{3}u^2 + 5 = 17$.

Solution:
Solution

	$\frac{2}{3}u^2 + 5 = 17$
Isolate the quadratic term.	$\frac{2}{3}u^2 = 12$
Multiply by $\frac{3}{2}$ to make the coefficient 1.	$\frac{3}{2} \cdot \frac{2}{3}u^2 = \frac{3}{2} \cdot 12$
Simplify.	$u^2 = 18$
Use the Square Root Property.	$u = \pm\sqrt{18}$
Simplify the radical.	$u = \pm\sqrt{9}\sqrt{2}$
Simplify.	$u = \pm 3\sqrt{2}$
Rewrite to show two solutions.	$u = 3\sqrt{2}, u = -3\sqrt{2}$

Check.

$$\begin{array}{cc}
\frac{2}{3}u^2+5=17 & \frac{2}{3}u^2+5=17 \\
\frac{2}{3}(3\sqrt{2})^2+5\overset{?}{=}17 & \frac{2}{3}(-3\sqrt{2})^2+5\overset{?}{=}17 \\
\frac{2}{3}\cdot 18+5\overset{?}{=}17 & \frac{2}{3}\cdot 18+5\overset{?}{=}17 \\
12+5\overset{?}{=}17 & 12+5\overset{?}{=}17 \\
17=17\checkmark & 17=17\checkmark
\end{array}$$

Note:
Exercise:

Problem: Solve: $\frac{1}{2}x^2+4=24$.

Solution:

$x=2\sqrt{10}, x=-2\sqrt{10}$

Note:
Exercise:

Problem: Solve: $\frac{3}{4}y^2-3=18$.

Solution:

$y=2\sqrt{7}, y=-2\sqrt{7}$

The solutions to some equations may have fractions inside the radicals. When this happens, we must rationalize the denominator.

Example:
Exercise:

Problem: Solve: $2c^2-4=45$.

Solution:

Solution

$2c^2 - 4 = 45$

Isolate the quadratic term. $2c^2 = 49$

Divide by 2 to make the coefficient 1. $\frac{2c^2}{2} = \frac{49}{2}$

Simplify. $c^2 = \frac{49}{2}$

Use the Square Root Property. $c = \pm\sqrt{\frac{49}{2}}$

Simplify the radical. $c = \pm\frac{\sqrt{49}}{\sqrt{2}}$

Rationalize the denominator. $c = \pm\frac{\sqrt{49}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}$

Simplify. $c = \pm\frac{7\sqrt{2}}{2}$

Rewrite to show two solutions. $c = \frac{7\sqrt{2}}{2}, \quad c = -\frac{7\sqrt{2}}{2}$

Check. We leave the check for you.

Note:
Exercise:

Problem: Solve: $5r^2 - 2 = 34$.

Solution:

$r = \frac{6\sqrt{5}}{5}, r = -\frac{6\sqrt{5}}{5}$

Note:
Exercise:

Problem: Solve: $3t^2 + 6 = 70$.

Solution:

$t = \frac{8\sqrt{3}}{3}, t = -\frac{8\sqrt{3}}{3}$

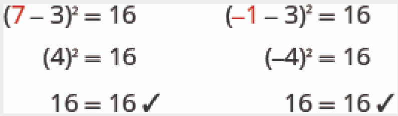
Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

We can use the Square Root Property to solve an equation like $(x-3)^2 = 16$, too. We will treat the whole binomial, $(x-3)$, as the quadratic term.

Example:
Exercise:

Problem: Solve: $(x-3)^2 = 16$.

Solution:
Solution

	$(x-3)^2 = 16$
Use the Square Root Property.	$x-3 = \pm\sqrt{16}$
Simplify.	$x-3 = \pm 4$
Write as two equations.	$x-3=4, x-3=-4$
Solve.	$x=7, x=-1$
Check.  $(7-3)^2 = 16$ $\quad (-1-3)^2 = 16$ $(4)^2 = 16$ $\quad (-4)^2 = 16$ $16 = 16\checkmark$ $\quad 16 = 16\checkmark$	

Note:
Exercise:

Problem: Solve: $(q+5)^2 = 1$.

Solution:

$q=-6, q=-4$

Note:
Exercise:

Problem: Solve: $(r-3)^2 = 25$.

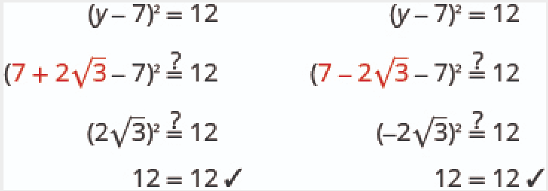
Solution:

$r = 8, r = -2$

Example:
Exercise:

Problem: Solve: $(y-7)^2 = 12$.

Solution:
Solution

	$(y-7)^2 = 12$
Use the Square Root Property.	$y - 7 = \pm\sqrt{12}$
Simplify the radical.	$y - 7 = \pm 2\sqrt{3}$
Solve for *y*.	$y = 7 \pm 2\sqrt{3}$
Rewrite to show two solutions.	$y = 7 + 2\sqrt{3}, y = 7 - 2\sqrt{3}$
Check.  $(y-7)^2 = 12$ $\quad$ $(y-7)^2 = 12$ $(7+2\sqrt{3}-7)^2 \overset{?}{=} 12$ $\quad$ $(7-2\sqrt{3}-7)^2 \overset{?}{=} 12$ $(2\sqrt{3})^2 \overset{?}{=} 12$ $\quad$ $(-2\sqrt{3})^2 \overset{?}{=} 12$ $12 = 12$ ✓ $\quad$ $12 = 12$ ✓	

Note:
Exercise:

Problem: Solve: $(a-3)^2 = 18$.

Solution:

$a=3+3\sqrt{2}, a=3-3\sqrt{2}$

Note:
Exercise:

Problem: Solve: $(b+2)^2=40$.

Solution:

$b=-2+2\sqrt{10}, b=-2-2\sqrt{10}$

Remember, when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

Example:
Exercise:

Problem: Solve: $\left(x-\frac{1}{2}\right)^2=\frac{5}{4}$.

Solution:
Solution

	$\left(x-\frac{1}{2}\right)^2 = \frac{5}{4}$
Use the Square Root Property.	$x-\frac{1}{2} = \pm\sqrt{\frac{5}{4}}$
Rewrite the radical as a fraction of square roots.	$x-\frac{1}{2} = \pm\frac{\sqrt{5}}{\sqrt{4}}$
Simplify the radical.	$x-\frac{1}{2} = \pm\frac{\sqrt{5}}{2}$
Solve for x.	$x = \frac{1}{2}\pm\frac{\sqrt{5}}{2}$
Rewrite to show two solutions.	$x=\frac{1}{2}+\frac{\sqrt{5}}{2}, \quad x=\frac{1}{2}-\frac{\sqrt{5}}{2}$
Check. We leave the check for you.	

Note:
Exercise:

Problem: Solve: $\left(x-\frac{1}{3}\right)^2=\frac{5}{9}$.

Solution:

$x=\frac{1}{3}+\frac{\sqrt{5}}{3}, x=\frac{1}{3}-\frac{\sqrt{5}}{3}$

Note:
Exercise:

Problem: Solve: $\left(y-\frac{3}{4}\right)^2=\frac{7}{16}$.

Solution:

$y=\frac{3}{4}+\frac{\sqrt{7}}{4}, y=\frac{3}{4}-\frac{\sqrt{7}}{4}$

We will start the solution to the next example by isolating the binomial.

Example:
Exercise:

Problem: Solve: $(x-2)^2+3=30$.

Solution:
Solution

$$(x-2)^2+3 = 30$$

Isolate the binomial term. $(x-2)^2 = 27$

Use the Square Root Property. $x-2 = \pm\sqrt{27}$

Simplify the radical. $x-2 = \pm 3\sqrt{3}$

Solve for x. $x = 2\pm 3\sqrt{3}$

Rewrite to show two solutions. $x=2+3\sqrt{3}, \quad x=2-3\sqrt{3}$

Check. We leave the check for you.

Note:
Exercise:

Problem: Solve: $(a-5)^2+4=24$.

Solution:

$a=5+2\sqrt{5}, a=5-2\sqrt{5}$

Note:
Exercise:

Problem: Solve: $(b-3)^2-8=24$.

Solution:

$b=3+4\sqrt{2}, b=3-4\sqrt{2}$

Example:
Exercise:

Problem: Solve: $(3v-7)^2=-12$.

Solution:
Solution

Use the Square Root Property.

$$\begin{aligned}(3v-7)^2 &= -12\\ 3v-7 &= \pm\sqrt{-12}\end{aligned}$$

The $\sqrt{-12}$ is not a real number. There is no real solution.

Note:
Exercise:

Problem: Solve: $(3r+4)^2=-8$.

Solution:

no real solution

Note:
Exercise:

Problem: Solve: $(2t-8)^2 = -10$.

Solution:

no real solution

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

Example:
Exercise:

Problem: Solve: $p^2 - 10p + 25 = 18$.

Solution:
Solution

The left side of the equation is a perfect square trinomial. We will factor it first.

$p^2 - 10p + 25 = 18$

Factor the perfect square trinomial. $(p-5)^2 = 18$

Use the Square Root Property. $p - 5 = \pm\sqrt{18}$

Simplify the radical. $p - 5 = \pm 3\sqrt{2}$

Solve for p. $p = 5 \pm 3\sqrt{2}$

Rewrite to show two solutions. $p = 5 + 3\sqrt{2}, \quad p = 5 - 3\sqrt{2}$

Check. We leave the check for you.

Note:
Exercise:

Problem: Solve: $x^2 - 6x + 9 = 12$.

Solution:

$x = 3 + 2\sqrt{3}, x = 3 - 2\sqrt{3}$

Note:
Exercise:

Problem: Solve: $y^2 + 12y + 36 = 32$.

Solution:

$y = -6 + 4\sqrt{2}, y = -6 - 4\sqrt{2}$

Example:
Exercise:

Problem: Solve: $4n^2 + 4n + 1 = 16$.

Solution:
Solution

Again, we notice the left side of the equation is a perfect square trinomial. We will factor it first.

	$4n^2 + 4n + 1 = 16$
Factor the perfect square trinomial.	$(2n + 1)^2 = 16$
Use the Square Root Property.	$2n + 1 = \pm\sqrt{16}$
Simplify the radical.	$2n + 1 = \pm 4$
Solve for *n*.	$2n = -1 \pm 4$
Divide each side by 2.	$\frac{2n}{2} = \frac{-1\pm4}{2}$ $n = \frac{-1\pm4}{2}$
Rewrite to show two solutions.	$n = \frac{-1+4}{2}, n = \frac{-1-4}{2}$
Simplify each equation.	$n = \frac{3}{2}, n = -\frac{5}{2}$
Check.	

$$\begin{aligned} 4n^2+4n+1 &= 16 \\ 4\left(\frac{3}{2}\right)^2+4\left(\frac{3}{2}\right)+1 &\overset{?}{=} 16 \\ 4\left(\frac{9}{4}\right)+4\left(\frac{3}{2}\right)+1 &\overset{?}{=} 16 \\ 9+6+1 &\overset{?}{=} 16 \\ 16 &= 16\checkmark \end{aligned}$$

$$\begin{aligned} 4n^2+4n+1 &= 16 \\ 4\left(-\frac{5}{2}\right)^2+4\left(-\frac{5}{2}\right)+1 &\overset{?}{=} 16 \\ 4\left(\frac{25}{4}\right)+4\left(-\frac{5}{2}\right)+1 &\overset{?}{=} 16 \\ 25-10+1 &\overset{?}{=} 16 \\ 16 &= 16\checkmark \end{aligned}$$

Note:
Exercise:

Problem: Solve: $9m^2-12m+4=25$.

Solution:

$m=7, m=-3$

Note:
Exercise:

Problem: Solve: $16n^2+40n+25=4$.

Solution:

$n=-\frac{3}{4}, n=-\frac{7}{4}$

Note:
Access these online resources for additional instruction and practice with solving quadratic equations:

- Solving Quadratic Equations: Solving by Taking Square Roots
- Using Square Roots to Solve Quadratic Equations
- Solving Quadratic Equations: The Square Root Method

Key Concepts

- Square Root Property
  If $x^2=k$, and $k\geq 0$, then $x=\sqrt{k}$ or $x=-\sqrt{k}$.

Practice Makes Perfect

Solve Quadratic Equations of the form $\mathbf{ax^2 = k}$ Using the Square Root Property

In the following exercises, solve the following quadratic equations.

Exercise:

Problem: $a^2 = 49$

Solution:

$a = \pm\,7$

Exercise:

Problem: $b^2 = 144$

Exercise:

Problem: $r^2 - 24 = 0$

Solution:

$r = \pm\,2\sqrt{6}$

Exercise:

Problem: $t^2 - 75 = 0$

Exercise:

Problem: $u^2 - 300 = 0$

Solution:

$u = \pm\,10\sqrt{3}$

Exercise:

Problem: $v^2 - 80 = 0$

Exercise:

Problem: $4m^2 = 36$

Solution:

$m = \pm\,3$

Exercise:

Problem: $3n^2 = 48$

Exercise:

Problem: $x^2 + 20 = 0$

Solution:

no real solution

Exercise:

Problem: $y^2 + 64 = 0$

Exercise:

Problem: $\frac{2}{5}a^2 + 3 = 11$

Solution:

$a = \pm\, 2\sqrt{5}$

Exercise:

Problem: $\frac{3}{2}b^2 - 7 = 41$

Exercise:

Problem: $7p^2 + 10 = 26$

Solution:

$p = \pm\, \frac{4\sqrt{7}}{7}$

Exercise:

Problem: $2q^2 + 5 = 30$

Solve Quadratic Equations of the Form $\mathbf{a(x - h)^2 = k}$ Using the Square Root Property

In the following exercises, solve the following quadratic equations.

Exercise:

Problem: $(x + 2)^2 = 9$

Solution:

$x = 1, x = -5$

Exercise:

Problem: $(y - 5)^2 = 36$

Exercise:

Problem: $(u-6)^2=64$

Solution:

$u=14, u=-2$

Exercise:

Problem: $(v+10)^2=121$

Exercise:

Problem: $(m-6)^2=20$

Solution:

$m=6\pm 2\sqrt{5}$

Exercise:

Problem: $(n+5)^2=32$

Exercise:

Problem: $\left(r-\frac{1}{2}\right)^2=\frac{3}{4}$

Solution:

$r=\frac{1}{2}\pm\frac{\sqrt{3}}{2}$

Exercise:

Problem: $\left(t-\frac{5}{6}\right)^2=\frac{11}{25}$

Exercise:

Problem: $(a-7)^2+5=55$

Solution:

$a=7\pm 5\sqrt{2}$

Exercise:

Problem: $(b-1)^2-9=39$

Exercise:

Problem: $(5c+1)^2 = -27$

Solution:

no real solution

Exercise:

Problem: $(8d-6)^2 = -24$

Exercise:

Problem: $m^2 - 4m + 4 = 8$

Solution:

$m = 2 \pm 2\sqrt{2}$

Exercise:

Problem: $n^2 + 8n + 16 = 27$

Exercise:

Problem: $25x^2 - 30x + 9 = 36$

Solution:

$x = -\frac{3}{5}, x = \frac{9}{5}$

Exercise:

Problem: $9y^2 + 12y + 4 = 9$

Mixed Practice

In the following exercises, solve using the Square Root Property.

Exercise:

Problem: $2r^2 = 32$

Solution:

$r = \pm 4$

Exercise:

Problem: $4t^2 = 16$

Exercise:

Problem: $(a-4)^2=28$

Solution:

$a=4\pm 2\sqrt{7}$

Exercise:

Problem: $(b+7)^2=8$

Exercise:

Problem: $9w^2-24w+16=1$

Solution:

$w=1, w=\frac{5}{3}$

Exercise:

Problem: $4z^2+4z+1=49$

Exercise:

Problem: $a^2-18=0$

Solution:

$a=\pm 3\sqrt{2}$

Exercise:

Problem: $b^2-108=0$

Exercise:

Problem: $\left(p-\frac{1}{3}\right)^2=\frac{7}{9}$

Solution:

$p=\frac{1}{3}\pm\frac{\sqrt{7}}{3}$

Exercise:

Problem: $\left(q-\frac{3}{5}\right)^2=\frac{3}{4}$

Exercise:

Problem: $m^2+12=0$

Solution:

no real solution

Exercise:

Problem: $n^2 + 48 = 0$

Exercise:

Problem: $u^2 - 14u + 49 = 72$

Solution:

$u = 7 \pm 6\sqrt{2}$

Exercise:

Problem: $v^2 + 18v + 81 = 50$

Exercise:

Problem: $(m - 4)^2 + 3 = 15$

Solution:

$m = 4 \pm 2\sqrt{3}$

Exercise:

Problem: $(n - 7)^2 - 8 = 64$

Exercise:

Problem: $(x + 5)^2 = 4$

Solution:

$x = -3, x = -7$

Exercise:

Problem: $(y - 4)^2 = 64$

Exercise:

Problem: $6c^2 + 4 = 29$

Solution:

$c = \pm \frac{5\sqrt{6}}{6}$

Exercise:

Problem: $2d^2 - 4 = 77$

Exercise:

Problem: $(x - 6)^2 + 7 = 3$

Solution:

no real solution

Exercise:

Problem: $(y - 4)^2 + 10 = 9$

Everyday Math

Exercise:

Problem:

Paola has enough mulch to cover 48 square feet. She wants to use it to make three square vegetable gardens of equal sizes. Solve the equation $3s^2 = 48$ to find s, the length of each garden side.

Solution:

4 feet

Exercise:

Problem:

Kathy is drawing up the blueprints for a house she is designing. She wants to have four square windows of equal size in the living room, with a total area of 64 square feet. Solve the equation $4s^2 = 64$ to find s, the length of the sides of the windows.

Writing Exercises

Exercise:

Problem: Explain why the equation $x^2 + 12 = 8$ has no solution.

Solution:

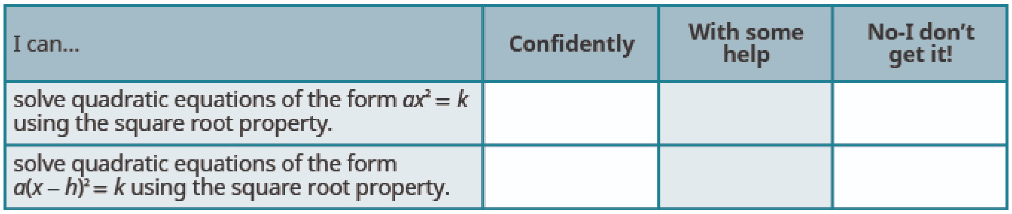
Answers will vary.

Exercise:

Problem: Explain why the equation $y^2 + 8 = 12$ has two solutions.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can…	Confidently	With some help	No-I don’t get it!
solve quadratic equations of the form $ax^2 = k$ using the square root property.			
solve quadratic equations of the form $a(x - h)^2 = k$ using the square root property.			

ⓑ If most of your checks were:

…confidently: Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help: This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no-I don’t get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

Glossary

quadratic equation

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$.

Square Root Property

The Square Root Property states that, if $x^2 = k$ and $k \geq 0$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

Solve Quadratic Equations by Completing the Square
By the end of this section, you will be able to:

- Complete the square of a binomial expression
- Solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square
- Solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square

Note:
Before you get started, take this readiness quiz. If you miss a problem, go back to the section listed and review the material.

1. Simplify $(x + 12)^2$.
   If you missed this problem, review [link].
2. Factor $y^2 - 18y + 81$.
   If you missed this problem, review [link].
3. Factor $5n^2 + 40n + 80$.
   If you missed this problem, review [link].

So far, we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called ‘completing the square.’

Complete The Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation $(y - 7)^2 = 12$ because the left side was a perfect square.
Equation:

$$\begin{aligned} (y-7)^2 &= 12 \\ y-7 &= \pm\sqrt{12} \\ y-7 &= \pm 2\sqrt{3} \\ y &= 7 \pm 2\sqrt{3} \end{aligned}$$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form $(x-k)^2$ in order to use the square root property.

Equation:

$$\begin{aligned} x^2 - 10x + 25 &= 18 \\ (x-5)^2 &= 18 \end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let’s study the binomial square pattern we have used many times. We will look at two examples.

Equation:

$$\begin{array}{ll} (x+9)^2 & (y-7)^2 \\ (x+9)\,(x+9) & (y-7)\ (y-7) \\ x^2+9x+9x+81 & y^2-7y-7y+49 \\ x^2+18x+81 & y^2-14y+49 \end{array}$$

Note:

Binomial Squares Pattern

If a, b are real numbers,

Equation:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\times(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

Equation:

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$\underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\times(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

We can use this pattern to “make” a perfect square.

We will start with the expression $x^2 + 6x$. Since there is a plus sign between the two terms, we will use the $(a+b)^2$ pattern.

Equation:

$$a^2 + 2ab + b^2 = (a+b)^2$$

Notice that the first term of $x^2 + 6x$ is a square, x^2.

We now know $a = x$.

What number can we add to $x^2 + 6x$ to make a perfect square trinomial?

$$\begin{array}{l} a^2 + 2ab + b^2 \\ x^2 + 6x + \underline{\quad} \end{array}$$

The middle term of the Binomial Squares Pattern, $2ab$, is twice the product of the two terms of the binomial. This means twice the product of x and some number is $6x$. So, two times some number must be six. The number we need is $\frac{1}{2} \cdot 6 = 3$. The second term in the binomial, b, must be 3.

$$\begin{array}{l} a^2 + \quad 2ab \quad + \; b^2 \\ x^2 + 2 \cdot 3 \cdot x + \_\_ \end{array}$$

We now know $b = 3$.

Now, we just square the second term of the binomial to get the last term of the perfect square trinomial, so we square three to get the last term, nine.

$$\begin{array}{l} a^2 + 2ab + b^2 \\ x^2 + 6x + 9 \end{array}$$

We can now factor to

$$\begin{array}{l} (a + b)^2 \\ (x + 3)^2 \end{array}$$

So, we found that adding nine to $x^2 + 6x$ ‘completes the square,’ and we write it as $(x + 3)^2$.

Note:
Complete a square.
To complete the square of $x^2 + bx$:

Identify b, the coefficient of x.
Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$.

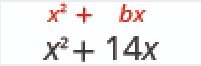
Example:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Then, write the result as a binomial square.

$x^2 + 14x$

Solution:
Solution

The coefficient of *x* is 14.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2} \cdot 14\right)^2$ $(7)^2$ 49	
Add 49 to the binomial to complete the square.	$x^2 + 14x + 49$
Rewrite as a binomial square.	$(x + 7)^2$

Note:

Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$y^2 + 12y$

Solution:

$(y + 6)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$z^2 + 8z$

Solution:

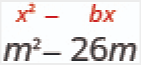
$(z + 4)^2$

Example:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Then, write the result as a binomial squared. $m^2 - 26m$

Solution:
Solution

The coefficient of *m* is −26.	 $x^2 -$ bx $m^2 - 26m$
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot(-26)\right)^2$ $(-13)^2$ 169	
Add 169 to the binomial to complete the square.	$m^2 - 26m + 169$
Rewrite as a binomial square.	$(m-13)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$a^2 - 20a$

Solution:

$(a-10)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

b^2-4b

Solution:

$(b-2)^2$

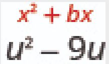
Example:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Then, write the result as a binomial squared.

u^2-9u

Solution:
Solution

The coefficient of *u* is −9.	 $x^2 + bx$ $u^2 - 9u$
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2} \cdot (-9)\right)^2$ $\left(-\frac{9}{2}\right)^2$ $\frac{81}{4}$	
Add $\frac{81}{4}$ to the binomial to complete the square.	$u^2 - 9u + \frac{81}{4}$
Rewrite as a binomial square.	$\left(u - \frac{9}{2}\right)^2$

Note:

Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$m^2 - 5m$

Solution:

$\left(m - \frac{5}{2}\right)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$n^2 + 13n$

Solution:

$\left(n + \frac{13}{2}\right)^2$

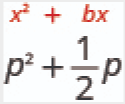
Example:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Then, write the result as a binomial squared.

$p^2 + \frac{1}{2}p$

Solution:
Solution

The coefficient of p is $\frac{1}{2}$.	 $\overset{x^2}{p^2} \overset{+}{+} \overset{bx}{\frac{1}{2}p}$

Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot\frac{1}{2}\right)^2$ $\left(\frac{1}{4}\right)^2$ $\frac{1}{16}$	
Add $\frac{1}{16}$ to the binomial to complete the square.	$p^2+\frac{1}{2}p+\frac{1}{16}$
Rewrite as a binomial square.	$\left(p+\frac{1}{4}\right)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$p^2+\frac{1}{4}p$

Solution:

$\left(p+\frac{1}{8}\right)^2$

Note:
Exercise:

Problem:

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$q^2 - \frac{2}{3}q$

Solution:

$\left(q - \frac{1}{3}\right)^2$

Solve Quadratic Equations of the Form $x^2 + bx + c = 0$ by completing the square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a quadratic equation by completing the square, too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2 + 6x = 40$ and we want to complete the square on the left, we will add nine to both sides of the equation.

$$\begin{aligned} x^2 + 6x &= 40 \\ x^2 + 6x + \_\_ &= 40 + \_\_ \\ x^2 + 6x + 9 &= 40 + 9 \end{aligned}$$

Then, we factor on the left and simplify on the right.

Equation:

$$(x+3)^2 = 49$$

Now the equation is in the form to solve using the Square Root Property. Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

Example:

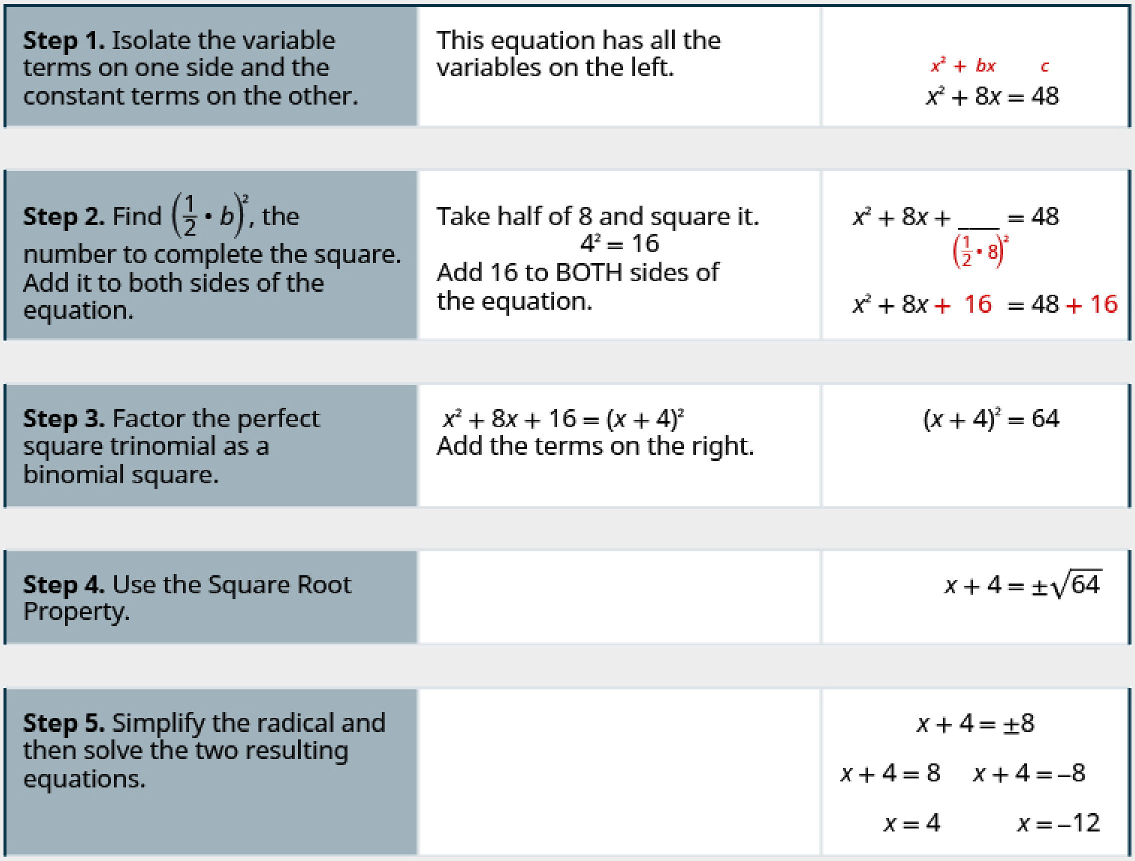
How To Solve a Quadratic Equation of the Form $x^2 + bx + c = 0$ by Completing the Square

Exercise:

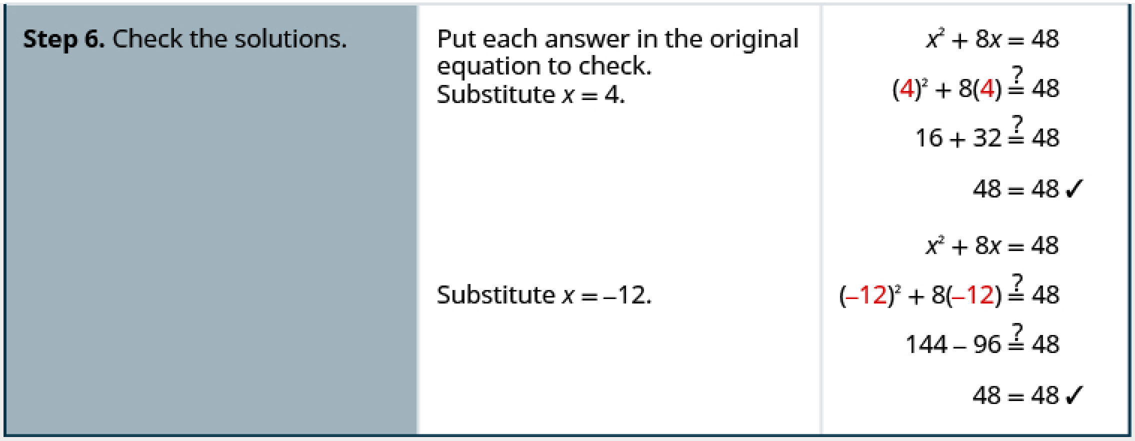
Problem: Solve $x^2 + 8x = 48$ by completing the square.

Solution:

Solution



Step 1. Isolate the variable terms on one side and the constant terms on the other.	This equation has all the variables on the left.	$\overset{x^2 + bx}{x^2 + 8x} = \overset{c}{48}$
Step 2. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.	Take half of 8 and square it. $4^2 = 16$ Add 16 to BOTH sides of the equation.	$x^2 + 8x + \underset{\left(\frac{1}{2} \cdot 8\right)^2}{\_\_\_\_} = 48$ $x^2 + 8x + 16 = 48 + 16$
Step 3. Factor the perfect square trinomial as a binomial square.	$x^2 + 8x + 16 = (x + 4)^2$ Add the terms on the right.	$(x + 4)^2 = 64$
Step 4. Use the Square Root Property.		$x + 4 = \pm\sqrt{64}$
Step 5. Simplify the radical and then solve the two resulting equations.		$x + 4 = \pm 8$ $x + 4 = 8 \quad x + 4 = -8$ $x = 4 \quad x = -12$



Step 6. Check the solutions.	Put each answer in the original equation to check. Substitute $x = 4$.	$x^2 + 8x = 48$ $(4)^2 + 8(4) \stackrel{?}{=} 48$ $16 + 32 \stackrel{?}{=} 48$ $48 = 48$ ✓
	Substitute $x = -12$.	$x^2 + 8x = 48$ $(-12)^2 + 8(-12) \stackrel{?}{=} 48$ $144 - 96 \stackrel{?}{=} 48$ $48 = 48$ ✓

Note:
Exercise:

Problem: Solve $c^2 + 4c = 5$ by completing the square.

Solution:

$c = -5, c = 1$

Note:
Exercise:

Problem: Solve $d^2 + 10d = -9$ by completing the square.

Solution:

$d = -9, d = -1$

Note:
Solve a quadratic equation of the form $x^2 + bx + c = 0$ by completing the square.

Isolate the variable terms on one side and the constant terms on the other.
Find $\left(\frac{1}{2} \cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.
Factor the perfect square trinomial as a binomial square.
Use the Square Root Property.
Simplify the radical and then solve the two resulting equations.
Check the solutions.

Example:
Exercise:

Problem: Solve $y^2 - 6y = 16$ by completing the square.

Solution:
Solution

The variable terms are on the left side.	$\overset{x^2-bx}{y^2-6y}=\overset{c}{16}$
Take half of -6 and square it. $\left(\frac{1}{2}(-6)\right)^2 = 9$	$y^2-6y+\underset{\left(\frac{1}{2}\cdot(-6)\right)^2}{\underline{\qquad\qquad}}=16$
Add 9 to both sides.	$y^2-6y+9=16+9$

Factor the perfect square trinomial as a binomial square.	$(y-3)^2 = 25$
Use the Square Root Property.	$y-3 = \pm\sqrt{25}$
Simplify the radical.	$y-3 = \pm 5$
Solve for *y*.	$y = 3 \pm 5$
Rewrite to show two solutions.	$y = 3+5,\ y = 3-5$
Solve the equations.	$y = 8,\ y = -2$
Check. $y^2 - 6y = 16$ $y^2 - 6y = 16$ $8^2 - 6 \cdot 8 \overset{?}{=} 16$ $(-2)^2 - 6(-2) \overset{?}{=} 16$ $64 - 48 \overset{?}{=} 16$ $4 + 12 \overset{?}{=} 16$ $16 = 16$ ✓ $16 = 16$ ✓	

Note:
Exercise:

Problem: Solve $r^2 - 4r = 12$ by completing the square.

Solution:

$r = -2, r = 6$

Note:
Exercise:

Problem: Solve $t^2 - 10t = 11$ by completing the square.

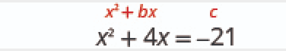
Solution:

$t = -1, t = 11$

Example:
Exercise:

Problem: Solve $x^2 + 4x = -21$ by completing the square.

Solution:
Solution

The variable terms are on the left side.	 $\overset{x^2+bx}{x^2+4x} = \overset{c}{-21}$

Take half of 4 and square it. $\left(\frac{1}{2}(4)\right)^2 = 4$	$x^2 + 4x + \underset{\left(\frac{1}{2}\cdot 4\right)^2}{\underline{\qquad}} = -21$
Add 4 to both sides.	$x^2 + 4x + 4 = -21 + 4$
Factor the perfect square trinomial as a binomial square.	$(x+2)^2 = -17$
Use the Square Root Property.	$x + 2 = \pm\sqrt{-17}$
We cannot take the square root of a negative number.	There is no real solution.

Note:
Exercise:

Problem: Solve $y^2 - 10y = -35$ by completing the square.

Solution:

no real solution

Note:
Exercise:

Problem: Solve $z^2 + 8z = -19$ by completing the square.

Solution:

no real solution

In the previous example, there was no real solution because $(x+k)^2$ was equal to a negative number.

Example:
Exercise:

Problem: Solve $p^2 - 18p = -6$ by completing the square.

Solution:
Solution

The variable terms are on the left side.	$\overset{x^2+bx}{P^2 - 18P} = \overset{c}{-6}$
Take half of -18 and square it. $\left(\frac{1}{2}(-18)\right)^2 = 81$	$P^2 - 18p + \underset{\left(\frac{1}{2}\cdot(-18)\right)^2}{\underline{\qquad\qquad}} = -6$
Add 81 to both sides.	$p^2 - 18p + 81 = -6 + 81$
Factor the perfect square trinomial as a binomial square.	$(p-9)^2 = 75$

Use the Square Root Property.	$p-9=\pm\sqrt{75}$
Simplify the radical.	$p-9=\pm 5\sqrt{3}$
Solve for *p*.	$p=9\pm 5\sqrt{3}$
Rewrite to show two solutions.	$p=9+5\sqrt{3},\ p=9-5\sqrt{3}$
Check. $p^2-18p=-6$ $\left(9+5\sqrt{3}\right)^2-18\left(9+5\sqrt{3}\right)\overset{?}{=}-6$ $81+90\sqrt{3}+75-162-90\sqrt{3}\overset{?}{=}-6$ $-6=-6\checkmark$ $p^2-18p=-6$ $\left(9-5\sqrt{3}\right)^2-18\left(9-5\sqrt{3}\right)\overset{?}{=}-6$ $81-90\sqrt{3}+75-162+90\sqrt{3}\overset{?}{=}-6$ $-6=-6\checkmark$	

Another way to check this would be to use a calculator. Evaluate p^2-18p for both of the solutions. The answer should be -6.

Note:
Exercise:

Problem: Solve $x^2-16x=-16$ by completing the square.

Solution:

$x=8\pm 4\sqrt{3}$

Note:
Exercise:

Problem: Solve $y^2 + 8y = 11$ by completing the square.

Solution:

$y = -4 \pm 3\sqrt{3}$

We will start the next example by isolating the variable terms on the left side of the equation.

Example:
Exercise:

Problem: Solve $x^2 + 10x + 4 = 15$ by completing the square.

Solution:
Solution

The variable terms are on the left side.	$x^2 + 10x + 4 = 15$
Subtract 4 to get the constant terms on the right side.	$x^2 + 10x \qquad = 11$
Take half of 10 and square it.	

$\left(\frac{1}{2}(10)\right)^2 = 25$	$x^2 + 10x + \underset{\left(\frac{1}{2}\cdot(10)\right)^2}{\underline{\qquad}} = 11$
Add 25 to both sides.	$x^2 + 10x + 25 = 11 + 25$
Factor the perfect square trinomial as a binomial square.	$(x+5)^2 = 36$
Use the Square Root Property.	$x + 5 = \pm\sqrt{36}$
Simplify the radical.	$x + 5 = \pm\sqrt{36}$
Solve for *x*.	$x = -5 \pm 6$
Rewrite to show two equations.	$x = -5 + 6, x = -5 - 6$
Solve the equations.	$x = 1, x = -11$
Check. $\begin{array}{rr} x^2 + 10x + 4 = 15 & x^2 + 10x + 4 = 15 \\ (1)^2 + 10(1) + 4 \overset{?}{=} 15 & (-11)^2 + 10(-11) + 4 \overset{?}{=} 15 \\ 1 + 10 + 4 \overset{?}{=} 15 & 121 - 110 + 4 \overset{?}{=} 15 \\ 15 = 15\checkmark & 15 = 15\checkmark \end{array}$	

Note:
Exercise:

Problem: Solve $a^2 + 4a + 9 = 30$ by completing the square.

Solution:

$a = -7, a = 3$

Note:
Exercise:

Problem: Solve $b^2 + 8b - 4 = 16$ by completing the square.

Solution:

$b = -10, b = -2$

To solve the next equation, we must first collect all the variable terms to the left side of the equation. Then, we proceed as we did in the previous examples.

Example:
Exercise:

Problem: Solve $n^2 = 3n + 11$ by completing the square.

Solution:
Solution

	$n^2 = 3n + 11$
Subtract 3*n* to get the variable terms on the left side.	$n^2 - 3n \quad = 11$
Take half of -3 and square it. $\left(\frac{1}{2}(-3)\right)^2 = \frac{9}{4}$	$n^2 - 3n + \underset{\left(\frac{1}{2}\cdot(-3)\right)^2}{\underline{\qquad}} = 11$
Add $\frac{9}{4}$ to both sides.	$n^2 - 3n + \frac{9}{4} = 11 + \frac{9}{4}$
Factor the perfect square trinomial as a binomial square.	$\left(n - \frac{3}{2}\right)^2 = \frac{44}{4} + \frac{9}{4}$
Add the fractions on the right side.	$\left(n - \frac{3}{2}\right)^2 = \frac{53}{4}$
Use the Square Root Property.	$n - \frac{3}{2} = \pm\sqrt{\frac{53}{4}}$
Simplify the radical.	$n - \frac{3}{2} = \pm\frac{\sqrt{53}}{2}$
Solve for *n.*	$n = \frac{3}{2} \pm \frac{\sqrt{53}}{2}$
Rewrite to show two equations.	$n = \frac{3}{2} + \frac{\sqrt{53}}{2},\ n = \frac{3}{2} - \frac{\sqrt{53}}{2}$

Check. We leave the check for you!	

Note:
Exercise:

Problem: Solve $p^2 = 5p + 9$ by completing the square.

Solution:

$p = \frac{5}{2} \pm \frac{\sqrt{61}}{2}$

Note:
Exercise:

Problem: Solve $q^2 = 7q - 3$ by completing the square.

Solution:

$q = \frac{7}{2} \pm \frac{\sqrt{37}}{2}$

Notice that the left side of the next equation is in factored form. But the right side is not zero, so we cannot use the Zero Product Property. Instead, we multiply the factors and then put the equation into the standard form to solve by completing the square.

Example:

Exercise:

Problem: Solve $(x-3)\,(x+5)=9$ by completing the square.

Solution:
Solution

	$(x-3)\,(x+5)=9$
We multiply binomials on the left.	$x^2+2x-15=9$
Add 15 to get the variable terms on the left side.	$x^2+2x\qquad=24$
Take half of 2 and square it. $\left(\frac{1}{2}(2)\right)^2=1$	$x^2+2x+\underset{\left(\frac{1}{2}\cdot(2)\right)^2}{\underline{\qquad}}=24$
Add 1 to both sides.	$x^2+2x+1=24+1$
Factor the perfect square trinomial as a binomial square.	$(x+1)^2=25$
Use the Square Root Property.	$x+1=\pm\sqrt{25}$
Solve for *x*.	$x=-1\pm5$

Rewite to show two solutions.	$x=-1+5, x=-1-5$
Simplify.	$x=4, x=-6$
Check. We leave the check for you!	

Note:
Exercise:

Problem: Solve $\left(c-2\right)\left(c+8\right)=7$ by completing the square.

Solution:

$c=-3\pm 4\sqrt{2}$

Note:
Exercise:

Problem: Solve $\left(d-7\right)\left(d+3\right)=56$ by completing the square.

Solution:

$d=-7,d=11$

Solve Quadratic Equations of the form $ax^2 + bx + c = 0$ by completing the square

The process of completing the square works best when the leading coefficient is one, so the left side of the equation is of the form $x^2 + bx + c$. If the x^2 term has a coefficient, we take some preliminary steps to make the coefficient equal to one.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

Example:
Exercise:

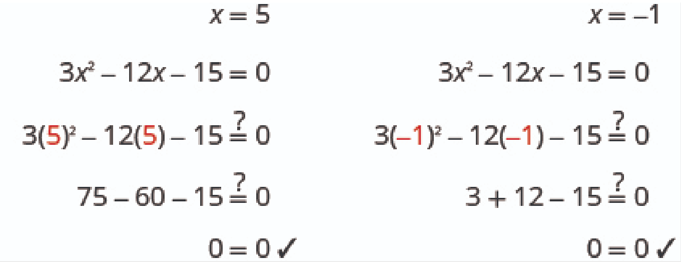
Problem: Solve $3x^2 - 12x - 15 = 0$ by completing the square.

Solution:
Solution

To complete the square, we need the coefficient of x^2 to be one. If we factor out the coefficient of x^2 as a common factor, we can continue with solving the equation by completing the square.

	$3x^2 - 12x - 15 = 0$
Factor out the greatest common factor.	$3(x^2 - 4x - 5) = 0$
Divide both sides by 3 to isolate the	

trinomial.	$\frac{3(x^2-4x-5)}{3}=\frac{0}{3}$
Simplify.	$x^2-4x-5=0$
Subtract 5 to get the constant terms on the right.	$x^2-4x \qquad =5$
Take half of 4 and square it. $\left(\frac{1}{2}(4)\right)^2=4$	$x^2-4x+\underset{\left(\frac{1}{2}\cdot(4)\right)^2}{\underline{\qquad}}=5$
Add 4 to both sides.	$x^2-4x+4=5+4$
Factor the perfect square trinomial as a binomial square.	$(x-2)^2=9$
Use the Square Root Property.	$x-2=\pm\sqrt{9}$
Solve for *x*.	$x-2=\pm 3$
Rewrite to show 2 solutions.	$x=2+3,\ x=2-3$
Simplify.	$x=5,\ x=-1$
Check.	



$$\begin{array}{rr} x = 5 & x = -1 \\ 3x^2 - 12x - 15 = 0 & 3x^2 - 12x - 15 = 0 \\ 3(5)^2 - 12(5) - 15 \overset{?}{=} 0 & 3(-1)^2 - 12(-1) - 15 \overset{?}{=} 0 \\ 75 - 60 - 15 \overset{?}{=} 0 & 3 + 12 - 15 \overset{?}{=} 0 \\ 0 = 0 \checkmark & 0 = 0 \checkmark \end{array}$$

Note:
Exercise:

Problem: Solve $2m^2 + 16m - 8 = 0$ by completing the square.

Solution:

$m = -4 \pm 2\sqrt{5}$

Note:
Exercise:

Problem: Solve $4n^2 - 24n - 56 = 8$ by completing the square.

Solution:

$n = -2, 8$

To complete the square, the leading coefficient must be one. When the leading coefficient is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient. This will give us a fraction for the second coefficient. We have already seen how to complete the square with fractions in this section.

Example:
Exercise:

Problem: Solve $2x^2 - 3x = 20$ by completing the square.

Solution:
Solution

Again, our first step will be to make the coefficient of x^2 be one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$2x^2 - 3x = 20$
Divide both sides by 2 to get the coefficient of x^2 to be 1.	$\frac{2x^2 - 3x}{2} = \frac{20}{2}$
Simplify.	$x^2 - \frac{3}{2}x = 10$
Take half of $-\frac{3}{2}$ and square it. $\left(\frac{1}{2}\left(-\frac{3}{2}\right)\right)^2 = \frac{9}{16}$	$x^2 - \frac{3}{2}x + \underset{\left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2}{\underline{\hspace{6em}}} = 10$

Add $\frac{9}{16}$ to both sides.	$x^2 - \frac{3}{2}x + \frac{9}{16} = 10 + \frac{9}{16}$
Factor the perfect square trinomial as a binomial square.	$\left(x - \frac{3}{4}\right)^2 = \frac{160}{16} + \frac{9}{16}$
Add the fractions on the right side.	$\left(x - \frac{3}{4}\right)^2 = \frac{169}{16}$
Use the Square Root Property.	$x - \frac{3}{4} = \pm\sqrt{\frac{169}{16}}$
Simplify the radical.	$x - \frac{3}{4} = \pm\frac{13}{4}$
Solve for *x*.	$x = \frac{3}{4} \pm \frac{13}{4}$
Rewrite to show 2 solutions.	$x = \frac{3}{4} + \frac{13}{4},\ x = \frac{3}{4} - \frac{13}{4}$
Simplify.	$x = 4,\ x = -\frac{5}{2}$
Check. We leave the check for you.	

Note:
Exercise:

Problem: Solve $3r^2 - 2r = 21$ by completing the square.

Solution:

$r = -\frac{7}{3}, r = 3$

Note:
Exercise:

Problem: Solve $4t^2 + 2t = 20$ by completing the square.

Solution:

$t = -\frac{5}{2}, t = 2$

Example:
Exercise:

Problem: Solve $3x^2 + 2x = 4$ by completing the square.

Solution:
Solution

Again, our first step will be to make the coefficient of x^2 be one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$3x^2+2x=4$
Divide both sides by 3 to make the coefficient of x^2 equal 1.	$\frac{3x^2+2x}{3}=\frac{4}{3}$
Simplify.	$x^2+\frac{2}{3}x=\frac{4}{3}$
Take half of $\frac{2}{3}$ and square it. $\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2=\frac{1}{9}$	$x^2+\frac{2}{3}x+\underset{\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2}{\underline{\qquad\qquad}}=\frac{4}{3}$
Add $\frac{1}{9}$ to both sides.	$x^2+\frac{2}{3}x+\frac{1}{9}=\frac{4}{3}+\frac{1}{9}$
Factor the perfect square trinomial as a binomial square.	$\left(x+\frac{1}{3}\right)^2=\frac{12}{9}+\frac{1}{9}$
Use the Square Root Property.	$x+\frac{1}{3}=\pm\sqrt{\frac{13}{9}}$
Simplify the radical.	$x+\frac{1}{3}=\pm\frac{\sqrt{13}}{3}$
Solve for *x*.	$x=-\frac{1}{3}\pm\frac{\sqrt{13}}{3}$
Rewrite to show 2 solutions.	$x=-\frac{1}{3}+\frac{\sqrt{13}}{3},\ x=-\frac{1}{3}-\frac{\sqrt{13}}{3}$

Check. We leave the check for you.	

Note:
Exercise:

Problem: Solve $4x^2 + 3x = 12$ by completing the square.

Solution:

$x = -\frac{3}{8} \pm \frac{\sqrt{201}}{8}$

Note:
Exercise:

Problem: Solve $5y^2 + 3y = 10$ by completing the square.

Solution:

$y = -\frac{3}{10} \pm \frac{\sqrt{209}}{10}$

Note:
Access these online resources for additional instruction and practice with solving quadratic equations by completing the square:

- Introduction to the method of completing the square
- How to Solve By Completing the Square

Key Concepts

- Binomial Squares Pattern If a, b are real numbers,
  $(a+b)^2 = a^2 + 2ab + b^2$

  $$\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\times(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

  $(a-b)^2 = a^2 - 2ab + b^2$

  $$\underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\times(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

- Complete a Square
  To complete the square of $x^2 + bx$:

  Identify b, the coefficient of x.
  Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
  Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$.

Practice Makes Perfect

Complete the Square of a Binomial Expression

In the following exercises, complete the square to make a perfect square trinomial. Then, write the result as a binomial squared.

Exercise:

Problem: $a^2 + 10a$

Solution:

$(a+5)^2$

Exercise:

Problem: b^2+12b

Exercise:

Problem: m^2+18m

Solution:

$(m+9)^2$

Exercise:

Problem: n^2+16n

Exercise:

Problem: m^2-24m

Solution:

$(m-12)^2$

Exercise:

Problem: n^2-16n

Exercise:

Problem: p^2-22p

Solution:

$(p-11)^2$

Exercise:

Problem: $q^2 - 6q$

Exercise:

Problem: $x^2 - 9x$

Solution:

$\left(x - \frac{9}{2}\right)^2$

Exercise:

Problem: $y^2 + 11y$

Exercise:

Problem: $p^2 - \frac{1}{3}p$

Solution:

$\left(p - \frac{1}{6}\right)^2$

Exercise:

Problem: $q^2 + \frac{3}{4}q$

Solve Quadratic Equations of the Form $\mathbf{x^2 + bx + c = 0}$ by Completing the Square

In the following exercises, solve by completing the square.

Exercise:

Problem: $v^2 + 6v = 40$

Solution:

$v = -10, v = 4$

Exercise:

Problem: $w^2 + 8w = 65$

Exercise:

Problem: $u^2 + 2u = 3$

Solution:

$u = -3, u = 1$

Exercise:

Problem: $z^2 + 12z = -11$

Exercise:

Problem: $c^2 - 12c = 13$

Solution:

$c = -1, c = 13$

Exercise:

Problem: $d^2 - 8d = 9$

Exercise:

Problem: $x^2 - 20x = 21$

Solution:

$x = -1, x = 21$

Exercise:

Problem: $y^2 - 2y = 8$

Exercise:

Problem: $m^2 + 4m = -44$

Solution:

no real solution

Exercise:

Problem: $n^2 - 2n = -3$

Exercise:

Problem: $r^2 + 6r = -11$

Solution:

no real solution

Exercise:

Problem: $t^2 - 14t = -50$

Exercise:

Problem: $a^2 - 10a = -5$

Solution:

$a = 5 \pm 2\sqrt{5}$

Exercise:

Problem: $b^2 + 6b = 41$

Exercise:

Problem: $u^2 - 14u + 12 = -1$

Solution:

$u = 1, u = 13$

Exercise:

Problem: $z^2 + 2z - 5 = 2$

Exercise:

Problem: $v^2 = 9v + 2$

Solution:

$v = \frac{9}{2} \pm \frac{\sqrt{89}}{2}$

Exercise:

Problem: $w^2 = 5w - 1$

Exercise:

Problem: $(x + 6)(x - 2) = 9$

Solution:

$x = -7, x = 3$

Exercise:

Problem: $(y + 9)(y + 7) = 79$

Solve Quadratic Equations of the Form $\mathbf{ax^2 + bx + c = 0}$ by Completing the Square

In the following exercises, solve by completing the square.

Exercise:

Problem: $3m^2 + 30m - 27 = 6$

Solution:

$m = -11, m = 1$

Exercise:

Problem: $2n^2 + 4n - 26 = 0$

Exercise:

Problem: $2c^2 + c = 6$

Solution:

$c = -2, c = \frac{3}{2}$

Exercise:

Problem: $3d^2 - 4d = 15$

Exercise:

Problem: $2p^2 + 7p = 14$

Solution:

$p = -\frac{7}{4} \pm \frac{\sqrt{161}}{4}$

Exercise:

Problem: $3q^2 - 5q = 9$

Everyday Math

Exercise:

Problem:

Rafi is designing a rectangular playground to have an area of 320 square feet. He wants one side of the playground to be four feet longer than the other side. Solve the equation $p^2 + 4p = 320$ for p, the length of one side of the playground. What is the length of the other side?

Solution:

16 feet, 20 feet

Exercise:

Problem:

Yvette wants to put a square swimming pool in the corner of her backyard. She will have a 3 foot deck on the south side of the pool and a 9 foot deck on the west side of the pool. She has a total area of 1080 square feet for the pool and two decks. Solve the equation $\left(s + 3\right)\left(s + 9\right) = 1080$ for s, the length of a side of the pool.

Writing Exercises

Exercise:

Problem:

Solve the equation $x^2 + 10x = -25$ ⓐ by using the Square Root Property and ⓑ by completing the square. ⓒ Which method do you prefer? Why?

Solution:

ⓐ -5 ⓑ -5 ⓒ Answers will vary.

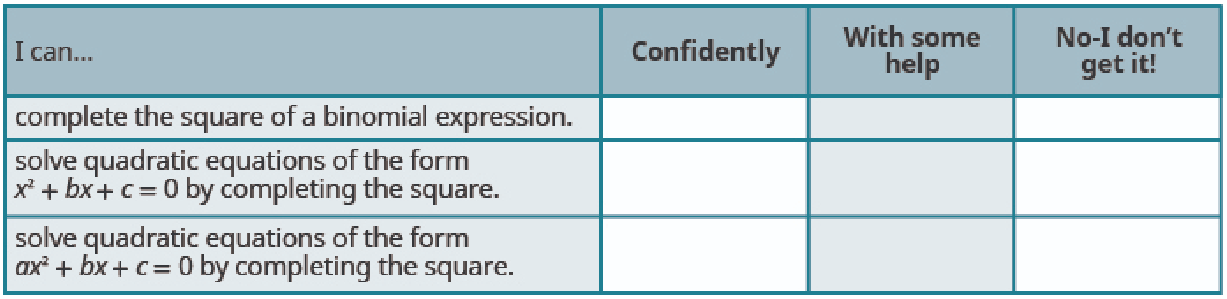
Exercise:

Problem:

Solve the equation $y^2 + 8y = 48$ by completing the square and explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
complete the square of a binomial expression.			
solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square.			
solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

completing the square

Completing the square is a method used to solve quadratic equations.

Solve Quadratic Equations Using the Quadratic Formula

By the end of this section, you will be able to:

- Solve quadratic equations using the quadratic formula
- Use the discriminant to predict the number of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\frac{-20-5}{10}$.
   If you missed this problem, review [link].
2. Simplify: $4+\sqrt{121}$.
   If you missed this problem, review [link].
3. Simplify: $\sqrt{128}$.
   If you missed this problem, review [link].

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering ‘isn’t there an easier way to do this?’ The answer is ‘yes.’ In this section, we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable ‘in general’ so that we would do the algebraic steps only once and then use the new formula to find the value of the specific variable. Now, we will go through the steps of completing the square in general to solve a quadratic equation for *x*. It may be helpful to look at one of the examples at the end of the last section where we solved an equation of the form $ax^2+bx+c=0$ as you read through the algebraic steps below, so you see them with numbers as well as ‘in general.’

We start with the standard form of a quadratic equation and solve it for x by completing the square.	$ax^2+bx+c = 0 \qquad a\neq 0$
Isolate the variable terms on one side.	$ax^2+bx = -c$
Make leading coefficient 1, by dividing by a.	$\frac{ax^2}{a}+\frac{b}{a}x = -\frac{c}{a}$
Simplify.	$x^2+\frac{b}{a}x = -\frac{c}{a}$
To complete the square, find $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2$ and add it to both sides of the equation. $\left(\frac{1}{2}\ \frac{b}{a}\right)^2=\frac{b^2}{4a^2}$	$x^2+\frac{b}{a}x+\frac{b^2}{4a^2} = -\frac{c}{a}+\frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$\left(x+\frac{b}{2a}\right)^2 = -\frac{c}{a}+\frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}-\frac{c\cdot 4a}{a\cdot 4a}$
Simplify.	$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$
Use the square root property.	$x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Simplify.	$x+\frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x = -\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

This last equation is the Quadratic Formula.

Note:
Quadratic Formula
The solutions to a quadratic equation of the form $ax^2+bx+c=0$, $a\neq 0$ are given by the formula:
Equation:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

To use the Quadratic Formula, we substitute the values of $a, b,$ and c into the expression on the right side of the formula. Then, we do all the math to simplify the expression. The result gives the solution(s) to the quadratic equation.

Example:

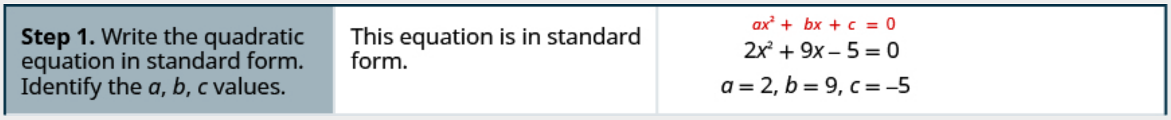
How to Solve a Quadratic Equation Using the Quadratic Formula

Exercise:

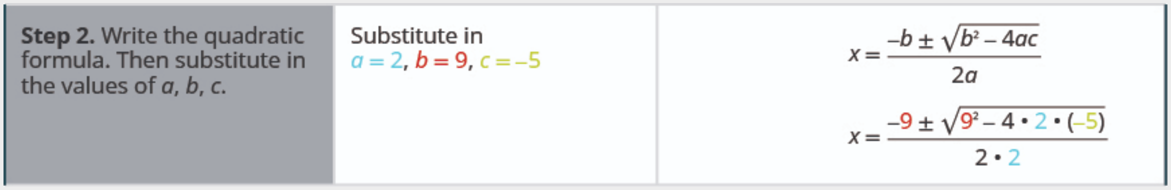
Problem: Solve $2x^2 + 9x - 5 = 0$ by using the Quadratic Formula.

Solution:

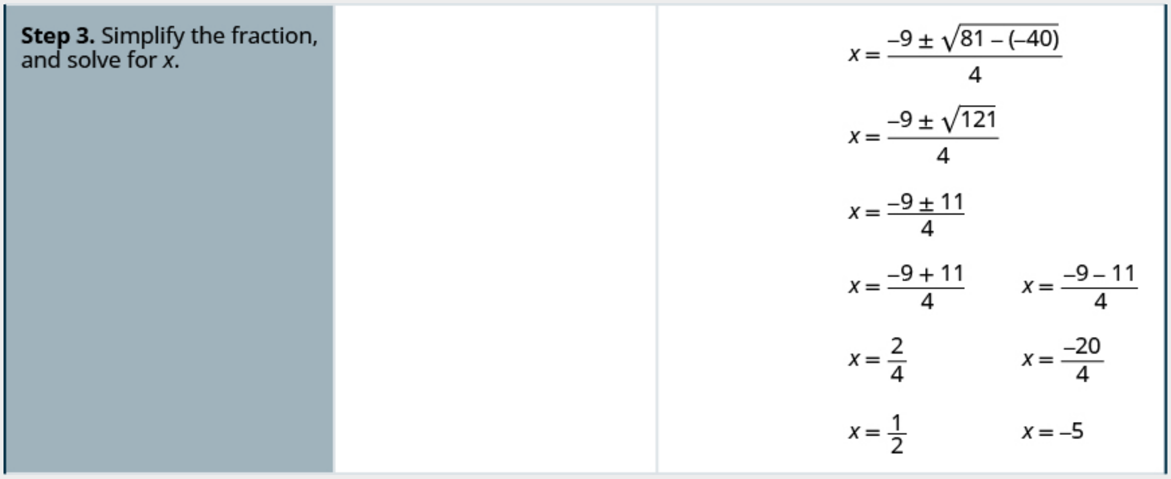
Solution



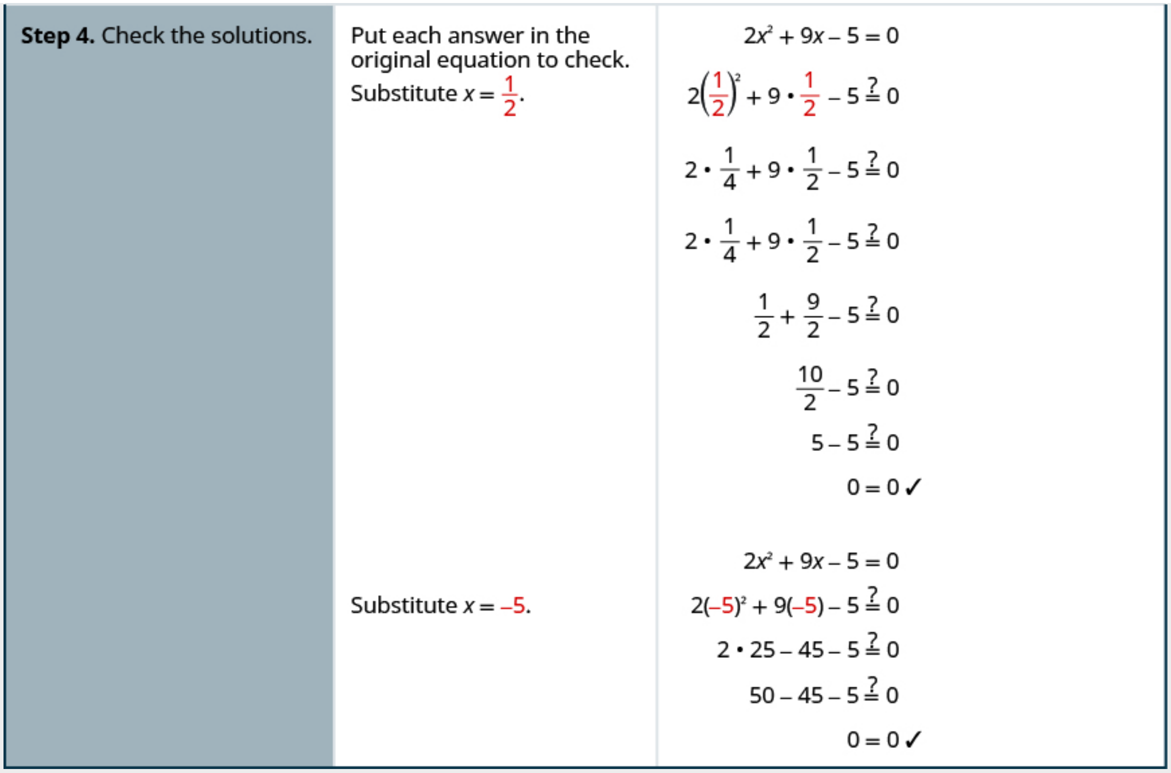
Step 1. Write the quadratic equation in standard form. Identify the *a*, *b*, *c* values.	This equation is in standard form.	$ax^2 + bx + c = 0$ $2x^2 + 9x - 5 = 0$ $a = 2, b = 9, c = -5$



Step 2. Write the quadratic formula. Then substitute in the values of *a*, *b*, *c*.	Substitute in $a = 2$, $b = 9$, $c = -5$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-9 \pm \sqrt{9^2 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2}$



Step 3. Simplify the fraction, and solve for *x*.		$x = \frac{-9 \pm \sqrt{81 - (-40)}}{4}$ $x = \frac{-9 \pm \sqrt{121}}{4}$ $x = \frac{-9 \pm 11}{4}$ $x = \frac{-9 + 11}{4} \quad x = \frac{-9 - 11}{4}$ $x = \frac{2}{4} \quad x = \frac{-20}{4}$ $x = \frac{1}{2} \quad x = -5$



Step 4. Check the solutions.	Put each answer in the original equation to check. Substitute $x = \frac{1}{2}$.	$2x^2 + 9x - 5 = 0$ $2\left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$ $\frac{10}{2} - 5 \stackrel{?}{=} 0$ $5 - 5 \stackrel{?}{=} 0$ $0 = 0$ ✓
	Substitute $x = -5$.	$2x^2 + 9x - 5 = 0$ $2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$ $2 \cdot 25 - 45 - 5 \stackrel{?}{=} 0$ $50 - 45 - 5 \stackrel{?}{=} 0$ $0 = 0$ ✓

Note:
Exercise:

Problem: Solve $3y^2 - 5y + 2 = 0$ by using the Quadratic Formula.

Solution:

$y = \frac{2}{3}, y = 1$

Note:
Exercise:

Problem: Solve $4z^2 + 2z - 6 = 0$ by using the Quadratic Formula.

Solution:

$z = -\frac{3}{2}, z = 1$

Note:
Solve a quadratic equation using the Quadratic Formula.

Write the Quadratic Formula in standard form. Identify the a,b, and c values.
Write the Quadratic Formula. Then substitute in the values of a,b, and c.
Simplify.
Check the solutions.

If you say the formula as you write it in each problem, you’ll have it memorized in no time. And remember, the Quadratic Formula is an equation. Be sure you start with ‘$x =$’.

Example:
Exercise:

Problem: Solve $x^2 - 6x + 5 = 0$ by using the Quadratic Formula.

Solution:
Solution

	$x^2 - 6x + 5 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $x^2 - 6x + 5 = 0$
Identify the *a*, *b*, *c* values.	$a = 1, b = -6, c = 5$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (5)}}{2 \cdot 1}$
Simplify.	$x = \frac{6 \pm \sqrt{36 - 20}}{2}$ $x = \frac{6 \pm \sqrt{16}}{2}$

	$x = \frac{6 \pm 4}{2}$
Rewrite to show two solutions.	$x = \frac{6+4}{2}, x = \frac{6-4}{2}$
Simplify.	$x = \frac{10}{2}, x = \frac{2}{2}$ $x = 5,\ x = 1$
Check. $x^2 - 6x + 5 = 0 \qquad x^2 - 6x + 5 = 0$ $5^2 - 6 \cdot 5 + 5 \overset{?}{=} 0 \qquad 1^2 - 6 \cdot 1 + 5 \overset{?}{=} 0$ $25 - 30 + 5 \overset{?}{=} 0 \qquad 1 - 6 + 5 \overset{?}{=} 0$ $0 = 0 \checkmark \qquad 0 = 0 \checkmark$	

Note:
Exercise:

Problem: Solve $a^2 - 2a - 15 = 0$ by using the Quadratic Formula.

Solution:

$a = -3, a = 5$

Note:
Exercise:

Problem: Solve $b^2 + 10b + 24 = 0$ by using the Quadratic Formula.

Solution:

$b = -6, b = -4$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the Quadratic Formula. If we get a radical as a solution, the final answer must have the radical in its simplified form.

Example:
Exercise:

Problem: Solve $4y^2-5y-3=0$ by using the Quadratic Formula.

Solution:
Solution

We can use the Quadratic Formula to solve for the variable in a quadratic equation, whether or not it is named ‘*x*’.

	$4y^2-5y-3=0$
This equation is in standard form.	$ax^2+bx+c=0$ $4y^2-5y-3=0$
Identify the *a*, *b*, *c* values.	$a=4,\ b=-5,\ c=-3$
Write the Quadratic Formula.	$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$y=\frac{-(-5)\pm\sqrt{(-5)^2-4\cdot4\cdot(-3)}}{2\cdot4}$
Simplify.	$y=\frac{5\pm\sqrt{25+48}}{8}$ $y=\frac{5\pm\sqrt{73}}{8}$
Rewrite to show two solutions.	$y=\frac{5+\sqrt{73}}{8},\ y=\frac{5-\sqrt{73}}{8}$

Check. We leave the check to you.	

Note:
Exercise:

Problem: Solve $2p^2 + 8p + 5 = 0$ by using the Quadratic Formula.

Solution:

$p = \frac{-4 \pm \sqrt{6}}{2}$

Note:
Exercise:

Problem: Solve $5q^2 - 11q + 3 = 0$ by using the Quadratic Formula.

Solution:

$q = \frac{11 \pm \sqrt{61}}{10}$

Example:
Exercise:

Problem: Solve $2x^2 + 10x + 11 = 0$ by using the Quadratic Formula.

Solution:
Solution

	$2x^2 + 10x + 11 = 0$
This equation is in standard form.	$\begin{array}{l} ax^2 + bx + c = 0 \\ 2x^2 + 10x + 11 = 0 \end{array}$
Identify the *a*, *b*, *c* values.	$a = 2, b = 10, c = 11$

Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$x = \frac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$
Simplify.	$x = \frac{-10 \pm \sqrt{100 - 88}}{4}$ $x = \frac{-10 \pm \sqrt{12}}{4}$
Simplify the radical.	$x = \frac{-10 \pm 4\sqrt{3}}{4}$
Factor out the common factor in the numerator.	$x = \frac{2(-5 \pm 2\sqrt{3})}{4}$
Remove the common factors.	$x = \frac{-5 \pm 2\sqrt{3}}{2}$
Rewrite to show two solutions.	$x = \frac{-5 + 2\sqrt{3}}{2}, x = \frac{-5 - 2\sqrt{3}}{2}$
Check. We leave the check to you.	

Note:
Exercise:

Problem: Solve $3m^2 + 12m + 7 = 0$ by using the Quadratic Formula.

Solution:

$m = \frac{-6 \pm \sqrt{15}}{3}$

Note:
Exercise:

Problem: Solve $5n^2 + 4n - 4 = 0$ by using the Quadratic Formula.

Solution:

$n = \frac{-2 \pm 2\sqrt{6}}{5}$

We cannot take the square root of a negative number. So, when we substitute a, b, and c into the Quadratic Formula, if the quantity inside the radical is negative, the quadratic equation has no real solution. We will see this in the next example.

Example:
Exercise:

Problem: Solve $3p^2 + 2p + 9 = 0$ by using the Quadratic Formula.

Solution:
Solution

This equation is in standard form.	$ax^2 + bx + c = 0$ $3p^2 + 2p + 9 = 0$
Identify the *a*, *b*, *c* values.	$a = 3, b = 2, c = 9$
Write the Quadratic Formula.	$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$p = \frac{-(2) \pm \sqrt{(2)^2 - 4 \cdot 3 \cdot (9)}}{2 \cdot 3}$
Simplify.	$p = \frac{-2 \pm \sqrt{4 - 108}}{6}$
Simplify the radical.	

	$p = \frac{-2 \pm \sqrt{-104}}{6}$
We cannot take the square root of a negative number.	There is no real solution.

Note:
Exercise:

Problem: Solve $4a^2 - 3a + 8 = 0$ by using the Quadratic Formula.

Solution:

no real solution

Note:
Exercise:

Problem: Solve $5b^2 + 2b + 4 = 0$ by using the Quadratic Formula.

Solution:

no real solution

The quadratic equations we have solved so far in this section were all written in standard form, $ax^2 + bx + c = 0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

Example:
Exercise:

Problem: Solve $x\left(x+6\right)+4=0$ by using the Quadratic Formula.

Solution:
Solution

	$x(x+6)+4=0$

Distribute to get the equation in standard form.	$x^2 + 6x + 4 = 0$
This equation is now in standard form.	$ax^2 + bx + c = 0$ $x^2 + 6x + 4 = 0$
Identify the *a*, *b*, *c* values.	$a = 1, b = 6, c = 4$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$x = \frac{-(6) \pm \sqrt{(6)^2 - 4 \cdot 1 \cdot (4)}}{2 \cdot 1}$
Simplify.	$x = \frac{-6 \pm \sqrt{36 - 16}}{2}$
Simplify inside the radical.	$x = \frac{-6 \pm \sqrt{20}}{2}$
Simplify the radical.	$x = \frac{-6 \pm 2\sqrt{5}}{2}$
Factor out the common factor in the numerator.	$x = \frac{2(-3 \pm 2\sqrt{5})}{2}$
Remove the common factors.	$x = -3 \pm 2\sqrt{5}$
Rewrite to show two solutions.	$x = -3 + 2\sqrt{5}, \quad x = -3 - 2\sqrt{5}$
Check. We leave the check to you.	

Note:

Exercise:

Problem: Solve $x\,(x+2)-5=0$ by using the Quadratic Formula.

Solution:

$x=-1\pm\sqrt{6}$

Note:
Exercise:

Problem: Solve $y\,(3y-1)-2=0$ by using the Quadratic Formula.

Solution:

$y=-\frac{2}{3},y=1$

When we solved linear equations, if an equation had too many fractions we ‘cleared the fractions’ by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions—to solve. We can use the same strategy with quadratic equations.

Example:
Exercise:

Problem: Solve $\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$ by using the Quadratic Formula.

Solution:
Solution

	$\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2+\frac{2}{3}u\right)=6\left(\frac{1}{3}\right)$
Multiply.	$3u^2+4u=2$

Subtract 2 to get the equation in standard form.	$ax^2 + bx + c = 0$ $3u^2 + 4u - 2 = 0$
Identify the *a*, *b*, *c* values.	$a = 3,\ b = 4,\ c = -2$
Write the Quadratic Formula.	$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$u = \frac{-(4) \pm \sqrt{(4)^2 - 4 \cdot 3 \cdot (-2)}}{2 \cdot 3}$
Simplify.	$u = \frac{-4 \pm \sqrt{16 + 24}}{6}$ $u = \frac{-4 \pm \sqrt{40}}{6}$
Simplify the radical.	$u = \frac{-4 \pm 2\sqrt{10}}{6}$
Factor out the common factor in the numerator.	$u = \frac{2(-2 \pm \sqrt{10})}{6}$
Remove the common factors.	$u = \frac{-2 \pm \sqrt{10}}{3}$
Rewrite to show two solutions.	$u = \frac{-2 \pm \sqrt{10}}{3},\ u = \frac{-2 - \sqrt{10}}{3}$
Check. We leave the check to you.	

Note:
Exercise:

Problem: Solve $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$ by using the Quadratic Formula.

Solution:

$c = \frac{2 \pm \sqrt{7}}{3}$

Note:
Exercise:

Problem: Solve $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{2}$ by using the Quadratic Formula.

Solution:

$d = \frac{2}{3}, d = 0$

Think about the equation $(x-3)^2 = 0$. We know from the Zero Products Principle that this equation has only one solution: $x = 3$.

We will see in the next example how using the Quadratic Formula to solve an equation with a perfect square also gives just one solution.

Example:
Exercise:

Problem: Solve $4x^2 - 20x = -25$ by using the Quadratic Formula.

Solution:
Solution

	$4x^2 - 20x = -25$
Add 25 to get the equation in standard form.	$ax^2 + bx + c = 0$ $4x^2 - 20x + 25 = 0$
Identify the *a*, *b*, *c* values.	$a = 4, b = -20, c = 25$

Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \frac{20 \pm \sqrt{400 - 400}}{8}$ $x = \frac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \frac{20}{8}$
Simplify the fraction.	$x = \frac{5}{2}$
Check. We leave the check to you.	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square?

Note:
Exercise:

Problem: Solve $r^2 + 10r + 25 = 0$ by using the Quadratic Formula.

Solution:

$r = -5$

Note:
Exercise:

Problem: Solve $25t^2 - 40t = -16$ by using the Quadratic Formula.

Solution:

$t = \frac{4}{5}$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two solutions, sometimes one solution, sometimes no real solutions. Is there a way to predict the number of solutions to a quadratic equation without actually solving the equation?

Yes, the quantity inside the radical of the Quadratic Formula makes it easy for us to determine the number of solutions. This quantity is called the discriminant.

Note:
Discriminant
In the Quadratic Formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, the quantity b^2-4ac is called the **discriminant**.

Let’s look at the discriminant of the equations in [link], [link], and [link], and the number of solutions to those quadratic equations.

	Quadratic Equation (in standard form)	**Discriminant b^2-4ac**	**Sign of the Discriminant**	**Number of real solutions**
[link]	$2x^2+9x-5=0$	$9^2-4\cdot 2\left(-5\right)=121$	+	2
[link]	$4x^2-20x+25=0$	$\left(-20\right)^2-4\cdot 4\cdot 25=0$	0	1
[link]	$3p^2+2p+9=0$	$2^2-4\cdot 3\cdot 9=-104$	−	0

When the discriminant is **positive** $\left(x=\frac{-b\pm\sqrt{+}}{2a}\right)$ the quadratic equation has **two solutions**.

When the discriminant is **zero** $\left(x=\frac{-b\pm\sqrt{0}}{2a}\right)$ the quadratic equation has **one solution**.

When the discriminant is **negative** $\left(x=\frac{-b\pm\sqrt{-}}{2a}\right)$ the quadratic equation has **no real solutions**.

Note:
Use the discriminant, b^2-4ac, to determine the number of solutions of a Quadratic Equation.
For a quadratic equation of the form $ax^2+bx+c=0$, $a\neq 0$,

- if $b^2-4ac>0$, the equation has two solutions.

- if $b^2 - 4ac = 0$, the equation has one solution.
- if $b^2 - 4ac < 0$, the equation has no real solutions.

Example:
Exercise:

Problem: Determine the number of solutions to each quadratic equation:

ⓐ $2v^2 - 3v + 6 = 0$ ⓑ $3x^2 + 7x - 9 = 0$ ⓒ $5n^2 + n + 4 = 0$ ⓓ $9y^2 - 6y + 1 = 0$

Solution:
Solution

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

ⓐ

	$2v^2 - 3v + 6 = 0$
The equation is in standard form, identify a, b, c.	$a = 2, b = -3, c = 6$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(3)^2 - 4 \cdot 2 \cdot 6$
Simplify.	$9 - 48$ -39

Because the discriminant is negative, there are no real solutions to the equation.

ⓑ

	$3x^2 + 7x - 9 = 0$
The equation is in standard form, identify a, b, c.	$a = 3, b = 7, c = -9$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(7)^2 - 4 \cdot 3 \cdot (-9)$
Simplify.	$49 + 108$ 157

Because the discriminant is positive, there are two solutions to the equation.

ⓒ

	$5n^2 + n + 4 = 0$
The equation is in standard form, identify $a, b,$ and c.	$a = 5, b = 1, c = 4$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(1)^2 - 4 \cdot 5 \cdot 4$
Simplify.	$1 - 80$ -79

Because the discriminant is negative, there are no real solutions to the equation.

ⓓ

	$9y^2 - 6y + 1 = 0$
The equation is in standard form, identify a, b, c.	$a = 9, b = -6, c = 1$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(-6)^2 - 4 \cdot 9 \cdot 1$
Simplify.	$36 - 36$ 0

Because the discriminant is 0, there is one solution to the equation.

Note:
Exercise:

Problem: Determine the number of solutions to each quadratic equation:

ⓐ $8m^2 - 3m + 6 = 0$ ⓑ $5z^2 + 6z - 2 = 0$ ⓒ $9w^2 + 24w + 16 = 0$ ⓓ $9u^2 - 2u + 4 = 0$

Solution:

ⓐ no real solutions ⓑ 2 ⓒ 1 ⓓ no real solutions

Note:
Exercise:

Problem: Determine the number of solutions to each quadratic equation:

ⓐ $b^2 + 7b - 13 = 0$ ⓑ $5a^2 - 6a + 10 = 0$ ⓒ $4r^2 - 20r + 25 = 0$ ⓓ $7t^2 - 11t + 3 = 0$

Solution:

ⓐ 2 ⓑ no real solutions ⓒ 1 ⓓ 2

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We have used four methods to solve quadratic equations:

- Factoring
- Square Root Property
- Completing the Square
- Quadratic Formula

You can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method to use.

Note:
Identify the most appropriate method to solve a Quadratic Equation.

Try **Factoring** first. If the quadratic factors easily, this method is very quick.
Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.
Use the **Quadratic Formula**. Any quadratic equation can be solved by using the Quadratic Formula.

What about the method of completing the square? Most people find that method cumbersome and prefer not to use it. We needed to include it in this chapter because we completed the square in general to derive the Quadratic Formula. You will also use the process of completing the square in other areas of algebra.

Example:
Exercise:

Problem: Identify the most appropriate method to use to solve each quadratic equation:

ⓐ $5z^2 = 17$ ⓑ $4x^2 - 12x + 9 = 0$ ⓒ $8u^2 + 6u = 11$

Solution:
Solution

ⓐ $5z^2 = 17$

Since the equation is in the $ax^2 = k$, the most appropriate method is to use the Square Root Property.

ⓑ $4x^2 - 12x + 9 = 0$

We recognize that the left side of the equation is a perfect square trinomial, and so Factoring will be the most appropriate method.

ⓒ $8u^2 + 6u = 11$

Put the equation in standard form. $8u^2 + 6u - 11 = 0$

While our first thought may be to try Factoring, thinking about all the possibilities for trial and error leads us to choose the Quadratic Formula as the most appropriate method

Note:
Exercise:

Problem: Identify the most appropriate method to use to solve each quadratic equation:

ⓐ $x^2 + 6x + 8 = 0$ ⓑ $(n - 3)^2 = 16$ ⓒ $5p^2 - 6p = 9$

Solution:

ⓐ factor ⓑ Square Root Property ⓒ Quadratic Formula

Note:
Exercise:

Problem: Identify the most appropriate method to use to solve each quadratic equation:

ⓐ $8a^2 + 3a - 9 = 0$ ⓑ $4b^2 + 4b + 1 = 0$ ⓒ $5c^2 = 125$

Solution:

ⓐ Quadratic Formula ⓑ factoring ⓒ Square Root Property

Note:
Access these online resources for additional instruction and practice with using the Quadratic Formula:

- Solving Quadratic Equations: Solving with the Quadratic Formula
- How to solve a quadratic equation in standard form using the Quadratic Formula (example)
- Solving Quadratic Equations using the Quadratic Formula—Example 3
- Solve Quadratic Equations using Quadratic Formula

Key Concepts

- **Quadratic Formula** The solutions to a quadratic equation of the form $ax^2 + bx + c = 0,\ a \neq 0$ are given by the formula:
  Equation:

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- **Solve a Quadratic Equation Using the Quadratic Formula**
  To solve a quadratic equation using the Quadratic Formula.

  Write the quadratic formula in standard form. Identify the a, b, c values.
  Write the quadratic formula. Then substitute in the values of a, b, c.
  Simplify.
  Check the solutions.

- **Using the Discriminant, $b^2 - 4ac$, to Determine the Number of Solutions of a Quadratic Equation**
  For a quadratic equation of the form $ax^2 + bx + c = 0,\ a \neq 0$,

  - if $b^2 - 4ac > 0$, the equation has 2 solutions.

  - if $b^2 - 4ac = 0$, the equation has 1 solution.
  - if $b^2 - 4ac < 0$, the equation has no real solutions.

- **To identify the most appropriate method to solve a quadratic equation:**

  Try Factoring first. If the quadratic factors easily this method is very quick.
  Try the Square Root Property next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.
  Use the Quadratic Formula. Any other quadratic equation is best solved by using the Quadratic Formula.

Practice Makes Perfect

Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.
Exercise:

Problem: $4m^2 + m - 3 = 0$

Solution:

$m = -1, m = \frac{3}{4}$

Exercise:

Problem: $4n^2 - 9n + 5 = 0$

Exercise:

Problem: $2p^2 - 7p + 3 = 0$

Solution:

$p = \frac{1}{2}, p = 3$

Exercise:

Problem: $3q^2 + 8q - 3 = 0$

Exercise:

Problem: $p^2 + 7p + 12 = 0$

Solution:

$p = -4, p = -3$

Exercise:

Problem: $q^2 + 3q - 18 = 0$

Exercise:

Problem: $r^2 - 8r - 33 = 0$

Solution:

$r = -3, r = 11$

Exercise:

Problem: $t^2 + 13t + 40 = 0$

Exercise:

Problem: $3u^2 + 7u - 2 = 0$

Solution:

$u = \frac{-7 \pm \sqrt{73}}{6}$

Exercise:

Problem: $6z^2 - 9z + 1 = 0$

Exercise:

Problem: $2a^2 - 6a + 3 = 0$

Solution:

$a = \frac{3 \pm \sqrt{3}}{2}$

Exercise:

Problem: $5b^2 + 2b - 4 = 0$

Exercise:

Problem: $2x^2 + 3x + 9 = 0$

Solution:

no real solution

Exercise:

Problem: $6y^2 - 5y + 2 = 0$

Exercise:

Problem: $v\left(v + 5\right) - 10 = 0$

Solution:

$v = \frac{-5 \pm \sqrt{65}}{2}$

Exercise:

Problem: $3w\left(w-2\right)-8=0$

Exercise:

Problem: $\frac{1}{3}m^2+\frac{1}{12}m=\frac{1}{4}$

Solution:

$m=-1, m=\frac{3}{4}$

Exercise:

Problem: $\frac{1}{3}n^2+n=-\frac{1}{2}$

Exercise:

Problem: $16c^2+24c+9=0$

Solution:

$c=-\frac{3}{4}$

Exercise:

Problem: $25d^2-60d+36=0$

Exercise:

Problem: $5m^2+2m-7=0$

Solution:

$m=-\frac{7}{5}, m=1$

Exercise:

Problem: $8n^2-3n+3=0$

Exercise:

Problem: $p^2-6p-27=0$

Solution:

$p=-3, p=9$

Exercise:

Problem: $25q^2+30q+9=0$

Exercise:

Problem: $4r^2+3r-5=0$

Solution:

$r=\frac{-3\pm\sqrt{89}}{8}$

Exercise:

Problem: $3t\left(t-2\right)=2$

Exercise:

Problem: $2a^2+12a+5=0$

Solution:

$a=\frac{-6\pm\sqrt{26}}{2}$

Exercise:

Problem: $4d^2-7d+2=0$

Exercise:

Problem: $\frac{3}{4}b^2+\frac{1}{2}b=\frac{3}{8}$

Solution:

$b=\frac{-2\pm\sqrt{11}}{6}$

Exercise:

Problem: $\frac{1}{9}c^2+\frac{2}{3}c=3$

Exercise:

Problem: $2x^2+12x-3=0$

Solution:

$x=\frac{-6\pm\sqrt{42}}{4}$

Exercise:

Problem: $16y^2+8y+1=0$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions to each quadratic equation.

Exercise:

Problem: ⓐ $4x^2 - 5x + 16 = 0$
ⓑ $36y^2 + 36y + 9 = 0$
ⓒ $6m^2 + 3m - 5 = 0$
ⓓ $18n^2 - 7n + 3 = 0$

Solution:

ⓐ no real solutions ⓑ 1
ⓒ 2 ⓓ no real solutions

Exercise:

Problem: ⓐ $9v^2 - 15v + 25 = 0$
ⓑ $100w^2 + 60w + 9 = 0$
ⓒ $5c^2 + 7c - 10 = 0$
ⓓ $15d^2 - 4d + 8 = 0$

Exercise:

Problem: ⓐ $r^2 + 12r + 36 = 0$
ⓑ $8t^2 - 11t + 5 = 0$
ⓒ $4u^2 - 12u + 9 = 0$
ⓓ $3v^2 - 5v - 1 = 0$

Solution:

ⓐ 1 ⓑ no real solutions
ⓒ 1 ⓓ 2

Exercise:

Problem: ⓐ $25p^2 + 10p + 1 = 0$
ⓑ $7q^2 - 3q - 6 = 0$
ⓒ $7y^2 + 2y + 8 = 0$
ⓓ $25z^2 - 60z + 36 = 0$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

Exercise:

Problem: ⓐ $x^2 - 5x - 24 = 0$
ⓑ $(y + 5)^2 = 12$
ⓒ $14m^2 + 3m = 11$

Solution:

ⓐ factor ⓑ square root
ⓒ Quadratic Formula

Exercise:

Problem: ⓐ $(8v+3)^2 = 81$
ⓑ $w^2 - 9w - 22 = 0$
ⓒ $4n^2 - 10 = 6$

Exercise:

Problem: ⓐ $6a^2 + 14 = 20$
ⓑ $\left(x - \frac{1}{4}\right)^2 = \frac{5}{16}$
ⓒ $y^2 - 2y = 8$

Solution:

ⓐ factor ⓑ square root
ⓒ factor

Exercise:

Problem: ⓐ $8b^2 + 15b = 4$
ⓑ $\frac{5}{9}v^2 - \frac{2}{3}v = 1$
ⓒ $\left(w + \frac{4}{3}\right)^2 = \frac{2}{9}$

Everyday Math

Exercise:

Problem:

A flare is fired straight up from a ship at sea. Solve the equation $16\left(t^2 - 13t + 40\right) = 0$ for t, the number of seconds it will take for the flare to be at an altitude of 640 feet.

Solution:

5 seconds, 8 seconds

Exercise:

Problem:

An architect is designing a hotel lobby. She wants to have a triangular window looking out to an atrium, with the width of the window 6 feet more than the height. Due to energy restrictions, the area of the window must be 140 square feet. Solve the equation $\frac{1}{2}h^2 + 3h = 140$ for h, the height of the window.

Writing Exercises

Exercise:

Problem: Solve the equation $x^2+10x=200$
ⓐ by completing the square
ⓑ using the Quadratic Formula
ⓒ Which method do you prefer? Why?

Solution:

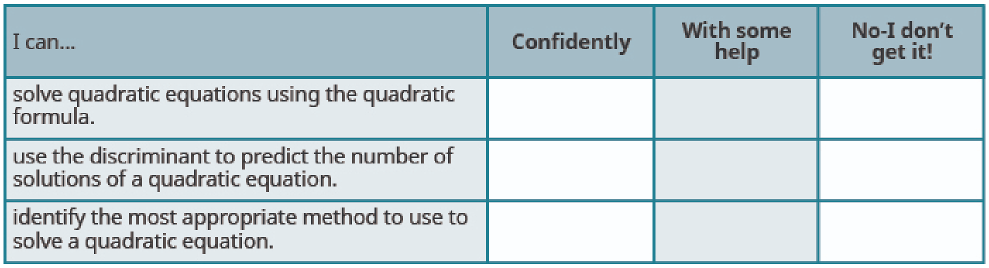
ⓐ $-20, 10$ ⓑ $-20, 10$
ⓒ answers will vary

Exercise:

Problem: Solve the equation $12y^2+23y=24$
ⓐ by completing the square
ⓑ using the Quadratic Formula
ⓒ Which method do you prefer? Why?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations using the quadratic formula.			
use the discriminant to predict the number of solutions of a quadratic equation.			
identify the most appropriate method to use to solve a quadratic equation.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

discriminant

In the Quadratic Formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ the quantity b^2-4ac is called the discriminant.

Solve Applications Modeled by Quadratic Equations
By the end of this section, you will be able to:

- Solve applications modeled by Quadratic Equations

Note:
Before you get started, take this readiness quiz.

1. The sum of two consecutive odd numbers is -100. Find the numbers.
   If you missed this problem, review [link].
2. The area of triangular mural is 64 square feet. The base is 16 feet. Find the height.
   If you missed this problem, review [link].
3. Find the length of the hypotenuse of a right triangle with legs 5 inches and 12 inches.
   If you missed this problem, review [link].

Solve Applications of the Quadratic Formula

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications. To get us started, we will copy our usual Problem Solving Strategy here so we can follow the steps.

Note:
Use the problem solving strategy.

Readthe problem. Make sure all the words and ideas are understood.
Identifywhat we are looking for.
Namewhat we are looking for. Choose a variable to represent that quantity.
Translateinto an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
Solvethe equation using good algebra techniques.
Checkthe answer in the problem and make sure it makes sense.
Answerthe question with a complete sentence.

We have solved number applications that involved **consecutive even integers** and **consecutive odd integers** by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n+2$. The next one would be $n+2+2$ or $n+4$. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

Equation:

Consecutive even integers

64, 66, 68

n	1st even integer
$n+2$	2nd consecutive even integer
$n+4$	3rd consecutive even integer

Consecutive odd integers

77, 79, 81

n	1st odd integer
$n+2$	2nd consecutive odd integer
$n+4$	3rd consecutive odd integer

Some applications of consecutive odd integers or consecutive even integers are modeled by quadratic equations. The notation above will be helpful as you name the variables.

Example:
Exercise:

Problem: The product of two consecutive odd integers is 195. Find the integers.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two consecutive odd integers.
Step 3. Name what we are looking for.	Let $n=$ the first odd integer. $n+2=$ the next odd integer
Step 4. Translate into an equation. State the problem in one sentence.	"The product of two consecutive odd integers is 195." The product of the first odd integer and the second odd integer is 195.
Translate into an equation	$n(n+2)=195$
Step 5. Solve the equation. Distribute.	$n^2+2n=195$
Subtract 195 to get the equation in standard form.	$ax^2 + bx + c = 0$ $n^2+2n-195=0$
Identify the *a*, *b*, *c* values.	

	$a = 1, b = 2, c = -195$
Write the quadratic equation.	$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*..	$n = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-195)}}{2 \cdot 1}$
Simplify.	$n = \frac{-2 \pm \sqrt{4 + 780}}{2}$ $n = \frac{-2 \pm \sqrt{784}}{2}$
Simplify the radical.	$n = \frac{-2 \pm 28}{2}$
Rewrite to show two solutions.	$n = \frac{-2 + 28}{2}, n = \frac{-2 - 28}{2}$
Solve each equation.	$n = \frac{26}{2}, n = \frac{-30}{2}$ $n = 13 \quad n = -15$
There are two values of *n* that are solutions. This will give us two pairs of consecutive odd integers for our solution.	First odd integer $n = 13$ next odd integer $n + 2$ $13 + 2$ 15
	First odd integer $n = -15$ next odd integer $n + 2$ $-15 + 2$ -13
Step 6. Check the answer. Do these pairs work? Are they consecutive odd integers? Is their product 195?	

	$13, 15, \text{ yes} \qquad -13, -15, \text{ yes}$ $13 \cdot 15 = 195, \text{ yes} \qquad -13(-15) = 195, \text{ yes}$
Step 7. Answer the question.	The two consecutive odd integers whose product is 195 are 13, 15, and −13, −15.

Note:
Exercise:

Problem: The product of two consecutive odd integers is 99. Find the integers.

Solution:

Two consecutive odd numbers whose product is 99 are 9 and 11, and -9 and -11.

Note:
Exercise:

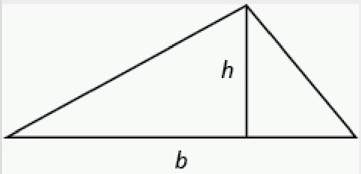
Problem: The product of two consecutive odd integers is 168. Find the integers.

Solution:

Two consecutive even numbers whose product is 168 are 12 and 14,and -12 and -14.

We will use the formula for the area of a triangle to solve the next example.

Note:
Area of a Triangle
For a triangle with base b and height h, the area, A, is given by the formula $A = \frac{1}{2}bh$.



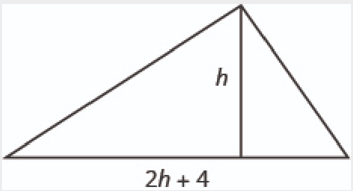
Recall that, when we solve geometry applications, it is helpful to draw the figure.

Example:
Exercise:

Problem:

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can have an area of 120 square feet and the architect wants the width to be 4 feet more than twice the height. Find the height and width of the window.

Solution:
Solution

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the height and width.
Step 3. Name what we are looking for.	Let $h =$ the height of the triangle. $2h + 4 =$ the width of the triangle
Step 4. Translate.	We know the area. Write the formula for the area of a triangle. $A = \frac{1}{2}bh$
Step 5. Solve the equation. Substitute in the values.	$120 = \frac{1}{2}(2h + 4)h$
Distribute.	$120 = h^2 + 2h$
This is a quadratic equation, rewrite it in	

standard form.	$ax^2 + bx + c = 0$ $h^2 + 2h - 120 = 0$
Solve the equation using the Quadratic Formula. Identify the *a*, *b*, *c* values.	$a = 1, b = 2, c = -120$
Write the quadratic equation.	$h = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*..	$h = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-120)}}{2 \cdot 1}$
Simplify.	$h = \frac{-2 \pm \sqrt{4 + 480}}{2}$ $h = \frac{-2 \pm \sqrt{484}}{2}$
Simplify the radical.	$h = \frac{-2 \pm 22}{2}$
Rewrite to show two solutions.	$h = \frac{-2 + 22}{2}, \quad h = \frac{-2 - 22}{2}$
Simplify.	$h = \frac{20}{2}, \quad h = \frac{-24}{2}$
Since h is the height of a window, a value of $h = -12$ does not make sense.	$h = 10 \quad \cancel{h = -12}$
	The height of the triangle: $h = 10$ The width of the triangle: $2h + 4$ $2 \cdot 10 + 4$ 24
Step 6. Check the answer. Does a triangle with a height 10 and width 24 have area	

120? Yes.	
Step 7. Answer the question.	The height of the triangular window is 10 feet and the width is 24 feet.

Notice that the solutions were integers. That tells us that we could have solved the equation by factoring.

When we wrote the equation in standard form, $h^2 + 2h - 120 = 0$, we could have factored it. If we did, we would have solved the equation $(h + 12)(h - 10) = 0$.

Note:
Exercise:

Problem:

Find the dimensions of a triangle whose width is four more than six times its height and has an area of 208 square inches.

Solution:

The height of the triangle is 8 inches and the width is 52 inches.

Note:
Exercise:

Problem:

If a triangle that has an area of 110 square feet has a height that is two feet less than twice the width, what are its dimensions?

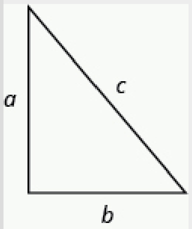
Solution:

The height of the triangle is 20 feet and the width is 11 feet.

In the two preceding examples, the number in the radical in the Quadratic Formula was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

The Pythagorean Theorem gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

Note:
Pythagorean Theorem
In any right triangle, where a and b are the lengths of the legs and c is the length of the hypotenuse,
$a^2 + b^2 = c^2$

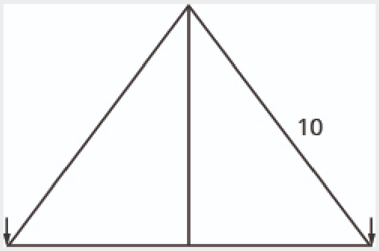


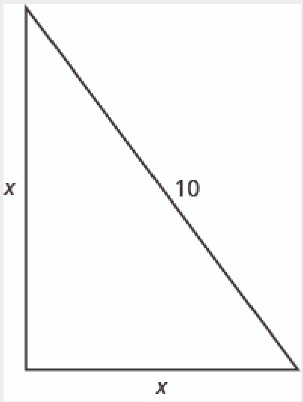
Example:
Exercise:

Problem:

Rene is setting up a holiday light display. He wants to make a ‘tree’ in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

Solution:
Solution

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the height of the pole.
Step 3. Name what we are looking for.	The distance from the base of the pole to either stake is the same as the height of the pole. Let

	$x =$ the height of the pole. $x =$ the distance from the pole to stake
Each side is a right triangle. We draw a picture of one of them.	
Step 4. Translate into an equation. We can use the Pythagorean Theorem to solve for *x*.	
Write the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Step 5. Solve the equation. Substitute.	$x^2 + x^2 = 10^2$
Simplify.	$2x^2 = 100$
Divide by 2 to isolate the variable.	$\frac{2x^2}{2} = \frac{100}{2}$
Simplify.	$x^2 = 50$
Use the Square Root Property.	$x = \pm\sqrt{50}$
Simplify the radical.	$x = \pm 5\sqrt{2}$
Rewrite to show two solutions.	$x = 5\sqrt{2}$ $\cancel{x = -5\sqrt{2}}$
Approximate this number to the nearest tenth with a calculator.	$x \approx 7.1$
Step 6. Check the answer. Check on your own in the Pythagorean Theorem.	
Step 7. Answer the question.	The pole should be about 7.1 feet tall.

Note:
Exercise:

Problem:

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth of a foot.

Solution:

The length of the shadow is 6.3 feet and the length of the flag pole is 18.9 ft.

Note:
Exercise:

Problem:

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

Solution:

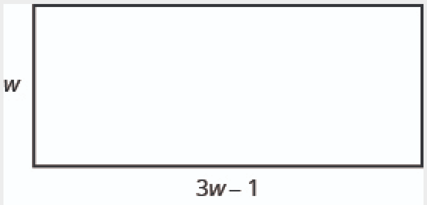
The distance to the opposite corner is 3.2.

Example:
Exercise:

Problem:

Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than three times the width. Find the length and width. Round to the nearest tenth of a foot.

Solution:
Solution

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the length and width.
Step 3. Name what we are looking for.	Let $w =$ the width of the rectangle. $3w - 1 =$ the length of the rectangle
Step 4. Translate into an equation. We know the area. Write the formula for the area of a rectangle.	$A = L \cdot W$
Step 5. Solve the equation. Substitute in the values.	$150 = (3w - 1)w$
Distribute.	$150 = 3w^2 - w$
This is a quadratic equation, rewrite it in standard form.	$ax^2 + bx + c = 0$ $3w^2 - w - 150 = 0$
Solve the equation using the Quadratic Formula.	
Identify the *a*, *b*, *c* values.	$a = 3, b = -1, c = -150$
Write the Quadratic Formula.	$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$w = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot (-150)}}{2 \cdot 3}$
Simplify.	$w = \frac{1 \pm \sqrt{1 + 1800}}{6}$

	$w = \frac{1 \pm \sqrt{1801}}{6}$
Rewrite to show two solutions.	$w = \frac{1 + \sqrt{1801}}{6},\ w = \frac{1 - \sqrt{1801}}{6}$
Approximate the answers using a calculator. We eliminate the negative solution for the width.	$w \approx 7.2, \qquad \cancel{w \approx -6.9}$ $\text{Width } w \approx 7.2$ $\text{Length } \approx 3w - 1$ $\approx 3(7.2) - 1$ ≈ 20.6
Step 6. Check the answer. Make sure that the answers make sense.	
Step 7. Answer the question.	The width of the rectangle is approximately 7.2 feet and the length 20.6 feet.

Note:
Exercise:

Problem:

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden. Round to the nearest tenth of a foot.

Solution:

The width of the garden is 11 feet and the length is 18 feet.

Note:
Exercise:

Problem:

A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length. What are the length and width of the tablecloth? Round to the nearest tenth of a foot.

Solution:

The width of the tablecloth is 6.8 feet and the length is 11.8 feet.

The height of a projectile shot upwards is modeled by a quadratic equation. The initial velocity, v_0, propels the object up until gravity causes the object to fall back down.

Note:
Projectile Motion
The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula:
Equation:

$$h = -16t^2 + v_0 t$$

We can use the formula for projectile motion to find how many seconds it will take for a firework to reach a specific height.

Example:
Exercise:

Problem:

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Solution:
Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of seconds,

	which is time.
Step 3. Name what we are looking for.	Let $t =$ the number of seconds.
Step 4. Translate into an equation.	Use the formula.
	$h = -16t^2 + v_0 t$
Step 5. Solve the equation. We know the velocity v_0 is 130 feet per second.	
The height is 260 feet. Substitute the values.	$260 = -16t^2 + 130t$
This is a quadratic equation, rewrite it in standard form.	$\begin{array}{l} ax^2 + bx + c = 0 \\ 16t^2 + 130t + 260 = 0 \end{array}$
Solve the equation using the Quadratic Formula.	
Identify the *a*, *b*, *c* values.	$a = 16, b = -130, c = 260$
Write the Quadratic Formula.	$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$t = \frac{-(-130) \pm \sqrt{(-130)^2 - 4 \cdot 16 \cdot (260)}}{2 \cdot 16}$
Simplify.	$t = \frac{130 \pm \sqrt{16{,}900 - 16{,}640}}{32}$ $t = \frac{130 \pm \sqrt{260}}{32}$
Rewrite to show two solutions.	$t = \frac{130 + \sqrt{260}}{32}, \; t = \frac{130 - \sqrt{260}}{32}$
Approximate the answers with a calculator.	$t \approx 4.6$ seconds, $t \approx 3.6$
Step 6. Check the answer.	

The check is left to you.	
Step 7. Answer the question.	The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds.

Note:
Exercise:

Problem:

An arrow is shot from the ground into the air at an initial speed of 108 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth of a second.

Solution:

The arrow will reach 180 on its way up in 3 seconds, and on the way down in 3.8 seconds.

Note:
Exercise:

Problem:

A man throws a ball into the air with a velocity of 96 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth of a second.

Solution:

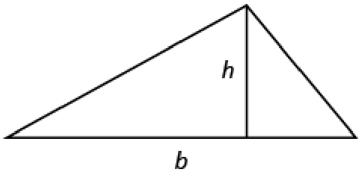
The ball will reach 48 feet on its way up in .6 seconds and on the way down in 5.5 seconds.

Note:
Access these online resources for additional instruction and practice with solving word problems using the quadratic equation:

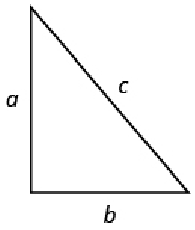
- General Quadratic Word Problems
- Word problem: Solve a projectile problem using a quadratic equation

Key Concepts

- **Area of a Triangle** For a triangle with base, b, and height, h, the area, A, is given by the formula: $A = \frac{1}{2}bh$



- **Pythagorean Theorem** In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypothenuse, $a^2 + b^2 = c^2$



- **Projectile motion** The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds can be modeled by the formula:
Equation:

$$h = -16t^2 + v_0t$$

Practice Makes Perfect

Solve Applications of the Quadratic Formula

In the following exercises, solve by using methods of factoring, the square root principle, or the Quadratic Formula. Round your answers to the nearest tenth.
Exercise:

Problem: The product of two consecutive odd numbers is 255. Find the numbers.

Solution:

Two consecutive odd numbers whose product is 255 are 15 and 17, and -15 and -17.

Exercise:

Problem: The product of two consecutive even numbers is 360. Find the numbers.

Exercise:

Problem: The product of two consecutive even numbers is 624. Find the numbers.

Solution:

Two consecutive even numbers whose product is 624 are 24 and 26, and -26 and -24.

Exercise:

Problem: The product of two consecutive odd numbers is 1023. Find the numbers.

Exercise:

Problem: The product of two consecutive odd numbers is 483. Find the numbers.

Solution:

Two consecutive odd numbers whose product is 483 are 21 and 23, and -21 and -23.

Exercise:

Problem: The product of two consecutive even numbers is 528. Find the numbers.

Exercise:

Problem:

A triangle with area 45 square inches has a height that is two less than four times the width. Find the height and width of the triangle.

Solution:

The width of the triangle is 5 inches and the height is 18 inches.

Exercise:

Problem:

The width of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the height and width of the triangle.

Exercise:

Problem:

The hypotenuse of a right triangle is twice the length of one of its legs. The length of the other leg is three feet. Find the lengths of the three sides of the triangle.

Solution:

The leg of the right triangle is 1.7 feet and the hypotenuse is 3.4 feet.

Exercise:

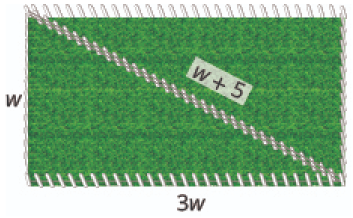
Problem:

The hypotenuse of a right triangle is 10 cm long. One of the triangle’s legs is three times the length of the other leg. Round to the nearest tenth. Find the lengths of the three sides of the triangle.

Exercise:

Problem:

A farmer plans to fence off sections of a rectangular corral. The diagonal distance from one corner of the corral to the opposite corner is five yards longer than the width of the corral. The length of the corral is three times the width. Find the length of the diagonal of the corral.



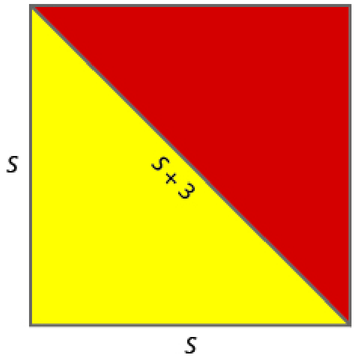
Solution:

The length of the fence is 7.1 units.

Exercise:

Problem:

Nautical flags are used to represent letters of the alphabet. The flag for the letter O consists of a yellow right triangle and a red right triangle which are sewn together along their hypotenuse to form a square. The adjoining side of the two triangles is three inches longer than a side of the flag. Find the length of the side of the flag.



Exercise:

Problem:

The length of a rectangular driveway is five feet more than three times the width. The area is 350 square feet. Find the length and width of the driveway.

Solution:

The width of the driveway is 10 feet and its length is 35 feet.

Exercise:

Problem:

A rectangular lawn has area 140 square yards. Its width that is six less than twice the length. What are the length and width of the lawn?

Exercise:

Problem:

A firework rocket is shot upward at a rate of 640 ft/sec. Use the projectile formula $h = -16t^2 + v_0 t$ to determine when the height of the firework rocket will be 1200 feet.

Solution:

The rocket will reach 1,200 feet on its way up in 2 seconds and on the way down in 38 seconds.

Exercise:

Problem:

An arrow is shot vertically upward at a rate of 220 feet per second. Use the projectile formula $h = -16t^2 + v_0 t$ to determine when height of the arrow will be 400 feet.

Everyday Math

Exercise:

Problem:

A bullet is fired straight up from a BB gun with initial velocity 1120 feet per second at an initial height of 8 feet. Use the formula $h=-16t^2+v_0t+8$ to determine how many seconds it will take for the bullet to hit the ground. (That is, when will $h=0$?)

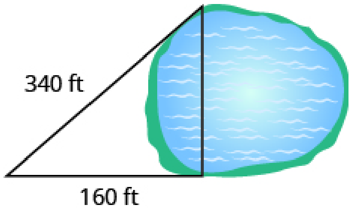
Solution:

70 seconds

Exercise:

Problem:

A city planner wants to build a bridge across a lake in a park. To find the length of the bridge, he makes a right triangle with one leg and the hypotenuse on land and the bridge as the other leg. The length of the hypotenuse is 340 feet and the leg is 160 feet. Find the length of the bridge.



Writing Exercises

Exercise:

Problem:

Make up a problem involving the product of two consecutive odd integers. Start by choosing two consecutive odd integers. ⓐ What are your integers? ⓑ What is the product of your integers? ⓒ Solve the equation $n\left(n+2\right)=p$, where p is the product you found in part (b). ⓓ Did you get the numbers you started with?

Solution:

ⓐ answers will vary
ⓑ answers will vary ⓒ answers will vary ⓓ answers will vary

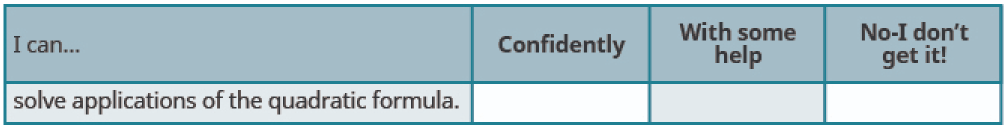
Exercise:

Problem:

Make up a problem involving the product of two consecutive even integers. Start by choosing two consecutive even integers. ⓐ What are your integers? ⓑ What is the product of your integers? ⓒ Solve the equation $n\left(n+2\right)=p$, where p is the product you found in part (b). ⓓ Did you get the numbers you started with?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
solve applications of the quadratic formula.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

consecutive even integers

Consecutive even integers are even integers that follow right after one another. If an even integer is represented by n, the next consecutive even integer is $n+2$, and the next after that is $n+4$.

consecutive odd integers

Consecutive odd integers are odd integers that follow right after one another. If an odd integer is represented by n, the next consecutive odd integer is $n+2$, and the next after that is $n+4$.

Graphing Quadratic Equations
By the end of this section, you will be able to:

- Recognize the graph of a quadratic equation in two variables
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic equations in two variables
- Solve maximum and minimum applications

Note:
Before you get started, take this readiness quiz.

1. Graph the equation $y=3x-5$ by plotting points.
   If you missed this problem, review [link].
2. Evaluate $2x^2+4x-1$ when $x=-3$.
   If you missed this problem, review [link].
3. Evaluate $-\frac{b}{2a}$ when $a=\frac{1}{3}$ and $b=\frac{5}{6}$.
   If you missed this problem, review [link].

Recognize the Graph of a Quadratic Equation in Two Variables

We have graphed equations of the form $Ax+By=C$. We called equations like this linear equations because their graphs are straight lines.

Now, we will graph equations of the form $y=ax^2+bx+c$. We call this kind of equation a quadratic equation in two variables.

Note:
Quadratic Equation in Two Variables
A **quadratic equation in two variables**, where $a, b,$ and c are real numbers and $a \neq 0$, is an equation of the form
Equation:

$$y=ax^2+bx+c$$

Just like we started graphing linear equations by plotting points, we will do the same for quadratic equations.

Let’s look first at graphing the quadratic equation $y=x^2$. We will choose integer values of x between -2 and 2 and find their y values. See [link].

$y = x^2$	
x	y
0	0
1	1
-1	1
2	4
-2	4

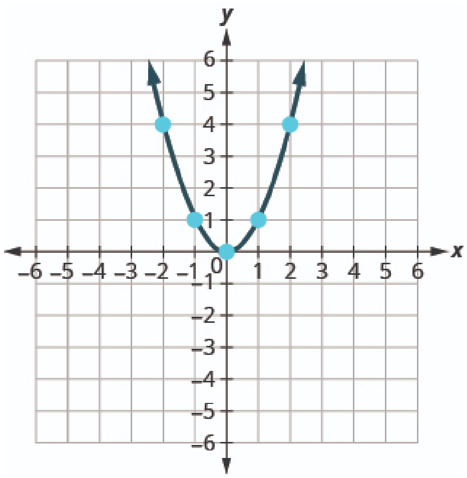
Notice when we let $x = 1$ and $x = -1$, we got the same value for y.

Equation:

$$\begin{array}{ll} y = x^2 & y = x^2 \\ y = 1^2 & y = (-1)^2 \\ y = 1 & y = 1 \end{array}$$

The same thing happened when we let $x = 2$ and $x = -2$.

Now, we will plot the points to show the graph of $y = x^2$. See [link].



The graph is not a line. This figure is called a **parabola**. Every quadratic equation has a graph that looks like this.

In [link] you will practice graphing a parabola by plotting a few points.

Example:

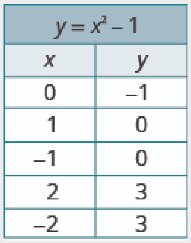
Exercise:

Problem: Graph $y = x^2 - 1$.

Solution:
Solution

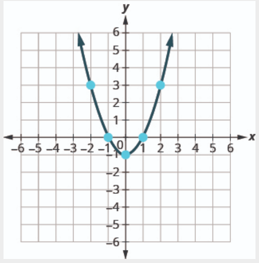
We will graph the equation by plotting points.

<table>
<tr><td>Choose integers values for <i>x</i>, substitute them into the equation and solve for <i>y</i>.</td><td></td></tr>
<tr><td>Record the values of the ordered pairs in the chart.</td><td>



<table>
<tr><th colspan="2">$y = x^2 - 1$</th></tr>
<tr><th>x</th><th>y</th></tr>
<tr><td>0</td><td>–1</td></tr>
<tr><td>1</td><td>0</td></tr>
<tr><td>–1</td><td>0</td></tr>
<tr><td>2</td><td>3</td></tr>
<tr><td>–2</td><td>3</td></tr>
</table>

</td></tr>
<tr><td>Plot the points, and then connect them with a smooth curve. The result will be the graph of the equation$y = x^2 - 1$.</td><td>

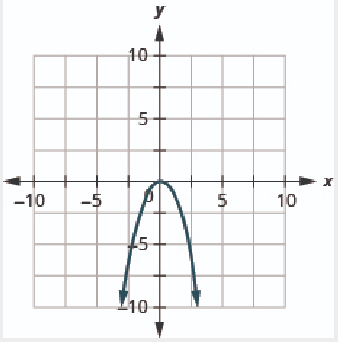


</td></tr>
</table>

Note:
Exercise:

Problem: Graph $y = -x^2$.

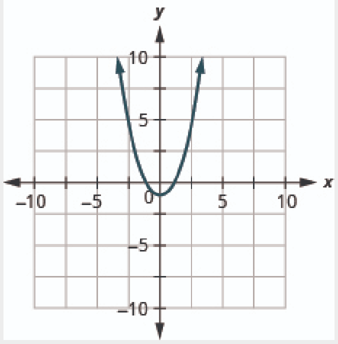
Solution:



Note:
Exercise:

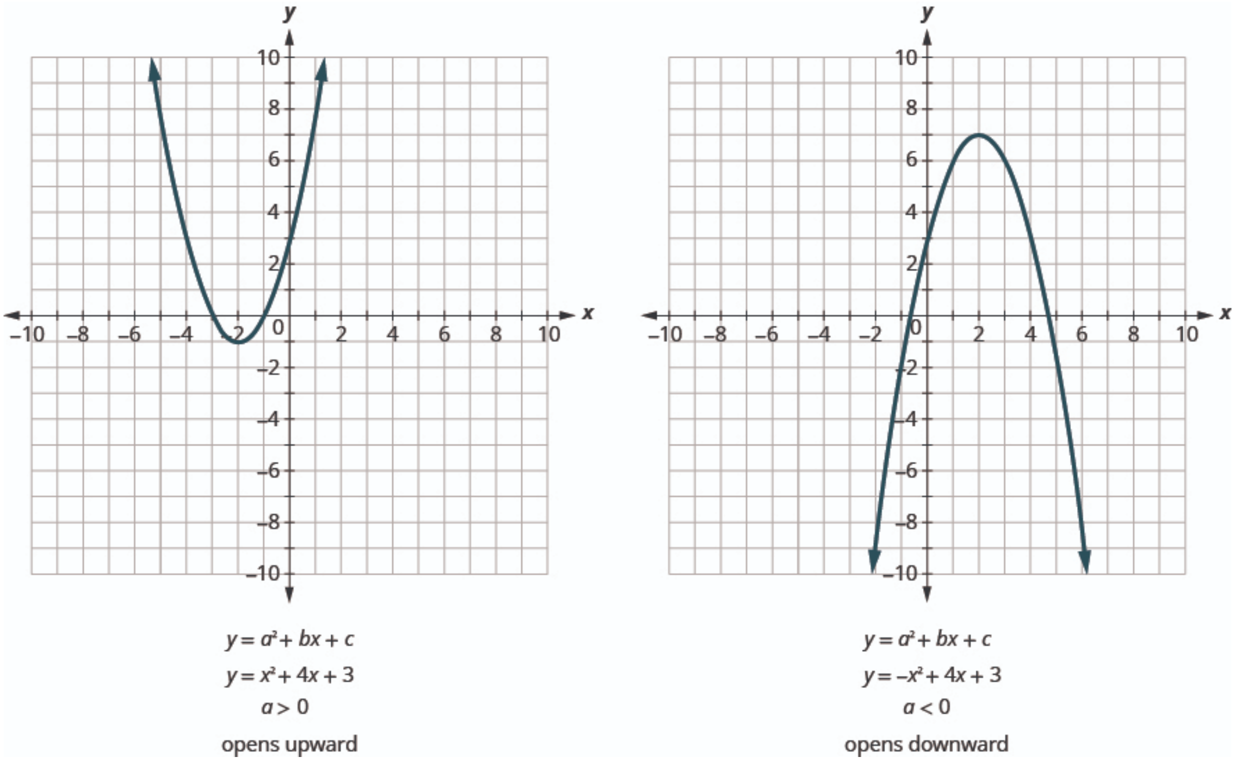
Problem: Graph $y = x^2 + 1$.

Solution:



How do the equations $y = x^2$ and $y = x^2 - 1$ differ? What is the difference between their graphs? How are their graphs the same?

All parabolas of the form $y = ax^2 + bx + c$ open upwards or downwards. See [link].

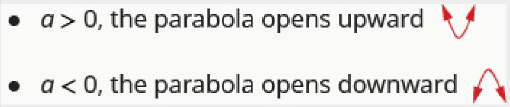


$y = a^2 + bx + c$
$y = x^2 + 4x + 3$
$a > 0$
opens upward

$y = a^2 + bx + c$
$y = -x^2 + 4x + 3$
$a < 0$
opens downward

Notice that the only difference in the two equations is the negative sign before the x^2 in the equation of the second graph in [link]. When the x^2 term is positive, the parabola opens upward, and when the x^2 term is negative, the parabola opens downward.

Note:
Parabola Orientation
For the quadratic equation $y = ax^2 + bx + c$, if:



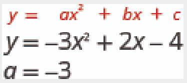
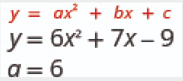
- $a > 0$, the parabola opens upward
- $a < 0$, the parabola opens downward

Example:
Exercise:

Problem: Determine whether each parabola opens upward or downward:

ⓐ $y = -3x^2 + 2x - 4$ ⓑ $y = 6x^2 + 7x - 9$

Solution:
Solution

ⓐ Find the value of "*a*".	 $y = ax^2 + bx + c$ $y = -3x^2 + 2x - 4$ $a = -3$ Since the “a” is negative, the parabola will open downward.
ⓑ Find the value of "*a*".	 $y = ax^2 + bx + c$ $y = 6x^2 + 7x - 9$ $a = 6$ Since the “a” is positive, the parabola will open upward.

Note:
Exercise:

Problem: Determine whether each parabola opens upward or downward:

ⓐ $y = 2x^2 + 5x - 2$ ⓑ $y = -3x^2 - 4x + 7$

Solution:

ⓐ up ⓑ down

Note:
Exercise:

Problem: Determine whether each parabola opens upward or downward:

ⓐ $y = -2x^2 - 2x - 3$ ⓑ $y = 5x^2 - 2x - 1$

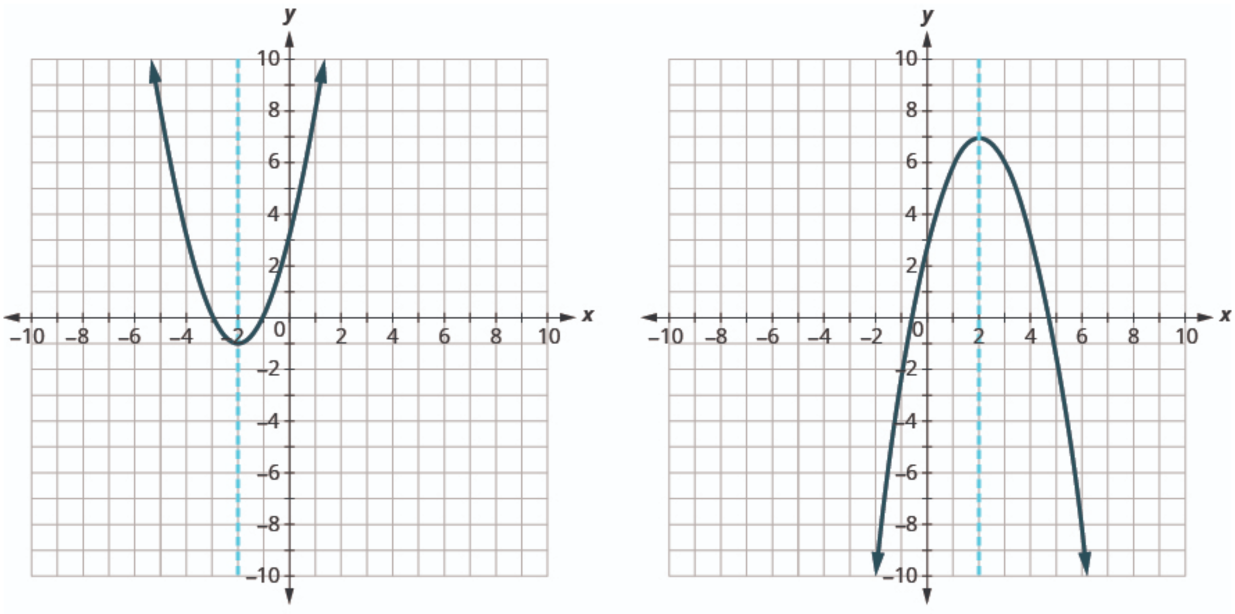
Solution:

ⓐ down ⓑ up

Find the Axis of Symmetry and Vertex of a Parabola

Look again at [link]. Do you see that we could fold each parabola in half and that one side would lie on top of the other? The ‘fold line’ is a line of symmetry. We call it the **axis of symmetry** of the parabola.

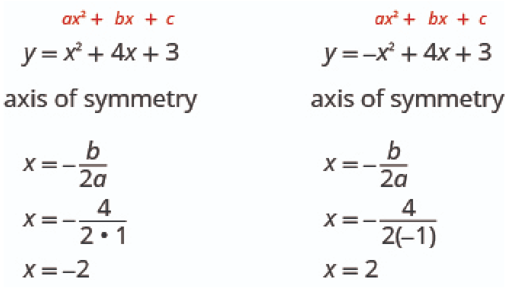
We show the same two graphs again with the axis of symmetry in red. See [link].



$y = x^2 + 4x + 3$ $y = -x^2 + 4x + 3$

The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of $y = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

So, to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula $x = -\frac{b}{2a}$.

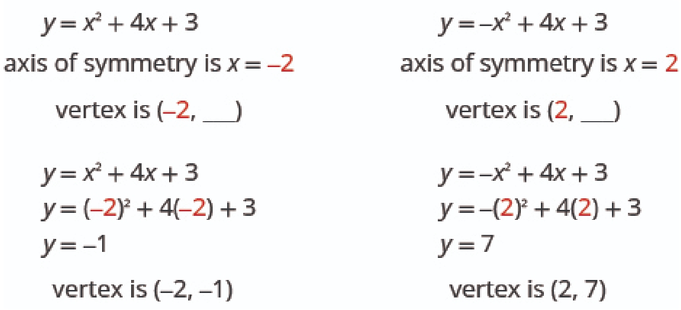


$ax^2 + bx + c$	$ax^2 + bx + c$
$y = x^2 + 4x + 3$	$y = -x^2 + 4x + 3$
axis of symmetry	axis of symmetry
$x = -\frac{b}{2a}$	$x = -\frac{b}{2a}$
$x = -\frac{4}{2 \cdot 1}$	$x = -\frac{4}{2(-1)}$
$x = -2$	$x = 2$

Look back at [link]. Are these the equations of the dashed red lines?

The point on the parabola that is on the axis of symmetry is the lowest or highest point on the parabola, depending on whether the parabola opens upwards or downwards. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its *x*-coordinate is $-\frac{b}{2a}$. To find the *y*-coordinate of the vertex, we substitute the value of the *x*-coordinate into the quadratic equation.



$y = x^2 + 4x + 3$	$y = -x^2 + 4x + 3$
axis of symmetry is $x = -2$	axis of symmetry is $x = 2$
vertex is $(-2, \_\_\_)$	vertex is $(2, \_\_\_)$
$y = x^2 + 4x + 3$ $y = (-2)^2 + 4(-2) + 3$ $y = -1$	$y = -x^2 + 4x + 3$ $y = -(2)^2 + 4(2) + 3$ $y = 7$
vertex is $(-2, -1)$	vertex is $(2, 7)$

Note:
Axis of Symmetry and Vertex of a Parabola
For a parabola with equation $y = ax^2 + bx + c$:

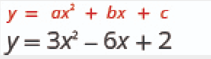
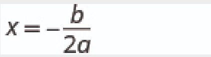
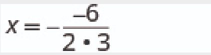
- The axis of symmetry of a parabola is the line $x = -\frac{b}{2a}$.
- The vertex is on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}$.

To find the *y*-coordinate of the vertex, we substitute $x = -\frac{b}{2a}$ into the quadratic equation.

Example:
Exercise:

Problem: For the parabola $y = 3x^2 - 6x + 2$ find: ⓐ the axis of symmetry and ⓑ the vertex.

Solution:
Solution

ⓐ	 $y = ax^2 + bx + c$ $y = 3x^2 - 6x + 2$
The axis of symmetry is the line $x = -\frac{b}{2a}$.	 $x = -\frac{b}{2a}$
Substitute the values of *a*, *b* into the equation.	 $x = -\frac{-6}{2 \cdot 3}$
Simplify.	$x = 1$

	The axis of symmetry is the line $x = 1$.
ⓑ	$y = 3x^2 - 6x + 2$
The vertex is on the line of symmetry, so its *x*-coordinate will be $x = 1$.	
Substitute $x = 1$ into the equation and solve for *y*.	$y = 3(1)^2 - 6(1) + 2$
Simplify.	$y = 3 \cdot 1 - 6 + 2$
This is the *y*-coordinate.	$y = -1$ The vertex is $(1, -1)$.

Note:
Exercise:

Problem: For the parabola $y = 2x^2 - 8x + 1$ find: ⓐ the axis of symmetry and ⓑ the vertex.

Solution:

ⓐ $x = 2$ ⓑ $(2, -7)$

Note:
Exercise:

Problem: For the parabola $y = 2x^2 - 4x - 3$ find: ⓐ the axis of symmetry and ⓑ the vertex.

Solution:

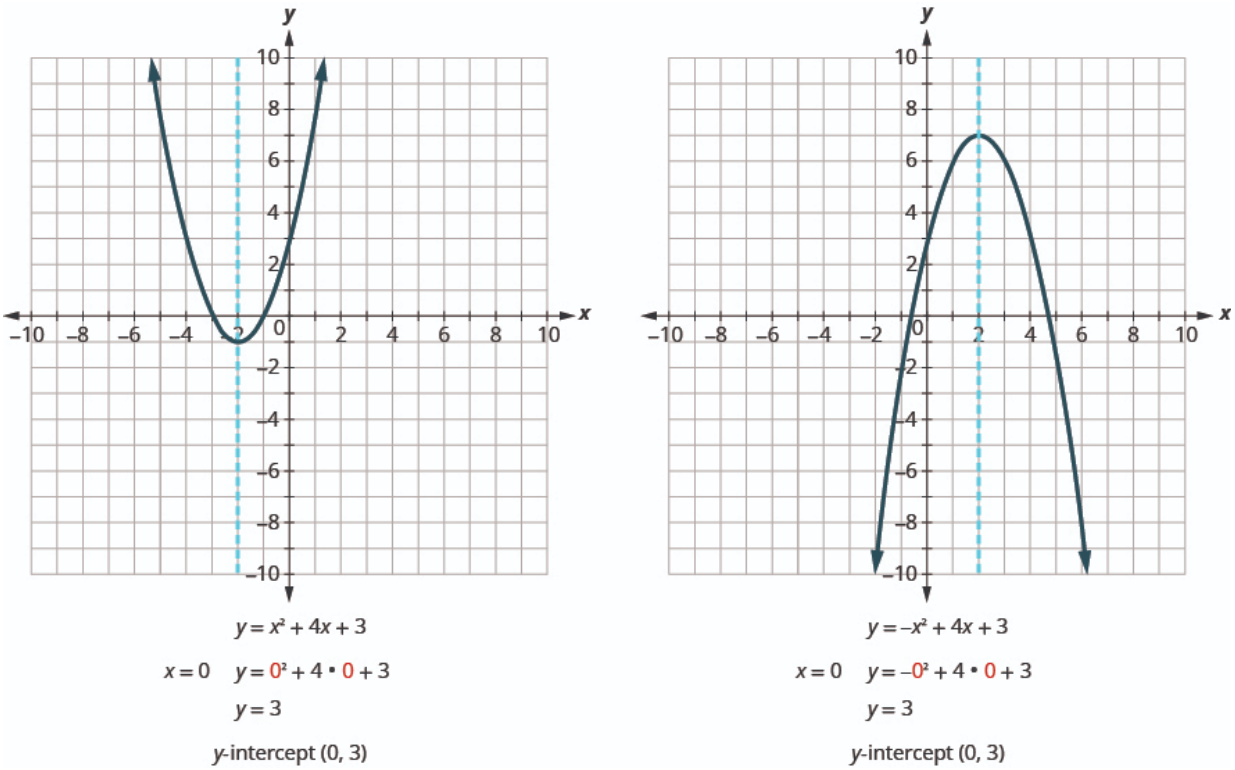
ⓐ $x = 1$ ⓑ $(1, -5)$

Find the Intercepts of a Parabola

When we graphed linear equations, we often used the *x*- and *y*-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the ***y*-intercept** the value of x is zero. So, to find the *y*-intercept, we substitute $x = 0$ into the equation.

Let’s find the *y*-intercepts of the two parabolas shown in the figure below.



$y = x^2 + 4x + 3$

$x = 0 \quad y = 0^2 + 4 \cdot 0 + 3$

$y = 3$

y-intercept (0, 3)

$y = -x^2 + 4x + 3$

$x = 0 \quad y = -0^2 + 4 \cdot 0 + 3$

$y = 3$

y-intercept (0, 3)

At an ***x*-intercept**, the value of y is zero. To find an *x*-intercept, we substitute $y = 0$ into the equation. In other words, we will need to solve the equation $0 = ax^2 + bx + c$ for x.

Equation:

$$y = ax^2 + bx + c$$
$$0 = ax^2 + bx + c$$

But solving quadratic equations like this is exactly what we have done earlier in this chapter.

We can now find the *x*-intercepts of the two parabolas shown in [link].

First, we will find the *x*-intercepts of a parabola with equation $y = x^2 + 4x + 3$.

	$y = x^2 + 4x + 3$

Let $y = 0$.	$0 = x^2 + 4x + 3$
Factor.	$0 = (x + 1)(x + 3)$
Use the zero product property.	$x + 1 = 0, x + 3 = 0$
Solve.	$x = -1, x = -3$
	The x intercepts are $(-1, 0)$ and $(-3, 0)$.

Now, we will find the x-intercepts of the parabola with equation $y = -x^2 + 4x + 3$.

	$y = -x^2 + 4x + 3$
Let $y = 0$.	$0 = -x^2 + 4x + 3$
This quadratic does not factor, so we use the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$a = -1, b = 4, c = 3$	$x = \frac{-4 \pm \sqrt{4^2 - 4(-1)(3)}}{2(-1)}$
Simplify.	$x = \frac{-4 \pm \sqrt{28}}{-2}$ $x = \frac{-4 \pm 2\sqrt{7}}{-2}$ $x = \frac{-2(2 \pm \sqrt{7})}{-2}$

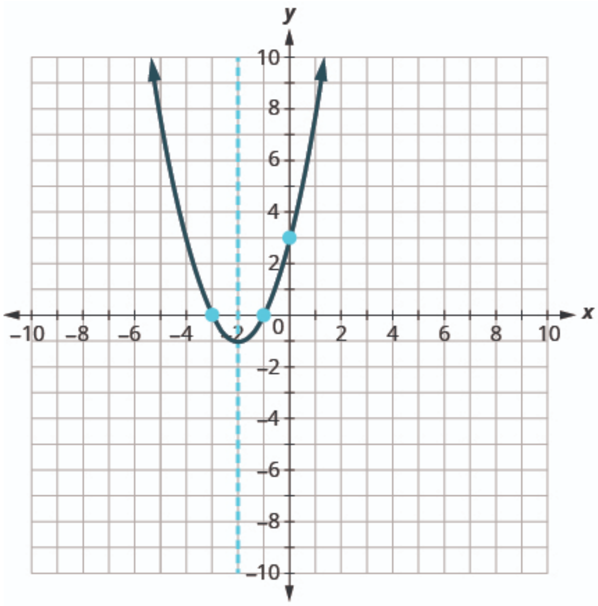
	$x = 2 \pm \sqrt{7}$
	The x intercepts are $\left(2 + \sqrt{7}, 0\right)$ and $\left(2 - \sqrt{7}, 0\right)$.

We will use the decimal approximations of the x-intercepts, so that we can locate these points on the graph.

Equation:

$$\left(2 + \sqrt{7}, 0\right) \approx (4.6, 0) \qquad \left(2 - \sqrt{7}, 0\right) \approx (-0.6, 0)$$

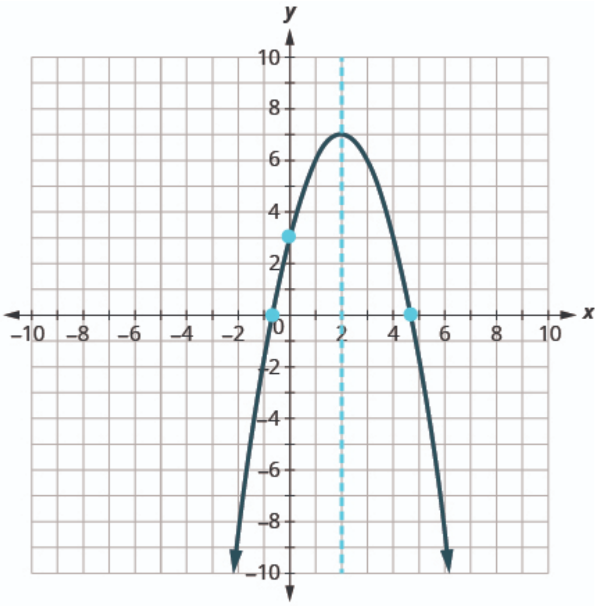
Do these results agree with our graphs? See [link].



$y = x^2 + 4x + 3$

y-intercept (0, 3)

x-intercepts (–1, 0) and (–3, 0)



$y = -x^2 + 4x + 3$

y-intercept (0, 3)

x-intercepts $(2 + \sqrt{7}, 0) \approx (4.6, 0)$
$(2 - \sqrt{7}, 0) \approx (-0.6, 0)$

Note:
Find the intercepts of a parabola.
To find the intercepts of a parabola with equation $y = ax^2 + bx + c$:
Equation:

y**-intercept**	x**-intercepts**
Let $x = 0$ and solve for y.	Let $y = 0$ and solve for x.

Example:
Exercise:

Problem: Find the intercepts of the parabola $y = x^2 - 2x - 8$.

Solution:
Solution

	$y = x^2 - 2x - 8$
To find the *y*-intercept, let $x = 0$ and solve for *y*.	$y = 0^2 - 2 \cdot 0 - 8$ $y = -8$
	When $x = 0$, then $y = -8$. The *y*-intercept is the point $(0, -8)$.
	$y = x^2 - 2x - 8$
To find the *x*-intercept, let $y = 0$ and solve for *x*.	$0 = x^2 - 2x - 8$
Solve by factoring.	$0 = (x - 4)(x + 2)$
	$0 = x - 4 \quad 0 = x + 2$ $4 = x \quad -2 = x$

When $y = 0$, then $x = 4$ or $x = -2$. The *x*-intercepts are the points $(4, 0)$ and $(-2, 0)$.

Note:
Exercise:

Problem: Find the intercepts of the parabola $y = x^2 + 2x - 8$.

Solution:

y: $(0,-8)$; x: $(-4,0),(2,0)$

Note:
Exercise:

Problem: Find the intercepts of the parabola $y=x^2-4x-12$.

Solution:

y: $(0,-12)$; x: $(6,0),(-2,0)$

In this chapter, we have been solving quadratic equations of the form $ax^2+bx+c=0$. We solved for x and the results were the solutions to the equation.

We are now looking at quadratic equations in two variables of the form $y=ax^2+bx+c$. The graphs of these equations are parabolas. The *x*-intercepts of the parabolas occur where $y=0$.

For example:
Equation:

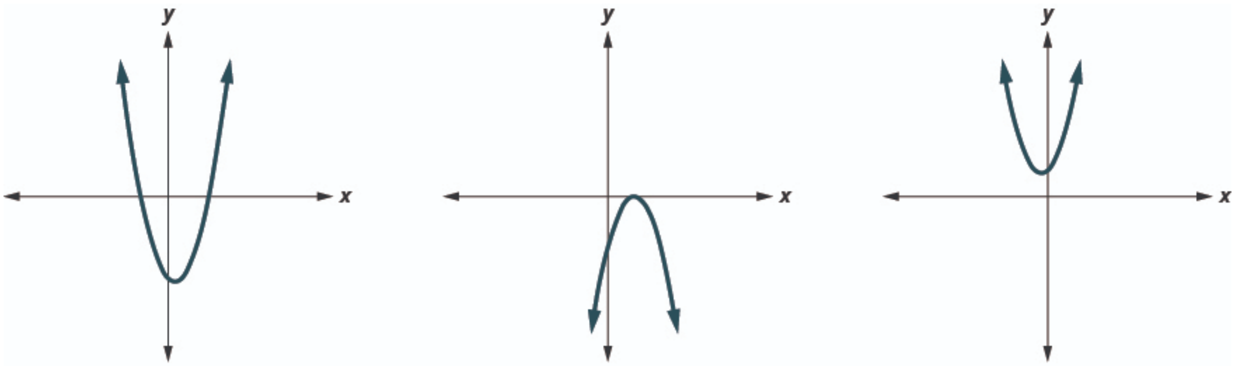
$$\begin{array}{c}\textbf{Quadratic equation}\\ \\ \begin{aligned} x^2-2x-15 &= 0\\ (x-5)(x+3) &= 0 \end{aligned}\\ \begin{aligned} x-5 &= 0 & \qquad x+3 &= 0\\ x &= 5 & x &= -3 \end{aligned}\end{array}$$

$$\begin{array}{c}\textbf{Quadratic equation in two variables}\\ y=x^2-2x-15\\ \text{let } y=0 \quad \begin{array}{l} 0=x^2-2x-15\\ 0=(x-5)(x+3) \end{array}\\ \begin{aligned} x-5 &= 0 & \qquad x+3 &= 0\\ x &= 5 & x &= -3 \end{aligned}\\ (5,0) \text{ and } (-3,0)\\ x\text{-intercepts}\end{array}$$

The solutions of the quadratic equation are the x values of the *x*-intercepts.

Earlier, we saw that quadratic equations have 2, 1, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the equations give the *x*-intercepts of the graphs, the number of *x*-intercepts is the same as the number of solutions.

Previously, we used the discriminant to determine the number of solutions of a quadratic equation of the form $ax^2+bx+c=0$. Now, we can use the discriminant to tell us how many *x*-intercepts there are on the graph.



$b^2 - 4ac > 0$
Two solutions
Two *x*-intercepts

$b^2 - 4ac = 0$
One solution
One *x*-intercept

$b^2 - 4ac < 0$
No real solution
No *x*-intercept

Before you start solving the quadratic equation to find the values of the *x*-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

Example:
Exercise:

Problem: Find the intercepts of the parabola $y = 5x^2 + x + 4$.

Solution:
Solution

	$y = 5x^2 + x + 4$
To find the *y*-intercept, let $x = 0$ and solve for *y*.	$y = 5 \cdot 0^2 + 0 + 4$ $y = 4$ When $x = 0$, then $y = 4$. The *y*-intercept is the point $(0, 4)$.
	$y = 5x^2 + x + 4$

To find the *x*-intercept, let $y = 0$ and solve for *x*.	$0 = 5x^2 + x + 4$
Find the value of the discriminant to predict the number of solutions and so *x*-intercepts.	$b^2 \quad - \quad 4ac$ $1^2 \quad - \quad 4 \cdot 5 \cdot 4$ $1 \quad - \quad 80$ -79
Since the value of the discriminant is negative, there is no real solution to the equation.	There are no *x*-intercepts.

Note:
Exercise:

Problem: Find the intercepts of the parabola $y = 3x^2 + 4x + 4$.

Solution:

y: $(0, 4)$; x: none

Note:
Exercise:

Problem: Find the intercepts of the parabola $y = x^2 - 4x - 5$.

Solution:

y: $(0, -5)$; x: $(5, 0)$ $(-1, 0)$

Example:
Exercise:

Problem: Find the intercepts of the parabola $y = 4x^2 - 12x + 9$.

Solution:
Solution

	$y = 4x^2 - 12x + 9$
To find the *y*-intercept, let $x = 0$ and solve for *y*.	$y = 4 \cdot 0^2 - 12 \cdot 0 + 9$ $y = 9$
	When $x = 0$, then $y = 9$. The *y*-intercept is the point $(0, 9)$.
	$y = 4x^2 - 12x + 9$
To find the *x*-intercept, let $y = 0$ and solve for *x*.	$0 = 4x^2 - 12x + 9$
Find the value of the discriminant to predict the number of solutions and so *x*-intercepts.	$b^2 - 4ac$ $2^2 - 4 \cdot 4 \cdot 9$ $144 - 144$ 0
	Since the value of the discriminant is 0, there is no real solution to the equation. So there is one *x*-intercept.
Solve the equation by factoring the perfect square trinomial.	$0 = (2x - 3)^2$
Use the Zero Product Property.	$0 = 2x - 3$
Solve for *x*.	$3 = 2x$ $\frac{3}{2} = x$
	When $y = 0$, then $\frac{3}{2} = x$.
	The *x*-intercept is the point $\left(\frac{3}{2}, 0\right)$.

Note:
Exercise:

Problem: Find the intercepts of the parabola $y=-x^2-12x-36$.

Solution:

y: $(0,-36)$; x: $(-6,0)$

Note:
Exercise:

Problem: Find the intercepts of the parabola $y=9x^2+12x+4$.

Solution:

y: $(0,4)$; x: $\left(-\frac{2}{3},0\right)$

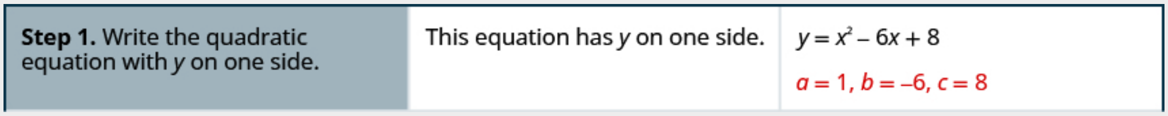
Graph Quadratic Equations in Two Variables

Now, we have all the pieces we need in order to graph a quadratic equation in two variables. We just need to put them together. In the next example, we will see how to do this.

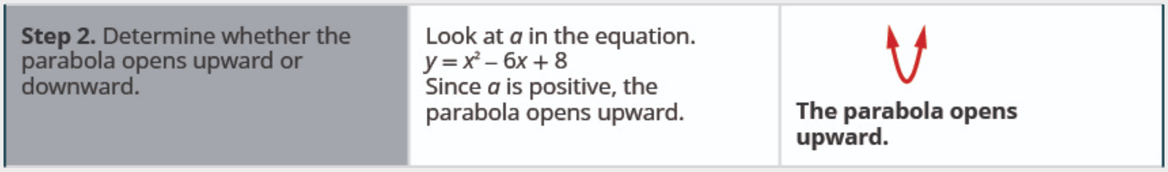
Example:
How To Graph a Quadratic Equation in Two Variables
Exercise:

Problem: Graph $y=x^2-6x+8$.

Solution:
Solution

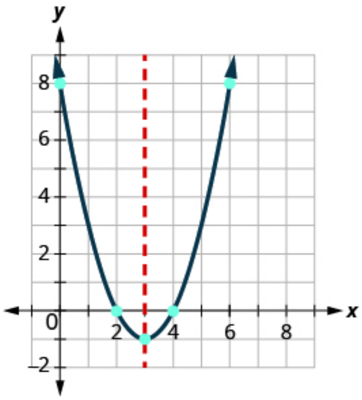


Step 1. Write the quadratic equation with y on one side.	This equation has y on one side.	$y=x^2-6x+8$ $a=1,\ b=-6,\ c=8$



Step 2. Determine whether the parabola opens upward or downward.	Look at a in the equation. $y=x^2-6x+8$ Since a is positive, the parabola opens upward.	**The parabola opens upward.**

Step 3. Find the axis of symmetry.	$y = x^2 - 6x + 8$ The axis of symmetry is the line $x = -\frac{b}{2a}$.	Axis of Symmetry $x = -\frac{b}{2a}$ $x = -\frac{(-6)}{2 \cdot 1}$ $x = 3$ **The axis of symmetry is the line $x = 3$.**
Step 4. Find the vertex.	The vertex is on the axis of symmetry. Substitute $x = 3$ into the equation and solve for y.	Vertex $y = x^2 - 6x + 8$ $y = (3)^2 - 6(3) + 8$ $y = -1$ **The vertex is (3, –1).**
Step 5. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	We substitute $x = 0$ into the equation.	y-intercept $y = x^2 - 6x + 8$ $y = (0)^2 - 6(0) + 8$ $y = 8$ **The y-intercept is (0, 8).**
	We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$.	Point symmetric to y-intercept **The point is (6, 8).**
Step 6. Find the x-intercepts.	We substitute $y = 0$ into the equation. We can solve this quadratic equation by factoring.	x-intercept $y = x^2 - 6x + 8$ $0 = x^2 - 6x + 8$ $0 = (x - 2)(x - 4)$ $x = 2$ or $x = 4$ **The x-intercepts are (2, 0) and (4, 0).**

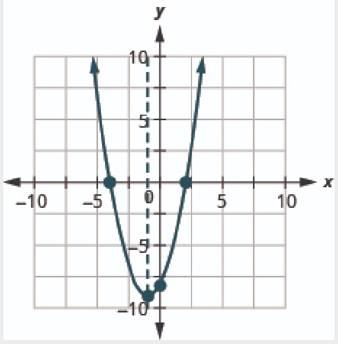
Step 7. Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the *y*-intercept. We connect these 5 points to sketch the parabola.	

Note:
Exercise:

Problem: Graph the parabola $y = x^2 + 2x - 8$.

Solution:

y: $(0, -8)$; x: $(2, 0), (-4, 0)$;
axis: $x = -1$; vertex: $(-1, -9)$;

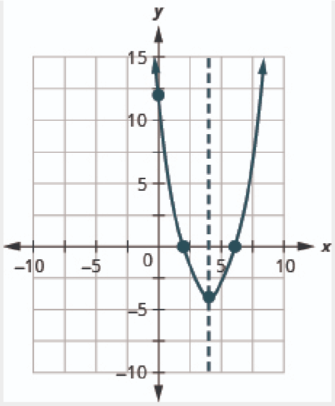


Note:
Exercise:

Problem: Graph the parabola $y = x^2 - 8x + 12$.

Solution:

y: $(0, 12)$; x: $(2, 0), (6, 0)$;
axis: $x = 4$; vertex: $(4, -4)$;



Note:
Graph a quadratic equation in two variables.

Write the quadratic equation with y on one side.
Determine whether the parabola opens upward or downward.
Find the axis of symmetry.
Find the vertex.
Find the*y*-intercept. Find the point symmetric to the*y*-intercept across the axis of symmetry.
Find the*x*-intercepts.
Graph the parabola.

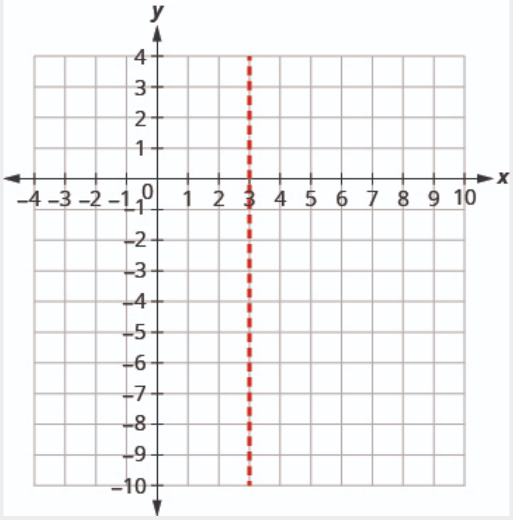
We were able to find the *x*-intercepts in the last example by factoring. We find the *x*-intercepts in the next example by factoring, too.

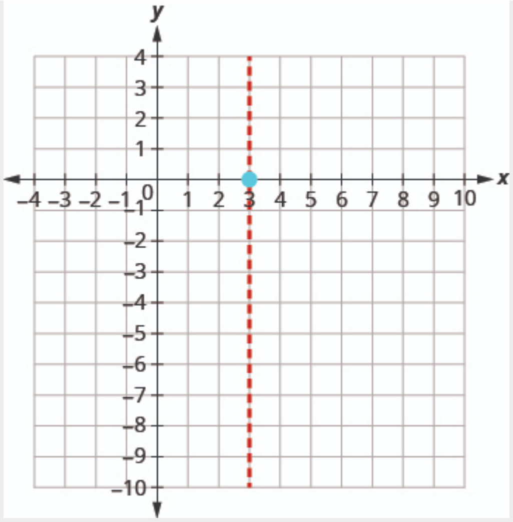
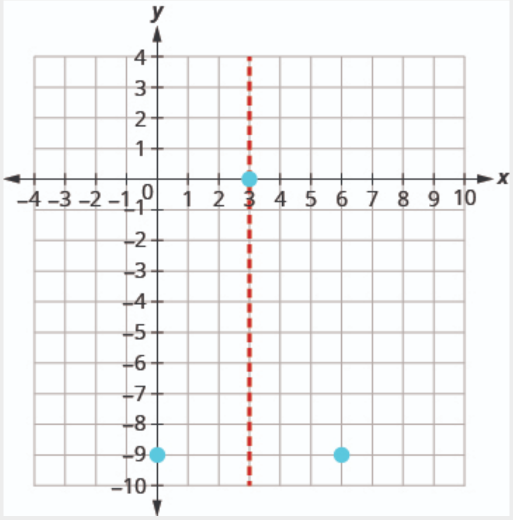
Example:
Exercise:

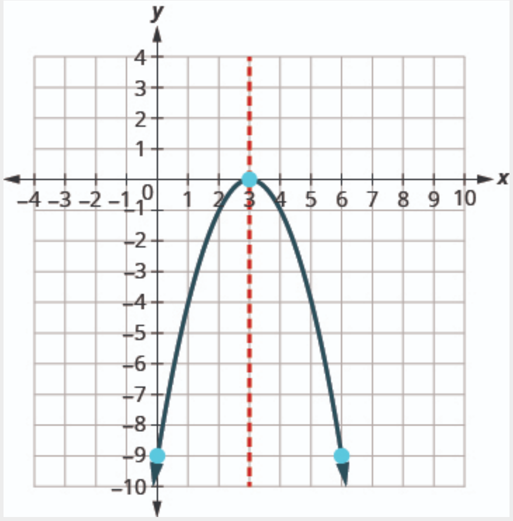
Problem: Graph $y = -x^2 + 6x - 9$.

Solution:
Solution

The equation *y* has on one side.	$y = ax^2 + bx + c$ $y = -x^2 + 6x - 9$

Since a is -1, the parabola opens downward. To find the axis of symmetry, find $x = -\frac{b}{2a}$.		$x = -\frac{b}{2a}$ $x = -\frac{6}{2(-1)}$ $x = 3$ The axis of symmetry is $x = 3$. The vertex is on the line $x = 3$. 
Find y when $x = 3$.		$y = -x^2 + 6x - 9$ $y = -3^2 + 6 \cdot 3 - 9$ $y = -9 + 18 - 9$ $y = 0$ The vertex is $(3, 0)$.

	
The *y*-intercept occurs when $x = 0$. Substitute $x = 0$. Simplify. The point $(0, -9)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -9)$. Point symmetric to the *y*-intercept is $(6, -9)$	$y = -x^2 + 6x - 9$ $y = -0^2 + 6 \cdot 0 - 9$ $y = -9$ The *y*-intercept is $(0, -9)$. 
The *x*-intercept occurs when $y = 0$.	$y = -x^2 + 6x - 9$
Substitute $y = 0$.	

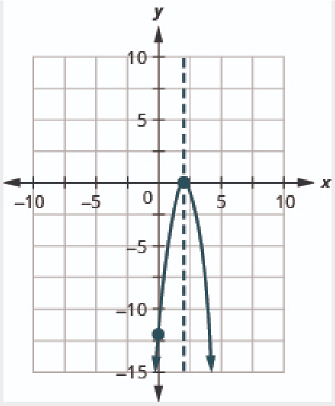
	$0 = -x^2 + 6x - 9$
Factor the GCF.	$0 = -(x^2 - 6x + 9)$
Factor the trinomial.	$0 = -(x - 3)^2$
Solve for *x*.	$x = 3$
Connect the points to graph the parabola.	

Note:
Exercise:

Problem: Graph the parabola $y = -3x^2 + 12x - 12$.

Solution:

y: $(0, -12)$; x: $(2, 0)$;
axis: $x = 2$; vertex: $(2, 0)$;

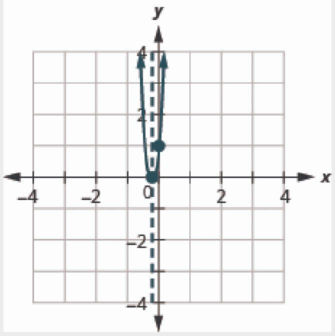


Note:
Exercise:

Problem: Graph the parabola $y=25x^2+10x+1$.

Solution:

y: $(0,1)$; x: $\left(-\frac{1}{5},0\right)$;
axis: $x=-\frac{1}{5}$; vertex: $\left(-\frac{1}{5},0\right)$;



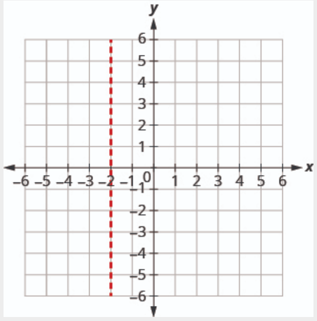
For the graph of $y=-x^2+6x-9$, the vertex and the *x*-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0=-x^2+6x-9$ is 0, so there is only one solution. That means there is only one *x*-intercept, and it is the vertex of the parabola.

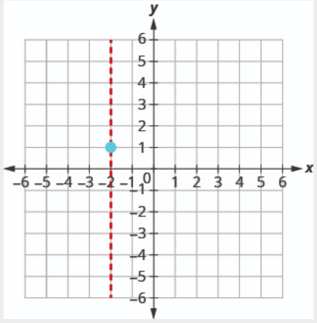
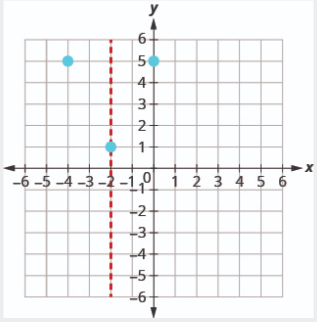
How many *x*-intercepts would you expect to see on the graph of $y=x^2+4x+5$?

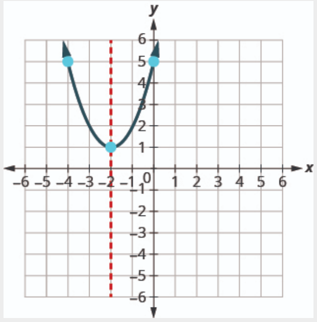
Example:
Exercise:

Problem: Graph $y = x^2 + 4x + 5$.

Solution:
Solution

The equation has *y* on one side.	$y = ax^2 + bx + c$ $y = x^2 + 4x + 5$
Since a is 1, the parabola opens upward.	
To find the axis of symmetry, find $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$ $x = -\frac{4}{2(1)}$ $x = -2$ The axis of symmetry is $x = -2$. 
The vertex is on the line $x = -2$.	
Find *y* when $x = -2$.	$y = x^2 + 4x + 5$

	$y = (-2)^2 + 4 \cdot (-2) + 5$ $y = 4 - 8 + 5$ $y = 1$ The vertex is $(-2, 1)$. 
The *y*-intercept occurs when $x = 0$. Substitute $x = 0$. Simplify. The point $(0, 5)$ is two units to the right of the line of symmetry. The point two units to the left of the line of symmetry is $(-4, 5)$.	$y = x^2 + 4x + 5$ $y = (0)^2 + 4(0) + 5$ $y = 5$ The *y*-intercept is $(0, 5)$.  Point symmetric to the *y*-intercept is $(-4, 5)$.
The *x*- intercept occurs when $y = 0$.	

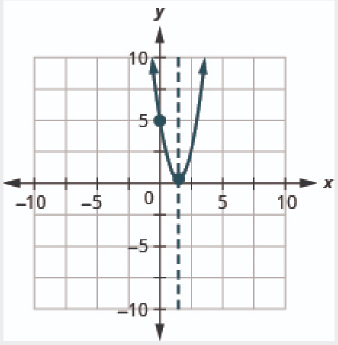
Substitute $y = 0$. Test the discriminant.	$y = x^2 + 4x + 5$ $0 = x^2 + 4x + 5$
	$b^2 - 4ac$ $4^2 - 4 \cdot 15$ $16 - 20$ -4
Since the value of the discriminant is negative, there is no solution and so no *x*- intercept. Connect the points to graph the parabola. You may want to choose two more points for greater accuracy.	

Note:
Exercise:

Problem: Graph the parabola $y = 2x^2 - 6x + 5$.

Solution:

y: $(0, 5)$; x: none;
axis: $x = \frac{3}{2}$; vertex: $\left(\frac{3}{2}, \frac{1}{2}\right)$;

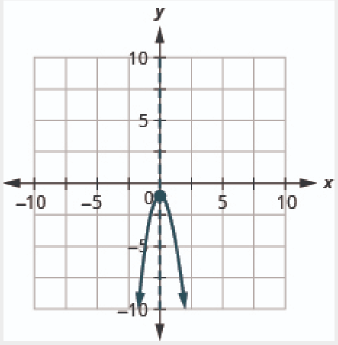


Note:
Exercise:

Problem: Graph the parabola $y=-2x^2-1$.

Solution:

y: $(0,-1)$; x: none;
axis: $x=0$; vertex: $(0,-1)$;



Finding the *y*-intercept by substituting $x=0$ into the equation is easy, isn't it? But we needed to use the Quadratic Formula to find the *x*-intercepts in [link]. We will use the Quadratic Formula again in the next example.

Example:
Exercise:

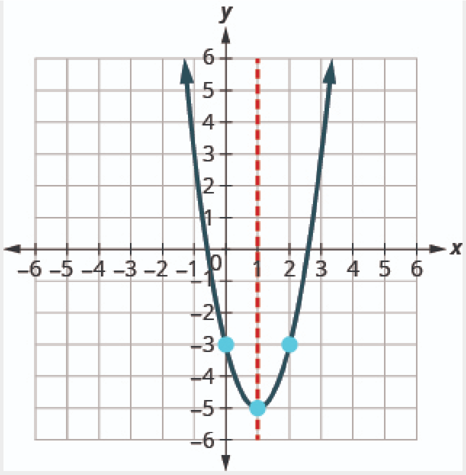
Problem: Graph $y=2x^2-4x-3$.

Solution:
Solution

	$y = ax^2 + bx + c$ $y = 2x^2 - 4x - 3$
The equation *y* has one side. Since *a* is 2, the parabola opens upward.	

To find the axis of symmetry, find $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$ $x = -\frac{-4}{2 \cdot 2}$ $x = 1$ The axis of symmetry is $x = 1$.
The vertex on the line $x = 1$.	$y = 2x^2 - 4x - 3$
Find *y* when $x = 1$.	$y = 2(1)^2 - 4 \cdot (1) - 3$ $y = 2 - 4 - 3$ $y = -5$ The vertex is $(1, -5)$.
The *y*-intercept occurs when $x = 0$.	$y = 2x^2 - 4x - 3$
Substitute $x = 0$.	$y = 2 \cdot 0^2 - 4 \cdot 0 - 3$
Simplify.	$y = -3$ The *y*-intercept is $(0, -3)$.
The point $(0, -3)$ is one unit to the left of the line of symmetry. The point one unit to the right of the line of symmetry is $(2, -3)$	Point symmetric to the *y*-intercept is $(2, -3)$.
The *x*-intercept occurs when $y = 0$.	$y = 2x^2 - 4x - 3$

Substitute $y = 0$.	$0 = 2x^2 - 4x - 3$
Use the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Substitute in the values of *a*, *b*, *c*.	$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot (-3)}}{2 \cdot 2}$
Simplify.	$x = \frac{4 \pm \sqrt{16 + 24}}{4}$
Simplify inside the radical.	$x = \frac{4 \pm \sqrt{40}}{4}$
Simplify the radical.	$x = \frac{4 \pm 2\sqrt{10}}{4}$
Factor the GCF.	$x = \frac{2(2 \pm \sqrt{10})}{4}$
Remove common factors.	$x = \frac{2 \pm \sqrt{10}}{2}$
Write as two equations.	$x = \frac{2 + \sqrt{10}}{2}, x = \frac{2 - \sqrt{10}}{2}$
Approximate the values.	$x \approx 2.5, x \approx -0.6$
	The approximate values of the *x*-intercepts are $(2.5, 0)$ and $(-0.6, 0)$.
Graph the parabola using the points found.	

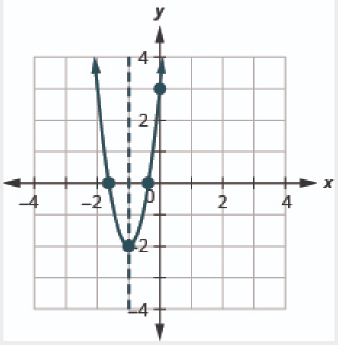


Note:
Exercise:

Problem: Graph the parabola $y = 5x^2 + 10x + 3$.

Solution:

y: $(0,3)$; x: $(-1.6,0),(-0.4,0)$;
axis: $x=-1$; vertex: $(-1,-2)$;

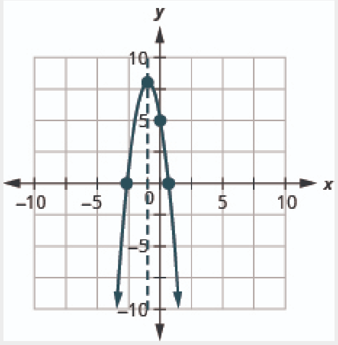


Note:
Exercise:

Problem: Graph the parabola $y = -3x^2 - 6x + 5$.

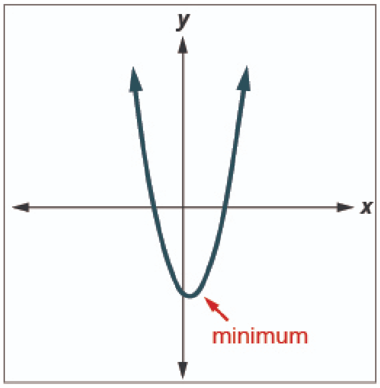
Solution:

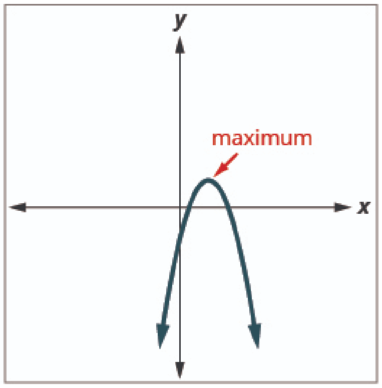
y: $(0,5)$; x: $(0.6,0), (-2.6,0)$;
axis: $x=-1$; vertex: $(-1,8)$;



Solve Maximum and Minimum Applications

Knowing that the vertex of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic equation. The *y*-coordinate of the vertex is the minimum *y*-value of a parabola that opens upward. It is the maximum *y*-value of a parabola that opens downward. See [link].





Note:
Minimum or Maximum Values of a Quadratic Equation
The ***y*-coordinate of the vertex** of the graph of a quadratic equation is the

- minimum value of the quadratic equation if the parabola opens upward.
- maximum value of the quadratic equation if the parabola opens downward.

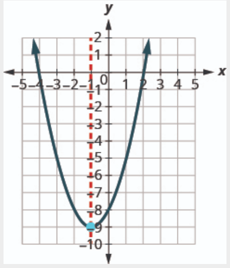
Example:

Exercise:

Problem: Find the minimum value of the quadratic equation $y = x^2 + 2x - 8$.

Solution:
Solution

	$y = x^2 + 2x - 8$
Since *a* is positive, the parabola opens upward.	
The quadratic equation has a minimum.	
Find the axis of symmetry.	$x = -\frac{b}{2a}$ $x = -\frac{2}{2 \cdot 1}$ $x = -1$ The axis of symmetry is $x = -1$.
The vertex is on the line $x = -1$.	$y = x^2 + 2x - 8$
Find *y* when $x = -1$.	$y = (-1)^2 + 2 \cdot (-1) - 8$ $y = 1 - 2 - 8$ $y = -9$ The vertex is $(-1, -9)$.

Since the parabola has a minimum, the *y*-coordinate of the vertex is the minimum *y*-value of the quadratic equation.	
The minimum value of the quadratic is -9 and it occurs when $x = -1$.	
Show the graph to verify the result.	

Note:
Exercise:

Problem: Find the maximum or minimum value of the quadratic equation $y = x^2 - 8x + 12$.

Solution:

The minimum value is -4 when $x = 4$.

Note:
Exercise:

Problem: Find the maximum or minimum value of the quadratic equation $y = -4x^2 + 16x - 11$.

Solution:

The maximum value is 5 when $x = 2$.

We have used the formula
Equation:

$$h = -16t^2 + v_0 t + h_0$$

to calculate the height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds.

This formula is a quadratic equation in the variable t, so its graph is a parabola. By solving for the coordinates of the vertex, we can find how long it will take the object to reach its maximum height. Then, we can calculate the maximum height.

Example:
Exercise:

Problem:

The quadratic equation $h=-16t^2+v_0t+h_0$ models the height of a volleyball hit straight upwards with velocity 176 feet per second from a height of 4 feet.

ⓐ How many seconds will it take the volleyball to reach its maximum height?
ⓑ Find the maximum height of the volleyball.

Solution:
Solution

$h=-16t^2+176t+4$

Since a is negative, the parabola opens downward.

The quadratic equation has a maximum.

ⓐ

Find the axis of symmetry.	$t=-\frac{b}{2a}$ $t=-\frac{176}{2(-16)}$ $t=5.5$ The axis of symmetry is $t=5.5$.
The vertex is on the line $t=5.5$.	The maximum occurs when $t=5.5$ seconds.

ⓑ

Find h when $t=5.5$.	$h=-16t^2+176t+4$ $h=-16(5.5)^2+176\cdot(5.5)+4$
Use a calculator to simplify.	$h=488$
	The vertex is $(5.5,488)$.

Since the parabola has a maximum, the h-coordinate of the vertex is the maximum y-value of the quadratic equation.	The maximum value of the quadratic is 488 feet and it occurs when $t = 5.5$ seconds.

Note:
Exercise:

Problem:

The quadratic equation $h = -16t^2 + 128t + 32$ is used to find the height of a stone thrown upward from a height of 32 feet at a rate of 128 ft/sec. How long will it take for the stone to reach its maximum height? What is the maximum height? Round answers to the nearest tenth.

Solution:

It will take 4 seconds to reach the maximum height of 288 feet.

Note:
Exercise:

Problem:

A toy rocket shot upward from the ground at a rate of 208 ft/sec has the quadratic equation of $h = -16t^2 + 208t$. When will the rocket reach its maximum height? What will be the maximum height? Round answers to the nearest tenth.

Solution:

It will take 6.5 seconds to reach the maximum height of 676 feet.

Note:
Access these online resources for additional instruction and practice graphing quadratic equations:

- Graphing Quadratic Functions
- How do you graph a quadratic function?
- Graphing Quadratic Equations

Key Concepts

- **The graph of every quadratic equation is a parabola.**
- **Parabola Orientation** For the quadratic equation $y = ax^2 + bx + c$, if

  - $a > 0$, the parabola opens upward.
  - $a < 0$, the parabola opens downward.

- **Axis of Symmetry and Vertex of a Parabola** For a parabola with equation $y = ax^2 + bx + c$:
  - The axis of symmetry of a parabola is the line $x = -\frac{b}{2a}$.
  - The vertex is on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}$.
  - To find the *y*-coordinate of the vertex we substitute $x = -\frac{b}{2a}$ into the quadratic equation.

- **Find the Intercepts of a Parabola** To find the intercepts of a parabola with equation $y = ax^2 + bx + c$:

y**-intercept**	x**-intercepts**
Let $x = 0$ and solve for y.	Let $y = 0$ and solve for x.

- **To Graph a Quadratic Equation in Two Variables**

  Write the quadratic equation with y on one side.
  Determine whether the parabola opens upward or downward.
  Find the axis of symmetry.
  Find the vertex.
  Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
  Find the *x*-intercepts.
  Graph the parabola.

- **Minimum or Maximum Values of a Quadratic Equation**
  - The *y*-**coordinate of the vertex** of the graph of a quadratic equation is the
  - **minimum** value of the quadratic equation if the parabola opens upward.
  - **maximum** value of the quadratic equation if the parabola opens downward.

Section Exercises

Practice Makes Perfect

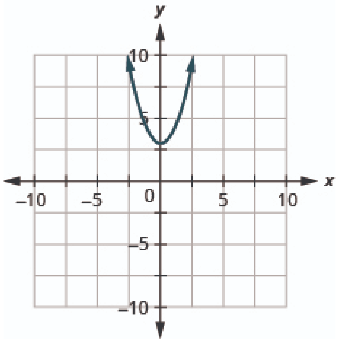
Recognize the Graph of a Quadratic Equation in Two Variables

In the following exercises, graph:

Exercise:

Problem: $y = x^2 + 3$

Solution:



Exercise:

Problem: $y = -x^2 + 1$

In the following exercises, determine if the parabola opens up or down.

Exercise:

Problem: $y = -2x^2 - 6x - 7$

Solution:

down

Exercise:

Problem: $y = 6x^2 + 2x + 3$

Exercise:

Problem: $y = 4x^2 + x - 4$

Solution:

up

Exercise:

Problem: $y = -9x^2 - 24x - 16$

Find the Axis of Symmetry and Vertex of a Parabola

In the following exercises, find ⓐ the axis of symmetry and ⓑ the vertex.

Exercise:

Problem: $y = x^2 + 8x - 1$

Solution:

ⓐ $x = -4$ ⓑ $(-4, -17)$

Exercise:

Problem: $y = x^2 + 10x + 25$

Exercise:

Problem: $y = -x^2 + 2x + 5$

Solution:

ⓐ $x = 1$ ⓑ $(1, 6)$

Exercise:

Problem: $y = -2x^2 - 8x - 3$

Find the Intercepts of a Parabola

In the following exercises, find the *x*- and *y*-intercepts.

Exercise:

Problem: $y = x^2 + 7x + 6$

Solution:

y: $(0, 6)$; x: $(-1, 0), (-6, 0)$

Exercise:

Problem: $y = x^2 + 10x - 11$

Exercise:

Problem: $y = -x^2 + 8x - 19$

Solution:

y: $(0, 19)$; x: none

Exercise:

Problem: $y = x^2 + 6x + 13$

Exercise:

Problem: $y = 4x^2 - 20x + 25$

Solution:

y: $(0, 25)$; x: $(\frac{5}{2}, 0)$

Exercise:

Problem: $y=-x^2-14x-49$

Graph Quadratic Equations in Two Variables

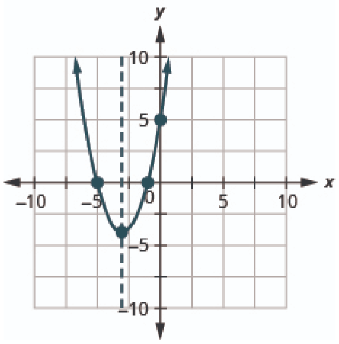
In the following exercises, graph by using intercepts, the vertex, and the axis of symmetry.

Exercise:

Problem: $y=x^2+6x+5$

Solution:

y: $(0,5)$; x: $(-1,0),(-5,0)$;
axis: $x=-3$; vertex: $(-3,-4)$



Exercise:

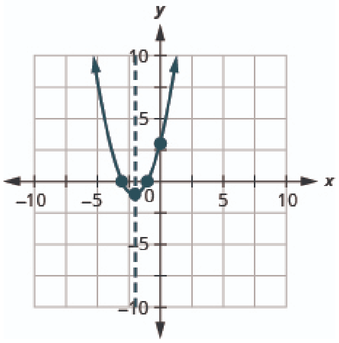
Problem: $y=x^2+4x-12$

Exercise:

Problem: $y=x^2+4x+3$

Solution:

y: $(0,3)$; x: $(-1,0),(-3,0)$;
axis: $x=-2$; vertex: $(-2,-1)$



Exercise:

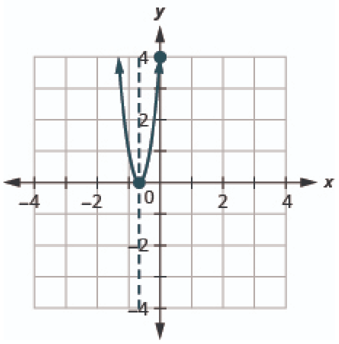
Problem: $y = x^2 - 6x + 8$

Exercise:

Problem: $y = 9x^2 + 12x + 4$

Solution:

y: $(0, 4)$ x: $(-\frac{2}{3}, 0)$;
axis: $x = -\frac{2}{3}$; vertex: $(-\frac{2}{3}, 0)$



Exercise:

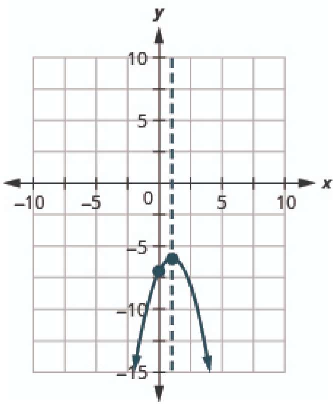
Problem: $y = -x^2 + 8x - 16$

Exercise:

Problem: $y = -x^2 + 2x - 7$

Solution:

y: $(0, -7)$; x: none;
axis: $x = 1$; vertex: $(1, -6)$



Exercise:

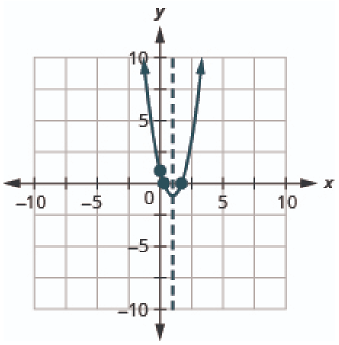
Problem: $y = 5x^2 + 2$

Exercise:

Problem: $y = 2x^2 - 4x + 1$

Solution:

y: $(0, 1)$; x: $(1.7, 0)$, $(0.3, 0)$;
axis: $x = 1$; vertex: $(1, -1)$



Exercise:

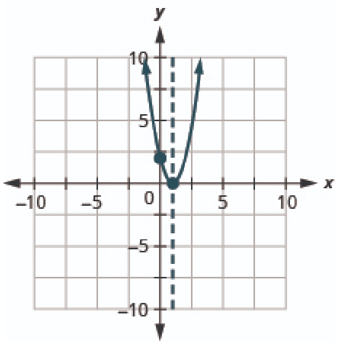
Problem: $y = 3x^2 - 6x - 1$

Exercise:

Problem: $y = 2x^2 - 4x + 2$

Solution:

y: $(0, 2)$ x: $(1, 0)$;
axis: $x = 1$; vertex: $(1, 0)$



Exercise:

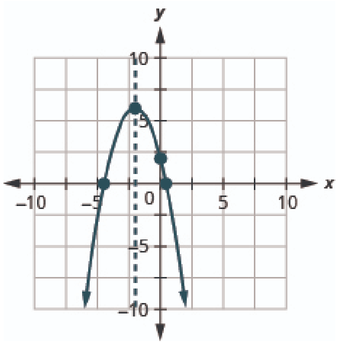
Problem: $y = -4x^2 - 6x - 2$

Exercise:

Problem: $y = -x^2 - 4x + 2$

Solution:

y: $(0, 2)$ x: $(-4.4, 0), (0.4, 0)$;
axis: $x = -2$; vertex: $(-2, 6)$



Exercise:

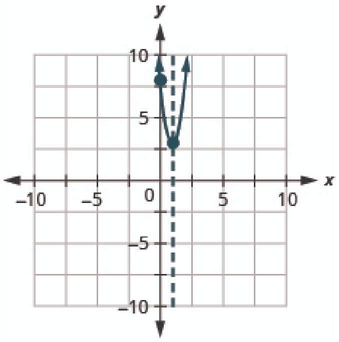
Problem: $y = x^2 + 6x + 8$

Exercise:

Problem: $y = 5x^2 - 10x + 8$

Solution:

y: $(0, 8)$; x: none;
axis: $x = 1$; vertex: $(1, 3)$



Exercise:

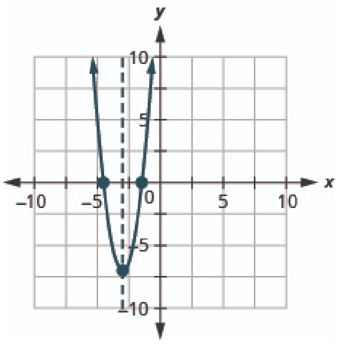
Problem: $y = -16x^2 + 24x - 9$

Exercise:

Problem: $y = 3x^2 + 18x + 20$

Solution:

y: $(0, 20)$ x: $(-4.5, 0), (-1.5, 0)$;
axis: $x = -3$; vertex: $(-3, -7)$



Exercise:

Problem: $y = -2x^2 + 8x - 10$

Solve Maximum and Minimum Applications

In the following exercises, find the maximum or minimum value.

Exercise:

Problem: $y = 2x^2 + x - 1$

Solution:

The minimum value is $-\frac{9}{8}$ when $x = -\frac{1}{4}$.

Exercise:

Problem: $y = -4x^2 + 12x - 5$

Exercise:

Problem: $y = x^2 - 6x + 15$

Solution:

The minimum value is 6 when $x = 3$.

Exercise:

Problem: $y = -x^2 + 4x - 5$

Exercise:

Problem: $y = -9x^2 + 16$

Solution:

The maximum value is 16 when $x = 0$.

Exercise:

Problem: $y = 4x^2 - 49$

In the following exercises, solve. Round answers to the nearest tenth.

Exercise:

Problem:

An arrow is shot vertically upward from a platform 45 feet high at a rate of 168 ft/sec. Use the quadratic equation $h = -16t^2 + 168t + 45$ to find how long it will take the arrow to reach its maximum height, and then find the maximum height.

Solution:

In 5.3 sec the arrow will reach maximum height of 486 ft.

Exercise:

Problem:

A stone is thrown vertically upward from a platform that is 20 feet high at a rate of 160 ft/sec. Use the quadratic equation $h = -16t^2 + 160t + 20$ to find how long it will take the stone to reach its maximum height, and then find the maximum height.

Exercise:

Problem:

A computer store owner estimates that by charging x dollars each for a certain computer, he can sell $40 - x$ computers each week. The quadratic equation $R = -x^2 + 40x$ is used to find the revenue, R, received when the selling price of a computer is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

Solution:

20 computers will give the maximum of $400 in receipts.

Exercise:

Problem:

A retailer who sells backpacks estimates that, by selling them for x dollars each, he will be able to sell $100 - x$ backpacks a month. The quadratic equation $R = -x^2 + 100x$ is used to find the R received when the selling price of a backpack is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

Exercise:

Problem:

A rancher is going to fence three sides of a corral next to a river. He needs to maximize the corral area using 240 feet of fencing. The quadratic equation $A = x\left(240 - 2x\right)$ gives the area of the corral, A, for the length, x, of the corral along the river. Find the length of the corral along the river that will give the maximum area, and then find the maximum area of the corral.

Solution:

The length of the side along the river of the corral is 120 feet and the maximum area is 7,200 sq ft.

Exercise:

Problem:

A veterinarian is enclosing a rectangular outdoor running area against his building for the dogs he cares for. He needs to maximize the area using 100 feet of fencing. The quadratic equation $A = x\left(100 - 2x\right)$ gives the area, A, of the dog run for the length, x, of the building that will border the dog run. Find the length of the building that should border the dog run to give the maximum area, and then find the maximum area of the dog run.

Everyday Math

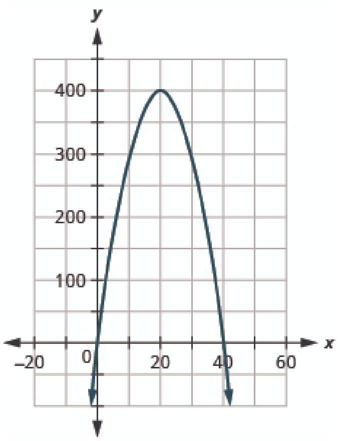
Exercise:

Problem:

In the previous set of exercises, you worked with the quadratic equation $R = -x^2 + 40x$ that modeled the revenue received from selling computers at a price of x dollars. You found the selling price that would give the maximum revenue and calculated the maximum revenue. Now you will look at more characteristics of this model.
ⓐ Graph the equation $R = -x^2 + 40x$. ⓑ Find the values of the *x*-intercepts.

Solution:

ⓐ



ⓑ $(0,0),(40,0)$

Exercise:

Problem:

In the previous set of exercises, you worked with the quadratic equation $R=-x^2+100x$ that modeled the revenue received from selling backpacks at a price of x dollars. You found the selling price that would give the maximum revenue and calculated the maximum revenue. Now you will look at more characteristics of this model.
ⓐ Graph the equation $R=-x^2+100x$. ⓑ Find the values of the *x*-intercepts.

Writing Exercises

Exercise:

Problem:

For the revenue model in [link] and [link], explain what the *x*-intercepts mean to the computer store owner.

Solution:

Answers will vary.

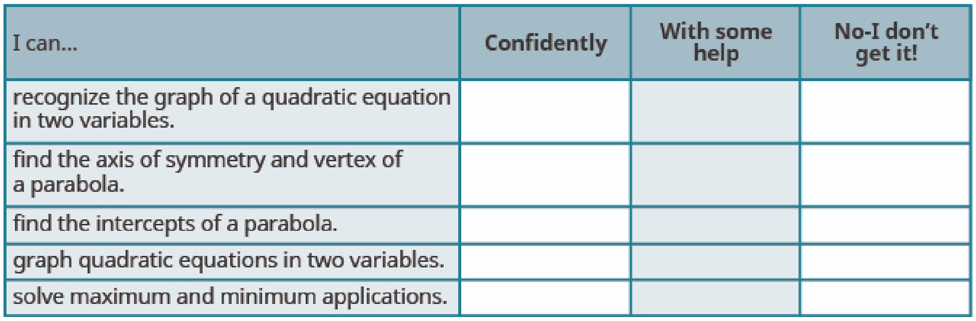
Exercise:

Problem:

For the revenue model in [link] and [link], explain what the *x*-intercepts mean to the backpack retailer.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



I can...	Confidently	With some help	No-I don't get it!
recognize the graph of a quadratic equation in two variables.			
find the axis of symmetry and vertex of a parabola.			
find the intercepts of a parabola.			
graph quadratic equations in two variables.			
solve maximum and minimum applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Chapter 10 Review Exercises

10.1 Solve Quadratic Equations Using the Square Root Property

In the following exercises, solve using the Square Root Property.

Exercise:

Problem: $x^2 = 100$

Solution:

$x = \pm 10$

Exercise:

Problem: $y^2 = 144$

Exercise:

Problem: $m^2 - 40 = 0$

Solution:

$m = \pm 2\sqrt{10}$

Exercise:

Problem: $n^2 - 80 = 0$

Exercise:

Problem: $4a^2 = 100$

Solution:

$a = \pm 5$

Exercise:

Problem: $2b^2 = 72$

Exercise:

Problem: $r^2 + 32 = 0$

Solution:

no solution

Exercise:

Problem: $t^2 + 18 = 0$

Exercise:

Problem: $\frac{4}{3}v^2 + 4 = 28$

Solution:

$v = \pm\, 3\sqrt{2}$

Exercise:

Problem: $\frac{2}{3}w^2 - 20 = 30$

Exercise:

Problem: $5c^2 + 3 = 19$

Solution:

$c = \pm\, \frac{4\sqrt{5}}{5}$

Exercise:

Problem: $3d^2 - 6 = 43$

In the following exercises, solve using the Square Root Property.

Exercise:

Problem: $(p - 5)^2 + 3 = 19$

Solution:

$p = 1, 9$

Exercise:

Problem: $(q + 4)^2 = 9$

Exercise:

Problem: $(u + 1)^2 = 45$

Solution:

$u = -1 \pm 3\sqrt{5}$

Exercise:

Problem: $(z - 5)^2 = 50$

Exercise:

Problem: $\left(x-\frac{1}{4}\right)^2=\frac{3}{16}$

Solution:

$x=\frac{1}{4}\pm\frac{\sqrt{3}}{4}$

Exercise:

Problem: $\left(y-\frac{2}{3}\right)^2=\frac{2}{9}$

Exercise:

Problem: $(m-7)^2+6=30$

Solution:

$m=7\pm 2\sqrt{6}$

Exercise:

Problem: $(n-4)^2-50=150$

Exercise:

Problem: $(5c+3)^2=-20$

Solution:

no solution

Exercise:

Problem: $(4c-1)^2=-18$

Exercise:

Problem: $m^2-6m+9=48$

Solution:

$m=3\pm 4\sqrt{3}$

Exercise:

Problem: $n^2+10n+25=12$

Exercise:

Problem: $64a^2+48a+9=81$

Solution:

$a = -\frac{3}{2}, \frac{3}{4}$

Exercise:

Problem: $4b^2 - 28b + 49 = 25$

10.2 Solve Quadratic Equations Using Completing the Square

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

Exercise:

Problem: $x^2 + 22x$

Solution:

$(x + 11)^2$

Exercise:

Problem: $y^2 + 6y$

Exercise:

Problem: $m^2 - 8m$

Solution:

$(m - 4)^2$

Exercise:

Problem: $n^2 - 10n$

Exercise:

Problem: $a^2 - 3a$

Solution:

$\left(a - \frac{3}{2}\right)^2$

Exercise:

Problem: $b^2 + 13b$

Exercise:

Problem: $p^2 + \frac{4}{5}p$

Solution:

$\left(p + \frac{2}{5}\right)^2$

Exercise:

Problem: $q^2 - \frac{1}{3}q$

In the following exercises, solve by completing the square.

Exercise:

Problem: $c^2 + 20c = 21$

Solution:

$c = 1, -21$

Exercise:

Problem: $d^2 + 14d = -13$

Exercise:

Problem: $x^2 - 4x = 32$

Solution:

$x = -4, 8$

Exercise:

Problem: $y^2 - 16y = 36$

Exercise:

Problem: $r^2 + 6r = -100$

Solution:

no solution

Exercise:

Problem: $t^2 - 12t = -40$

Exercise:

Problem: $v^2 - 14v = -31$

Solution:

$v = 7 \pm 3\sqrt{2}$

Exercise:

Problem: $w^2 - 20w = 100$

Exercise:

Problem: $m^2 + 10m - 4 = -13$

Solution:

$m = -9, -1$

Exercise:

Problem: $n^2 - 6n + 11 = 34$

Exercise:

Problem: $a^2 = 3a + 8$

Solution:

$a = \frac{3}{2} \pm \frac{\sqrt{41}}{2}$

Exercise:

Problem: $b^2 = 11b - 5$

Exercise:

Problem: $(u + 8)\,(u + 4) = 14$

Solution:

$u = -6 \pm 2\sqrt{2}$

Exercise:

Problem: $(z - 10)\,(z + 2) = 28$

Exercise:

Problem: $3p^2 - 18p + 15 = 15$

Solution:

$p = 0, 6$

Exercise:

Problem: $5q^2+70q+20=0$

Exercise:

Problem: $4y^2-6y=4$

Solution:

$y=-\frac{1}{2},2$

Exercise:

Problem: $2x^2+2x=4$

Exercise:

Problem: $3c^2+2c=9$

Solution:

$c=-\frac{1}{3}\pm\frac{2\sqrt{7}}{3}$

Exercise:

Problem: $4d^2-2d=8$

10.3 Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

Exercise:

Problem: $4x^2-5x+1=0$

Solution:

$x=\frac{1}{4},1$

Exercise:

Problem: $7y^2+4y-3=0$

Exercise:

Problem: $r^2-r-42=0$

Solution:

$r=-6,7$

Exercise:

Problem: $t^2 + 13t + 22 = 0$

Exercise:

Problem: $4v^2 + v - 5 = 0$

Solution:

$v = -\frac{5}{4}, 1$

Exercise:

Problem: $2w^2 + 9w + 2 = 0$

Exercise:

Problem: $3m^2 + 8m + 2 = 0$

Solution:

$m = \frac{-4 \pm \sqrt{10}}{3}$

Exercise:

Problem: $5n^2 + 2n - 1 = 0$

Exercise:

Problem: $6a^2 - 5a + 2 = 0$

Solution:

no real solution

Exercise:

Problem: $4b^2 - b + 8 = 0$

Exercise:

Problem: $u\left(u - 10\right) + 3 = 0$

Solution:

$u = 5 \pm \sqrt{22}$

Exercise:

Problem: $5z(z - 2) = 3$

Exercise:

Problem: $\frac{1}{8}p^2 - \frac{1}{5}p = -\frac{1}{20}$

Solution:

$p = \frac{4 \pm \sqrt{6}}{5}$

Exercise:

Problem: $\frac{2}{5}q^2 + \frac{3}{10}q = \frac{1}{10}$

Exercise:

Problem: $4c^2 + 4c + 1 = 0$

Solution:

$c = -\frac{1}{2}$

Exercise:

Problem: $9d^2 - 12d = -4$

In the following exercises, determine the number of solutions to each quadratic equation.

Exercise:

Problem:

ⓐ $9x^2 - 6x + 1 = 0$
ⓑ $3y^2 - 8y + 1 = 0$
ⓒ $7m^2 + 12m + 4 = 0$
ⓓ $5n^2 - n + 1 = 0$

Solution:

ⓐ 1 ⓑ 2 ⓒ 2 ⓓ none

Exercise:

Problem:

ⓐ $5x^2 - 7x - 8 = 0$
ⓑ $7x^2 - 10x + 5 = 0$
ⓒ $25x^2 - 90x + 81 = 0$
ⓓ $15x^2 - 8x + 4 = 0$

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation.

Exercise:

Problem:

ⓐ $16r^2 - 8r + 1 = 0$
ⓑ $5t^2 - 8t + 3 = 9$
ⓒ $3(c + 2)^2 = 15$

Solution:

ⓐ factor ⓑ Quadratic Formula ⓒ square root

Exercise:

Problem:

ⓐ $4d^2 + 10d - 5 = 21$
ⓑ $25x^2 - 60x + 36 = 0$
ⓒ $6(5v - 7)^2 = 150$

10.4 Solve Applications Modeled by Quadratic Equations

In the following exercises, solve by using methods of factoring, the square root principle, or the quadratic formula.

Exercise:

Problem: Find two consecutive odd numbers whose product is 323.

Solution:

Two consecutive odd numbers whose product is 323 are 17 and 19, and -17 and -19.

Exercise:

Problem: Find two consecutive even numbers whose product is 624.

Exercise:

Problem:

A triangular banner has an area of 351 square centimeters. The length of the base is two centimeters longer than four times the height. Find the height and length of the base.

Solution:

The height of the banner is 13 cm and the length of the side is 54 cm.

Exercise:

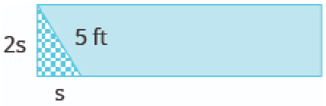
Problem:

Julius built a triangular display case for his coin collection. The height of the display case is six inches less than twice the width of the base. The area of the of the back of the case is 70 square inches. Find the height and width of the case.

Exercise:

Problem:

A tile mosaic in the shape of a right triangle is used as the corner of a rectangular pathway. The hypotenuse of the mosaic is 5 feet. One side of the mosaic is twice as long as the other side. What are the lengths of the sides? Round to the nearest tenth.



Solution:

The lengths of the sides of the mosaic are 2.2 and 4.4 feet.

Exercise:

Problem:

A rectangular piece of plywood has a diagonal which measures two feet more than the width. The length of the plywood is twice the width. What is the length of the plywood's diagonal? Round to the nearest tenth.

Exercise:

Problem:

The front walk from the street to Pam's house has an area of 250 square feet. Its length is two less than four times its width. Find the length and width of the sidewalk. Round to the nearest tenth.

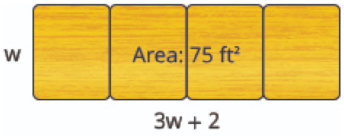
Solution:

The width of the front walk is 8.1 feet and its length is 30.8 feet.

Exercise:

Problem:

For Sophia's graduation party, several tables of the same width will be arranged end to end to give a serving table with a total area of 75 square feet. The total length of the tables will be two more than three times the width. Find the length and width of the serving table so Sophia can purchase the correct size tablecloth. Round answer to the nearest tenth.



Exercise:

Problem:

A ball is thrown vertically in the air with a velocity of 160 ft/sec. Use the formula $h = -16t^2 + v_0t$ to determine when the ball will be 384 feet from the ground. Round to the nearest tenth.

Solution:

The ball will reach 384 feet on its way up in 4 seconds and on the way down in 6 seconds.

Exercise:

Problem:

A bullet is fired straight up from the ground at a velocity of 320 ft/sec. Use the formula $h=-16t^2+v_0t$ to determine when the bullet will reach 800 feet. Round to the nearest tenth.

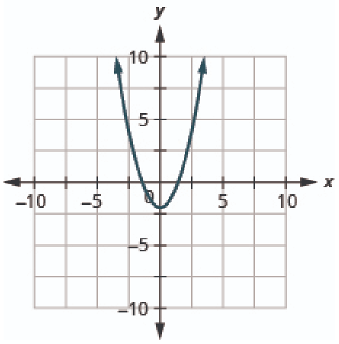
10.5 Graphing Quadratic Equations in Two Variables

In the following exercises, graph by plotting point.

Exercise:

Problem: Graph $y=x^2-2$

Solution:



Exercise:

Problem: Graph $y=-x^2+3$

In the following exercises, determine if the following parabolas open up or down.

Exercise:

Problem: $y=-3x^2+3x-1$

Solution:

down

Exercise:

Problem: $y=5x^2+6x+3$

Exercise:

Problem: $y=x^2+8x-1$

Solution:

up

Exercise:

Problem: $y = -4x^2 - 7x + 1$

In the following exercises, find ⓐ the axis of symmetry and ⓑ the vertex.

Exercise:

Problem: $y = -x^2 + 6x + 8$

Solution:

ⓐ $x = 3$ ⓑ $(3, 17)$

Exercise:

Problem: $y = 2x^2 - 8x + 1$

In the following exercises, find the *x*- and *y*-intercepts.

Exercise:

Problem: $y = x^2 - 4x + 5$

Solution:

y: $(0, 5)$; x: $(5, 0), (-1, 0)$

Exercise:

Problem: $y = x^2 - 8x + 15$

Exercise:

Problem: $y = x^2 - 4x + 10$

Solution:

y: $(0, 10)$; x: none

Exercise:

Problem: $y = -5x^2 - 30x - 46$

Exercise:

Problem: $y = 16x^2 - 8x + 1$

Solution:

y: $(0, 1)$; x: $\left(\frac{1}{4}, 0\right)$

Exercise:

Problem: $y = x^2 + 16x + 64$

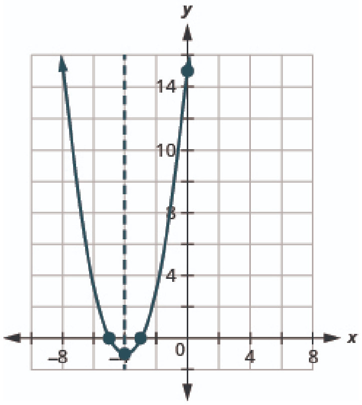
In the following exercises, graph by using intercepts, the vertex, and the axis of symmetry.

Exercise:

Problem: $y = x^2 + 8x + 15$

Solution:

y: $(0, 15)$; x: $(-3, 0), (-5, 0)$;
axis: $x = -4$; vertex: $(-4, -1)$



Exercise:

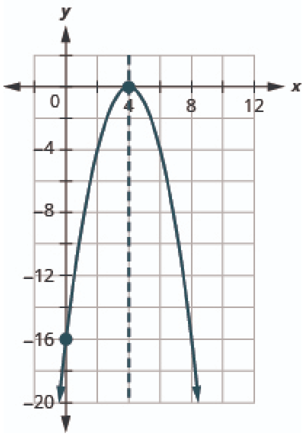
Problem: $y = x^2 - 2x - 3$

Exercise:

Problem: $y = -x^2 + 8x - 16$

Solution:

y: $(0, -16)$; x: $(4, 0)$;
axis: $x = 4$; vertex: $(4, 0)$



Exercise:

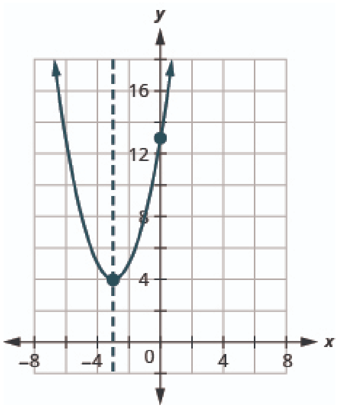
Problem: $y = 4x^2 - 4x + 1$

Exercise:

Problem: $y = x^2 + 6x + 13$

Solution:

y: $(0, 13)$; x: none;
axis: $x = -3$; vertex: $(-3, 4)$



Exercise:

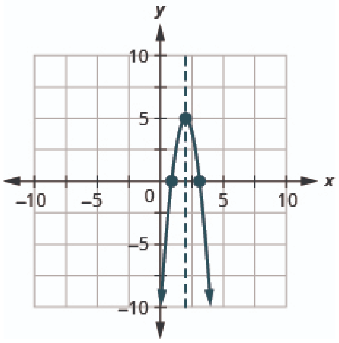
Problem: $y = -2x^2 - 8x - 12$

Exercise:

Problem: $y = -4x^2 + 16x - 11$

Solution:

y: $(0, -11)$ x: $(3.1, 0), (0.9, 0)$;
axis: $x = 2;$ vertex: $(2, 5)$



Exercise:

Problem: $y = x^2 + 8x + 10$

In the following exercises, find the minimum or maximum value.

Exercise:

Problem: $y = 7x^2 + 14x + 6$

Solution:

The minimum value is -1 when $x = -1$.

Exercise:

Problem: $y = -3x^2 + 12x - 10$

In the following exercises, solve. Rounding answers to the nearest tenth.

Exercise:

Problem:

A ball is thrown upward from the ground with an initial velocity of 112 ft/sec. Use the quadratic equation $h = -16t^2 + 112t$ to find how long it will take the ball to reach maximum height, and then find the maximum height.

Solution:

In 3.5 seconds the ball is at its maximum height of 196 feet.

Exercise:

Problem:

A daycare facility is enclosing a rectangular area along the side of their building for the children to play outdoors. They need to maximize the area using 180 feet of fencing on three sides of the yard. The quadratic equation $A = -2x^2 + 180x$ gives the area, A, of the yard for the length, x, of the building that will border the yard. Find the length of the building that should border the yard to maximize the area, and then find the maximum area.

Practice Test

Exercise:

Problem: Use the Square Root Property to solve the quadratic equation: $3(w+5)^2 = 27$.

Solution:

$w = -2, -8$

Exercise:

Problem: Use Completing the Square to solve the quadratic equation: $a^2 - 8a + 7 = 23$.

Exercise:

Problem: Use the Quadratic Formula to solve the quadratic equation: $2m^2 - 5m + 3 = 0$.

Solution:

$m = 1, \frac{3}{2}$

Solve the following quadratic equations. Use any method.

Exercise:

Problem: $8v^2 + 3 = 35$

Exercise:

Problem: $3n^2 + 8n + 3 = 0$

Solution:

$n = \frac{-4 \pm \sqrt{7}}{3}$

Exercise:

Problem: $2b^2 + 6b - 8 = 0$

Exercise:

Problem: $x\left(x+3\right) + 12 = 0$

Solution:

no real solution

Exercise:

Problem: $\frac{4}{3}y^2 - 4y + 3 = 0$

Use the discriminant to determine the number of solutions of each quadratic equation.

Exercise:

Problem: $6p^2 - 13p + 7 = 0$

Solution:

2

Exercise:

Problem: $3q^2 - 10q + 12 = 0$

Solve by factoring, the Square Root Property, or the Quadratic Formula.

Exercise:

Problem: Find two consecutive even numbers whose product is 360.

Solution:

Two consecutive even number are -20 and -18 and 18 and 20.

Exercise:

Problem:

The length of a diagonal of a rectangle is three more than the width. The length of the rectangle is three times the width. Find the length of the diagonal. (Round to the nearest tenth.)

For each parabola, find ⓐ which ways it opens, ⓑ the axis of symmetry, ⓒ the vertex, ⓓ the *x*- and *y*-intercepts, and ⓔ the maximum or minimum value.

Exercise:

Problem: $y = 3x^2 + 6x + 8$

Solution:

ⓐ up ⓑ $x = -1$ ⓒ $(-1, 5)$ ⓓ y: $(0, 8)$; x: none ⓔ minimum value of 5 when $x = -1$

Exercise:

Problem: $y = x^2 - 4$

Exercise:

Problem: $y=x^2+10x+24$

Solution:

ⓐ up ⓑ $x=-5$ ⓒ $(-5,-1)$ ⓓ y; $(0,24)$; x: $(-6,0)$, $(-4,0)$ ⓔ minimum value of -5 when $x=-1$

Exercise:

Problem: $y=-3x^2+12x-8$

Exercise:

Problem: $y=-x^2-8x+16$

Solution:

ⓐ down ⓑ $x=-4$
ⓒ $(-4,32)$ ⓓ y; $(0,16)$; x: $(-9.7,0)$, $(1.7,0)$
ⓔ maximum value of 32 when $x=-4$

Graph the following parabolas by using intercepts, the vertex, and the axis of symmetry.

Exercise:

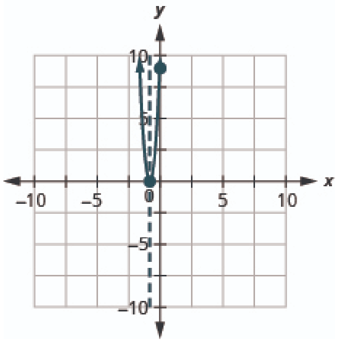
Problem: $y=2x^2+6x+2$

Exercise:

Problem: $y=16x^2+24x+9$

Solution:

y: $(0,9)$; x: $\left(-\frac{3}{4},0\right)$;
axis: $x=-\frac{3}{4}$; vertex: $\left(-\frac{3}{4},0\right)$



Solve.

Exercise:

Problem:

A water balloon is launched upward at the rate of 86 ft/sec. Using the formula $h = -16t^2 + 86t$, find how long it will take the balloon to reach the maximum height and then find the maximum height. Round to the nearest tenth.

Glossary

axis of symmetry

The axis of symmetry is the vertical line passing through the middle of the parabola $y = ax^2 + bx + c$.

parabola

The graph of a quadratic equation in two variables is a parabola.

quadratic equation in two variables

A quadratic equation in two variables, where $a, b, \text{ and } c$ are real numbers and $a \neq 0$ is an equation of the form $y = ax^2 + bx + c$.

vertex

The point on the parabola that is on the axis of symmetry is called the *vertex* of the parabola; it is the lowest or highest point on the parabola, depending on whether the parabola opens upwards or downwards.

x-intercepts of a parabola

The *x*-intercepts are the points on the parabola where $y = 0$.

y-intercept of a parabola

The *y*-intercept is the point on the parabola where $x = 0$.

Introduction class="introduction"

Hydroponic systems allow botanists to grow crops without land. (credit: “Izhamwong”/Wikimedia Commons)

As the world population continues to grow, food supplies are becoming less able to meet the increasing demand. At the same time, available resources of fertile soil for growing plants is dwindling. One possible solution—grow plants without soil. Botanists around the world are expanding the potential of hydroponics, which is the process of growing plants without soil. To provide the plants with the nutrients they need, the botanists keep careful growth records. Some growth is described by the types of functions you will explore in this chapter—exponential and logarithmic. You will evaluate and graph these functions, and solve equations using them.

Evaluate and Graph Exponential Functions

By the end of this section, you will be able to:

- Graph exponential functions
- Use exponential models in applications

Note:
Before you get started, take this readiness quiz.

1. Simplify: $\left(\frac{x^3}{x^2}\right)$.
   If you missed this problem, review [link].
2. Evaluate: ⓐ 2^0 ⓑ $\left(\frac{1}{3}\right)^0$.
   If you missed this problem, review [link].
3. Evaluate: ⓐ 2^{-1} ⓑ $\left(\frac{1}{3}\right)^{-1}$.
   If you missed this problem, review [link].

Graph Exponential Functions

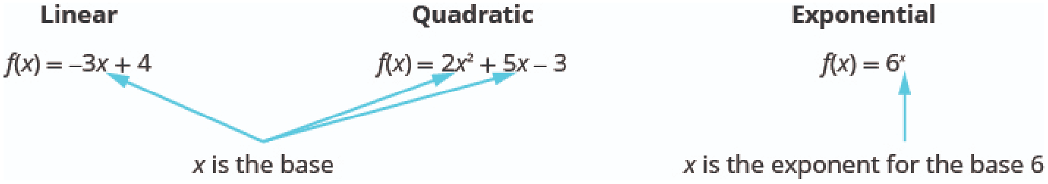
The functions we have studied so far do not give us a model for many naturally occurring phenomena. From the growth of populations and the spread of viruses to radioactive decay and compounding interest, the models are very different from what we have studied so far. These models involve exponential functions.

An **exponential function** is a function of the form $f\left(x\right)=b^x$ where $b>0$ and $b\ne 1$.

Note:
Exponential Function
An exponential function, where $b>0$ and $b\ne 1$, is a function of the form
Equation:

$$f\left(x\right)=b^x$$

Notice that in this function, the variable is the exponent. In our functions so far, the variables were the base.



Our definition says $b \neq 1$. If we let $b = 1$, then $f(x) = b^x$ becomes $f(x) = 1^x$. Since $1^x = 1$ for all real numbers, $f(x) = 1$. This is the constant function.

Our definition also says $b > 0$. If we let a base be negative, say -4, then $f(x) = (-4)^x$ is not a real number when $x = \frac{1}{2}$.

Equation:

$$\begin{aligned} f(x) &= (-4)^x \\ f\left(\tfrac{1}{2}\right) &= (-4)^{\frac{1}{2}} \\ f\left(\tfrac{1}{2}\right) &= \sqrt{-4} \quad \text{not a real number} \end{aligned}$$

In fact, $f(x) = (-4)^x$ would not be a real number any time x is a fraction with an even denominator. So our definition requires $b > 0$.

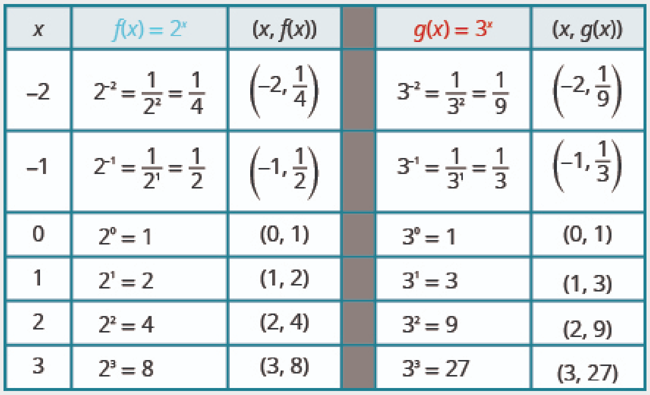
By graphing a few exponential functions, we will be able to see their unique properties.

Example:
Exercise:

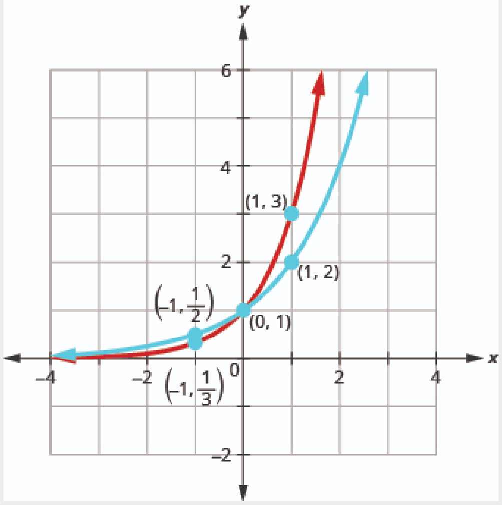
Problem: On the same coordinate system graph $f(x) = 2^x$ and $g(x) = 3^x$.

Solution:

We will use point plotting to graph the functions.



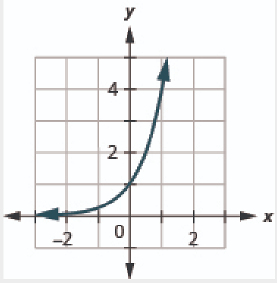
x	$f(x) = 2^x$	$(x, f(x))$		$g(x) = 3^x$	$(x, g(x))$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$		$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$		$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	$2^0 = 1$	$(0, 1)$		$3^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$		$3^1 = 3$	$(1, 3)$
2	$2^2 = 4$	$(2, 4)$		$3^2 = 9$	$(2, 9)$
3	$2^3 = 8$	$(3, 8)$		$3^3 = 27$	$(3, 27)$



Note:
Exercise:

Problem: Graph: $f(x) = 4^x$.

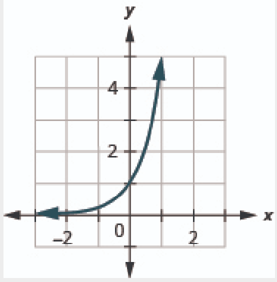
Solution:



Note:
Exercise:

Problem: Graph: $g(x) = 5^x$.

Solution:



If we look at the graphs from the previous Example and Try Its, we can identify some of the properties of exponential functions.

The graphs of $f\left(x\right)=2^x$ and $g\left(x\right)=3^x$, as well as the graphs of $f\left(x\right)=4^x$ and $g\left(x\right)=5^x$, all have the same basic shape. This is the shape we expect from an exponential function where $b>1$..

We notice, that for each function, the graph contains the point $(0,1)$. This make sense because $b^0=1$ for any $b>1$.

The graph of each function, $f\left(x\right)=b^x$ also contains the point $(1,b)$. The graph of $f\left(x\right)=2^x$ contained $(1,2)$ and the graph of $g\left(x\right)=3^x$ contained $(1,3)$. This makes sense as $b^1=b$.

Notice too, the graph of each function $f\left(x\right)=b^x$ also contains the point $\left(-1,\frac{1}{b}\right)$. The graph of $f\left(x\right)=2^x$ contained $\left(-1,\frac{1}{2}\right)$ and the graph of $g\left(x\right)=3^x$ contained $\left(-1,\frac{1}{3}\right)$. This makes sense as $b^{-1}=\frac{1}{b}$.

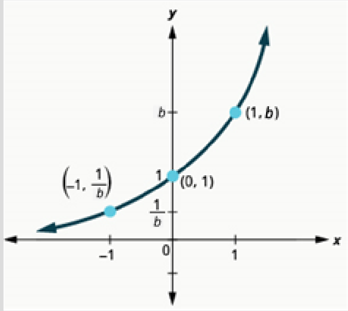
What is the domain for each function? From the graphs we can see that the domain is the set of all real numbers. There is no restriction on the domain. We write the domain in interval notation as $(-\infty,\infty)$.

Look at each graph. What is the range of the function? The graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0,\infty)$.

Whenever a graph of a function approaches a line but never touches it, we call that line an **asymptote**. For the exponential functions we are looking at, the graph approaches the x-axis very closely but will never cross it, we call the line $y=0$, the x-axis, a horizontal asymptote.

Note:
Properties of the Graph of $f\left(x\right)=b^x$ when $b>1$

Domain	$(-\infty,\infty)$
Range	$(0,\infty)$

x-intercept	None
y-intercept	$(0, 1)$
Contains	$(1, b),\ \left(-1, \frac{1}{b}\right)$
Asymptote	x-axis, the line $y = 0$
	

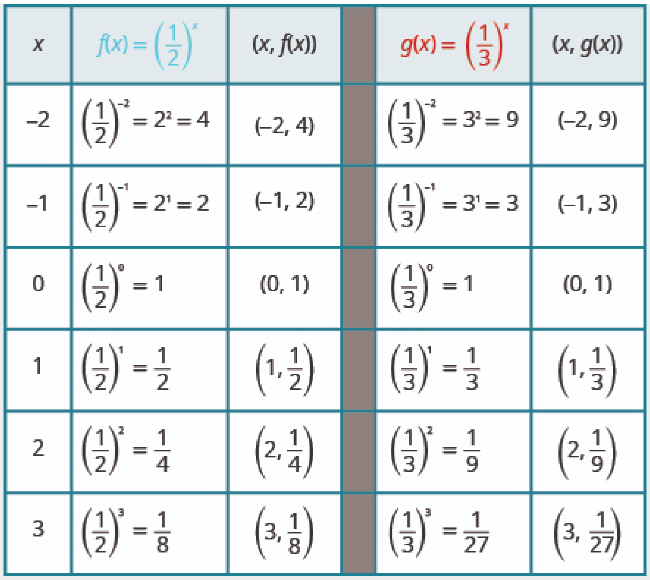
Our definition of an exponential function $f(x) = b^x$ says $b > 0$, but the examples and discussion so far has been about functions where $b > 1$. What happens when $0 < b < 1$? The next example will explore this possibility.

Example:
Exercise:

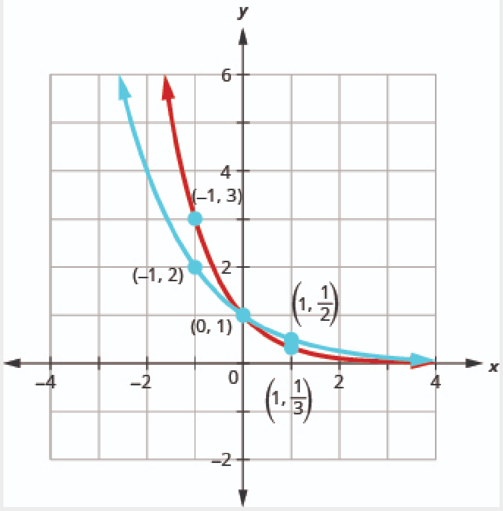
Problem: On the same coordinate system, graph $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$.

Solution:

We will use point plotting to graph the functions.



x	$f(x) = \left(\frac{1}{2}\right)^x$	$(x, f(x))$	$g(x) = \left(\frac{1}{3}\right)^x$	$(x, g(x))$
-2	$\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$	$(-2, 4)$	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	$(-2, 9)$
-1	$\left(\frac{1}{2}\right)^{-1} = 2^1 = 2$	$(-1, 2)$	$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	$(-1, 3)$
0	$\left(\frac{1}{2}\right)^0 = 1$	$(0, 1)$	$\left(\frac{1}{3}\right)^0 = 1$	$(0, 1)$
1	$\left(\frac{1}{2}\right)^1 = \frac{1}{2}$	$\left(1, \frac{1}{2}\right)$	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(1, \frac{1}{3}\right)$
2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$	$\left(2, \frac{1}{4}\right)$	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(2, \frac{1}{9}\right)$
3	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$	$\left(3, \frac{1}{8}\right)$	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$\left(3, \frac{1}{27}\right)$

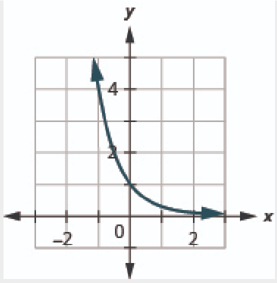


Note:

Exercise:

Problem: Graph: $f(x) = \left(\frac{1}{4}\right)^x$.

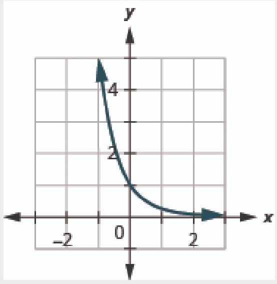
Solution:



Note:
Exercise:

Problem: Graph: $g\left(x\right)={\left(\frac{1}{5}\right)}^{x}$.

Solution:



Now let’s look at the graphs from the previous Example and Try Its so we can now identify some of the properties of exponential functions where $0<b<1$.

The graphs of $f\left(x\right)={\left(\frac{1}{2}\right)}^{x}$ and $g\left(x\right)={\left(\frac{1}{3}\right)}^{x}$ as well as the graphs of $f\left(x\right)={\left(\frac{1}{4}\right)}^{x}$ and $g\left(x\right)={\left(\frac{1}{5}\right)}^{x}$ all have the same basic shape. While this is the shape we expect from an exponential function where $0<b<1$, the graphs go down from left to right while the previous graphs, when $b>1$, went from up from left to right.

We notice that for each function, the graph still contains the point (0, 1). This make sense because $b^{0}=1$ for any $b>0$.

As before, the graph of each function, $f\left(x\right)=b^{x}$, also contains the point $\left(1,b\right)$. The graph of $f\left(x\right)={\left(\frac{1}{2}\right)}^{x}$ contained $\left(1,\frac{1}{2}\right)$ and the graph of $g\left(x\right)={\left(\frac{1}{3}\right)}^{x}$ contained $\left(1,\frac{1}{3}\right)$. This makes sense as $b^{1}=b$.

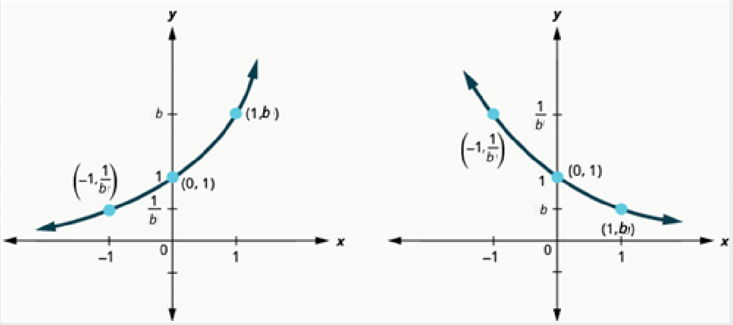
Notice too that the graph of each function, $f(x) = b^x$, also contains the point $\left(-1, \frac{1}{b}\right)$. The graph of $f(x) = \left(\frac{1}{2}\right)^x$ contained $(-1, 2)$ and the graph of $g(x) = \left(\frac{1}{3}\right)^x$ contained $(-1, 3)$. This makes sense as $b^{-1} = \frac{1}{b}$ for any $b > 0$.

What is the domain and range for each function? From the graphs we can see that the domain is the set of all real numbers and we write the domain in interval notation as $(-\infty, \infty)$. Again, the graph never hits the x-axis. Thus, the range is the set of all positive numbers. We write the range in interval notation as $(0, \infty)$.

We will summarize these properties in the chart below. Which also include when $b > 1$.

Note:
Properties of the Graph of $f(x) = a^x$

when $b > 1$		**when** $0 < b < 1$	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
x-intercept	none	x-intercept	none
y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
Contains	$(1, b)$, $\left(-1, \frac{1}{b}\right)$	Contains	$(1, b)$, $\left(-1, \frac{1}{b}\right)$
Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing



It is important for us to notice that both of these graphs are one-to-one, as they both pass the horizontal line test. This means the exponential function will have an inverse. We will look at this later.

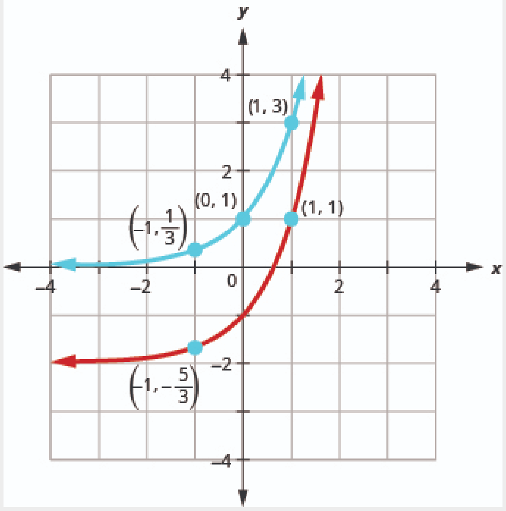
Example:
Exercise:

Problem: On the same coordinate system graph $f\left(x\right)=3^x$ and $g\left(x\right)=3^x-2$.

Solution:

We will use point plotting to graph the functions.

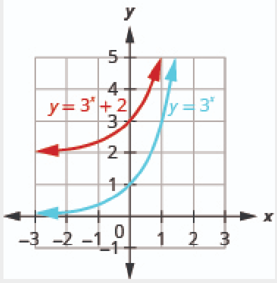
x	$f(x)=3^x$	$(x, g(x))$		$g(x)=3^x-2$	$(x, g(x))$
-2	$3^{-2}=\frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$		$3^{-2}-2=\frac{1}{9}-2=-\frac{17}{9}$	$\left(-2, -\frac{17}{9}\right)$
-1	$3^{-1}=\frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$		$3^{-1}-2=\frac{1}{3}-2=-\frac{5}{3}$	$\left(-1, -\frac{5}{3}\right)$
0	$3^0=1$	$(0, 1)$		$3^0-2=1-2=-1$	$(0, -1)$
1	$3^1=3$	$(1, 3)$		$3^1-2=3-2=1$	$(1, 1)$
2	$3^2=9$	$(2, 9)$		$3^2-2=9-2=7$	$(2, 8)$



Note:
Exercise:

Problem: On the same coordinate system, graph: $f\left(x\right)=3^x$ and $g\left(x\right)=3^x+2$.

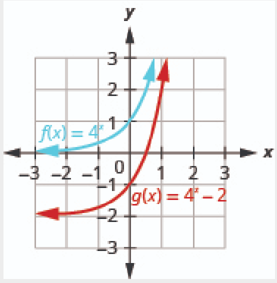
Solution:



Note:
Exercise:

Problem: On the same coordinate system, graph: $f\left(x\right) = 4^{x}$ and $g\left(x\right) = 4^{x} - 2$.

Solution:



All of our exponential functions have had either an integer or a rational number as the base. We will now look at an exponential function with an irrational number as the base.

Before we can look at this exponential function, we need to define the irrational number, e. This number is used as a base in many applications in the sciences and business that are modeled by exponential functions. The number is defined as the value of $\left(1 + \frac{1}{n}\right)^{n}$ as n gets larger and larger. We say, as n approaches infinity, or increases without bound. The table shows the value of $\left(1 + \frac{1}{n}\right)^{n}$ for several values of n.

n	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
5	2.48832
10	2.59374246
100	2.704813829…
1,000	2.716923932…
10,000	2.718145927…
100,000	2.718268237…
1,000,000	2.718280469…
1,000,000,000	2.718281827…

Equation:

$$e \approx 2.718281827$$

The number e is like the number π in that we use a symbol to represent it because its decimal representation never stops or repeats. The irrational number e is called the **natural base**.

Note:

Natural Base e

The number e is defined as the value of $\left(1+\frac{1}{n}\right)^n$, as n increases without bound. We say, as n approaches infinity,

Equation:

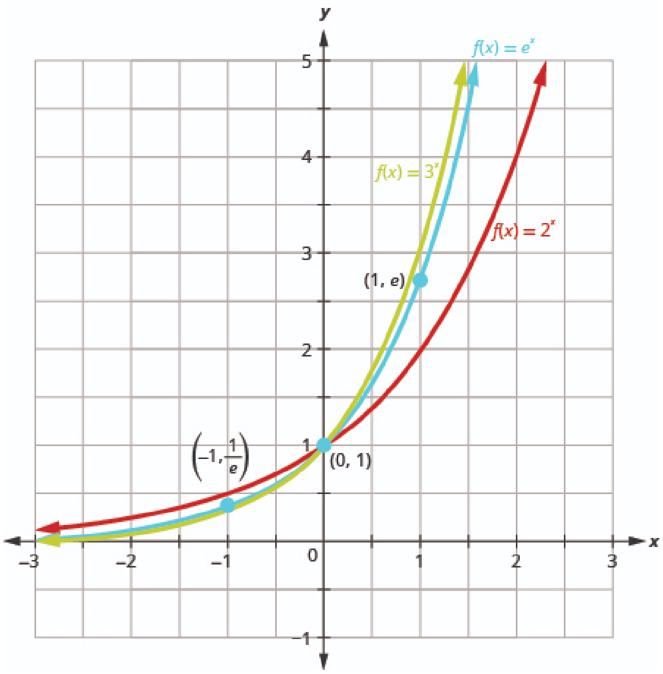
$$e \approx 2.718281827...$$

Note:

The Natural Exponential Function

The exponential function whose base is e, $f\left(x\right)=e^x$ is called the **natural exponential function**.

Let’s graph the function $f\left(x\right)=e^x$ on the same coordinate system as $g\left(x\right)=2^x$ and $h\left(x\right)=3^x$.



Notice that the graph of $f(x) = e^x$ is "between" the graphs of $g(x) = 2^x$ and $h(x) = 3^x$. Does this make sense as $2 < e < 3$?

Use Exponential Models in Applications

Exponential functions model many situations. If you own a bank account, you have experienced the use of an exponential function. There are two formulas that are used to determine the balance in the account when interest is earned. If a principal, P, is invested at an interest rate, r (in decimal form), for t years, the new balance, A, will depend on how often the interest is compounded. If the interest is compounded n times a year we use the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, we use the formula $A = Pe^{rt}$. These are the formulas for **compound interest**.

Note:
Compound Interest
For a principal, P, invested at an interest rate, r (in decimal form), for t years, the new balance, A, is:
Equation:

$$\begin{array}{ll} A = P\left(1 + \frac{r}{n}\right)^{nt} & \text{when compounded } n \text{ times a year.} \\ A = Pe^{rt} & \text{when compounded continuously.} \end{array}$$

As you work with the interest formulas, it is often helpful to identify the values of the variables first and then substitute them into the formula.

Example:
Exercise:

Problem:

A total of $10,000 was invested in a college fund for a new grandchild. If the interest rate is 5%, how much will be in the account in 18 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Solution:

Identify the values of each variable in the formulas. Remember to express the percent as a decimal.

$$\begin{aligned} A &= ? \\ P &= \$10{,}000 \\ r &= 0.05 \\ t &= 18 \text{ years} \end{aligned}$$

ⓐ

For quarterly compounding, $n = 4.$ There are 4 quarters in a year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10{,}000\left(1 + \frac{0.05}{4}\right)^{4 \cdot 18}$$

Compute the amount. Be careful to consider the order of operations as you enter the expression into your calculator.

$$A = \$24{,}459.20$$

ⓑ

For monthly compounding, $n = 12.$ There are 12 months in a year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10{,}000\left(1 + \frac{0.05}{12}\right)^{12 \cdot 18}$$

Compute the amount.

$$A = \$24{,}550.08$$

ⓒ

For compounding continuously,

$$A = Pe^{rt}$$

Substitute the values in the formula.

$$A = 10{,}000e^{0.05 \cdot 18}$$

Compute the amount.

$$A = \$24{,}596.03$$

Note:
Exercise:

Problem:

Angela invested $\$15{,}000$ in a savings account. If the interest rate is 4%, how much will be in the account in 10 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Solution:

ⓐ $\$22{,}332.96$
ⓑ $\$22{,}362.49$ ⓒ $\$22{,}377.37$

Note:
Exercise:

Problem:

Allan invested \$10,000 in a mutual fund. If the interest rate is 5%, how much will be in the account in 15 years by each method of compounding?

ⓐ compound quarterly

ⓑ compound monthly

ⓒ compound continuously

Solution:

ⓐ \$21,071.81 ⓑ \$21,137.04
ⓒ \$21,170.00

Other topics that are modeled by exponential functions involve growth and decay. Both also use the formula $A = Pe^{rt}$ we used for the growth of money. For growth and decay, generally we use A_0, as the original amount instead of calling it P, the principal. We see that **exponential growth** has a positive rate of growth and **exponential decay** has a negative rate of growth.

Note:
Exponential Growth and Decay
For an original amount, A_0, that grows or decays at a rate, r, for a certain time, t, the final amount, A, is:
Equation:

$$A = A_0e^{rt}$$

Exponential growth is typically seen in the growth of populations of humans or animals or bacteria. Our next example looks at the growth of a virus.

Example:
Exercise:

Problem:

Chris is a researcher at the Center for Disease Control and Prevention and he is trying to understand the behavior of a new and dangerous virus. He starts his experiment with 100 of the virus that grows at a rate of 25% per hour. He will check on the virus in 24 hours. How many viruses will he find?

Solution:

Identify the values of each variable in the formulas.	$A = ?$
Be sure to put the percent in decimal form.	$A_0 = 100$
Be sure the units match—the rate is per hour and	$r = 0.25/\text{hour}$
the time is in hours.	$t = 24 \text{ hours}$
Substitute the values in the formula: $A = A_0e^{rt}$.	$A = 100e^{0.25 \cdot 24}$
Compute the amount.	$A = 40{,}342.88$
Round to the nearest whole virus.	$A = 40{,}343$
	The researcher will find 40,343 viruses.

Note:
Exercise:

Problem:

Another researcher at the Center for Disease Control and Prevention, Lisa, is studying the growth of a bacteria. She starts his experiment with 50 of the bacteria that grows at a rate of 15% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

Solution:

She will find 166 bacteria.

Note:

Exercise:

Problem:

Maria, a biologist is observing the growth pattern of a virus. She starts with 100 of the virus that grows at a rate of 10% per hour. She will check on the virus in 24 hours. How many viruses will she find?

Solution:

She will find 1,102 viruses.

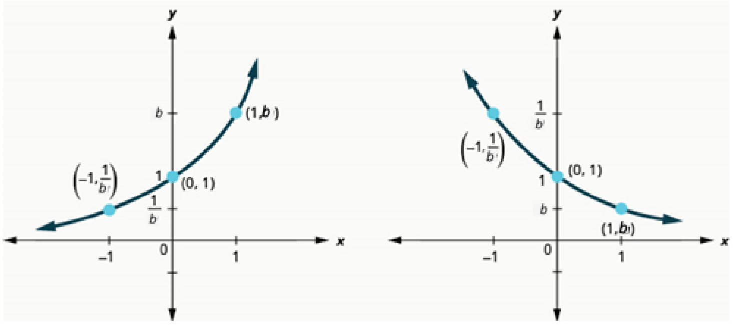
Note:
Access these online resources for additional instruction and practice with evaluating and graphing exponential functions.

- Graphing Exponential Functions
- Applications of Exponential Functions
- Continuously Compound Interest
- Radioactive Decay and Exponential Growth

Key Concepts

- **Properties of the Graph of $f(x) = b^x$:**

when $b > 1$		**when $0 < b < 1$**	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
x-intercept	none	x-intercept	none
y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
Contains	$(1, b),\ \left(-1, \frac{1}{b}\right)$	Contains	$(1, b),\ \left(-1, \frac{1}{b}\right)$
Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing



- **One-to-One Property of Exponential Equations:**
  For $b > 0$ and $b \neq 1$,
  Equation:

$$A = A_0 e^{rt}$$

- **Compound Interest:** For a principal, P, invested at an interest rate, r, (in decimal form) for t years, the new balance, A, is

$$\begin{array}{ll} A = P\left(1 + \frac{r}{n}\right)^{nt} & \text{when compounded } n \text{ times a year.} \\ A = Pe^{rt} & \text{when compounded continuously.} \end{array}$$

- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, r, (in decimal form) for a certain time t, the final amount,A, is $A = A_0 e^{rt}$.

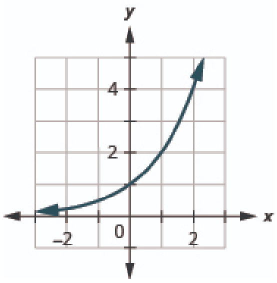
Practice Makes Perfect

Graph Exponential Functions

In the following exercises, graph each exponential function.
Exercise:

Problem: $f\left(x\right) = 2^x$

Solution:



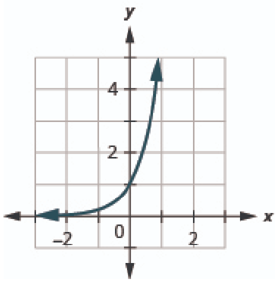
Exercise:

Problem: $g\left(x\right)=3^x$

Exercise:

Problem: $f\left(x\right)=6^x$

Solution:



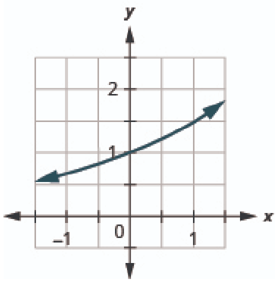
Exercise:

Problem: $g\left(x\right)=7^x$

Exercise:

Problem: $f\left(x\right)=(1.5)^x$

Solution:



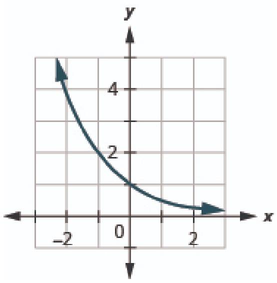
Exercise:

Problem: $g\left(x\right)=(2.5)^x$

Exercise:

Problem: $f\left(x\right)=\left(\frac{1}{2}\right)^x$

Solution:



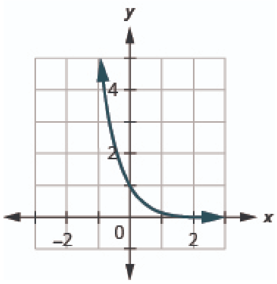
Exercise:

Problem: $g\left(x\right)=\left(\frac{1}{3}\right)^{x}$

Exercise:

Problem: $f\left(x\right)=\left(\frac{1}{6}\right)^{x}$

Solution:



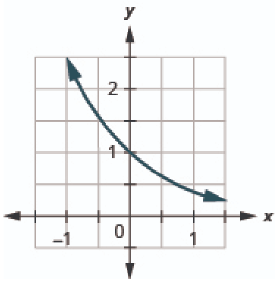
Exercise:

Problem: $g\left(x\right)=\left(\frac{1}{7}\right)^{x}$

Exercise:

Problem: $f\left(x\right)=\left(0.4\right)^{x}$

Solution:



Exercise:

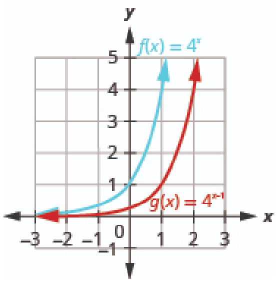
Problem: $g\left(x\right)=\left(0.6\right)^{x}$

In the following exercises, graph each function in the same coordinate system.

Exercise:

Problem: $f\left(x\right)=4^{x},\ g\left(x\right)=4^{x-1}$

Solution:



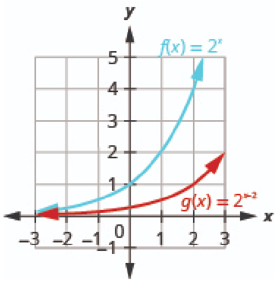
Exercise:

Problem: $f\left(x\right)=3^{x},\ g\left(x\right)=3^{x-1}$

Exercise:

Problem: $f\left(x\right)=2^{x},\ g\left(x\right)=2^{x-2}$

Solution:



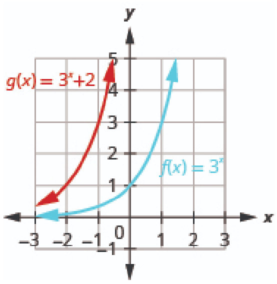
Exercise:

Problem: $f\left(x\right)=2^{x},\ g\left(x\right)=2^{x+2}$

Exercise:

Problem: $f\left(x\right)=3^{x},\ g\left(x\right)=3^{x}+2$

Solution:



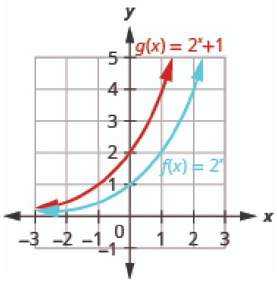
Exercise:

Problem: $f\left(x\right)=4^x,\ g\left(x\right)=4^x+2$

Exercise:

Problem: $f\left(x\right)=2^x,\ g\left(x\right)=2^x+1$

Solution:



Exercise:

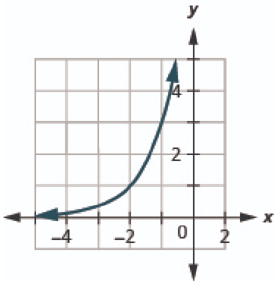
Problem: $f\left(x\right)=2^x,\ g\left(x\right)=2^x-1$

In the following exercises, graph each exponential function.

Exercise:

Problem: $f\left(x\right)=3^{x+2}$

Solution:



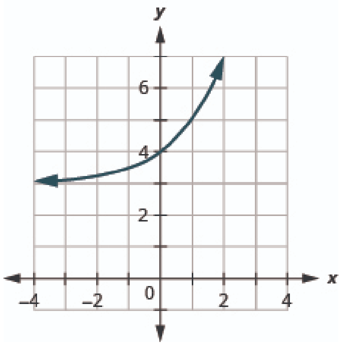
Exercise:

Problem: $f\left(x\right)=3^{x-2}$

Exercise:

Problem: $f\left(x\right)=2^{x}+3$

Solution:



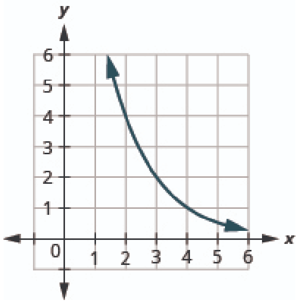
Exercise:

Problem: $f\left(x\right)=2^{x}-3$

Exercise:

Problem: $f\left(x\right)=\left(\frac{1}{2}\right)^{x-4}$

Solution:



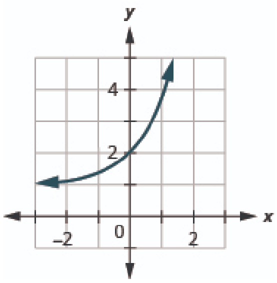
Exercise:

Problem: $f\left(x\right)=\left(\frac{1}{2}\right)^{x}-3$

Exercise:

Problem: $f\left(x\right)=e^{x}+1$

Solution:



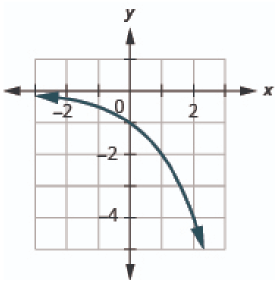
Exercise:

Problem: $f\left(x\right)=e^{x-2}$

Exercise:

Problem: $f\left(x\right)=-2^{x}$

Solution:



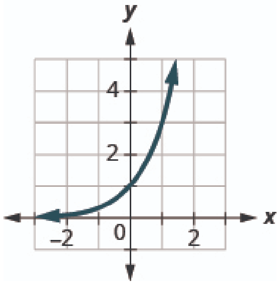
Exercise:

Problem: $f\left(x\right)=3^{x}$

In the following exercises, match the graphs to one of the following functions: ⓐ 2^{x} ⓑ 2^{x+1} ⓒ 2^{x-1} ⓓ $2^{x}+2$ ⓔ $2^{x}-2$ ⓕ 3^{x}

Exercise:

Problem:

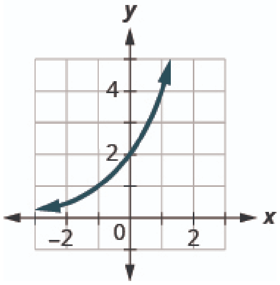


Solution:

ⓕ

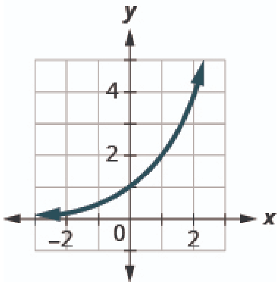
Exercise:

Problem:



Exercise:

Problem:

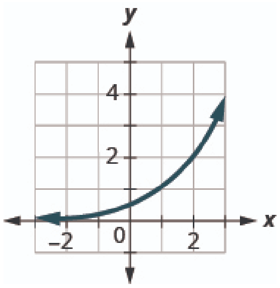


Solution:

ⓐ

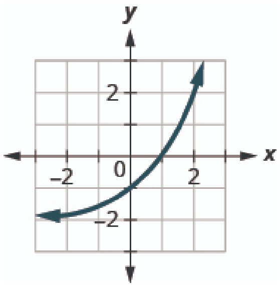
Exercise:

Problem:



Exercise:

Problem:

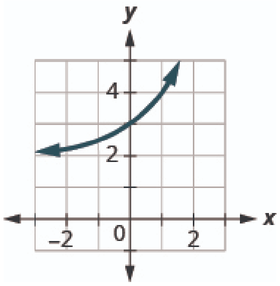


Solution:

ⓔ

Exercise:

Problem:



Use exponential models in applications

In the following exercises, use an exponential model to solve.

Exercise:

Problem:

Edgar accumulated $\$5,000$ in credit card debt. If the interest rate is 20% per year, and he does not make any payments for 2 years, how much will he owe on this debt in 2 years by each method of compounding?
ⓐ compound quarterly
ⓑ compound monthly
ⓒ compound continuously

Solution:

ⓐ $\$7,387.28$ ⓑ $\$7,434.57$ ⓒ $\$7,459.12$

Exercise:

Problem:

Cynthia invested $\$12,000$ in a savings account. If the interest rate is 6%, how much will be in the account in 10 years by each method of compounding?
ⓐ compound quarterly
ⓑ compound monthly
ⓒ compound continuously

Exercise:

Problem:

Rochelle deposits $\$5,000$ in an IRA. What will be the value of her investment in 25 years if the investment is earning 8% per year and is compounded continuously?

Solution:

$\$36,945.28$

Exercise:

Problem:

Nazerhy deposits $\$8,000$ in a certificate of deposit. The annual interest rate is 6% and the interest will be compounded quarterly. How much will the certificate be worth in 10 years?

Exercise:

Problem:

A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. He starts his experiment with 100 of the bacteria that grows at a rate of 6% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

Solution:

162 bacteria

Exercise:

Problem:

A biologist is observing the growth pattern of a virus. She starts with 50 of the virus that grows at a rate of 20% per hour. She will check on the virus in 24 hours. How many viruses will she find?

Exercise:

Problem:

In the last ten years the population of Indonesia has grown at a rate of 1.12% per year to 258,316,051. If this rate continues, what will be the population in 10 more years?

Solution:

288,929,825

Exercise:

Problem:

In the last ten years the population of Brazil has grown at a rate of 0.9% per year to 205,823,665. If this rate continues, what will be the population in 10 more years?

Writing Exercises

Exercise:

Problem:

Explain how you can distinguish between exponential functions and polynomial functions.

Solution:

Answers will vary.

Exercise:

Problem: Compare and contrast the graphs of $y=x^2$ and $y=2^x$.

Exercise:

Problem:

What happens to an exponential function as the values of x decreases? Will the graph ever cross the y-axis? Explain.

Solution:

Answers will vary.

Glossary

asymptote
A line which a graph of a function approaches closely but never touches.

exponential function
An exponential function, where $b>0$ and $b\neq 1$, is a function of the form $f\left(x\right)=b^x$.

natural base
The number e is defined as the value of $\left(1+\frac{1}{n}\right)^n$, as n gets larger and larger. We say, as n increases without bound, $e\approx 2.718281827...$

natural exponential function
The natural exponential function is an exponential function whose base is e: $f\left(x\right)=e^x$. The domain is $\left(-\infty,\infty\right)$ and the range is $\left(0,\infty\right)$.

Logarithmic Functions

In this section, you will:

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce)

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes[footnote]. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings, [footnote] like those shown in [link]. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a

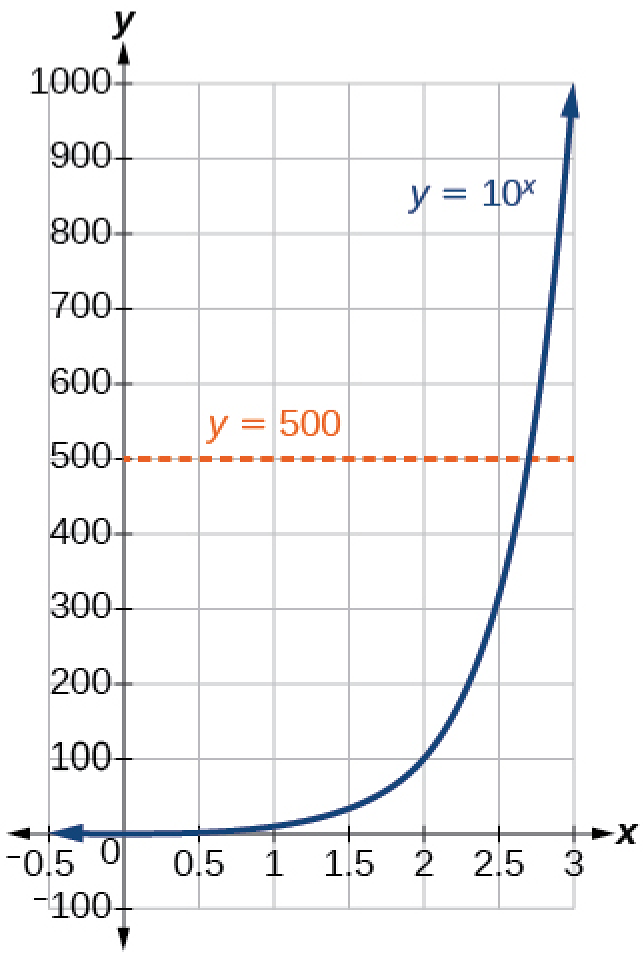
7.0 on the Richter Scale[footnote] whereas the Japanese earthquake registered a 9.0.[footnote]
http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/#summary. Accessed 3/4/2013.
http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/#summary. Accessed 3/4/2013.
http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/. Accessed 3/4/2013.
http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/#details. Accessed 3/4/2013.

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is $10^{8-4} = 10^4 = 10{,}000$ times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in [link], to better estimate the solution.



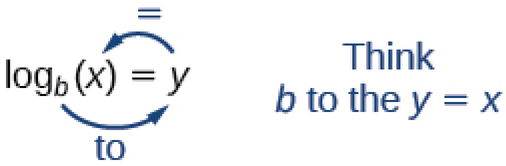
Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in [link] passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a logarithmic function of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

We read a logarithmic expression as, "The logarithm with base b of x is equal to y," or, simplified, "log base b of x is y." We can also say, "b raised to the power of y is x," because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as "log base 2 of 32 is 5."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:
Equation:

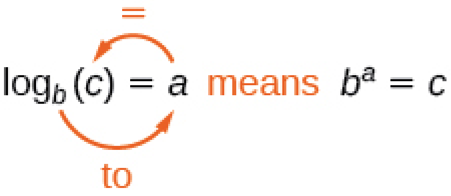
$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1$$

Note that the base b is always positive.



Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:



Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log_b(x)$ and $y = b^x$ are inverse functions.

Note:
Definition of the Logarithmic Function
A **logarithm** base b of a positive number x satisfies the following definition.

For $x > 0, b > 0, b \neq 1,$

Equation:

$$y = \log_b(x) \text{ is equivalent to } b^y = x$$

where,

- we read $\log_b(x)$ as, “the logarithm with base b of x” or the “log base b of $x.''$
- the logarithm y is the exponent to which b must be raised to get $x.$

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $(0, \infty).$
- the range of the logarithm function with base b is $(-\infty, \infty).$

Note:

Can we take the logarithm of a negative number?

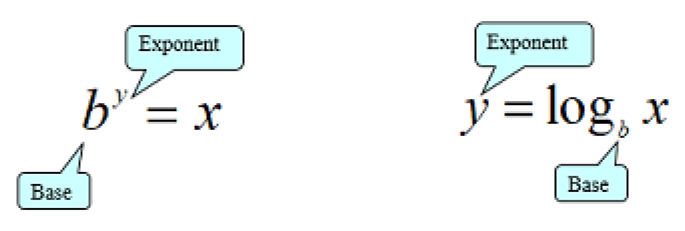
No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

Note:

Given an equation in logarithmic form $\log_b(x) = y$, convert it to exponential form.

1. Examine the equation $y = \log_b x$ and identify $b, y,$ and$x.$
2. Rewrite $\log_b x = y$ as $b^y = x.$

The following diagram may help you identify the base and exponent when converting between exponential and logarithmic forms of an equation:



Example:
Exercise:

Problem:
Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$
b. $\log_3(9) = 2$

Solution:

First, identify the values of b, y, and x. Then, write the equation in the form $b^y = x$.

a. $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$

Here, $b = 6$, $y = \frac{1}{2}$, and $x = \sqrt{6}$. Therefore, the equation $\log_6\left(\sqrt{6}\right) = \frac{1}{2}$ is equivalent to

$$6^{\frac{1}{2}} = \sqrt{6}.$$

b. $\log_3(9) = 2$

Here, $b=3, y=2,$ and $x=9$. Therefore, the equation $\log_3(9)=2$ is equivalent to $3^2=9$.

Note:
Exercise:

Problem:

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1{,}000{,}000)=6$
b. $\log_5(25)=2$

Solution:

a. $\log_{10}(1,000,000)=6$ is equivalent to $10^6=1,000,000$
b. $\log_5(25)=2$ is equivalent to $5^2=25$

Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b, exponent y, and output x. Then we write $y=\log_b(x)$.

Example:
Exercise:

Problem:
Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-4} = \frac{1}{10{,}000}$

Solution:

First, identify the values of $b, y, \text{and} x$. Then, write the equation in the form $y = \log_b(x)$.

a. $2^3 = 8$

   Here, $b = 2$, $y = 3$, and $x = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. $5^2 = 25$

   Here, $b = 5$, $y = 2$, and $x = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c. $10^{-4} = \frac{1}{10{,}000}$

   Here, $b = 10$, $y = -4$, and $x = \frac{1}{10{,}000}$. Therefore, the equation $10^{-4} = \frac{1}{10{,}000}$ is equivalent to $\log_{10}\left(\frac{1}{10{,}000}\right) = -4$.

Note:
Exercise:

Problem:

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$

b. $5^3 = 125$
c. $2^{-1} = \frac{1}{2}$

Solution:

a. $3^2 = 9$ is equivalent to $\log_3(9) = 2$
b. $5^3 = 125$ is equivalent to $\log_5(125) = 3$
c. $2^{-1} = \frac{1}{2}$ is equivalent to $\log_2(\frac{1}{2}) = -1$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, “To what exponent must 2 be raised in order to get 8?” Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, “To what exponent must 7 be raised in order to get 49?” We know $7^2 = 49$. Therefore, $\log_7 49 = 2$
- We ask, “To what exponent must 3 be raised in order to get 27?” We know $3^3 = 27$. Therefore, $\log_3 27 = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let’s evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, “To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$? ” We know $2^2 = 4$ and $3^2 = 9$,so $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$. Therefore, $\log_{\frac{2}{3}} \left(\frac{4}{9}\right) = 2$.

Note:

Given a logarithm of the form $y = \log_b(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of $b: \ b^y = x$.
2. Use previous knowledge of powers of b to identify y by asking, “To what exponent should b be raised in order to get x?”

Example:
Exercise:

Problem:
Solving Logarithms Mentally

Solve $y = \log_4(64)$ without using a calculator.

Solution:

First we rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, “To what exponent must 4 be raised in order to get 64?”

We know
Equation:

$$4^3 = 64$$

Therefore,
Equation:

$$\log_4(64) = 3$$

Note:
Exercise:

Problem: Solve $y = \log_{121}(11)$ without using a calculator.

Solution:

$\log_{121}(11) = \frac{1}{2}$ (recalling that $\sqrt{121} = (121)^{\frac{1}{2}} = 11$)

Example:
Exercise:

Problem:
Evaluating the Logarithm of a Reciprocal

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

Solution:

First we rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$. Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$?"

We know $3^3 = 27$, but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-n} = \frac{1}{b^n}$. We use this information to write
Equation:

$$\begin{aligned} 3^{-3} &= \frac{1}{3^3} \\ &= \frac{1}{27} \end{aligned}$$

Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

Note:
Exercise:

Problem: Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

Solution:

$\log_2\left(\frac{1}{32}\right) = -5$

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x).$ We call a base-10 logarithm a **common logarithm**. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

Note:
Definition of the Common Logarithm
A **common logarithm** is a logarithm with base $10.$ We write $\log_{10}(x)$ simply as $\log(x).$ The common logarithm of a positive number x satisfies the following definition.
For $x > 0,$
Equation:

$$y = \log(x) \text{ is equivalent to } 10^y = x$$

We read $\log(x)$ as, “the logarithm with base 10 of x ” or “log base 10 of $x.$ ”
The logarithm y is the exponent to which 10 must be raised to get $x.$

Example:
Exercise:

Problem:
Converting from Common Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log\left(\sqrt{10}\right) = \frac{1}{2}$
b. $\log(100) = 2$

Solution:

First, identify the values of $b, y, \text{and} x.$ Then, write the equation in the form $b^y = x.$

a. $\log\left(\sqrt{10}\right) = \frac{1}{2}$

Here, $b = 10, y = \frac{1}{2}$, and $x = \sqrt{10}.$ Therefore, the equation $\log\left(\sqrt{10}\right) = \frac{1}{2}$ is equivalent to

$$10^{\frac{1}{2}} = \sqrt{10}.$$

b. $\log(100) = 2$

Here, $b = 10, y = 2,$ and $x = 100.$ Therefore, the equation $\log(100) = 2$ is equivalent to $10^2 = 100.$

Note:
Given a common logarithm of the form $y = \log(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of $10: \ 10^y = x$.
2. Use previous knowledge of powers of 10 to identify y by asking, “To what exponent must 10 be raised in order to get x?”

Example:
Exercise:

Problem:
Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

Solution:

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, “To what exponent must 10 be raised in order to get 1000?” We know

Equation:

$$10^3 = 1000$$

Therefore, $\log(1000) = 3$.

Note:
Exercise:

Problem: Evaluate $y = \log(1{,}000{,}000)$.

Solution:

$\log(1,000,000) = 6$

Note:

Given a common logarithm with the form $y = \log(x)$,evaluate it using a calculator.

1. Press **[LOG]**.
2. Enter the value given for x, followed by **[ )]**.
3. Press **[ENTER]**.

Example:

Exercise:

Problem:

Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Solution:

- Press **[LOG]**.
- Enter 321, followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$.

Analysis

Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100)$ and $\log(1000)$. This gives us the following:

Equation:

$$\begin{array}{ccccc} 100 & < & 321 & < & 1000 \\ 2 & < & 2.5065 & < & 3 \end{array}$$

Note:
Exercise:

Problem:

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Solution:

$\log(123) \approx 2.0899$

Example:
Exercise:

Problem:
Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution:

We begin by rewriting the exponential equation in logarithmic form.
Equation:

$$\begin{aligned} 10^x &= 500 \\ \log(500) &= x \qquad \text{Use the definition of the common log.} \end{aligned}$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter 500,followed by **[ )]**.

- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699$.

The difference in magnitudes was about 2.699.

Note:
Exercise:

Problem:

The amount of energy released from one earthquake was $8{,}500$ times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution:

The difference in magnitudes was about 3.929.

Using Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has its own notation, $\ln(x)$.

Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln 1 = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

Note:

Definition of the Natural Logarithm

A **natural logarithm** is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

For $x > 0$,

Equation:

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read $\ln(x)$ as, “the logarithm with base e of x” or “the natural logarithm of x.”

The logarithm y is the exponent to which e must be raised to get x.

Example:

Exercise:

Problem:

Converting from Natural Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\ln(M) = \frac{1}{2}$
b. $\ln(95) = K$

Solution:

First, identify the values of $b, y,$ and x. Then, write the equation in the form $b^y = x$.

a. $\ln(M) = \frac{1}{2}$

Here, $b = e, y = \frac{1}{2}$, and $x =$ M. Therefore, the equation $\ln(M) = \frac{1}{2}$ is equivalent to

$$e^{\frac{1}{2}} = M.$$

b. $\ln(95) = K$

Here, $b = e, y = K,$ and $x = 95.$ Therefore, the equation $\ln(95) = K$ is equivalent to

$$e^K = 95.$$

Note:
Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator.

1. Press **[LN]**.
2. Enter the value given for x, followed by **[ )]**.
3. Press **[ENTER]**.

Example:
Exercise:

Problem:
Evaluating a Natural Logarithm Using a Calculator

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

Solution:

- Press **[LN]**.
- Enter 500,followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$

Note:
Exercise:

Problem: Evaluate $\ln(-500)$.

Solution:

It is not possible to take the logarithm of a negative number in the set of real numbers.

Note:
Access this online resource for additional instruction and practice with logarithms.

- Introduction to Logarithms

Key Equations

Definition of the logarithmic function	For $x>0, b>0, b\neq 1$, $y=\log_b(x)$ if and only if $b^y=x$.
Definition of the common logarithm	For $x>0$, $y=\log(x)$ if and only if $10^y=x$.
Definition of the natural logarithm	For $x>0$, $y=\ln(x)$ if and only if $e^y=x$.

Key Concepts

- The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.
- Logarithmic equations can be written in an equivalent exponential form, using the definition of a logarithm. See [link].
- Exponential equations can be written in their equivalent logarithmic form using the definition of a logarithm See [link].
- Logarithmic functions with base b can be evaluated mentally using previous knowledge of powers of $b.$ See [link] and [link].
- Common logarithms can be evaluated mentally using previous knowledge of powers of $10.$ See [link].
- When common logarithms cannot be evaluated mentally, a calculator can be used. See [link].
- Real-world exponential problems with base 10 can be rewritten as a common logarithm and then evaluated using a calculator. See [link].
- Natural logarithms can be evaluated using a calculator [link].

Section Exercises

Verbal

Exercise:

Problem:

What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b x = y$ for $b > 0, b \neq 1.$

Solution:

A logarithm is an exponent. Specifically, it is the exponent to which a base b is raised to produce a given value. In the expressions given, the base b has the same value. The exponent, y, in the expression b^y can also be written as the logarithm, $\log_b x$, and the value of x is the result of raising b to the power of y.

Exercise:

Problem:

How is the logarithmic function $f(x) = \log_b x$ related to the exponential function $g(x) = b^x$? What is the result of composing these two functions?

Exercise:

Problem:

How can the logarithmic equation $\log_b x = y$ be solved for x using the properties of exponents?

Solution:

Since the equation of a logarithm is equivalent to an exponential equation, the logarithm can be converted to the exponential equation $b^y = x$, and then properties of exponents can be applied to solve for x.

Exercise:

Problem:

Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

Exercise:

Problem:

Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

Solution:

The natural logarithm is a special case of the logarithm with base b in that the natural log always has base e. Rather than notating the natural logarithm as $\log_e(x)$, the notation used is $\ln(x)$.

Algebraic

For the following exercises, rewrite each equation in exponential form.

Exercise:

Problem: $\log_4(q) = m$

Exercise:

Problem: $\log_a(b) = c$

Solution:

$a^c = b$

Exercise:

Problem: $\log_{16}(y) = x$

Exercise:

Problem: $\log_x(64) = y$

Solution:

$x^y = 64$

Exercise:

Problem: $\log_y(x) = -11$

Exercise:

Problem: $\log_{15}(a) = b$

Solution:

$15^b = a$

Exercise:

Problem: $\log_{y}(137) = x$

Exercise:

Problem: $\log_{13}(142) = a$

Solution:

$13^{a} = 142$

Exercise:

Problem: $\log(v) = t$

Exercise:

Problem: $\ln(w) = n$

Solution:

$e^{n} = w$

For the following exercises, rewrite each equation in logarithmic form.

Exercise:

Problem: $4^{x} = y$

Exercise:

Problem: $c^{d} = k$

Solution:

$\log_{c}(k) = d$

Exercise:

Problem: $m^{-7} = n$

Exercise:

Problem: $19^{x} = y$

Solution:

$\log_{19} y = x$

Exercise:

Problem: $x^{-\frac{10}{13}} = y$

Exercise:

Problem: $n^{4} = 103$

Solution:

$\log_{n}(103) = 4$

Exercise:

Problem: $\left(\frac{7}{5}\right)^{m} = n$

Exercise:

Problem: $y^{x} = \frac{39}{100}$

Solution:

$\log_{y}\left(\frac{39}{100}\right) = x$

Exercise:

Problem: $10^{a} = b$

Exercise:

Problem: $e^k = h$

Solution:

$\ln(h) = k$

For the following exercises, use the definition of common and natural logarithms to simplify.

Exercise:

Problem: $\log(100^8)$

Exercise:

Problem: $\log(10^{32})$

Solution:

32

Exercise:

Problem: $2\log(.0001)$

Exercise:

Problem: $\ln(e^{1.06})$

Solution:

1.06

Numeric

For the following exercises, evaluate the base b logarithmic expression without using a calculator.

Exercise:

Problem: $\log_3\left(\frac{1}{27}\right)$

Exercise:

Problem: $\log_6(\sqrt{6})$

Solution:

$\frac{1}{2}$

Exercise:

Problem: $\log_2\left(\frac{1}{8}\right)+4$

Exercise:

Problem: $6\log_8(4)$

Solution:

4

For the following exercises, evaluate the common logarithmic expression without using a calculator.

Exercise:

Problem: $\log(10,000)$

Exercise:

Problem: $\log(0.001)$

Solution:

-3

Exercise:

Problem: $\log(1) + 7$

Exercise:

Problem: $2\log(100^{-3})$

Solution:

-12

For the following exercises, evaluate the natural logarithmic expression without using a calculator.

Exercise:

Problem: $\ln(e^{\frac{1}{3}})$

Exercise:

Problem: $\ln(1)$

Solution:

0

Exercise:

Problem: $\ln(e^{-0.225}) - 3$

Exercise:

Problem: $25\ln(e^{\frac{2}{5}})$

Solution:

10

Technology

For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.

Exercise:

Problem: $\log(0.04)$

Exercise:

Problem: $\ln(15)$

Solution:

2.708

Exercise:

Problem: $\ln\left(\frac{4}{5}\right)$

Exercise:

Problem: $\log(\sqrt{2})$

Solution:

0.151

Exercise:

Problem: $\ln(\sqrt{2})$

Extensions

Exercise:

Problem:

Is $x = 0$ in the domain of the function $f(x) = \log(x)$? If so, what is the value of the function when $x = 0$? Verify the result.

Solution:

No, the function has no defined value for $x = 0$. To verify, suppose $x = 0$ is in the domain of the function $f(x) = \log(x)$. Then there is some number n such that $n = \log(0)$. Rewriting as an exponential equation gives: $10^n = 0$, which is impossible since no such real number n exists. Therefore, $x = 0$ is *not* the domain of the function $f(x) = \log(x)$.

Exercise:

Problem:

Is $f(x) = 0$ in the range of the function $f(x) = \log(x)$? If so, for what value of x? Verify the result.

Exercise:

Problem:

Is there a number x such that $\ln x = 2$? If so, what is that number? Verify the result.

Solution:

Yes. Suppose there exists a real number x such that $\ln x = 2$. Rewriting as an exponential equation gives $x = e^2$, which is a real number. To verify, let $x = e^2$. Then, by definition, $\ln(x) = \ln\left(e^2\right) = 2$.

Real-World Applications

Exercise:

Problem:

The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$, where f is the “f-stop” setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?

Exercise:

Problem:

Refer to the previous exercise. Suppose the light meter on a camera indicates an EI of -2, and the desired exposure time is 16 seconds. What should the f-stop setting be?

Solution:

2

Exercise:

Problem:

The intensity levels I of two earthquakes measured on a seismograph can be compared by the formula $\log\frac{I_1}{I_2} = M_1 - M_2$ where M is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0.[footnote] How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number.
http://earthquake.usgs.gov/earthquakes/world/historical.php. Accessed 3/4/2014.

Glossary

common logarithm

the exponent to which 10 must be raised to get x; $\log_{10}(x)$ is written simply as $\log(x)$.

logarithm

the exponent to which b must be raised to get x; written $y=\log_b(x)$

natural logarithm

the exponent to which the number e must be raised to get x; $\log_e(x)$ is written as $\ln(x)$.

Using the Change-of-Base Formula to Evaluate Logarithms
By the end of this section, you will be able to:

- Use the Change-of-Base Formula to evaluate logarithms with bases other than 10 or e.

Note:
Before you get started, take this readiness quiz.

1. Evaluate: ⓐ a^0 ⓑ a^1.
   If you missed this problem, review [link].
2. Write with a rational exponent: $\sqrt[3]{x^2y}$.
   If you missed this problem, review [link].
3. Round to three decimal places: 2.5646415.
   If you missed this problem, review [link].

Use the Change-of-Base Formula

To evaluate a logarithm with any base other than 10 or e, we can use the Change-of-Base Formula. Understanding why this formula works is beyond the scope of this course, but may be covered, along with many properties of logarithms, in your next college algebra course.

The Change-of-Base Formula introduces a new base b. This can be any base b we want where $b > 0, b \neq 1$. Because our calculators have keys for logarithms base 10 and base e, we will rewrite the Change-of-Base Formula with the new base as 10 or e.

Note:
Change-of-Base Formula
For any logarithmic base b, where $b > 0, b \neq 1$, and $M > 0$,

Equation:

$$\log_b M = \frac{\log M}{\log b} \quad \text{or} \quad \log_b M = \frac{\ln M}{\ln b}$$

$$\text{new base 10} \qquad\qquad \text{new base e}$$

As you can see, you can choose to use either the "log" button or the "ln" button on your calculator to evaluate (find the value of) a logarithm with a base other than 10 or e. The important thing to remember is that you divide the log of the input by the log of the base and that you must use the same log button for both.

When we use a calculator to find the logarithm value, we usually round to three decimal places. This gives us an approximate value and so we use the approximately equal symbol $(\approx)$.

Example:
Exercise:

Problem: Rounding to three decimal places, approximate $\log_4 35$.

Solution:

	$\log_4 35$
Use the Change-	

of-Base Formula.	$\log_b M = \frac{\log M}{\log b}$ or $\log_b M = \frac{\ln M}{\ln b}$
Identify b and M. Choose log base 10 to evaluate the log.	$\log_4 35 = \frac{\log 35}{\log 4}$
Enter the expression $\frac{\log 35}{\log 4}$ in the calculator using the log button for base 10. Round to three decimal places.	$\log_4 35 \approx 2.565$

Note:
Exercise:

Problem: Rounding to three decimal places, approximate $\log_3 42$.

Solution:

3.402

Note:
Exercise:

Problem: Rounding to three decimal places, approximate $\log_5 46$.

Solution:

2.379

Note:
Access these online resources for additional instruction and practice with using the change-of-base formula.

- Change of Base

Key Concepts

- **Change-of-Base Formula**

  For any logarithmic base b, where $b > 0, b \neq 1$, and $M > 0$,
  Equation:

  $$\log_b M = \frac{\log M}{\log b} \quad \text{or} \quad \log_b M = \frac{\ln M}{\ln b}$$

  $$\text{new base 10} \qquad \text{new base e}$$

Practice Makes Perfect

Use the Change-of-Base Formula

In the following exercises, use the Change-of-Base Formula, rounding to three decimal places, to approximate each logarithm.

Exercise:

Problem: $\log_3 42$

Exercise:

Problem: $\log_5 78$

Solution:

2.707

Exercise:

Problem: $\log_{12} 87$

Exercise:

Problem: $\log_{15} 93$

Solution:

1.674

Exercise:

Problem: $\log_{\sqrt{2}} 17$

Exercise:

Problem: $\log_{\sqrt{3}} 21$

Solution:

5.542

Writing Exercises

Exercise:

Problem:

Explain how to find the value of $\log_7 15$ using your calculator.

Solution:

Answers will vary.